

Two-Dimensional Turbulence of Dilute Polymer Solutions

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We investigate theoretically and numerically the effect of polymer additives on two-dimensional turbulence by means of a viscoelastic model. We provide compelling evidence that, at vanishingly small concentrations, such that the polymers are passively transported, the probability distribution of polymer elongation has a power law tail: Its slope is related to the statistics of finite-time Lyapunov exponents of the flow, in quantitative agreement with theoretical predictions. We show that at finite concentrations and sufficiently large elasticity the polymers react on the flow with manifold consequences: Velocity fluctuations are drastically depleted, as observed in soap film experiments; the velocity statistics becomes strongly intermittent; the distribution of finite-time Lyapunov exponents shifts to lower values, signaling the reduction of Lagrangian chaos.

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Since the discovery of the conspicuous drag reduction obtained by dissolving minute amounts of long chain molecules in a liquid, turbulence of dilute polymer solutions has attracted a lot of attention in view of its industrial applications (see, e.g., Refs. [1–3]). The fluid mechanics of polymer solutions is appropriately described by viscoelastic models that are able to reproduce the rheological behavior and many other experimental observations [4]. For example, it has been shown by Sureshkumar *et al.* that the drag reduction effect can be captured by numerical simulations of the channel flow of viscoelastic fluids [5]. Although the parameters used in those simulations do not match the experimental ones, the qualitative agreement is remarkable, and all the hallmarks of the turbulent flow of polymer solutions are recovered in numerical experiments.

Following this premise, it is natural to ask whether a two-dimensional viscoelastic model can reproduce the recent results by Amarouchene and Kellay [6] showing that the turbulent flow of soap films is spectacularly affected by polymer additives (see also Refs. [7,8]). Here we show that this is indeed the case, and that the suppression of large-scale velocity fluctuations observed experimentally has a simple theoretical explanation. However, the influence of polymers is not limited to the depletion of mean square velocity, which is a genuinely two-dimensional effect. In the viscoelastic case, we observe a strong intermittency, with exponential tails of the velocity probability density. As for the Lagrangian statistics, we show that the values of finite-time Lyapunov exponents lower significantly upon polymer addition, which therefore reduces the chaoticity of the flow. These effects are expected to be independent of the space dimensionality, and thus relevant to three-dimensional turbulence as well.

We also investigate the limit of vanishingly small polymer concentrations, in which the polymer molecules have no influence on the advecting flow. In this case, the

velocity field evolves according to the two-dimensional Navier-Stokes equation with friction, and is therefore smooth at scales smaller than the injection length scale [9,10]. For passive polymers, space dimensionality plays only a minor role, and our system is an instance of a generic random smooth flow to which the theory of passive polymers developed by Chertkov [11] and Balkovsky *et al.* [12,13] applies. We check this theory against our numerical results, and find an excellent quantitative agreement.

To describe the dynamics of a dilute polymer solution, we adopt the linear viscoelastic model (Oldroyd-B),

$$\partial_t \mathbf{u} + (\mathbf{u} \cdot \nabla) \mathbf{u} = -\nabla p + \nu \Delta \mathbf{u} + \frac{2\eta\nu}{\tau} \nabla \cdot \boldsymbol{\sigma} - \alpha \mathbf{u} + \mathbf{f}, \quad (1)$$

$$\partial_t \boldsymbol{\sigma} + (\mathbf{u} \cdot \nabla) \boldsymbol{\sigma} = (\nabla \mathbf{u})^T \cdot \boldsymbol{\sigma} + \boldsymbol{\sigma} \cdot (\nabla \mathbf{u}) - 2 \frac{(\boldsymbol{\sigma} - I)}{\tau}. \quad (2)$$

The velocity field \mathbf{u} is incompressible, the symmetric matrix $\boldsymbol{\sigma}$ is the conformation tensor of polymer molecules, and its trace $\text{tr} \boldsymbol{\sigma}$ is a measure of their elongation [14]. The parameter τ is the (slowest) polymer relaxation time. The energy source \mathbf{f} is a large-scale random, zero-mean, statistically homogeneous and isotropic, solenoidal vector field. The pressure term $-\nabla p$ ensures incompressibility of the velocity field. The matrix of velocity gradients is defined as $(\nabla \mathbf{u})_{ij} = \partial_i u_j$ and I is the unit tensor. The solvent viscosity is denoted by ν and η is the zero-shear contribution of polymers to the total solution viscosity $\nu_t = \nu(1 + \eta)$. The dissipative term $-\alpha \mathbf{u}$ models the mechanical friction between the soap film and the surrounding air [15], and plays a prominent role in the energy budget of Newtonian two-dimensional turbulence [16]. It should be remarked that a model that describes more accurately the polymer dynamics is the FENE-P

model, which accounts for the nonlinear character of polymer elasticity, culminating in a finite molecular extensibility [4]. Although here we limit ourselves to the linear case because it allows a simpler theoretical treatment, our conclusions apply to the nonlinear case as well, provided that the maximal polymer elongation is very large compared to the equilibrium length.

Passive polymers.—The effect of polymer concentration n is included in Eq. (1) through the parameter $\eta \propto n$. In the limit $\eta = 0$ polymers are passively transported and stretched by Newtonian two-dimensional turbulence. The flow is driven at the largest scales and develops an enstrophy cascade towards the small scales, while the inverse energy flux is immediately halted by friction. The ensuing velocity field is therefore everywhere smooth. We briefly recall that according to Refs. [11–13] the statistics of stretched polymers in random smooth flows is expected to depend critically on the value of the Weissenberg number, here defined as $Wi = \lambda_N \tau$, where λ_N is the maximum Lyapunov exponent of the Newtonian flow. At $Wi < 1$, the polymer molecules spend most of the time in a coiled state, and stretch occasionally by a considerable amount. The theory predicts a power law tail for the probability density function of $\text{tr } \boldsymbol{\sigma}$, i.e., the square polymer elongation

$$p(\text{tr } \boldsymbol{\sigma}) \sim (\text{tr } \boldsymbol{\sigma})^{-1-q} \quad \text{for } \text{tr } \boldsymbol{\sigma} \gg \text{tr } l. \quad (3)$$

The exponent q is related to the probability of finite-time Lyapunov exponents $P(\gamma, t) \propto \exp[-tS(\gamma)]$ via the equation $L_{2q} = 2q/\tau$, where $L_{2q} = \max_{\gamma}[2q\gamma - S(\gamma)]$ is the generalized Lyapunov exponent of order $2q$, and $S(\gamma)$ is the Cramér rate function (see, e.g., Ref. [17]). The convexity of $S(\gamma)$ ensures the positivity of q for $Wi < 1$. Since the distribution of polymer elongations is not accessible experimentally, in order to validate the theory it is necessary to resort to numerical simulations. Eckhardt *et al.* in Ref. [18] have given the first evidence of a power law tail for the probability distribution function of polymer elongation in three-dimensional shear turbulence. As shown in Fig. 1, in our two-dimensional simulations we observe a neat power law as well. In order to check whether the observed exponent coincides with the predicted one, we have also performed direct numerical simulations of particle trajectories, and measured the probability distribution of finite-time Lyapunov exponents, thereby obtaining the expected q . The numerical result is in close agreement with theory. As the Weissenberg number exceeds unity, the probability distribution of the conformation tensor becomes unstationary and all moments $\langle (\text{tr } \boldsymbol{\sigma})^n \rangle$ grow exponentially in time. This “coil-stretch” transition signals the breakdown of linear passive theory. Accounting for the nonlinear elastic modulus of polymer molecules allows one to recover a stationary statistics and to develop a consistent theory of passive polymers at all Weissenberg numbers [20]. In the

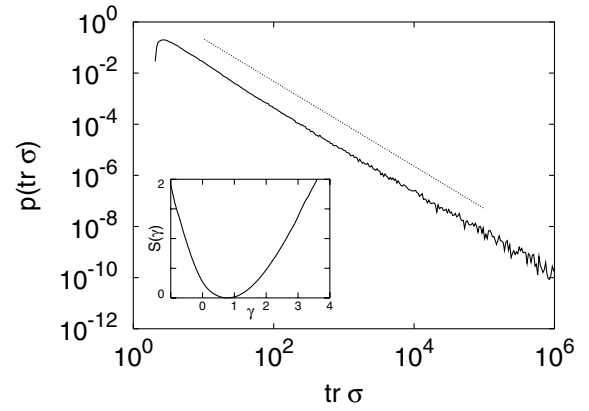


FIG. 1. Power law tail of the probability density function of polymer square elongation, in the passive case $\eta = 0$. The Weissenberg number is $Wi = 0.4$, quite below the coil-stretch transition. The power law $(\text{tr } \boldsymbol{\sigma})^{-1-q}$ with the value $q = 0.66$ (numerically obtained from the relation $L_{2q} = 2q/\tau$) is drawn for comparison. In the inset is the corresponding Cramér function $S(\gamma)$. Its minimum is $S(\lambda_N) = 0$, with $\lambda_N \approx 0.8$. Data have been obtained by direct numerical simulation of the equation of the conformation tensor (2) by a Lagrangian method (see, e.g., [19] and references therein), while Eq. (1) has been solved in a doubly periodic box by a pseudospectral code at resolution 256^2 . The velocity field is driven by a Gaussian, homogeneous, isotropic δ correlated in time forcing, with correlation length $L \approx 4$. The Reynolds number is $Re = u_{\text{rms}}L/\nu \approx 4000$.

following we do not pursue that approach, but we rather focus on a different mechanism that limits polymer elongation: the feedback of polymers on the advecting flow.

Active polymers.—When $\eta > 0$, polymers can affect significantly the velocity dynamics, provided that they are sufficiently elongated—a condition that is met at $Wi > 1$. This strong feedback regime is characterized in two dimensions by a suppression of large-scale velocity fluctuations (see Fig. 2), an effect first observed in soap film experiments [6]. In Fig. 3, we present the time evolution of the total kinetic energy of the system, showing that after polymer injection a drastic depletion of kinetic energy occurs. This should be contrasted with the three-dimensional case where, on the opposite, velocity fluctuations are larger in the viscoelastic case than in the Newtonian one [22].

The suppression of velocity fluctuations by polymer additives in two-dimensional turbulence can be easily explained in the context of the randomly driven viscoelastic model (1) and (2). Indeed, the average kinetic energy balance in the statistically stationary state reads

$$F = \epsilon + \frac{2\eta\nu}{\tau^2} (\langle \text{tr } \boldsymbol{\sigma} \rangle - \text{tr } l) + \alpha \langle |\mathbf{u}|^2 \rangle, \quad (4)$$

where $\epsilon = \nu \langle |\nabla \mathbf{u}|^2 \rangle$ is the viscous dissipation and F is the average energy input, which is flow-independent for a Gaussian, δ -correlated random forcing \mathbf{f} . To obtain

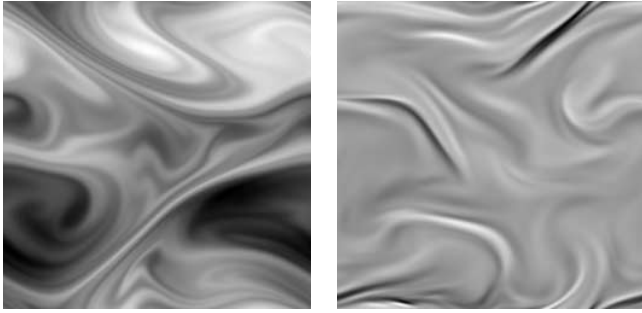


FIG. 2. Snapshots of the vorticity field $\nabla \times \mathbf{u}$ in the Newtonian (left) and in the viscoelastic case with strong feedback (right). Notice the suppression of large-scale structures in the latter case. The fields are obtained by a fully dealiased pseudospectral simulation of Eqs. (1) and (2) at resolution 256^2 . The viscosity is $\nu = 1.5 \times 10^{-3}$, $\eta = 0.2$, the relaxation time is $\tau = 4$, and the energy input is $F = 3.5$. As customary, an artificial stress-diffusivity term is added to Eq. (2) to prevent numerical instabilities [21]. The corresponding Schmidt number is $Sc = 0.25$.

Eq. (4), we multiply Eq. (1) by \mathbf{u} , add to it the trace of Eq. (2) times $\eta\nu/\tau$, and average over space and time. Since in two dimensions kinetic energy flows towards large scales, it is mainly drained by friction, and viscous dissipation is vanishingly small in the limit of very large Reynolds numbers [16]. Neglecting ϵ and observing that in the Newtonian case ($\eta = 0$) the balance (4) yields $F = \alpha \langle |\mathbf{u}|^2 \rangle_N$, we obtain

$$\langle |\mathbf{u}|^2 \rangle = \langle |\mathbf{u}|^2 \rangle_N - \frac{2\eta\nu}{\alpha\tau^2} (\langle \text{tr } \boldsymbol{\sigma} \rangle - \text{tr } I). \quad (5)$$

As a consequence of incompressibility and chaoticity of the flow, it can be shown from Eq. (2) that $\text{tr } \boldsymbol{\sigma} \geq \text{tr } I$, and we finally have $\langle |\mathbf{u}|^2 \rangle \leq \langle |\mathbf{u}|^2 \rangle_N$, in agreement with numerical results. This simple energy balance argument can be generalized to nonlinear elastic models. As viscosity tends to zero, the average polymer elongation increases so

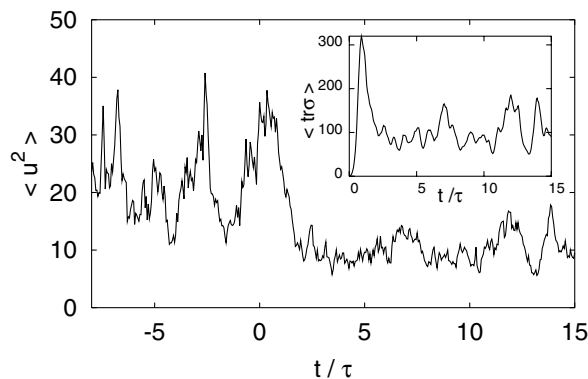


FIG. 3. Dilute polymers reduce the level of velocity fluctuations $\int |\mathbf{u}(\mathbf{x}, t)|^2 dx$. Polymers are introduced in the flow at $t = 0$. In the inset, the mean square elongation $\int \text{tr } \boldsymbol{\sigma}(\mathbf{x}, t) dx$ as a function of time.

as to compensate for the factor ν in Eq. (5), resulting in a finite effect also in the infinite Re limit. Since energy is essentially dissipated by linear friction, the depletion of $\langle |\mathbf{u}|^2 \rangle$ entails immediately the reduction of energy dissipation. The main difference between two-dimensional “friction reduction” and three-dimensional drag reduction resides in the length scales involved in the energy drain—large scales in 2D vs small scales in 3D.

The effect of polymer additives cannot be merely represented by a rescaling of velocity fluctuations by a given factor. In Fig. 4, we show the probability distribution of a velocity component, u_x . The choice of the x direction is immaterial by virtue of statistical isotropy. In the Newtonian case, the distribution is remarkably close to the sub-Gaussian density $N \exp(-c|u_x|^3)$ stemming from the balance between forcing and nonlinear terms in the Navier-Stokes equation, in agreement with the prediction by Falkovich and Lebedev [23]. On the contrary, the distribution in the viscoelastic case is markedly super-Gaussian, with approximately exponential tails. This strong intermittency in the velocity dynamics is due to the alternation of quiescent low-velocity phases ruled by polymer feedback and bursting events where inertial nonlinearities take over.

Dilute polymers also alter significantly the distribution of finite-time Lyapunov exponents $P(\gamma, t)$. In Fig. 5, the Cramér rate function $S(\gamma) \propto t^{-1} \ln P(\gamma, t)$ is shown for the Newtonian and for the viscoelastic case. Since in the former situation the Lyapunov exponent λ_N is greater than $1/\tau$, were the polymers passive all moments of elongation would grow exponentially fast. However, the feedback can damp stretching so effectively that after polymer addition λ lies below $1/\tau$. This implies a strong reduction of Lagrangian chaos and a decreased mixing

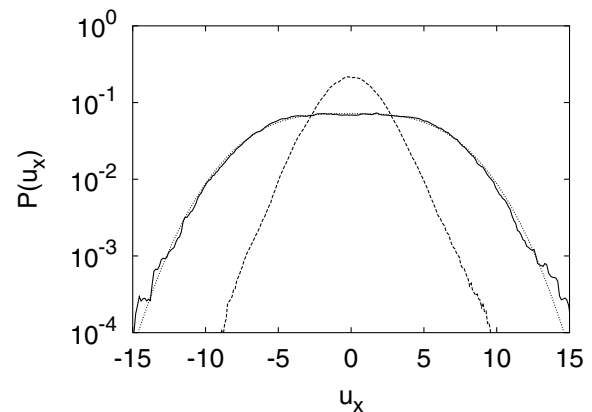


FIG. 4. Intermittency of velocity fluctuations induced by polymer additives. The probability density function $P(u_x)$ of the velocity component u_x for the Newtonian (solid line) and for the viscoelastic case with strong feedback (dashed line). Same parameters as in Fig. 2. Also shown is the distribution $\Gamma(2/3)3^{3/2} \exp(-c|u_x|^3)/(4\pi c)$ with $c = 2.1 \times 10^{-3}$ (dotted line).

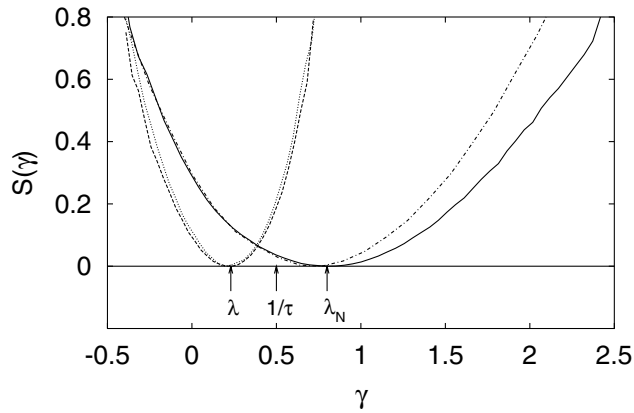


FIG. 5. Finite-time Lyapunov exponents decrease in the presence of polymers. The Cramér rate function $S(\gamma)$ for the Newtonian (solid line) and for the viscoelastic case with strong feedback ($Wi = \lambda_N \tau = 1.6$, dashed line). Viscosity $\nu = 6 \times 10^{-3}$, relaxation time $\tau = 2$, $\eta = 0.2$ (dashed line), $\eta = 2$ (dotted line). For the sake of completeness, we also show $S(\gamma)$ for a mild feedback case ($Wi = 0.4$, $\eta = 0.2$, dash-dotted line). In the latter case, the Lyapunov exponent is practically identical to the Newtonian value, and polymers affect only the right tail of $S(\gamma)$ reducing appreciably the probability of large stretching events $\gamma \gg \lambda_N$.

efficiency. The lowering of $\lambda\tau$ below unity would seem to contradict the statement that strong feedback takes place only at $Wi > 1$. Actually there is no inconsistency, since the critical value $Wi = 1$ holds for passive polymers only. For active polymers, the presence of correlations between the conformation tensor and the stretching exponents can indeed lower significantly the threshold. For a discussion of the differences between active and passive transport, see Ref. [24].

Finally, we discuss the influence of polymer concentration on the properties of the flow. As shown in Fig. 5, the Cramér functions at two very different values of η are practically indistinguishable. The level of velocity fluctuations (not shown) appears to be independent of concentration as well. This property follows from the viscoelastic equations assuming that the term $\propto 1/\tau$ in Eq. (2) can be neglected if polymers are substantially stretched. In that case the dynamics of the conformation tensor is invariant under rescaling by a constant factor, allowing to absorb η in the definition of σ , and making the velocity dynamics independent of concentration. This observation poses an interesting question: Can there be a concentration-dependent onset of friction or drag reduction within a *linear* viscoelastic model? Since polymers are increasingly stretched as $\eta \rightarrow 0$, there are two alternatives: Either this is a singular limit, and the passive case is not recovered but for η strictly equal to zero, or the feedback is not uniquely ruled by polymer elongation, and there are other relevant mechanisms for polymer activity, e.g., via the creation of strong gradients of the

conformation tensor. The limit of vanishingly small yet finite η is very demanding at the computational level and its investigation will require further numerical work.

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