# BRIEF COMMUNICATIONS

*The purpose of this Brief Communications section is to present important research results of more limited scope than regular articles appearing in Physics of Fluids. Submission of material of a peripheral or cursory nature is strongly discouraged. Brief Communications cannot exceed four printed pages in length, including space allowed for title, figures, tables, references, and an abstract limited to about 100 words.*

# Predictability of the inverse energy cascade in 2D turbulence

G. Boffetta and S. Musacchio

*Dipartimento di Fisica Generale, Università di Torino, Via Pietro Giuria 1, 10125 Torino, Italy and INFM Sezione di Torino Università, Torino, Italy*

The predictability problem in the inverse energy cascade of two-dimensional turbulence is addressed by means of high resolution direct numerical simulations. The growth rate as a function of the error level is determined by means of a finite size extension of the Lyapunov exponent. For errors within the inertial range, the linear growth of the error energy, predicted by dimensional argument, is verified with great accuracy. Our numerical findings quantitatively confirm the results of the classical TFM closure approximation. © *2001 American Institute of Physics.*
[DOI: 10.1063/1.1350877]

Unpredictability is an essential property of turbulent flows. Turbulence is characterized by a large number of degrees of freedom interacting with nonlinear dynamics. Thus turbulence is chaotic (and hence unpredictable), but the standard approach of dynamical system theory is not sufficient to characterize predictability in turbulence.[1]

In fully developed turbulence, the maximum Lyapunov exponent diverges, with the Reynolds number thus being very large for typical turbulent flows.[2] Nevertheless, a large value of the Lyapunov exponent does not imply automatically short time predictability. A familiar example is the atmosphere dynamics: convective motions in the atmosphere make the small scale features unpredictable after 1 h or less, but large scale dynamics can be predicted for several days, as it is demonstrated by weather forecasting. This effect, which can be called ''strong chaos with weak butterfly effect,'' arises in systems possessing many characteristic scales and times. From this point of view, turbulence probably represents the example most extensively studied.

The first attempts at the study of predictability in turbulence date back to the pioneering work of Lorenz[1] and to the Kraichnan and Leith papers.[3,4] On the basis of closure approximations, it was possible to obtain quantitative predictions on the evolution of the error in different turbulent situations, both in two and three dimensions.

A more recent approach to the problem is based on dynamical system theory. Chaotic properties and predictability of turbulent flow have been extensively investigated in simplified models of turbulence, called shell models, with particular emphasis on the relations with intermittency.[5–9] Because predictability experiments in fully developed turbulence are numerically rather expensive, a similar study on direct numerical simulations of Navier–Stokes equations is still lacking.

In this communication we address the predictability problem for two-dimensional turbulence by means of high resolution direct numerical simulations. Turbulence is generated in the inverse cascade regime where a robust energy cascade is observed.[10] The absence of intermittency corrections makes the problem simpler than in the three-dimensional case: velocity statistics (energy spectrum, structure functions) are found to be in close agreement with self-similar theory à la Kolmogorov.

The model equation is the two-dimensional Navier–Stokes equation written for the scalar vorticity $\omega(\mathbf{r},t) = -\Delta\psi(\mathbf{r},t)$ with generalized dissipation and linear friction

$$\partial_t\omega + J(\omega,\psi) = (-1)^{p+1}\nu_p\Delta^p\omega - \alpha\omega - f, \qquad (1)$$

where $J$ represents the Jacobian with the stream function $\psi$ and the velocity is $\mathbf{u} = (\partial_y\psi, -\partial_x\psi)$. $p$ is the order of the dissipation; $p=1$ for ordinary dissipation, $p>1$ for hyperviscosity. As it is customary in numerical simulations, we use hyperviscous dissipation ($p=8$) in order to extend the inertial range. Although this can affect the small scale features of the vorticity field,[11] in our simulations dissipation is not involved in the cascade and has simply the role of removing entropy at small scales. The friction term in (1) removes energy at large scales: it is necessary in order to avoid Bose–Einstein condensation on the gravest mode[12] and to obtain a stationary state. Energy is injected into the system by a random forcing $\delta$-correlated in time which is active on a shell of wavenumbers around $k_f$ only. Numerical integration of (1) is performed by a standard pseudo-spectral
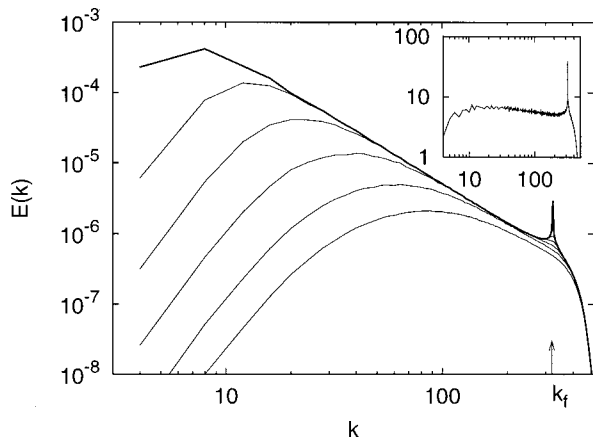
FIG. 1. Stationary energy spectrum $E(k)$ (thick line) and error spectrum $E_\Delta(k,t)$ at time $t=0.1, 0.2, 0.4, 0.8, 1.6$. $k_f=320$ is the forcing wavenumber. In the inset we plot the compensated spectrum $\epsilon^{-2/3}k^{5/3}E(k)$.



FIG. 2. Average error energy $\langle E_\Delta(t)\rangle$ growth. Dashed line represents closure prediction (5), dotted line is the saturation value $E$. The initial exponential growth is emphasized by the lin-log plot in the inset.

code fully dealiased with second-order Adams–Bashforth time stepping on a doubly periodic square domain with resolution $N=1024$.

Stationary turbulent flow is obtained after a very long simulation starting from a zero initial vorticity field. At stationarity one observes a wide inertial range with a well developed Kolmogorov energy spectrum $E(k)=C\epsilon^{2/3}k^{-5/3}$ (Fig. 1). Structure functions in physical space are found in agreement with the self-similar Kolmogorov theory.[10]

Starting from a configuration of the velocity field $\mathbf{u}_1(\mathbf{r},0)$ in the stationary state, one generates a second (perturbed) configuration

$$\mathbf{u}_2(\mathbf{r},0)=\mathbf{u}_1(\mathbf{r},0)+\sqrt{2}\,\delta\mathbf{u}(\mathbf{r},0), \qquad (2)$$

in which the initial error $\delta\mathbf{u}(\mathbf{r},0)$ is very small (the factor $\sqrt{2}$ is only for normalization convenience). The two configurations are integrated in time according to (1) and the evolution of the error $\delta\mathbf{u}(\mathbf{r},t)$ is computed according to (2). Of course, because we are interested in studying the error growth induced by the turbulent dynamics, we use the same realization of random forcing in both simulations.

From (2) one defines the error energy and the error energy spectrum as[4,13]

$$E_\Delta(t)=\int_0^\infty E_\Delta(k,t)\,dk=\frac{1}{2}\int|\delta\mathbf{u}(\mathbf{r},t)|^2 d^2r. \qquad (3)$$

Normalization in (2) ensures that $E_\Delta(k,t)\rightarrow E(k)$ for uncorrelated fields (i.e., for $t\rightarrow\infty$). Assuming that the initial error can be considered infinitesimal, the magnitude of the difference field starts growing exponentially and $E_\Delta(t)\simeq E_\Delta(0)\exp(2\lambda t)$ where $\lambda$ is the maximum Lyapunov exponent of the system.[14] The error growth in this stage is confined at the faster scales in the inertial range, corresponding in our model to the scales close to the forcing wavenumber $k_f$, while at larger scales the two fields remain correlated (see Fig. 1). At larger times, when $E_\Delta(k_f,t)$ becomes comparable with $E(k_f)$, the exponential growth terminates, because the two fields are completely decorrelated at small scales ($k\lesssim k_f$). The error growth continues at larger scales in the inertial range, where the two fields are still correlated, and an algebraic regime sets in. The dimensional prediction
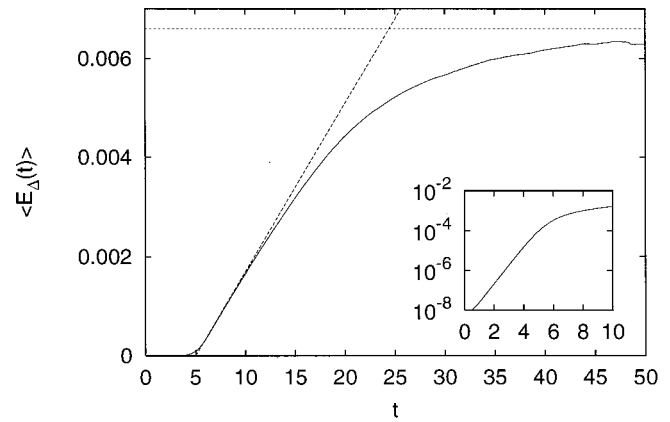
proposed by Lorenz[1] assumes that the time it takes for the error to induce a complete uncertainty at wavenumber $k$ is proportional to the characteristic time at that scale, $t\propto\tau(k)$. Within the Kolmogorov framework, $\tau(k)\simeq\epsilon^{-1/3}k^{-2/3}$. Reverting this dimensional expression one can say that at fixed time the error have reached the scale $k_E(t)\simeq\epsilon^{-1/2}t^{-3/2}$. At larger scales the error is still very small in comparison with the typical energy, while at smaller scale the two fields are completely decorrelated. Thus at each time we have a characteristic scale $k_E(t)$ which divide uncorrelated scales from correlated ones:

$$E_\Delta(k,t)=\begin{cases}0 & \text{if } k<k_E(t),\\ E(k) & \text{if } k>k_E(t).\end{cases} \qquad (4)$$

By inserting (4) in (3), using the Kolmogorov spectrum for $E(k)$ and assuming the dimensional expression for $k_E(t)$ one ends with the prediction[1,9]

$$E_\Delta(t)=G\epsilon t. \qquad (5)$$

The numerical constant $G$ in (5) can be obtained only by repeating the argument more formally within a closure framework.[3,4,15] The physical meaning of $G$ is the ratio of the
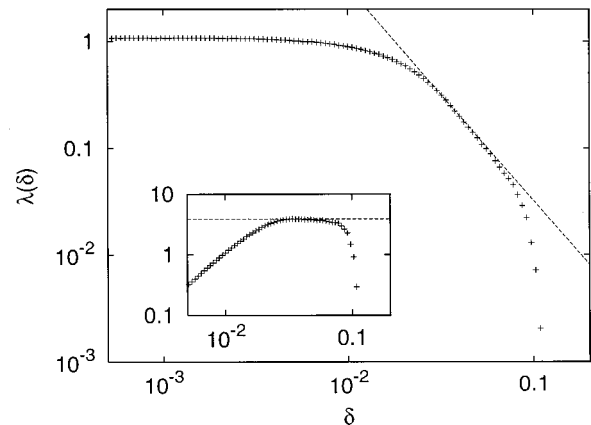


FIG. 3. Finite size Lyapunov exponent $\lambda(\delta)$ as a function of velocity uncertainty $\delta$. The asymptotic constant value for $\delta\rightarrow0$ is the maximum Lyapunov exponent of the turbulent flow. Dashed line represent prediction (7). In the inset we show in the compensated plot $\lambda(\delta)\delta^2/\epsilon$. The line represent the fit to the constant $A\simeq3.9$.

rate of uncorrelated energy production to the rate of energy injected by the forcing and transferred to large scales $\epsilon$.

In Fig. 2 we plot the time evolution of the error energy $\langle E_\Delta(t) \rangle$ obtained from direct numerical simulations averaged over 20 realizations. The exponential regime is clearly visible at small time, while the linear regime (5) is barely observable, making the precise determination of $G$ difficult.

The dimensional predictability argument given above can be rephrased in a language more close to dynamical systems by introducing the finite size Lyapunov exponent (FSLE) analysis. FSLE is a generalization of the Lyapunov exponent to finite size errors, which was recently proposed for the analysis of systems with many characteristic scales.[8] In a nutshell, one computes the ''error doubling time'' $T_r(\delta)$, i.e., the time it takes for an error of size $\delta$ to grow a factor $r$ (for $r=2$ we have actually a doubling time). The FSLE is defined in terms of the average doubling time as

$$\lambda(\delta) = \frac{1}{\langle T_r(\delta) \rangle} \ln r. \tag{6}$$

It is easy to show that definition (6) reduces to the standard Lyapunov exponent $\lambda$ in the infinitesimal error limit $\delta \to 0$.[8] For finite error, the FSLE measures the effective error growth rate at error size $\delta$. Let us remark that taking averages at fixed time, as in (5), is not the same as averaging at fixed error size, as in (6). This is particularly true in the case of intermittent systems, in which strong fluctuations of the error in different realizations can hide scaling laws like (5).[8] From a numerical point of view, the computation of $\lambda(\delta)$ is not more expensive than the computation of the Lyapunov exponent with a standard algorithm.

The same dimensional argument leading to (5) can be used for predicting the behavior of the FSLE in the inertial range. Taking $\delta = \sqrt{E_\Delta}$ as error, one easily obtains

$$\lambda(\delta) = A \epsilon \delta^{-2}. \tag{7}$$

The constant $A$, which again is not determined by dimensional arguments, relates the energy flux in the cascade to the rate of error growth. In the absence of intermittency and for $r \simeq 1$, the two constants in (5) and (7) are related by $A = (r - 1)/\log r \, G$.

The scaling (7), which can be shown to be unaffected by possible intermittency corrections (as in 3D turbulence[8]), is valid within the inertial range $u(k_f) < \delta < U$ where $u(k_f)$ is the typical velocity fluctuation at forcing wavenumber and $U \simeq \sqrt{2E}$ is the typical large scale velocity. At large errors $\delta \simeq U$, we expect error saturation, $E_\Delta \to E$ and thus $\lambda(\delta) \to 0$.

Figure 3 shows the FSLE computed from our simulations. For small errors $\delta < u(k_f)$ [corresponding to an error spectrum $E_\Delta(k_f, t) \ll E(k_f)$] we observe the convergence of $\lambda(\delta)$ to the leading Lyapunov exponent. Its value is essentially the inverse of the smallest characteristic time in the system and represents the growth rate of the most unstable features. At larger $\delta > 10^{-2}$ we clearly see the transition to the inertial range scaling (7). At further large $\delta \simeq U \simeq 0.1$, $\lambda(\delta)$ falls down to zero in correspondence of error saturation.

In order to emphasize scaling (7), in Fig. 3 we also show the compensation of $\lambda(\delta)$ with $\epsilon \delta^{-2}$. Prediction (7) is verified with very high accuracy which allows us to determine the value of $A \simeq 3.9 \pm 0.1$. With the present value of $r \simeq 1.12$, this corresponds to a value $G \simeq 4.1$. The physical picture we obtain is that the creation of uncorrelated energy in the inertial range due to chaotic dynamics is about four times faster than the energy transfer rate.

Our numerical result is in remarkable agreement with the old prediction obtained within the test field model closure[4] which gives $G = 4.19$. At least from the point of view of predictability, two-dimensional turbulence thus seems to be very well captured by low-order closure scheme. As a consequence we can exclude, on the basis of our numerical findings, the existence of intermittency effects in the inverse cascade of error. This is a result which is probably of more general interest than the specific problem discussed in this communication.

[1] E. N. Lorenz, ''The predictability of a flow which possesses many scales of motion,'' Tellus **21**, 289 (1969).

[2] D. Ruelle, ''Microscopic fluctuations and turbulence,'' Phys. Lett. A **72**, 81 (1979).

[3] C. E. Leith, ''Atmospheric predictability and two-dimensional turbulence,'' J. Atmos. Sci. **28**, 145 (1971).

[4] C. E. Leith and R. H. Kraichnan, ''Predictability of turbulent flows,'' J. Atmos. Sci. **29**, 1041 (1972).

[5] K. Ohkitani and M. Yamada, ''Temporal intermittency in the energy cascade process and local Lyapunov analysis in fully-developed model turbulence,'' Prog. Theor. Phys. **81**, 329 (1989).

[6] M. H. Jensen, G. Paladin, and A. Vulpiani, ''Intermittency in a cascade model for three-dimensional turbulence,'' Phys. Rev. A **43**, 798 (1991).

[7] E. Aurell, G. Boffetta, A. Crisanti, P. Frick, G. Paladin, and A. Vulpiani, ''Statistical mechanics of shell models for 2D-turbulence,'' Phys. Rev. E **50**, 4705 (1994).

[8] E. Aurell, G. Boffetta, A. Crisanti, G. Paladin, and A. Vulpiani, ''Growth of non-infinitesimal perturbations in turbulence,'' Phys. Rev. Lett. **77**, 1262 (1996); ''Predictability in the large: An extension of the concept of Lyapunov exponent,'' J. Phys. A **30**, 1 (1997).

[9] T. Bohr, M. H. Jensen, G. Paladin, and A. Vulpiani, *Dynamical Systems Approach to Turbulence* (Cambridge University Press, Cambridge, UK, 1998).

[10] G. Boffetta, A. Celani, and M. Vergassola, ''Inverse energy cascade in two-dimensional turbulence: Deviations from Gaussian behavior,'' Phys. Rev. E **61**, R29 (2000).

[11] V. Borue, ''Spectral exponents of enstrophy cascade in stationary two-dimensional homogeneous turbulence,'' Phys. Rev. Lett. **71**, 3967 (1993).

[12] L. M. Smith and V. Yakhot, ''Bose condensation and small-scale structure generation in a random force driven 2D turbulence,'' Phys. Rev. Lett. **71**, 352 (1993).

[13] M. Lesieur, *Turbulence in Fluids*, 3rd ed. (Kluwer, Dordrecht, 1997).

[14] Actually, the growth rate for the energy is the generalized Lyapunov exponent $L(2)$, but one has simply $L(2) = 2\lambda$ in nonintermittent systems (Ref. 9).

[15] O. Metais and M. Lesieur, ''Statistical predictability of decaying turbulence,'' J. Atmos. Sci. **43**, 857 (1986).