Operazioni sui campi vettoriali

Operatore *nabla* (come un vettore)

$$\nabla = \hat{\mathbf{i}}\frac{\partial}{\partial x} + \hat{\mathbf{j}}\frac{\partial}{\partial y} + \hat{\mathbf{k}}\frac{\partial}{\partial z}$$

Applicato a un campo scalare (ossia una funzione *f(x,y,z)*) produce un campo vettoriale, il *gradiente* di *f:*

$$\nabla f(x,y,z) = \hat{\mathbf{i}}\frac{\partial f}{\partial x} + \hat{\mathbf{j}}\frac{\partial f}{\partial y} + \hat{\mathbf{k}}\frac{\partial f}{\partial z} = \mathbf{A}(x,y,z)$$

Significato di gradiente: vettore che indica la direzione di max. variazione di *f*, e l'entita' della variazione

Campo elettrostatico:

$$\nabla V(x,y,z) = \hat{\boldsymbol{i}}\frac{\partial V}{\partial x} + \hat{\boldsymbol{j}}\frac{\partial V}{\partial x} + \hat{\boldsymbol{k}}\frac{\partial V}{\partial x} = -\boldsymbol{E}(x,y,z)$$

Divergenza

Applichiamo l'operatore nabla ad un campo vettoriale

$$\nabla \cdot \mathbf{A}(x,y,z) = \left(\hat{\mathbf{i}} \frac{\partial}{\partial x} + \hat{\mathbf{j}} \frac{\partial}{\partial y} + \hat{\mathbf{k}} \frac{\partial}{\partial z} \right) \left(\hat{\mathbf{i}} A_x + \hat{\mathbf{j}} A_y + \hat{\mathbf{k}} A_z \right)$$

$$\rightarrow \nabla \cdot \mathbf{A} = \frac{\partial A_x}{\partial x} + \frac{\partial A_y}{\partial y} + \frac{\partial A_z}{\partial z}$$

perche'

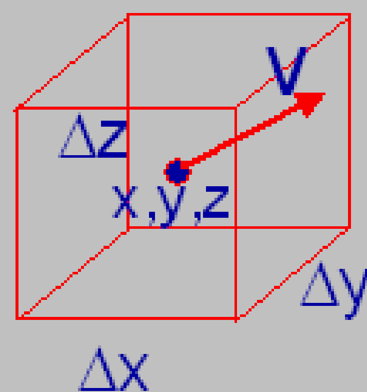
$$\hat{\mathbf{i}} \cdot \hat{\mathbf{i}} = \hat{\mathbf{j}} \cdot \hat{\mathbf{j}} = \hat{\mathbf{k}} \cdot \hat{\mathbf{k}} = 1, \hat{\mathbf{i}} \cdot \hat{\mathbf{j}} = \hat{\mathbf{j}} \cdot \hat{\mathbf{k}} = \hat{\mathbf{i}} \cdot \hat{\mathbf{k}} = 0$$

Produce una funzione scalare, la ***divergenza*** del campo vettoriale

Significato geometrico della divergenza

Divergenza:

$$\lim_{volume \to 0} \frac{\Phi_S(\mathbf{V})}{volume} = \nabla \cdot \mathbf{V}$$



Infatti, per un elemento di volume cubico:

$$\Phi_{\Delta y \Delta z}(\mathbf{V}) = \mathbf{V}(x + \frac{1}{2}\Delta x, y, z) \cdot (\Delta y \Delta z)\hat{\mathbf{i}} - \mathbf{V}(x - \frac{1}{2}\Delta x, y, z) \cdot (\Delta y \Delta z)\hat{\mathbf{i}}$$

$$\mathbf{V}(x + \frac{1}{2}\Delta x, y, z) \approx \mathbf{V}(x, y, z) + \left(\frac{\partial V_x}{\partial x}\frac{1}{2}\Delta x\right)\hat{\mathbf{i}} + \left(\frac{\partial V_y}{\partial x}\frac{1}{2}\Delta x\right)\hat{\mathbf{j}} + \left(\frac{\partial V_z}{\partial x}\frac{1}{2}\Delta x\right)\hat{\mathbf{k}}$$

$$\mathbf{V}(x - \frac{1}{2}\Delta x, y, z) \approx \mathbf{V}(x, y, z) - \left(\frac{\partial V_x}{\partial x}\frac{1}{2}\Delta x\right)\hat{\mathbf{i}} - \left(\frac{\partial V_y}{\partial x}\frac{1}{2}\Delta x\right)\hat{\mathbf{j}} - \left(\frac{\partial V_z}{\partial x}\frac{1}{2}\Delta x\right)\hat{\mathbf{k}}$$

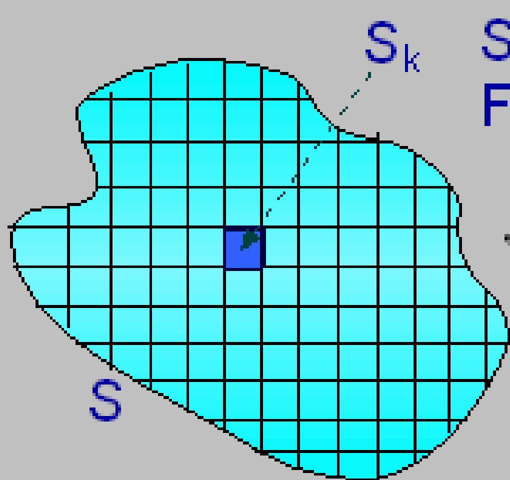
$$\to \Phi_{\Delta y \Delta z}(\mathbf{V}) \approx \frac{\partial V_x}{\partial x}\Delta x \Delta y \Delta z$$

$$\to \Phi_{cubo}(\mathbf{V}) \approx \left(\frac{\partial V_x}{\partial x} + \frac{\partial V_y}{\partial y} + \frac{\partial V_z}{\partial z}\right)\Delta x \Delta y \Delta z$$

$$\to \lim_{volume \to 0} \frac{\Phi_{cubo}(\mathbf{V})}{volume} = \frac{\partial V_x}{\partial x} + \frac{\partial V_y}{\partial y} + \frac{\partial V_z}{\partial x} = \nabla \cdot \mathbf{V}$$

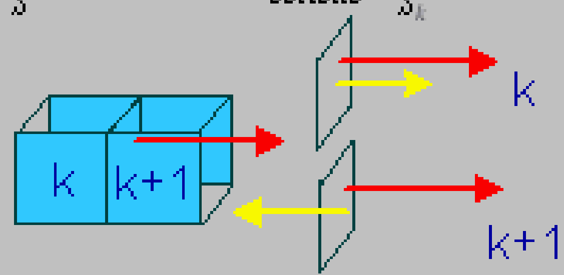
Teorema della divergenza-1

Volume qualsiasi, limitato da una superficie S:
flusso di un campo vettoriale attraverso S



Suddivisione in cellette
Flusso:

$$\oiint_S \mathbf{V}\cdot d\mathbf{A} = \sum_{cellette} \oiint_{S_k} \mathbf{V}\cdot d\mathbf{A}$$



Contributi di cellette contigue si annullano
Resta solo la superficie esterna

$$\sum_{cellette}\oiint_{S_k}\mathbf{V}\cdot d\mathbf{A} = \sum_{cellette}\oiint_{S_k}\mathbf{V}\cdot d\mathbf{A}\,\frac{\Delta v_k}{\Delta v_k} = \sum_{cellette}\oiint_{S_k}\frac{\mathbf{V}\cdot d\mathbf{A}}{\Delta v_k}\,\Delta v_k$$

Teorema della divergenza-2

Proprieta' generale di un campo vettoriale:

$$\sum_{cellette} \oiint_{S_k} \mathbf{V} \cdot d\mathbf{A} = \sum_{cellette} \oiint_{S_k} \mathbf{V} \cdot d\mathbf{A} \frac{dv_k}{dv_k}$$

$$= \sum_{cellette} \frac{\oiint_{S_k} \mathbf{V} \cdot d\mathbf{A}}{dv_k} dv_k = \sum_{cellette} \nabla \cdot \mathbf{V} dv_k \rightarrow \iiint_{volume} \nabla \cdot \mathbf{V} dv$$

$$\rightarrow \oiint_{Superficie} \mathbf{V} \cdot d\mathbf{A} = \iiint_{volume} \nabla \cdot \mathbf{V} dv$$

Per il campo elettrico

$$\oiint_{Superficie} \mathbf{E} \cdot d\mathbf{A} = \frac{q}{\varepsilon_0} = \frac{1}{\varepsilon_0} \iiint_{volume} \rho dv = \iiint_{volume} \nabla \cdot \mathbf{E} dv$$

$\rightarrow \nabla \cdot \mathbf{E} = \frac{\rho}{\varepsilon_0}$ I eq. di Maxwell

Esplicitamente: $\frac{\partial E_x}{\partial x} + \frac{\partial E_y}{\partial y} + \frac{\partial E_z}{\partial z} = \frac{\rho}{\varepsilon_0}$

Rotore

Divergenza: "prodotto interno"

$$\nabla \cdot \mathbf{V}$$

Rotore: "prodotto esterno"

$$\nabla \times \mathbf{V}$$

Esplicitamente:

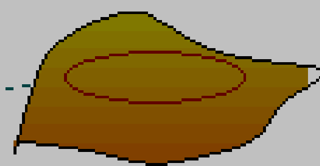
$$\nabla \times \mathbf{V} = \begin{vmatrix} \hat{\mathbf{i}} & \hat{\mathbf{j}} & \hat{\mathbf{k}} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ V_x & V_y & V_z \end{vmatrix}$$

$$= \left(\frac{\partial V_z}{\partial y} - \frac{\partial V_y}{\partial z} \right) \hat{\mathbf{i}} + \left(\frac{\partial V_x}{\partial z} - \frac{\partial V_z}{\partial x} \right) \hat{\mathbf{j}} + \left(\frac{\partial V_y}{\partial x} - \frac{\partial V_x}{\partial y} \right) \hat{\mathbf{k}}$$

Applicato a un vettore,produce un vettore

Significato geometrico del rotore

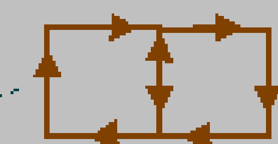
Superficie S
Curva L



Circuitazione del campo lungo una curva chiusa L che racchiude una superficie S

Definizione: $$(\mathbf{rotore})_{\mathbf{n}} = \lim_{S \to 0} \frac{\oint_L \mathbf{V} \cdot d\mathbf{s}}{S} = (\nabla \times \mathbf{V})_{\mathbf{n}}$$

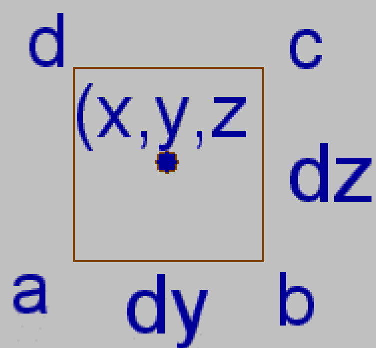
Infatti:



$$\oint_L \mathbf{V} \cdot d\mathbf{s} = \sum_{quadratini} \oint_{L_k} \mathbf{V} \cdot d\mathbf{s}$$

perche' i lati interni si compensano

Elemento di superficie



$$d\mathbf{s}_{ab} = dy\,\mathbf{j}$$
$$d\mathbf{s}_{bc} = dz\,\mathbf{k}$$
$$d\mathbf{s}_{cd} = -dy\,\mathbf{j}$$
$$d\mathbf{s}_{da} = -dz\,\mathbf{k}$$

$$\oint_{ab} \boldsymbol{A}\cdot d\boldsymbol{s} \approx \left[A_y(x,y,z) - \frac{\partial A_y}{\partial z}\frac{1}{2}\Delta z\right]\Delta y$$

$$\oint_{bc} \boldsymbol{A}\cdot d\boldsymbol{s} \approx \left[A_z(x,y,z) + \frac{\partial A_z}{\partial y}\frac{1}{2}\Delta y\right]\Delta z$$

$$\oint_{cd} \boldsymbol{A}\cdot d\boldsymbol{s} \approx -\left[A_y(x,y,z) + \frac{\partial A_y}{\partial z}\frac{1}{2}\Delta z\right]\Delta y$$ Segmento: *cd*

$$\oint \boldsymbol{A}\cdot d\boldsymbol{s} \approx -\left[A_z(x,y,z) - \frac{\partial A_z}{\partial y}\frac{1}{2}\Delta y\right]\Delta z$$ Segmento: *da*

$$\rightarrow \oint_{abcda} \boldsymbol{A}\cdot d\boldsymbol{s} \approx \left[\frac{\partial A_z}{\partial y} - \frac{\partial A_y}{\partial z}\right]\underbrace{\Delta y \Delta z}_{S_k} \rightarrow \frac{1}{S_k}\oint_{abcda} \boldsymbol{A}\cdot d\boldsymbol{s} \approx \left[\frac{\partial A_z}{\partial y} - \frac{\partial A_y}{\partial z}\right]$$

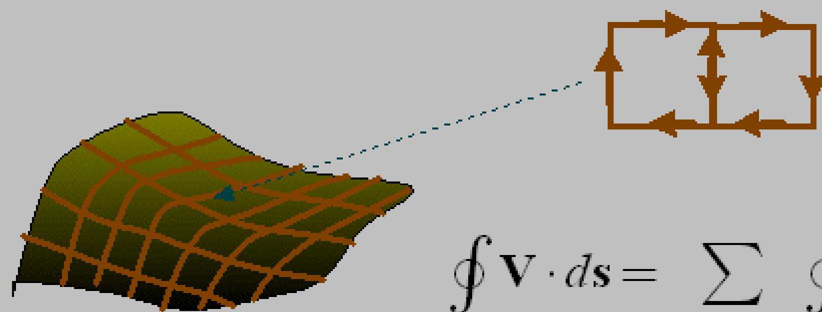
Quindi

$$\text{rotore di } A = \nabla \times A$$

Teorema del rotore

$$\oint_L \mathbf{V}\cdot d\mathbf{s} = \iint_S (\nabla\times\mathbf{V})\cdot d\mathbf{A}$$

Infatti:



$$\oint_L \mathbf{V}\cdot d\mathbf{s} = \sum_{quadratini} \oint_{L_k} \mathbf{V}\cdot d\mathbf{s}$$

perche' i lati interni si compensano

$$\oint_L \boldsymbol{A}\cdot d\boldsymbol{s} = \sum_k \oint_{L_k} \boldsymbol{A}\cdot d\boldsymbol{s} \approx \sum_k \frac{1}{S_k}\oint_{L_k} \boldsymbol{A}\cdot d\boldsymbol{s}\, S_k = \sum_k \oint_{L_k} \frac{\boldsymbol{A}\cdot d\boldsymbol{s}}{S_k} \qquad \underbrace{S_k}_{\text{area proiettata}}$$

$$\rightarrow \oint_L \boldsymbol{A}\cdot d\boldsymbol{s} = \sum_k \iint_{S_k} (\nabla\times\boldsymbol{A})\cdot d\boldsymbol{S}_k = \iint_S (\nabla\times\boldsymbol{A})\cdot d\boldsymbol{S}$$

Per il campo elettro*statico*:

$$\oint_L \boldsymbol{E}\cdot d\boldsymbol{s} = \iint_S (\nabla\times\boldsymbol{E})\cdot d\boldsymbol{S}$$

$\rightarrow \nabla\times\boldsymbol{E} = 0$ II equazione di Maxwell

(in assenza di correnti variabili)

Prime 2 equazioni di Maxwell

$$\oiint_S \mathbf{E} \cdot d\mathbf{A} = \frac{q}{\varepsilon_0} \quad \text{forma integrale}$$

$$\nabla \cdot \mathbf{E} = \frac{\rho}{\varepsilon_0} \quad \text{forma differenziale}$$

$$\oint_L \mathbf{E} \cdot d\mathbf{s} = 0 \quad \text{forma integrale}$$

$$\nabla \times \mathbf{E} = 0 \quad \text{forma differenziale}$$

La seconda e' in forma provvisoria: solo per campi statici