

Eq. fondamentali dei c. statici nel vuoto:

C. elettrostatico:

$$\left. \begin{array}{l} \nabla \times \mathbf{E} = 0 \\ \nabla \cdot \mathbf{E} = \frac{\rho}{\varepsilon_0} \end{array} \right\} \rightarrow \mathbf{E} \text{ irrotazionale}$$

→ Esiste  $\phi$  t.c.  $\mathbf{E} = -\nabla \phi$

C. magnetostatico:

$$\left. \begin{array}{l} \nabla \times \mathbf{B} = \mu_0 \mathbf{j} \\ \nabla \cdot \mathbf{B} = 0 \end{array} \right\} \rightarrow \mathbf{B} \text{ solenoidale}$$

→ Non esiste potenziale scalare per  $\mathbf{B}$  (tranne quando  $\mathbf{j} = 0$ )

→ Esiste potenziale vettore per  $\mathbf{B}$ :

$$\mathbf{B} = \nabla \times \mathbf{A}$$

OK perché  $\nabla \cdot \mathbf{B} = \nabla \cdot (\nabla \times \mathbf{A}) = 0$ , identicamente nulla

$\mathbf{A}$  non univocamente determinato:

$$\mathbf{B} = \nabla \times \mathbf{A} \Rightarrow \mathbf{A} \rightarrow \mathbf{A}' = \mathbf{A} + \nabla \psi \text{ lascia invariato } \mathbf{B}$$

$$\text{Infatti: } \nabla \times \mathbf{A}' = \nabla \times (\mathbf{A} + \nabla \psi) = \nabla \times \mathbf{A} = \mathbf{B}$$

$\mathbf{A}' = \mathbf{A} + \nabla \psi$  trasformazione di gauge,  $\psi$  funzione di  $(x, y, z)$

$\mathbf{B}$  fisico: entra nell'espressione della f. magnetica sulle cariche in moto

$\mathbf{A}$  non fisico (in fisica classica): solo un mezzo matematico

→ Gauge diversi (=diverse scelte di  $\psi$ ) tutti equivalenti

Nota: ancora teo. di Helmholtz !

$\mathbf{A}$  come sempre definito da divergenza e rotore;

rotore definito, divergenza libera

→ Diversa  $\nabla \cdot \mathbf{A} \leftrightarrow$  Diverso gauge

Legame fra  $\mathbf{A}$  e correnti:

$$\nabla \times \mathbf{B} = \mu_0 \mathbf{j} \rightarrow \nabla \times (\nabla \times \mathbf{A}) = \mu_0 \mathbf{j}$$

$$\nabla \times (\nabla \times \mathbf{A}) = \nabla (\nabla \cdot \mathbf{A}) - \nabla^2 \mathbf{A}$$

$$\rightarrow \nabla (\nabla \cdot \mathbf{A}) - \nabla^2 \mathbf{A} = \mu_0 \mathbf{j}, \text{ eq. della magnetostatica}$$

Scelta del gauge:

$$\nabla \cdot \mathbf{A} = 0, \text{ gauge di Coulomb, di Gauss, solenoidale...}$$

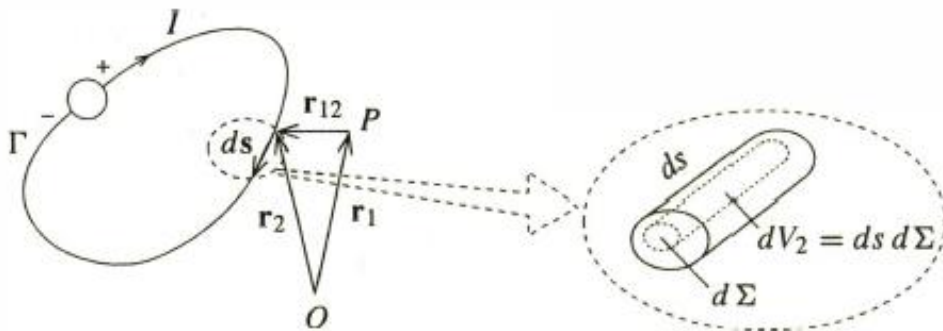
$$\rightarrow \nabla^2 \mathbf{A} = -\mu_0 \mathbf{j}$$

3 eq. di Poisson, una per ogni componente cartesiana

Sol. generale:

$$\mathbf{A}(\mathbf{r}_1) = \frac{\mu_0}{4\pi} \int_{V_2} \frac{\mathbf{j}(\mathbf{r}_2)}{|\mathbf{r}_1 - \mathbf{r}_2|} dV_2$$

Caso di correnti in conduttori filiformi:



$$I = \int_{\Sigma} \mathbf{j} \cdot \hat{\mathbf{n}} d\Sigma = \int_{\Sigma} j d\Sigma$$

$\mathbf{j} \neq 0$  solo dentro il filo  $\rightarrow$  Integrale esteso al solo volume del filo

$$\rightarrow dV_2 = d\Sigma ds$$

Se il filo e' sottile  $|\mathbf{r}_1 - \mathbf{r}_2| \gg \text{diam. filo}$

$$\rightarrow \mathbf{A}(\mathbf{r}_1) \simeq \frac{\mu_0}{4\pi} \oint_{\Gamma} ds \int_{\Sigma} \frac{\mathbf{j}(\mathbf{r}_2)}{|\mathbf{r}_1 - \mathbf{r}_2|} d\Sigma \simeq \frac{\mu_0}{4\pi} \oint_{\Gamma} \frac{ds}{|\mathbf{r}_1 - \mathbf{r}_2|} \int_{\Sigma} \mathbf{j}(\mathbf{r}_2) d\Sigma$$

Poiche'  $\mathbf{j} \parallel ds$

$$\rightarrow \mathbf{A}(\mathbf{r}_1) \simeq \frac{\mu_0}{4\pi} \oint_{\Gamma} \frac{ds}{|\mathbf{r}_1 - \mathbf{r}_2|} \underbrace{\int_{\Sigma} j(\mathbf{r}_2) d\Sigma}_I \rightarrow \mathbf{A}(\mathbf{r}_1) \simeq \frac{\mu_0 I}{4\pi} \oint_{\Gamma} \frac{ds}{|\mathbf{r}_1 - \mathbf{r}_2|}$$

Proprieta' di  $\mathbf{A}$  :

$$\Phi_{\Sigma}(\mathbf{B}) = \int_{\Sigma} \mathbf{B} \cdot \hat{\mathbf{n}} d\Sigma = \int_{\Sigma} (\nabla \times \mathbf{A}) \cdot \hat{\mathbf{n}} d\Sigma = \oint_{\Gamma} \mathbf{A} \cdot d\mathbf{s}, \quad \Gamma \text{ curva chiusa che delimita } \Sigma$$

$$\Phi_{\Sigma}(\mathbf{A}) = \oint_{\Sigma} \mathbf{A} \cdot \hat{\mathbf{n}} d\Sigma = \int_V (\nabla \cdot \mathbf{A}) dV = 0, \quad \Sigma \text{ sup. chiusa che delimita } V$$

Esempi: Si possono dedurre facilmente nei casi in cui esista un problema di elettrostatica matematicamente equivalente

$$\frac{\rho}{\epsilon_0} \leftrightarrow \mu_0 j_i$$

1) Filo rettilineo, corrente  $I \leftrightarrow$  Filo rettilineo, dens. di carica  $\lambda$

$$\phi(r') = -\frac{\lambda}{2\pi\epsilon_0} \ln r'$$

$\lambda$  dens. lineare di carica  $\rightarrow \lambda \Delta l = \rho S \Delta l \rightarrow \lambda = \rho S$

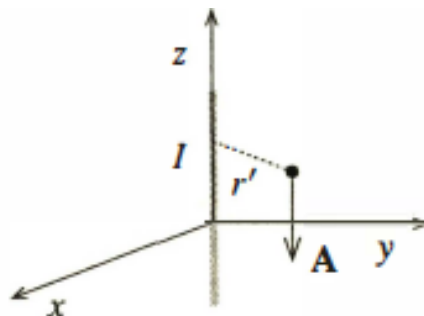
$j_z$  dens. di corrente  $\rightarrow j_z S = I$

$$\rightarrow \frac{\lambda}{S\epsilon_0} \leftrightarrow \mu_0 \frac{I}{S} \rightarrow \frac{\lambda}{\epsilon_0} \leftrightarrow \mu_0 I$$

$$\rightarrow A_z(r') = -\frac{\mu_0 I}{2\pi} \ln r', \quad A_x = A_y = 0, \quad r' \text{ dist. punto dal filo}$$

$\rightarrow \mathbf{A} \parallel \mathbf{j}$

NB  $\mathbf{r}' = (x', y', 0) = (x, y, 0)$

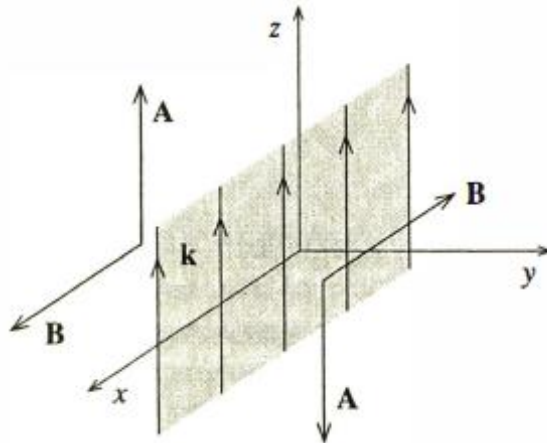


$$\mathbf{B} = \nabla \times \mathbf{A} \rightarrow B_z = 0$$

$$\rightarrow \begin{cases} B_x = \frac{\partial A_z}{\partial y} - \frac{\partial A_y}{\partial z} = \frac{\partial A_z}{\partial y} = -\frac{\mu_0 I}{2\pi} \frac{\partial \ln r'}{\partial y} = -\frac{\mu_0 I}{2\pi} \frac{y}{r'^2} \\ B_y = \frac{\partial A_x}{\partial z} - \frac{\partial A_z}{\partial x} = -\frac{\partial A_z}{\partial x} = +\frac{\mu_0 I}{2\pi} \frac{\partial \ln r'}{\partial x} = +\frac{\mu_0 I}{2\pi} \frac{x}{r'^2} \end{cases} \rightarrow \mathbf{B} \cdot \mathbf{r}' = 0$$

→ Linee di campo: circonferenze concentriche al filo

2) Piano di corrente, densità superficiale  $\mathbf{k} = k\hat{\mathbf{z}}$



Piano carico:

$$\phi = -\frac{\sigma|y|}{2\epsilon_0}, \quad \sigma \text{ dens. superficiale di carica}$$

$\phi$  indipendente da  $x, z$

Spessore piano carico:  $s$

$$\rightarrow \Delta q = \rho s \Delta S = \sigma \Delta S \rightarrow \rho s = \sigma$$

Spessore piano di corrente:  $s$ , direzione  $y$

$$\rightarrow \Delta I = j_z s \Delta x = k \Delta x \rightarrow j_z s = k$$

$$\rightarrow \frac{\sigma}{s\epsilon_0} = \mu_0 \frac{k}{s} \rightarrow \frac{\sigma}{\epsilon_0} = \mu_0 k$$

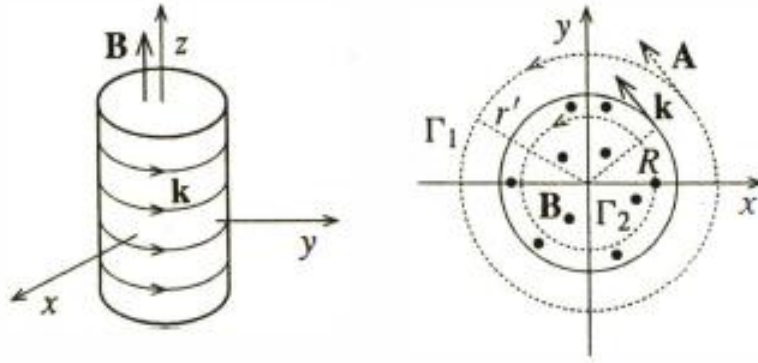
$$\rightarrow A_z = -\frac{\mu_0 k |y|}{2}, A_x = A_y = 0 \rightarrow \mathbf{A} \parallel \mathbf{j}$$

$$\mathbf{B} = \nabla \times \mathbf{A} = (B_x, 0, 0)$$

$$B_x = \frac{\partial A_z}{\partial y} - \frac{\partial A_y}{\partial z} = \mp \frac{\mu_0 k}{2}$$

$\rightarrow$  Linee di campo: rette  $\parallel$  al piano,  $\perp$  a  $\mathbf{k}$

3) Solenoide indefinito: densita' superficiale  $\mathbf{k} = nI\hat{\mathbf{u}}_\theta$



Uso della proprieta' di  $\mathbf{A}$  :

$$\oint_{\Gamma} \mathbf{A} \cdot d\mathbf{s} = \Phi_{\Sigma}(\mathbf{B}), \mathbf{A} \parallel d\mathbf{s} \text{ per simmetria} \rightarrow \mathbf{A} \parallel \mathbf{k}$$

$r' > R$ :

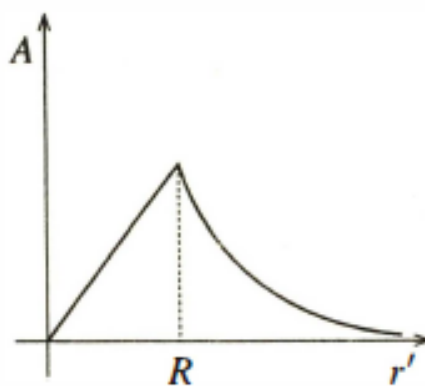
$$\rightarrow 2\pi r' A = B\pi R^2 = \mu_0 k \pi R^2$$

$$\rightarrow A(r') = \frac{\mu_0 k R^2}{2r'}$$

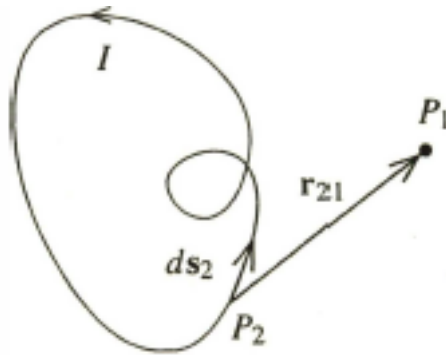
$r' < R$ :

$$\rightarrow 2\pi r' A = B\pi r'^2 = \mu_0 k \pi r'^2$$

$$\rightarrow A(r') = \frac{\mu_0 k r'}{2}$$



Legge di Ampere-Laplace (o di Biot-Savart) derivata da **A**



$$\mathbf{B}(x_1, y_1, z_1) = \nabla \times \mathbf{A}(x_1, y_1, z_1) = \frac{\mu_0}{4\pi} \nabla_{(1)} \times \int_{V_2} \frac{\mathbf{j}(x_2, y_2, z_2)}{|\mathbf{r}_2 - \mathbf{r}_1|} dx_2 dy_2 dz_2, V_2 \text{ vol. circuito}$$

$$\rightarrow \mathbf{B}(x_1, y_1, z_1) = \frac{\mu_0}{4\pi} \int_{V_2} \nabla_{(1)} \times \left[ \frac{\mathbf{j}(x_2, y_2, z_2)}{|\mathbf{r}_2 - \mathbf{r}_1|} \right] dx_2 dy_2 dz_2$$

$$\mathbf{B}(1) \equiv \mathbf{B}(x_1, y_1, z_1), \mathbf{j}(1) \equiv \mathbf{j}(x_1, y_1, z_1) \text{ etc}$$

$$\rightarrow B_x(1) = \frac{\partial A_z}{\partial y_1} - \frac{\partial A_y}{\partial z_1} = \frac{\mu_0}{4\pi} \int_{V_2} \left[ j_z(2) \frac{\partial}{\partial y_1} \frac{1}{|\mathbf{r}_2 - \mathbf{r}_1|} - j_y(2) \frac{\partial}{\partial z_1} \frac{1}{|\mathbf{r}_2 - \mathbf{r}_1|} \right] dx_2 dy_2 dz_2$$

$$\frac{\partial}{\partial y_1} \frac{1}{|\mathbf{r}_2 - \mathbf{r}_1|} = \frac{\partial}{\partial y_1} \frac{1}{\left[ (x_2 - x_1)^2 + (y_2 - y_1)^2 + (z_2 - z_1)^2 \right]^{1/2}}$$

$$\rightarrow \frac{\partial}{\partial y_1} \frac{1}{|\mathbf{r}_2 - \mathbf{r}_1|} = \frac{y_2 - y_1}{\left[ (x_2 - x_1)^2 + (y_2 - y_1)^2 + (z_2 - z_1)^2 \right]^{3/2}} = \frac{y_2 - y_1}{|\mathbf{r}_2 - \mathbf{r}_1|^3}$$

$$\rightarrow \frac{\partial}{\partial z_1} \frac{1}{|\mathbf{r}_2 - \mathbf{r}_1|} = \frac{z_2 - z_1}{\left[ (x_2 - x_1)^2 + (y_2 - y_1)^2 + (z_2 - z_1)^2 \right]^{3/2}} = \frac{z_2 - z_1}{|\mathbf{r}_2 - \mathbf{r}_1|^3}$$

$$\rightarrow B_x(1) = \frac{\mu_0}{4\pi} \int_{V_2} \left[ j_z(P_2) \frac{y_2 - y_1}{|\mathbf{r}_2 - \mathbf{r}_1|^3} - j_y(P_2) \frac{z_2 - z_1}{|\mathbf{r}_2 - \mathbf{r}_1|^3} \right] dx_2 dy_2 dz_2$$

$$\rightarrow B_x(1) = \frac{\mu_0}{4\pi} \int_{V_2} \left[ \frac{\mathbf{j}(2) \times (\mathbf{r}_2 - \mathbf{r}_1)}{|\mathbf{r}_2 - \mathbf{r}_1|^3} \right]_x dx_2 dy_2 dz_2$$

$$\rightarrow \mathbf{B}(1) = \frac{\mu_0}{4\pi} \int_{V_2} \frac{\mathbf{j}(2) \times (\mathbf{r}_2 - \mathbf{r}_1)}{|\mathbf{r}_2 - \mathbf{r}_1|^3} dx_2 dy_2 dz_2$$

Caso di conduttori filiformi: v. prima

$$I = jS$$

$$\rightarrow \mathbf{B}(1) = \frac{\mu_0 I}{4\pi} \int_{\Gamma} \frac{d\mathbf{s}_2 \times (\mathbf{r}_2 - \mathbf{r}_1)}{|\mathbf{r}_2 - \mathbf{r}_1|^3} = -\frac{\mu_0 I}{4\pi} \int_{\Gamma} \frac{(\mathbf{r}_2 - \mathbf{r}_1) \times d\mathbf{s}_2}{|\mathbf{r}_2 - \mathbf{r}_1|^3}, \text{ legge di Ampere-Laplace}$$

$$\rightarrow d\mathbf{B}(1) = -\frac{\mu_0 I}{4\pi} \frac{(\mathbf{r}_2 - \mathbf{r}_1) \times d\mathbf{s}_2}{|\mathbf{r}_2 - \mathbf{r}_1|^3}, \text{ legge elementare di Laplace}$$