Eq. fondamentali dei c. statici nel vuoto:

C. elettrostatico:

$$\left.\begin{array}{l}\nabla\times\mathbf{E}=0\\ \nabla\cdot\mathbf{E}=\dfrac{\rho}{\varepsilon_0}\end{array}\right\}\rightarrow\mathbf{E}\text{ irrotazionale}$$

$\rightarrow$ Esiste ϕ t.c. $\mathbf{E}=-\nabla\phi$

C. magnetostatico:

$$\left.\begin{array}{l}\nabla\times\mathbf{B}=\mu_0\mathbf{j}\\ \nabla\cdot\mathbf{B}=0\end{array}\right\}\rightarrow\mathbf{B}\text{ solenoidale}$$

$\rightarrow$ Non esiste potenziale scalare per $\mathbf{B}$ (tranne quando $\mathbf{j}=0$)

$\rightarrow$ Esiste potenziale vettore per $\mathbf{B}$:

$\mathbf{B}=\nabla\times\mathbf{A}$

OK perche' $\nabla\cdot\mathbf{B}=\nabla\cdot(\nabla\times\mathbf{A})=0,$ identicamente nulla

$\mathbf{A}$ non univocamente determinato:

$\mathbf{B}=\nabla\times\mathbf{A}\Rightarrow\mathbf{A}\rightarrow\mathbf{A}'=\mathbf{A}+\nabla\psi$ lascia invariato $\mathbf{B}$

Infatti: $\nabla\times\mathbf{A}'=\nabla\times(\mathbf{A}+\nabla\psi)=\nabla\times\mathbf{A}=\mathbf{B}$

$\mathbf{A}'=\mathbf{A}+\nabla\psi$ trasformazione di gauge, ψ funzione di (x,y,z)

$\mathbf{B}$ fisico: entra nell'espressione della f. magnetica sulle cariche in moto

$\mathbf{A}$ non fisico (in fisica classica): solo un mezzo matematico

$\rightarrow$ Gauge diversi (=diverse scelte di ψ) tutti equivalenti

Nota: ancora teo. di Helmholtz !

$\mathbf{A}$ come sempre definito da divergenza e rotore;

rotore definito, divergenza libera

$\rightarrow$ Diversa $\nabla\cdot\mathbf{A}\leftrightarrow$ Diverso gauge

Legame fra $\mathbf{A}$ e correnti:

$\nabla\times\mathbf{B}=\mu_0\mathbf{j}\rightarrow\nabla\times(\nabla\times\mathbf{A})=\mu_0\mathbf{j}$

$\nabla\times(\nabla\times\mathbf{A})=\nabla(\nabla\cdot\mathbf{A})-\nabla^2\mathbf{A}$

$\rightarrow\nabla(\nabla\cdot\mathbf{A})-\nabla^2\mathbf{A}=\mu_0\mathbf{j},$ eq. della magnetostatica

Scelta del gauge:

$\nabla\cdot\mathbf{A}=0,$ gauge di Coulomb, di Gauss, solenoidale...

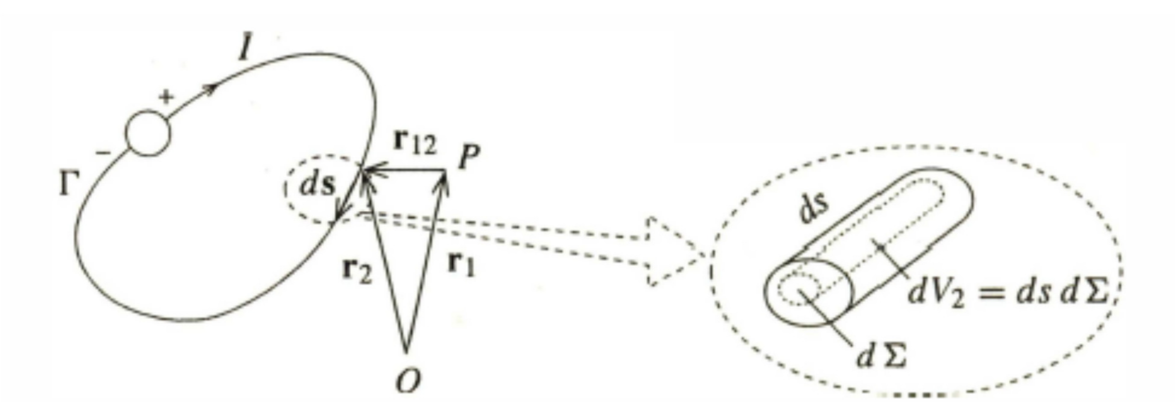
$\rightarrow\nabla^2\mathbf{A}=-\mu_0\mathbf{j}$

3 eq. di Poisson, una per ogni componente cartesiana

Sol. generale:

$$\mathbf{A}(\mathbf{r}_1)=\frac{\mu_0}{4\pi}\int_{V_2}\frac{\mathbf{j}(\mathbf{r}_2)}{|\mathbf{r}_1-\mathbf{r}_2|}dV_2$$

Caso di correnti in conduttori filiformi:



$I = \int_\Sigma \mathbf{j} \cdot \hat{\mathbf{n}} d\Sigma = \int_\Sigma j d\Sigma$

$\mathbf{j} \neq 0$ solo dentro il filo $\rightarrow$ Integrale esteso al solo volume del filo

$\rightarrow dV_2 = d\Sigma ds$

Se il filo e' sottile $|\mathbf{r}_1 - \mathbf{r}_2| \gg$ diam. filo

$$\rightarrow \mathbf{A}(\mathbf{r}_1) \simeq \frac{\mu_0}{4\pi} \oint_\Gamma ds \int_\Sigma \frac{\mathbf{j}(\mathbf{r}_2)}{|\mathbf{r}_1 - \mathbf{r}_2|} d\Sigma \simeq \frac{\mu_0}{4\pi} \oint_\Gamma \frac{ds}{|\mathbf{r}_1 - \mathbf{r}_2|} \int_\Sigma \mathbf{j}(\mathbf{r}_2) d\Sigma$$

Poiche' $\mathbf{j} \parallel d\mathbf{s}$

$$\rightarrow \mathbf{A}(\mathbf{r}_1) \simeq \frac{\mu_0}{4\pi} \oint_\Gamma \frac{d\mathbf{s}}{|\mathbf{r}_1 - \mathbf{r}_2|} \underbrace{\int_\Sigma j(\mathbf{r}_2) d\Sigma}_{I}$$

$$\rightarrow \mathbf{A}(\mathbf{r}_1) \simeq \frac{\mu_0 I}{4\pi} \oint_\Gamma \frac{d\mathbf{s}}{|\mathbf{r}_1 - \mathbf{r}_2|}$$

Proprieta' di $\mathbf{A}$:

$\Phi_\Sigma(\mathbf{B}) = \int_\Sigma \mathbf{B} \cdot \hat{\mathbf{n}} d\Sigma = \int_\Sigma (\nabla \times \mathbf{A}) \cdot \hat{\mathbf{n}} d\Sigma = \oint_\Gamma \mathbf{A} \cdot d\mathbf{s},$ Γ curva chiusa che delimita Σ

$\Phi_\Sigma(\mathbf{A}) = \oint_\Sigma \mathbf{A} \cdot \hat{\mathbf{n}} d\Sigma = \int_V (\nabla \cdot \mathbf{A}) dV = 0,$ Σ sup. chiusa che delimita V

Esempi: Si possono dedurre facilmente nei casi in cui esista
un problema di elettrostatica matematicamente equivalente

$$\frac{\rho}{\varepsilon_0} \leftrightarrow \mu_0 j_i$$

Filo rettilineo percorso da corrente $I \leftrightarrow$ Filo rettilineo con dens. di carica λ

$$\phi(r') = -\frac{\lambda}{2\pi\varepsilon_0}\ln r'$$

λ dens. lineare di carica $\rightarrow \lambda\Delta l = \rho S\Delta l \rightarrow \lambda = \rho S$

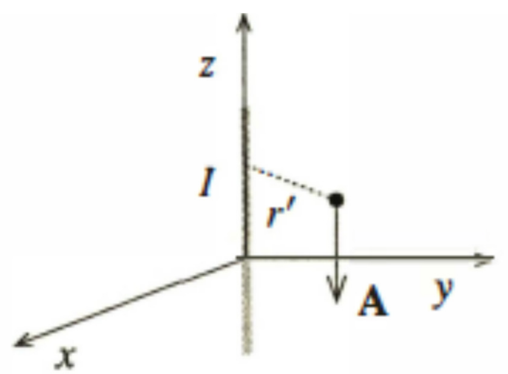
j_z dens. di corrente $\rightarrow j_z S = I$

$$\rightarrow \frac{\lambda}{S\varepsilon_0} \leftrightarrow \mu_0\frac{I}{S} \rightarrow \frac{\lambda}{\varepsilon_0} \leftrightarrow \mu_0 I$$

$\rightarrow A_z(r') = -\frac{\mu_0 I}{2\pi}\ln r', A_x = A_y = 0, \quad r'$ dist. punto dal filo

$\rightarrow \mathbf{A} \parallel \mathbf{j}$

NB $\mathbf{r}' = (x', y', 0) = (x, y, 0)$

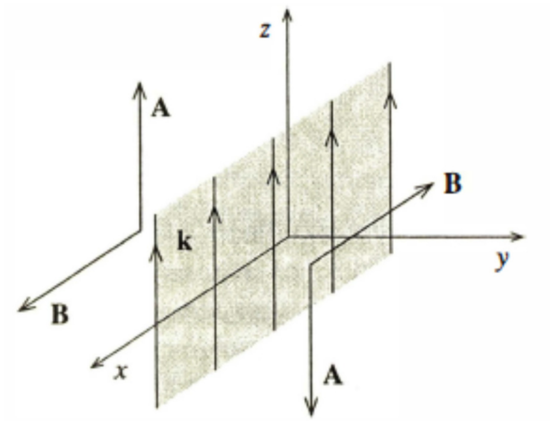


$$\mathbf{B} = \nabla \times \mathbf{A} \rightarrow B_z = 0$$

$$\rightarrow \begin{cases} B_x = \frac{\partial A_z}{\partial y} - \frac{\partial A_y}{\partial z} = \frac{\partial A_z}{\partial y} = -\frac{\mu_0 I}{2\pi}\frac{\partial \ln r'}{\partial y} = -\frac{\mu_0 I}{2\pi}\frac{y}{r'^2} \\ B_y = \frac{\partial A_x}{\partial z} - \frac{\partial A_z}{\partial x} = -\frac{\partial A_z}{\partial x} = +\frac{\mu_0 I}{2\pi}\frac{\partial \ln r'}{\partial x} = +\frac{\mu_0 I}{2\pi}\frac{x}{r'^2} \end{cases} \rightarrow \mathbf{B}\cdot\mathbf{r}' = 0$$

$\rightarrow$ Linee di campo: circonferenze concentriche al filo

Piano di corrente: densita' superficiale $\mathbf{k} = k\hat{\mathbf{z}}$



Piano carico:

$$\phi = -\frac{\sigma|y|}{2\varepsilon_0}, \quad \sigma \text{ dens. superficiale di carica}$$

ϕ indipendente da x, z

Spessore piano carico: s

$\rightarrow \Delta q = \rho s \Delta S = \sigma \Delta S \rightarrow \rho s = \sigma$

Spessore piano di corrente: s, direzione y

$\rightarrow \Delta I = j_z s \Delta x = k \Delta x \rightarrow j_z s = k$

$$\rightarrow \frac{\sigma}{s\varepsilon_0} = \mu_0 \frac{k}{s} \rightarrow \frac{\sigma}{\varepsilon_0} = \mu_0 k$$

$$\rightarrow A_z = -\frac{\mu_0 k|y|}{2}, A_x = A_y = 0$$

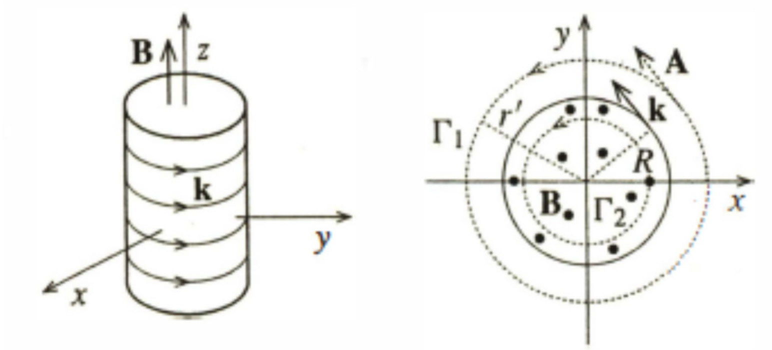
$\rightarrow \mathbf{A} \parallel \mathbf{j}$

$\mathbf{B} = \nabla \times \mathbf{A} = (B_x, 0, 0)$

$$B_x = \frac{\partial A_y}{\partial z} - \frac{\partial A_z}{\partial y} = \mp \frac{\mu_0 k}{2}$$

$\rightarrow$ Linee di campo: rette $\parallel$ al piano, $\perp$ a $\mathbf{k}$

Solenoide indefinito: densita' superficiale $\mathbf{k} = nI\hat{\mathbf{u}}_\theta$



Uso della proprieta' di $\mathbf{A}$:

$\oint_\Gamma \mathbf{A} \cdot d\mathbf{s} = \Phi_\Sigma(\mathbf{B})$, $\mathbf{A} \parallel d\mathbf{s}$ per simmetria $\rightarrow \mathbf{A} \parallel \mathbf{k}$

$r' > R$:

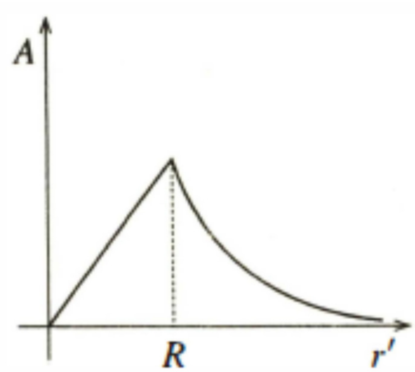
$\rightarrow 2\pi r' A = B\pi R^2 = \mu_0 k \pi R^2$

$\rightarrow A(r') = \dfrac{\mu_0 k R^2}{2r'}$

$r' < R$:

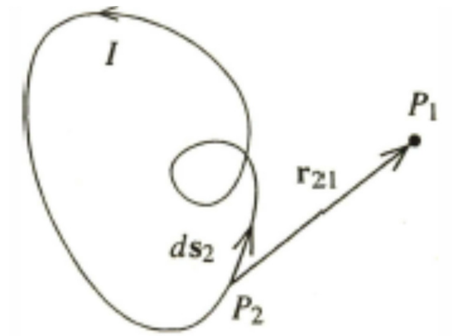
$\rightarrow 2\pi r' A = B\pi r'^2 = \mu_0 k \pi r'^2$

$\rightarrow A(r') = \dfrac{\mu_0 k r'}{2}$



C. magnetico determinato direttamente dalle correnti:

$\mathbf{B}$ determinato nel punto P_1 da tutti gli elementi di corrente del circuito



$$\mathbf{B}(x_1,y_1,z_1)=\nabla\times\mathbf{A}(x_1,y_1,z_1)=\frac{\mu_0}{4\pi}\nabla_{(1)}\times\int_{V_2}\frac{\mathbf{j}(x_2,y_2,z_2)}{|\mathbf{r}_2-\mathbf{r}_1|}dx_2dy_2dz_2, V_2 \text{ vol. circuito}$$

$\mathbf{B}(1)\equiv\mathbf{B}(x_1,y_1,z_1), \mathbf{j}(1)\equiv\mathbf{j}(x_1,y_1,z_1)$ etc

$$\to B_x(1)=\frac{\partial A_z}{\partial y_1}-\frac{\partial A_y}{\partial z_1}=B_x(1)=\frac{\mu_0}{4\pi}\int_{V_2}\left[j_z(2)\frac{\partial}{\partial y_1}\frac{1}{|\mathbf{r}_2-\mathbf{r}_1|}-j_y(2)\frac{\partial}{\partial z_1}\frac{1}{|\mathbf{r}_2-\mathbf{r}_1|}\right]dx_2dy_2dz_2$$

$$\frac{\partial}{\partial y_1}\frac{1}{|\mathbf{r}_2-\mathbf{r}_1|}=\frac{\partial}{\partial y_1}\frac{1}{\left[(x_2-x_1)^2+(y_2-y_1)^2+(z_2-z_1)^2\right]^{1/2}}$$

$$\to\frac{\partial}{\partial y_1}\frac{1}{|\mathbf{r}_2-\mathbf{r}_1|}=\frac{y_2-y_1}{\left[(x_2-x_1)^2+(y_2-y_1)^2+(z_2-z_1)^2\right]^{3/2}}=\frac{y_2-y_1}{|\mathbf{r}_2-\mathbf{r}_1|^3}$$

$$\to\frac{\partial}{\partial z_1}\frac{1}{|\mathbf{r}_2-\mathbf{r}_1|}=\frac{z_2-z_1}{\left[(x_2-x_1)^2+(y_2-y_1)^2+(z_2-z_1)^2\right]^{3/2}}=\frac{z_2-z_1}{|\mathbf{r}_2-\mathbf{r}_1|^3}$$

$$\to B_x(1)=\frac{\mu_0}{4\pi}\int_{V_2}\left[j_z(P_2)\frac{y_2-y_1}{|\mathbf{r}_2-\mathbf{r}_1|^3}-j_y(P_2)\frac{z_2-z_1}{|\mathbf{r}_2-\mathbf{r}_1|^3}\right]dx_2dy_2dz_2$$

$$\to B_x(1)=\frac{\mu_0}{4\pi}\int_{V_2}\left[\frac{\mathbf{j}(2)\times(\mathbf{r}_2-\mathbf{r}_1)}{|\mathbf{r}_2-\mathbf{r}_1|^3}\right]_x dx_2dy_2dz_2$$

$$\to\mathbf{B}(1)=\frac{\mu_0}{4\pi}\int_{V_2}\frac{\mathbf{j}(2)\times(\mathbf{r}_2-\mathbf{r}_1)}{|\mathbf{r}_2-\mathbf{r}_1|^3}dx_2dy_2dz_2$$

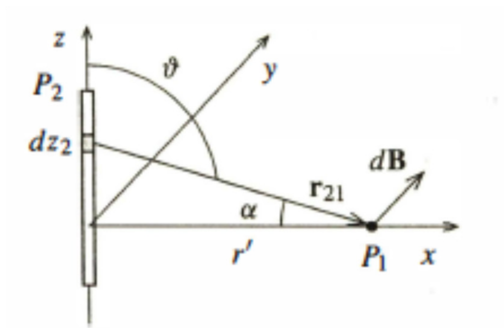
Caso di conduttori filiformi: v. prima

$I = jS$

$\rightarrow \mathbf{B}(1) = \frac{\mu_0 I}{4\pi}\int_\Gamma \frac{d\mathbf{s}_2 \times (\mathbf{r}_2 - \mathbf{r}_1)}{|\mathbf{r}_2 - \mathbf{r}_1|^3} = -\frac{\mu_0 I}{4\pi}\int_\Gamma \frac{(\mathbf{r}_2 - \mathbf{r}_1)\times d\mathbf{s}_2}{|\mathbf{r}_2 - \mathbf{r}_1|^3}$, legge di Ampere-Laplace

$\rightarrow d\mathbf{B}(1) = -\frac{\mu_0 I}{4\pi}\frac{(\mathbf{r}_2 - \mathbf{r}_1)\times d\mathbf{s}_2}{|\mathbf{r}_2 - \mathbf{r}_1|^3}$, I legge elementare di Laplace

Esempio: filo rettilineo



Simmetria cilindrica: $|\mathbf{B}| = $cost a distanza dal filo costante

$\rightarrow$Punto P_1 sull'asse x

$$\rightarrow \begin{cases} \mathbf{e}_{21} = (\sin\theta, 0, \cos\theta) \\ d\mathbf{s}_2 = (0, 0, dz_2) \end{cases}$$

$$\rightarrow dB = \frac{\mu_0}{4\pi}\frac{I dz_2 \sin\theta}{r_{21}^2} = \frac{\mu_0}{4\pi}\frac{I dz_2 \cos\alpha}{r_{21}^2}$$

$$r_{21} = \frac{r'}{\cos\alpha}$$

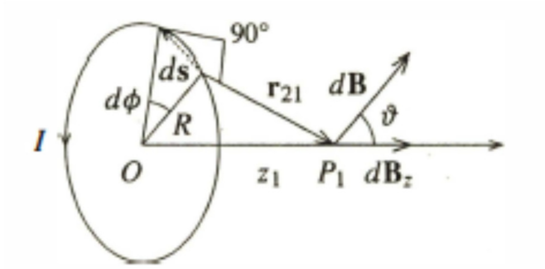
$$z_2 = r'\tan\alpha \rightarrow dz_2 = \frac{r'}{\cos^2\alpha}d\alpha$$

$$\rightarrow dB = \frac{\mu_0}{4\pi}\frac{I dz_2 \cos\alpha}{r_{21}^2} = \frac{\mu_0 I}{4\pi}\frac{\cos^3\alpha}{r'^2}\frac{r'}{\cos^2\alpha}d\alpha = \frac{\mu_0 I}{4\pi}\frac{\cos\alpha}{r'}d\alpha$$

C. totale in P_1:

$$B = \int_{-\pi/2}^{+\pi/2}\frac{\mu_0 I}{4\pi}\frac{\cos\alpha}{r'}d\alpha = \frac{\mu_0 I}{4\pi r'}\sin\alpha\Big|_{-\pi/2}^{+\pi/2} = \frac{\mu_0 I}{2\pi r'}$$

Esempio: Spira circolare (punti sull'asse)



Simmetria circolare: C. totale = Σ contributi elementari dalla spira

Punti sull'asse:

Componente $\perp$ all'asse da' risultato 0 quando integrata sulla spira (elementi opposti danno contributi opposti)

$\rightarrow \mathbf{B} \parallel$ asse $\rightarrow B \equiv B_z$

$dB_z = \frac{\mu_0 I}{4\pi}\frac{ds}{r_{21}^2}\cos\theta$ contributo di un elemento di spira

$$r_{21}^2 = R^2 + z^2$$

$$\cos\theta = \frac{R}{\sqrt{R^2+z^2}}$$

$$\rightarrow dB_z = \frac{\mu_0 I}{4\pi}\frac{ds}{r_{21}^2}\cos\theta = \frac{\mu_0 I}{4\pi}\frac{ds}{R^2+z^2}\frac{R}{\sqrt{R^2+z^2}} = \frac{\mu_0 I}{4\pi}\frac{R}{\left(R^2+z^2\right)^{3/2}}ds$$

$ds = Rd\phi$ elemento di spira

$$\rightarrow dB_z = \frac{\mu_0 I}{4\pi}\frac{R^2}{\left(R^2+z^2\right)^{3/2}}d\phi$$

$$\rightarrow B = B_z = \int_0^{2\pi}\frac{\mu_0 I}{4\pi}\frac{R^2}{\left(R^2+z^2\right)^{3/2}}d\phi = \frac{\mu_0 I}{2}\frac{R^2}{\left(R^2+z^2\right)^{3/2}}$$