

Elementary Particles I

1 – Symmetries

Mostly C, P, T

Symmetry, Invariance, Conservation

Deep connection between:

Physical Symmetry

(Geometrical, Kinematical, Dynamical, ...)

Invariance vs Coordinate Transformations

(Space-time, Internal, ..)

Conservation Laws

(Physical Observables)

Old subject, finally cast into modern framework by Emmy Noether (1918)

Classical Particle Dynamics

Lagrangian formalism

$$\frac{\partial}{\partial t} \left(\frac{\partial L}{\partial \dot{q}_i} \right) - \frac{\partial L}{\partial q_i} = 0 \quad \text{Lagrange's equations}$$

$$\frac{\partial L}{\partial q_i} = 0 \rightarrow \frac{\partial}{\partial t} \left(\frac{\partial L}{\partial \dot{q}_i} \right) = 0 \rightarrow p_i \equiv \frac{\partial L}{\partial \dot{q}_i} = \text{const}$$

Hamiltonian formalism

$$\dot{p}_i = -\frac{\partial H}{\partial q_i} \quad \text{Hamilton's equations}$$

$$\rightarrow H \text{ not depending on } q_i \Rightarrow p_i = \text{const}$$

Symmetry = Irrelevance of ... to the dynamics of a closed system

Symmetry	Invariance	Conservation
Frame origin	Space translation	Total momentum
Time origin	Time translation	Total energy
Frame orientation	Space rotation	Total angular momentum
Frame velocity	Lorentz transformation	CM velocity

Extension to Classical Fields

Take any physical system, *including fields*:

Can describe its motion by the same methods (Lagrangian or Hamiltonian) in terms of a *Principle of Minimum Action*

Then Noether's Theorem states that

For every continuous transformation of the field functions and coordinates which leaves the action unchanged, there is a definite combination of the field functions and their derivatives which is conserved

This is called a *conserved current*

Main point:

Conservation laws must include contributions from the fields, besides particles

Continuous vs Discrete Symmetry

Previous examples: *Continuous* symmetry operations

Any given operation S depends on a number of *continuous parameters*

Any given operation S can be thought as *continuously evolving* from the identity operation

Example: S = Shift by \mathbf{a} of the frame origin

Depends on 3 parameters a_x, a_y, a_z

$$S(\mathbf{a}) \xrightarrow{a_x, a_y, a_z \rightarrow 0} I$$

What are *discrete* symmetry operations?

They do *not* depend on any parameter

They are *intrinsically separated* from the identity operation

Example: S = Axis inversion

No parameter, no continuous evolution from identity

$$x \rightarrow -x, y \rightarrow -y, z \rightarrow -z$$

Quantum Mechanics - I

Link between invariance and conservation in QM

For any observable Q :

$$\langle Q \rangle = \langle \psi | Q | \psi \rangle \rightarrow \frac{d\langle Q \rangle}{dt} = \left\langle \frac{\partial \psi}{\partial t} \middle| Q | \psi \right\rangle + \langle \psi | \frac{\partial Q}{\partial t} | \psi \rangle + \langle \psi | Q \left| \frac{\partial \psi}{\partial t} \right\rangle$$

From Schrodinger equation:

$$i \left| \frac{\partial \psi}{\partial t} \right\rangle = H | \psi \rangle, \quad -i \left\langle \frac{\partial \psi}{\partial t} \right| = \langle \psi | H^\dagger$$

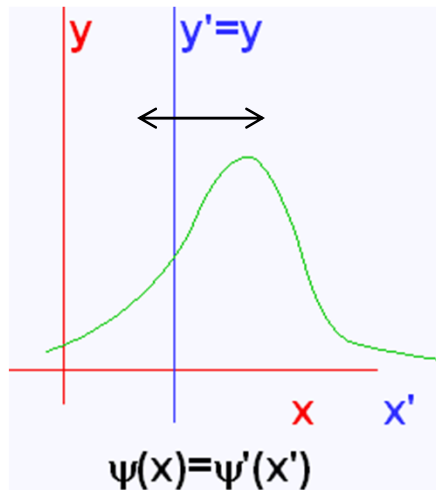
$$\frac{d\langle Q \rangle}{dt} = \left\langle \frac{\partial \psi}{\partial t} \middle| Q | \psi \right\rangle + \langle \psi | \frac{\partial Q}{\partial t} | \psi \rangle + \langle \psi | Q \left| \frac{\partial \psi}{\partial t} \right\rangle$$

$$= i \langle \psi | H^\dagger Q | \psi \rangle + \langle \psi | \frac{\partial Q}{\partial t} | \psi \rangle - i \langle \psi | Q H | \psi \rangle$$

$$H^\dagger = H \rightarrow \frac{d\langle Q \rangle}{dt} = i \langle \psi | \left([H, Q] + \frac{\partial Q}{\partial t} \right) | \psi \rangle \rightarrow \left. \begin{array}{l} [H, Q] = 0 \\ \frac{\partial Q}{\partial t} = 0 \end{array} \right\} \rightarrow \langle Q \rangle = \text{constant}$$

Quantum Mechanics - II

Coordinate transformation S induces a transformation in state (e.g. wave function) space



Just meaning:

One and same state

Different coordinates x, x' in S, S'

$$\rightarrow \psi(x) = \psi'(x')$$

One state, two wave functions in S, S'

Example:

$$S : x \rightarrow x' = S(x) = x - a$$

$$S^{-1} : x' \rightarrow x = S^{-1}(x') = x' + a$$

$$\psi'(x') = \psi(x) \leftrightarrow \psi'[S(x)] = \psi(x)$$

$$\rightarrow \psi'[S^{-1}(S(x))] = \psi(S^{-1}(x))$$

$$\rightarrow \psi'(x) = \psi[S^{-1}(x)] \neq \psi(x)$$

Indeed, for example:

$$\psi(x) = Ne^{-\frac{x^2}{2\sigma^2}}$$

$$\psi'(x') = \psi[S^{-1}(x')] = Ne^{-\frac{(x'+a)^2}{2\sigma^2}} \left(= Ne^{-\frac{(x-a+a)^2}{2\sigma^2}} = Ne^{-\frac{x^2}{2\sigma^2}} \right)$$

$$\psi'(x') = Ne^{-\frac{(x'+a)^2}{2\sigma^2}} \neq Ne^{-\frac{x'^2}{2\sigma^2}} = \psi(x')$$

Quantum Mechanics - III

Unitary transformation

$$U : \psi(x) \rightarrow \psi'(x) = \psi[S^{-1}(x)] = U[\psi(x)]$$

In term of state vectors:

$$|\psi'\rangle = U|\psi\rangle \rightarrow U \text{ unitary: } U^\dagger = U^{-1}$$

Take $A =$ Any operator

$$\langle \psi | A | \psi \rangle = \langle \psi' | A' | \psi' \rangle \text{ defining transformed operator}$$

$$\langle \psi' | A' | \psi' \rangle = \langle \psi | U^\dagger A' U | \psi \rangle$$

$$\langle \psi | A | \psi \rangle = \langle \psi' | A' | \psi' \rangle = \langle \psi | U^\dagger A' U | \psi \rangle \rightarrow A = U^\dagger A' U \rightarrow A' = U A U^\dagger$$

When A invariant wrt U

$$A = A'$$

$$\rightarrow U A U^\dagger = A$$

$$\rightarrow [U, A] = 0 \text{ when } U \text{ is a symmetry operator for } A$$

$$\text{By taking } A=H: \quad [U, H] = 0 \rightarrow \langle U \rangle = \text{const}$$

Scalar products must be preserved by U .

Unitary is actually too restrictive:

U anti-unitary is also OK (see later)

Summary

Coordinate transformation S (possibly including *internal* coordinates)

Continuous, Discrete

Examples:

Space translation (C)

Axis inversion (D)

Corresponding ‘Relativity Principles’

Example:

The laws of physics are the same for all the observers, irrespective of the chosen origin of their reference frames

Induced transformation U on state space

Unitary, Antiunitary

$\langle U \rangle = \text{constant of motion}$

Discrete Symmetries

Coordinate transformations not evolving continuously from identity

Very important in Particle Physics:

Space Inversion

Time Reversal

Charge Conjugation

Leading to:

Quantum Numbers

Laws of Conservation

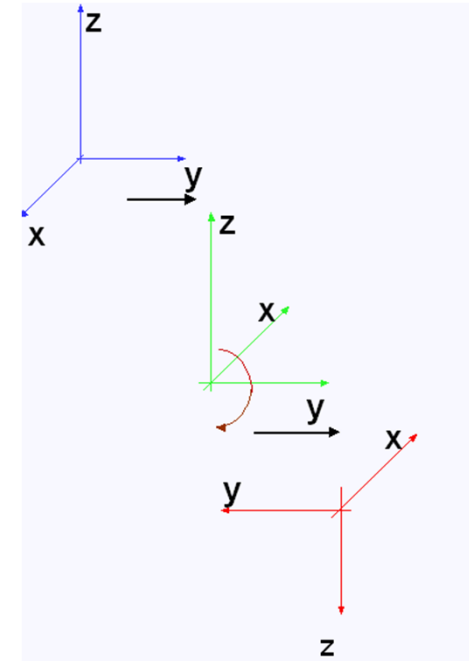
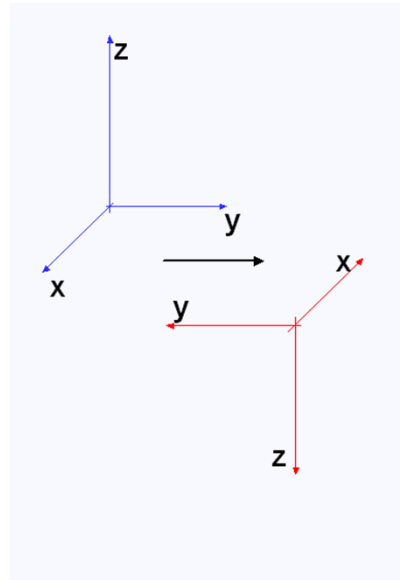
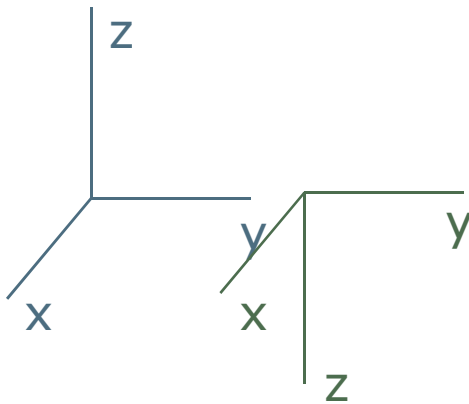
Selection Rules

Constraints on Interactions

Parity in Classical Physics - I

Two, non equivalent ways of choosing a Cartesian frame in 3D

In 3D: 3 axis inversions equivalent to 1 axis (= mirror) inversion * Rotation



Non equivalent = Not connected by a rotation
Connected by 3 axis inversions

Parity in Classical Physics - II

Define coordinate symmetry operation

$$P: \mathbf{r} \rightarrow \mathbf{r}' = -\mathbf{r}$$

Corresponding 'Relativity Principle'

The laws of physics are the same for all the observers, irrespective of the chosen chirality of their reference frames

Transformations of physical quantities follow:

$\mathbf{r} \rightarrow -\mathbf{r}$	position
$t \rightarrow t$	time
$\mathbf{p} \rightarrow -\mathbf{p}$	3-momentum
$E \rightarrow E$	energy
$\mathbf{L} \rightarrow \mathbf{L}$	angular momentum

No conservation law!

P not connected to identical transformation

Parity in Classical Physics - III

General taxonomy of physical quantities (observables):

With respect to rotations

With respect to reflections

‘Parity’ of corresponding Hermitian operator:

	True	Pseudo
Scalar	$+1$	-1
Vector	-1	$+1$
Rank N Tensor	$(-1)^N$	$(-1)^{N+1}$

Parity in Classical Physics - IV

Parity behavior of selected electromagnetic quantities:

$\mathbf{j}(\mathbf{r}, t) \rightarrow -\mathbf{j}(-\mathbf{r}, t)$ current density

$\rho(\mathbf{r}, t) \rightarrow \rho(-\mathbf{r}, t)$ charge density

$\mathbf{E}(\mathbf{r}, t) \rightarrow -\mathbf{E}(-\mathbf{r}, t)$ electric field

$\mathbf{B}(\mathbf{r}, t) \rightarrow \mathbf{B}(-\mathbf{r}, t)$ magnetic field

Note

All of this is indeed conventional, as based on our definition of the electric charge as a *scalar*.

What actually matters is that *force* is a polar vector

Then

$$\left. \begin{array}{l} \mathbf{F} = q\mathbf{E} \\ \mathbf{F} = q\mathbf{v} \times \mathbf{B} \\ \mathbf{F}, \mathbf{v} \text{ polar} \end{array} \right\} \rightarrow \left\{ \begin{array}{l} \text{either } q \text{ scalar, } \mathbf{E} \text{ polar, } \mathbf{B} \text{ axial} \\ \text{or } q \text{ pseudoscalar, } \mathbf{E} \text{ axial, } \mathbf{B} \text{ polar} \end{array} \right.$$

Parity in Quantum Mechanics - I

Coordinate transformation as before: $P : \mathbf{r} \rightarrow \mathbf{r}' = P(\mathbf{r}) = -\mathbf{r}$

Induced transformation in state space: unitary operator U_P

A) Commutation relations with position, momentum, angular momentum

$$\begin{aligned} U_P \mathbf{r} |\psi\rangle &= U_P \mathbf{r} \sum |\mathbf{r}\rangle \langle \mathbf{r} | \psi \rangle = U_P \sum \mathbf{r} |\mathbf{r}\rangle \langle \mathbf{r} | \psi \rangle = \sum U_P \mathbf{r} |\mathbf{r}\rangle \langle \mathbf{r} | \psi \rangle \\ &= \sum (-\mathbf{r}) |-\mathbf{r}\rangle \langle -\mathbf{r} | \psi \rangle = (-\mathbf{r}) U_P |\psi\rangle = -\mathbf{r} U_P |\psi\rangle \\ &\rightarrow U_P \mathbf{r} U_P^{-1} = -\mathbf{r} \rightarrow U_P \mathbf{r} = -\mathbf{r} U_P \end{aligned}$$

$$U_P U_T(\delta \mathbf{r}) = U_P (1 - \mathbf{p} \cdot \delta \mathbf{r})$$

$$U_P (1 - \mathbf{p} \cdot \delta \mathbf{r}) = (1 + \mathbf{p} \cdot \delta \mathbf{r}) U_P$$

$$\rightarrow -U_P \mathbf{p} \cdot \delta \mathbf{r} = \mathbf{p} \cdot \delta \mathbf{r} U_P \rightarrow -U_P \mathbf{p} = \mathbf{p} U_P$$

$$U_P \hat{\mathbf{L}} = \hat{\mathbf{L}} U_P$$

Summary:

U_P anticommutes with \mathbf{r}, \mathbf{p} , commutes with \mathbf{L}

Parity in Quantum Mechanics - II

B) Action of U_P on states

$$U_P : |\psi\rangle \rightarrow |\psi'\rangle \equiv U_P |\psi\rangle$$

Same physical state described by two different kets in the two frames

Take position eigenstate:

$$U_P : |\mathbf{r}\rangle \rightarrow |\mathbf{r}'\rangle \equiv U_P |\mathbf{r}\rangle = \eta |-\mathbf{r}\rangle, \quad \eta \text{ arbitrary phase} = e^{i\alpha}$$

Take generic state:

$$U_P : |\psi\rangle \rightarrow |\psi'\rangle \equiv U_P |\psi\rangle$$

$$\text{Expand into position eigenstates: } |\psi\rangle = \sum_{\mathbf{r}} |\mathbf{r}\rangle \langle \mathbf{r} | \psi \rangle$$

$$U_P |\psi\rangle = U_P \sum_{\mathbf{r}} |\mathbf{r}\rangle \langle \mathbf{r} | \psi \rangle = \sum_{\mathbf{r}} U_P |\mathbf{r}\rangle \langle \mathbf{r} | U_P^{-1} U_P \psi \rangle = \sum_{\mathbf{r}} [U_P |\mathbf{r}\rangle \langle \mathbf{r} | U_P^{-1}] U_P \psi \rangle$$

$$= \sum_{\mathbf{r}'} |\mathbf{r}'\rangle \langle \mathbf{r}' | U_P \psi \rangle = \sum_{\mathbf{r}'} |\mathbf{r}'\rangle \langle \mathbf{r}' | \psi' \rangle = |\psi'\rangle = \sum_{\mathbf{r}} |-\mathbf{r}\rangle \langle -\mathbf{r} | \psi' \rangle$$

$$\rightarrow \langle \mathbf{r}' | \psi' \rangle = \langle \mathbf{r} | \psi \rangle \rightarrow \langle \mathbf{r} | \psi' \rangle = \langle -\mathbf{r} | \psi \rangle$$

Talking wave functions: $\psi'(\mathbf{r}) = \psi(-\mathbf{r})$ (as shown before)

Would yield similar results by expanding into momentum eigenstates

Parity in Quantum Mechanics - III

C) Fundamental property of parity operator

$$P:\mathbf{r} \rightarrow \mathbf{r}' = P(\mathbf{r}) = -\mathbf{r}, P:\mathbf{r}' \rightarrow \mathbf{r}'' = P(\mathbf{r}') = -\mathbf{r}' = \mathbf{r} \rightarrow P^2 = I$$

$$U_P|\psi\rangle = |\psi'\rangle, U_P|\psi'\rangle = U_P(U_P|\psi\rangle) = U_P^2|\psi\rangle \rightarrow U_P^2 = \eta^2 I$$

Take η real $\rightarrow \eta = \pm 1$

(Can be shown to be always possible)

$$\left. \begin{array}{l} U_P^\dagger = U_P^{-1}, U_P \text{ unitary} \\ U_P^2 = I \end{array} \right\} \rightarrow U_P^2 = U_P U_P = I = U_P U_P^{-1} = U_P U_P^\dagger \rightarrow U_P^\dagger = U_P \quad U_P \text{ Hermitian}$$

$\eta \equiv \eta_P$ parity quantum number

$\rightarrow U_P$ eigenstates: $U_P|a\rangle = \pm|a\rangle$

Consequences:

$[H, U_P] = 0 \rightarrow$ stationary states have definite η_P - when not degenerate

$[H, U_P] = 0 \rightarrow \eta_P =$ constant of motion

Parity in Quantum Mechanics - IV

Parity for a composite system (just meaning with several degrees of freedom)

$|a\rangle$ compound state of subsystems 1 and 2, parity eigenstate

$U_P^{(1)}, U_P^{(2)}$ parity operators for 1,2

$$U_P^{(1)}|a\rangle = \eta_P^{(1)}|a\rangle, U_P^{(2)}|a\rangle = \eta_P^{(2)}|a\rangle$$

$$U_P^{(2)}[U_P^{(1)}|a\rangle] = U_P^{(2)}[\eta_P^{(1)}|a\rangle] = \eta_P^{(2)}\eta_P^{(1)}|a\rangle \rightarrow \eta_P \text{ multiplicative quantum number}$$

Quantum numbers (i.e., conserved quantities) are usually *additive*: ???

Reason: For continuous symmetries the unitary operators are not Hermitian

U unitary $\rightarrow U = e^{iaH} \simeq 1 + iaH$, H Hermitian; $\lim_{a \rightarrow 0} U = 1 \rightarrow$ Use infinitesimal generators

Example: Translations

$$U_a^{(1)} \simeq 1 + ia\mathbf{p}^{(1)}, U_a^{(2)} \simeq 1 + ia\mathbf{p}^{(2)} \rightarrow U_a^{(2)}(U_a^{(1)}|\psi\rangle) \simeq (1 + ia\mathbf{p}^{(2)})[(1 + ia\mathbf{p}^{(1)})|\psi\rangle]$$

$$\rightarrow U_a^{(2)}(U_a^{(1)}|\psi\rangle) \simeq (1 + ia\mathbf{p}^{(2)} + ia\mathbf{p}^{(1)})|\psi\rangle = [1 + ia(\mathbf{p}^{(2)} + \mathbf{p}^{(1)})]|\psi\rangle$$

U Hermitian \rightarrow Constant of motion = $U \rightarrow$ Multiplicative

U not Hermitian \rightarrow Constant of motion \sim “Logarithm” of $U \dots \rightarrow$ Additive

The Parity Quantum Number

As a quantum number, parity may or may not be conserved

Experimental fact:

All interactions, except weak interaction, do conserve parity

To the extent we can neglect weak interaction, all *stationary, non degenerate* states are parity eigenstates

Scattering states: similar but not identical to stationary states

Being \sim momentum eigenstates $[\mathbf{p}, U_p] \neq 0 \rightarrow$ Parity not defined

Stable particles and resonances: bound states

Angular momentum eigenstates $[\mathbf{L}, U_p] = 0 \rightarrow$ Parity defined

Orbital vs Intrinsic Parity

Parity quantum number introduced by making reference to orbital motion.

For angular momentum eigenstates, take position representation:

$$\psi(\mathbf{r}) = Nf(r)Y_l^m(\theta, \varphi)$$

$$P:\mathbf{r} \rightarrow -\mathbf{r} \Rightarrow Y_l^m(\theta, \varphi) \rightarrow Y_l^m(\theta - \pi, \varphi + \pi) = (-1)^l Y_l^m(\theta, \varphi)$$

Therefore

$$\eta_P^{(orb)} = (-1)^l$$

This is known as *orbital parity*.

Orbital parity, like angular momentum, is *frame dependent*

Arbitrary, extra phase factor: *State intrinsic parity*

Not relevant in non-relativistic regime:

Cannot be measured in non-relativistic processes, where particle numbers never change

Intrinsic Parity - I

Orbital parity not sufficient to deal with relativistic processes

Reason: *Particles are created and annihilated*

Best clarified first by a non relativistic example

Take a reaction between 2 nuclei, ignore nucleon spin



Angular momentum conservation:

$$\mathbf{L}_{TOT}^{(in)} = \mathbf{L}_A^{(CM-A)} + \mathbf{L}_B^{(CM-B)} + \mathbf{L}_{AB}^{(CM-AB)}$$

$$P_{TOT}^{in} = (-1)^{L_{TOT}^{(in)}} = \underbrace{(-1)^{L_A^{(CM-A)}}}_{P_A} \underbrace{(-1)^{L_B^{(CM-B)}}}_{P_B} \underbrace{(-1)^{L_{AB}^{(CM-AB)}}}_{P_{ORB}} \leftarrow \begin{array}{l} L_A, L_B \text{ originated by internal nuclear motion} \\ L_{AB} \text{ orbital motion} \end{array}$$

$$\mathbf{L}_{TOT}^{(out)} = \mathbf{L}_C^{(CM-C)} + \mathbf{L}_D^{(CM-D)} + \mathbf{L}_{CD}^{(CM-CD)}$$

$$P_{TOT}^{out} = (-1)^{L_{TOT}^{(out)}} = \underbrace{(-1)^{L_C^{(CM-C)}}}_{P_C} \underbrace{(-1)^{L_D^{(CM-D)}}}_{P_D} \underbrace{(-1)^{L_{CD}^{(CM-CD)}}}_{P_{ORB}} \leftarrow \text{Same for final state}$$

If we don't know A, B, C, D are composite systems, 'intrinsic' nuclear parity cannot be ignored, or the process would violate parity whenever $L_{CD} \neq L_{AB}$

Intrinsic Parity - II

Constituent intrinsic parity not originating from internal structure
 Just a drop of QFT to show the origin of intrinsic parity

1) One component, Boson field $\phi(\mathbf{r}, t)$

Expand $\phi(\mathbf{r})$ into creation+annihilation operators

$$\phi(\mathbf{r}, t) = \sum_{\mathbf{p}} \frac{1}{\sqrt{2EV}} \left[a_{\mathbf{p}} e^{-ip \cdot r} + a_{\mathbf{p}}^{\dagger} e^{+ip \cdot r} \right], \quad p \cdot r = (E, \mathbf{p}) \cdot (t, \mathbf{r}) = Et - \mathbf{p} \cdot \mathbf{r}$$

Apply parity operator:

$$U_P \phi U_P^{-1} = \eta_P \phi(-\mathbf{r}, t) = \sum_{\mathbf{p}} \frac{1}{\sqrt{2EV}} \left[U_P a_{\mathbf{p}} U_P^{-1} e^{-i(Et - \mathbf{p} \cdot \mathbf{r})} + U_P a_{\mathbf{p}}^{\dagger} U_P^{-1} e^{+i(Et - \mathbf{p} \cdot \mathbf{r})} \right]$$

η_P : Intrinsic parity of ϕ operator

Expand $\phi(-\mathbf{r})$ into creation+annihilation operators

$$\begin{aligned} \eta_P \phi(-\mathbf{r}, t) &= \eta_P \sum_{\mathbf{p}} \frac{1}{\sqrt{2EV}} \left[a_{\mathbf{p}} e^{-i(Et - \mathbf{p} \cdot (-\mathbf{r}))} + a_{\mathbf{p}}^{\dagger} e^{+i(Et - \mathbf{p} \cdot (-\mathbf{r}))} \right] \\ &= \sum_{\mathbf{p}} \frac{1}{\sqrt{2EV}} \left[\eta_P a_{-\mathbf{p}} e^{-i(Et - (-\mathbf{p}) \cdot \mathbf{r})} + \eta_P a_{-\mathbf{p}}^{\dagger} e^{+i(Et - (-\mathbf{p}) \cdot \mathbf{r})} \right] \rightarrow \begin{cases} U_P a_{\mathbf{p}} U_P^{-1} = \eta_P a_{-\mathbf{p}} \\ U_P a_{\mathbf{p}}^{\dagger} U_P^{-1} = \eta_P a_{-\mathbf{p}}^{\dagger} \end{cases} \\ &\rightarrow U_P |\mathbf{p}\rangle = U_P a_{\mathbf{p}}^{\dagger} U_P^{-1} U_P |0\rangle = \eta_P a_{-\mathbf{p}}^{\dagger} |0\rangle = \eta_P |-\mathbf{p}\rangle; U_P |\mathbf{p} = 0\rangle = \eta_P |\mathbf{p} = 0\rangle \end{aligned}$$

Different states

Same state

Intrinsic Parity - III

Non-Hermitian field:

$$\varphi^\dagger(\mathbf{r}, t) = \sum_{\mathbf{p}} \frac{1}{\sqrt{2EV}} [b_{\mathbf{p}} e^{-ip \cdot r} + b_{\mathbf{p}}^\dagger e^{+ip \cdot r}]$$

$$\rightarrow \begin{cases} U_P \varphi^\dagger U_P^{-1} = \eta_P^* \varphi^\dagger(-\mathbf{r}, t) \\ U_P \varphi^\dagger U_P^{-1} = \sum_{\mathbf{p}} \frac{1}{\sqrt{2EV}} [U_P b_{\mathbf{p}} U_P^{-1} e^{-i(Et - \mathbf{p} \cdot \mathbf{r})} + U_P b_{\mathbf{p}}^\dagger U_P^{-1} e^{+i(Et - \mathbf{p} \cdot \mathbf{r})}] \end{cases}$$

$$\rightarrow \eta_P^* \sum_{\mathbf{p}} \frac{1}{\sqrt{2EV}} [b_{\mathbf{p}} e^{-i(Et - \mathbf{p} \cdot (-\mathbf{r}))} + b_{\mathbf{p}}^\dagger e^{+i(Et - \mathbf{p} \cdot (-\mathbf{r}))}] = \sum_{\mathbf{p}} \frac{1}{\sqrt{2EV}} [\eta_P b_{-\mathbf{p}} e^{-i(Et - (-\mathbf{p}) \cdot \mathbf{r})} + \eta_P b_{-\mathbf{p}}^\dagger e^{+i(Et - (-\mathbf{p}) \cdot \mathbf{r})}]$$

$$\rightarrow \begin{cases} U_P b_{\mathbf{p}} U_P^{-1} = \eta_P^* b_{-\mathbf{p}} \\ U_P b_{\mathbf{p}}^\dagger U_P^{-1} = \eta_P^* b_{-\mathbf{p}}^\dagger \end{cases}$$

$$\rightarrow U_P |\mathbf{p}\rangle = U_P b_{\mathbf{p}}^\dagger U_P^{-1} U_P |0\rangle = \eta_P^* b_{-\mathbf{p}}^\dagger |0\rangle = \eta_P^* |-\mathbf{p}\rangle; U_P |\mathbf{p} = 0\rangle = \eta_P^* |\mathbf{p} = 0\rangle$$

Therefore, for boson particle-antiparticle:

$$\rightarrow \eta_b \eta_{\bar{b}} = \eta_P \eta_P^* = 1$$

Intrinsic Parity - IV

Intrinsic parity of a Dirac particle:

$$(\boldsymbol{\alpha} \cdot \mathbf{p} + \beta m) \psi = E \psi \xrightarrow{P} (\boldsymbol{\alpha} \cdot (-\mathbf{p}) + \beta m) \psi_P = E \psi_P$$

$$\beta (\boldsymbol{\alpha} \cdot (-\mathbf{p}) + \beta m) \psi_P = E \beta \psi_P$$

$$\beta \boldsymbol{\alpha} = -\boldsymbol{\alpha} \beta \rightarrow ((-\boldsymbol{\alpha}) \cdot (-\mathbf{p}) + \beta m) \beta \psi_P = E \beta \psi_P$$

$\rightarrow (\boldsymbol{\alpha} \cdot \mathbf{p} + \beta m) \beta \psi_P = E \beta \psi_P \rightarrow \beta \psi_P$ satisfies Dirac eq.

$$\beta = \gamma^0 = \begin{pmatrix} I & 0 \\ 0 & -I \end{pmatrix}$$

Plane wave solutions:

$$\psi_1(\mathbf{r}, t) = \begin{pmatrix} \varphi \\ \frac{\boldsymbol{\sigma} \cdot \mathbf{p}}{E + m} \varphi \end{pmatrix} e^{-i(Et - \mathbf{p} \cdot \mathbf{r})}, \quad \psi_2(\mathbf{r}, t) = \begin{pmatrix} \chi \\ \frac{\boldsymbol{\sigma} \cdot \mathbf{p}}{E + m} \chi \end{pmatrix} e^{-i(Et - \mathbf{p} \cdot \mathbf{r})},$$

$$\psi_3(\mathbf{r}, t) = \begin{pmatrix} -\frac{\boldsymbol{\sigma} \cdot \mathbf{p}}{E + m} \varphi \\ \varphi \end{pmatrix} e^{-i(-Et - \mathbf{p} \cdot \mathbf{r})}, \quad \psi_4(\mathbf{r}, t) = \begin{pmatrix} -\frac{\boldsymbol{\sigma} \cdot \mathbf{p}}{E + m} \chi \\ \chi \end{pmatrix} e^{-i(-Et - \mathbf{p} \cdot \mathbf{r})}$$

Intrinsic Parity - V

Rest frame: $\mathbf{p} = 0$, $E = m$

$$\psi_1 = \begin{pmatrix} \varphi \\ 0 \end{pmatrix} e^{-imt} \rightarrow \gamma^0 \begin{pmatrix} \varphi \\ 0 \end{pmatrix} e^{-imt} = \begin{pmatrix} \varphi \\ 0 \end{pmatrix} e^{-imt} = \psi_1$$

$$\psi_2 = \begin{pmatrix} \chi \\ 0 \end{pmatrix} e^{-imt} \rightarrow \gamma^0 \begin{pmatrix} \chi \\ 0 \end{pmatrix} e^{-imt} = \begin{pmatrix} \chi \\ 0 \end{pmatrix} e^{-imt} = \psi_2$$

$$\psi_3 = \begin{pmatrix} 0 \\ \varphi \end{pmatrix} e^{-i(-m)t} \rightarrow \gamma^0 \begin{pmatrix} 0 \\ \varphi \end{pmatrix} e^{-i(-m)t} = \begin{pmatrix} 0 \\ -\varphi \end{pmatrix} e^{-i(-m)t} = -\psi_3$$

$$\psi_4 = \begin{pmatrix} 0 \\ \chi \end{pmatrix} e^{-i(-m)t} \rightarrow \gamma^0 \begin{pmatrix} 0 \\ \chi \end{pmatrix} e^{-i(-m)t} = \begin{pmatrix} 0 \\ -\chi \end{pmatrix} e^{-i(-m)t} = -\psi_4$$

→ Generic frame:

$$\begin{cases} U_P \psi(\mathbf{r}, t) U_P^{-1} = \eta_P \gamma^0 \psi(-\mathbf{r}, t) \\ U_P \bar{\psi}(\mathbf{r}, t) U_P^{-1} = \eta_P^* \bar{\psi}(-\mathbf{r}, t) \gamma^0 \end{cases}$$

Therefore, for fermion particle-antiparticle:

$$\rightarrow \eta_f \eta_{\bar{f}} = \eta_P (-\eta_P^*) = -1$$

Intrinsic Parity - VI

Absolute intrinsic parity *not defined* for most particle states

Baryon particles: Baryonic number exactly conserved by all interactions

Charged particles: Electric charge as above

→ *Superselection rules*, limiting the quantum Superposition Principle:

Cannot superpose states with different values of exactly conserved quantum numbers

(Also true for integer/half integer spin; *almost* true for other quantum numbers as well, *by neglecting weak interactions*: Strangeness, Charm, Leptonic family numbers, ..)

Then phase factor unconstrained, can be fixed by convention

→ Take it *real*

Intrinsic Parity - VII

→ $U_P^2 = I$ for bosons → $\eta_P = \pm 1 = \eta_P^*$

→ Boson, Antiboson : Same intrinsic parity

Can't take $U_P^2 = I$ for fermions:

Funny behavior of fermion states under rotations

$$U_R(\hat{n}, \theta)|\psi\rangle = e^{i\mathbf{J}\cdot\hat{n}\theta}|\psi\rangle \rightarrow U_R(\hat{z}, 2\pi)|\psi\rangle = e^{iJ_z 2\pi}|\psi\rangle = e^{i\frac{1}{2}2\pi}|\psi\rangle = -|\psi\rangle$$

(← Compare integer/half integer superselection rule)

$$\left. \begin{array}{l} I = U_R(\hat{z}, 4\pi) = U_R^2(\hat{z}, 2\pi) \\ U_P^2 = U_R(\hat{z}, 2\pi) \end{array} \right\} \rightarrow I = U_P^4 \text{ for Fermions} \rightarrow \eta_P = \pm 1, \textcircled{\pm i}$$

Funny result! Not relevant in the Standard Model (Maybe *relevant* beyond..)

Take $\eta_P = \pm 1$ by convention

→ Fermion, Antifermion: Opposite intrinsic parity

General convention for Leptons & Quarks

<i>+ve parity particles</i>	<i>-ve parity antiparticles</i>
Quarks	Antiquarks
Leptons	Antileptons

Intrinsic Parity - VIII

Observe:

$$\varphi \text{ Hermitian} \rightarrow \varphi = \varphi^\dagger \rightarrow \eta_P^* = \eta_P \rightarrow \eta_P \text{ real} \rightarrow \eta_P = \pm 1$$

True for *really neutral* particles (*Photon, π^0 , J/ψ , ...*):

All additive quantum numbers = 0, like vacuum state

(Charge, Baryonic/Leptonic number, Strangeness, ...)

Indeed, such particles can be singly emitted and absorbed without violating any conservation law

Absolute parity *is* defined and measurable upon defining vacuum state parity (usually = +1)

Intrinsic Parity - Photon

Classical EM field

Standard convention:

$$\left. \begin{array}{l} P: \mathbf{E} \rightarrow -\mathbf{E} \\ P: \mathbf{B} \rightarrow \mathbf{B} \\ \mathbf{E} = -\frac{\partial \mathbf{A}}{\partial t} - \nabla \varphi \\ \mathbf{B} = \nabla \times \mathbf{A} \end{array} \right\} \rightarrow \left\{ \begin{array}{l} P: \mathbf{A} \rightarrow -\mathbf{A} \\ P: \varphi \rightarrow \varphi \end{array} \right.$$

\mathbf{A} : Vector field

φ : Not needed for radiation field

Quantum EM field

\mathbf{A} : Vector field

Photon spin = 1

→ *Negative intrinsic parity*

Interesting question: EM interactions conserve parity

→ *Photons from reactions, decays should have defined, total parity*

Photon Total Parity

What is the photon parity in a given state?

Can expand any \mathbf{A} into creation & annihilation operators

$$\mathbf{A}(\mathbf{r}, t) = \sum_{\mathbf{k}} \sum_{i=1}^2 \frac{1}{\sqrt{|\mathbf{k}|V}} \left[a_i(\mathbf{k}) \boldsymbol{\varepsilon}_i e^{i(\mathbf{p}_\gamma \cdot \mathbf{x} - E_\gamma t)} + a_i^\dagger(\mathbf{k}) \boldsymbol{\varepsilon}_i^* e^{-i(\mathbf{p}_\gamma \cdot \mathbf{x} - E_\gamma t)} \right]$$

But: $a_i(\mathbf{k})$ do not commute with parity. Expand instead into J, m C&A operators

$$\mathbf{A}(\mathbf{r}, t) = \sum_{J, m} \sum_{L=J-1}^{J+1} \sum_{i=1}^2 \frac{1}{\sqrt{|\mathbf{k}|V}} \left[a_i(J, L, m) \boldsymbol{\varepsilon}_i Y_{Jm}^L(\theta, \varphi) + a_i^\dagger(J, L, m) \boldsymbol{\varepsilon}_i^* Y_{Jm}^{L*}(\theta, \varphi) \right]$$

Photon states with defined total parity: states with given J, L, m .

Indeed:

$$\mathbf{J} = \mathbf{L} + \underbrace{\mathbf{S}}_{=1} \rightarrow J = L, L \pm 1 \rightarrow L = J, J \pm 1$$

$$\rightarrow \eta_P = (-1)^L (-1) = (-1)^{L+1} = (-1)^J, (-1)^{J \pm 1}$$

$$\rightarrow \begin{cases} \eta_P = (-1)^J & \text{Electric} \\ \eta_P = (-1)^{J \pm 1} & \text{Magnetic} \end{cases} \quad \text{2 kinds of photons for any given } J$$

Intrinsic Parity - Scalars

Really neutral states decaying into two photons

Example: π^0 , $spin = 0$

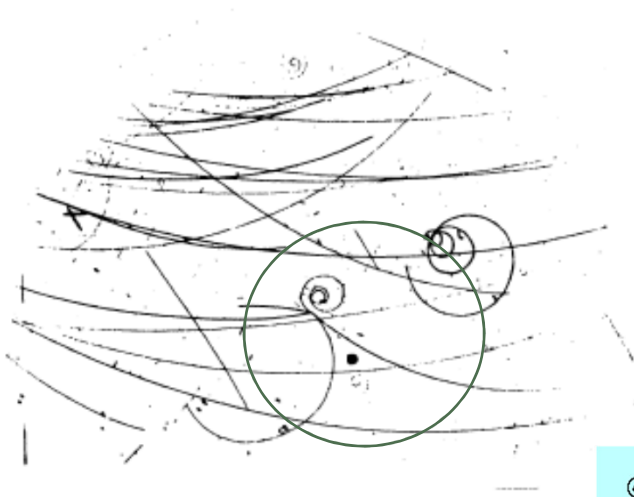
$$\pi^0 \rightarrow \gamma\gamma$$

Parity conserved by EM interaction

Final state: 2 creation operators $\propto \boldsymbol{\varepsilon}_1, \boldsymbol{\varepsilon}_2$

Can determine parity by looking at distribution of the angle between $\boldsymbol{\varepsilon}_1, \boldsymbol{\varepsilon}_2$

Observe: Double Dalitz decays $\pi^0 \rightarrow e^+e^-e^+e^-$



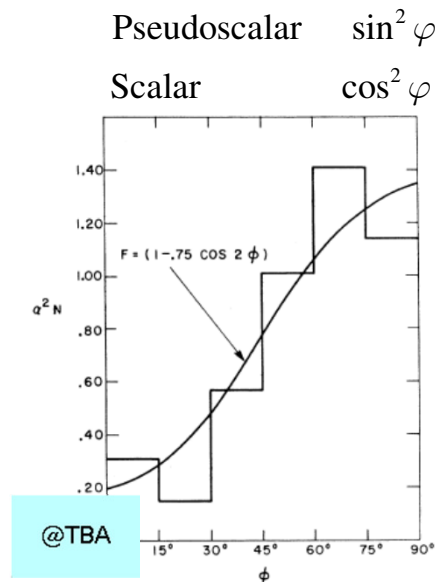
π^- stopping in liq. H_2

$$\pi^- + p \rightarrow \pi^0 + n \rightarrow e^+e^-e^+e^- n$$

$$\boldsymbol{\varepsilon}_1 \perp \text{plane}_1(\mathbf{p}_+, \mathbf{p}_-)$$

$$\boldsymbol{\varepsilon}_2 \perp \text{plane}_2(\mathbf{p}_+, \mathbf{p}_-)$$

@TBA



@TBA

Intrinsic Parity- Vectors

Most interesting result:
Yang Theorem

A vector ($s=1$) state cannot decay into 2 photons

Consequence of:

Angular momentum conservation

Parity conservation

Bose symmetry for identical photons

Gauge invariance of QED (transversality of photons)

Examples:

$\rho, \omega, \phi, J/\psi, Y, Z^0, ..$

Intrinsic Parity – Mesons

Examples

Charged Pion

$\pi^- + d \rightarrow n + n$ at low energy

$$\eta_P^{\pi^-} (\eta_P^N)^2 (-1)^L = (\eta_P^N)^2 (-1)^{L'}$$

Initial state: $L = 0 \rightarrow J = 0 \oplus 1 = 1$

Final state: $J' = 1$, two identical fermions

$L = 0(\text{symm}) \rightarrow S = 1(\text{symm}) \rightarrow \text{Symm}$ KO

$L = 1(\text{antisymm})$

$$\rightarrow S = \begin{cases} 0(\text{antisymm}) \rightarrow \text{Symm} & \text{KO} \\ 1(\text{symm}) \rightarrow \text{Antisymm} & \text{OK} \end{cases}$$

\rightarrow P-wave, triplet $\rightarrow L' = 1$

$$\rightarrow \eta_P^{\pi^-} (-1)^0 = (-1)^1 \rightarrow \eta_P^{\pi^-} = -1$$

by taking $\eta_P^n = \eta_P^p = +1$ by convention

Charged Kaon

Hypernucleus
One neutron $\rightarrow \Lambda^0$

$K^- + {}^4\text{He} \rightarrow \pi^0 + {}^4_\Lambda\text{H}$ at low energy

$$\eta_P^{K^-} \eta_P^{4\text{He}} (-1)^L = \eta_P^{\pi^0} \eta_P^{{}^4_\Lambda\text{H}} (-1)^{L'}$$

Initial state: $L = 0, S_{K^-} = 0 \rightarrow J = 0$

Final state: $J' = 0, S_{{}^4_\Lambda\text{H}} = 0 \rightarrow L' = 0$

$$\rightarrow \eta_P^{K^-} \cdot 1 \cdot (-1)^0 = (-1) \cdot 1 \cdot (-1)^0 \rightarrow \eta_P^{K^-} = -1$$

by taking $\eta_P^{\Lambda^0} = \eta_P^p = +1$ by convention

$$= (+1)^4 (-1)^0 = +1$$

Ground state $L=0$

Two-Particle States

Fermion-Antifermion: CM frame

$$\eta_P^{\bar{f}f} = \underbrace{(-1)}_{\text{intrinsic}} \underbrace{(-1)^l}_{\text{orbital}} = (-1)^{l+1}$$

Boson-Antiboson: CM frame

$$\eta_P^{\bar{b}b} = \underbrace{(\eta_P)^2}_{\text{intrinsic}} \underbrace{(-1)^l}_{\text{orbital}} = (-1)^l$$

Photon-photon: CM frame

Take single photon states with defined helicity:

$$U_P |\mathbf{k}, R; -\mathbf{k}, R\rangle = |-\mathbf{k}, L; \mathbf{k}, L\rangle \quad \text{not a } U_P \text{ eigenstate}$$

$$U_P |\mathbf{k}, L; -\mathbf{k}, L\rangle = |-\mathbf{k}, R; \mathbf{k}, R\rangle \quad \text{not a } U_P \text{ eigenstate}$$

$$U_P |\mathbf{k}, R; -\mathbf{k}, L\rangle = |-\mathbf{k}, L; \mathbf{k}, R\rangle \equiv |\mathbf{k}, R; -\mathbf{k}, L\rangle \quad U_P \text{ eigenstate, } \eta_P = +1, J_3 = +2$$

$$U_P |\mathbf{k}, L; -\mathbf{k}, R\rangle = |-\mathbf{k}, R; \mathbf{k}, L\rangle \equiv |\mathbf{k}, L; -\mathbf{k}, R\rangle \quad U_P \text{ eigenstate, } \eta_P = +1, J_3 = -2$$

$$|\mathbf{k}, R; -\mathbf{k}, R\rangle \pm |\mathbf{k}, L; -\mathbf{k}, L\rangle \quad U_P \text{ eigenstates, } \eta_P = \pm 1, J_3 = 0$$

Charge Conjugation - I

Charge conjugation operation: similar to parity

But: Operating on ‘internal’, rather than spatial, coordinates

Change sign of all additive quantum numbers Q, B, S, C, \dots

Like parity, unitary transformation in state space:

$$C: |\psi\rangle \equiv |\alpha; Q, B, S, C, \dots\rangle \rightarrow |\psi'\rangle = U_C |\psi\rangle \equiv \eta_C |\alpha; -Q, -B, -S, -C, \dots\rangle$$

[a : All other quantum numbers, like energy, position, momentum, angular momentum, ...]

Corresponding ‘Relativity Principle’ (to be checked by experiment)

The laws of physics are the same for all the observers, irrespective of the chosen positive direction of their ‘charge axis’

Experimental fact:

All interactions, except weak interaction, do conserve charge parity

Charge Conjugation - II

Commutation relations with observables

U_C commutes with all observables, except those yielding additive quantum numbers :

$$[U_C, \mathbf{r}] = [U_C, \mathbf{p}] = [U_C, \mathbf{L}] = 0$$

$$U_C Q = -Q U_C, U_C B = -B U_C, U_C S = -S U_C, \dots$$

Action on states

$$U_C : |Q\rangle \rightarrow |Q'\rangle \equiv U_C |Q\rangle = \eta_C |-Q\rangle, \eta_C \text{ arbitrary phase} = e^{i\alpha}$$

Charge Conjugation - III

C. Fundamental property of charge conjugation operator

$$C: Q \rightarrow Q' = C(Q) = -Q, C: Q' \rightarrow Q'' = C(Q') = -Q' = Q \rightarrow C^2 = I$$

$$U_c |\psi\rangle = |\psi'\rangle, U_c |\psi'\rangle = U_c (U_c |\psi\rangle) = U_c^2 |\psi\rangle \rightarrow U_c^2 = \eta^2 I$$

$$\left. \begin{array}{l} U_c^\dagger = U_c^{-1}, U_c \text{ unitary} \\ U_c^2 = I \end{array} \right\} \rightarrow U_c^2 = U_c U_c = I = U_c U_c^{-1} = U_c U_c^\dagger \rightarrow U_c^\dagger = U_c$$

→ U_c hermitian

→ U_p eigenvalues are real

→ $\eta = e^{i\alpha} = \pm 1 \equiv \eta_c$ charge parity quantum number

→ U_c eigenstates: $U_c |a\rangle = \pm |a\rangle$

Very few particles are U_c eigenstates:

As before, must have

$$Q=B=S=C=\dots=0$$

Applications - I

$$j_\mu = (\rho, \mathbf{j}) \xrightarrow{C} (-\rho, -\mathbf{j}) = -j_\mu$$

$$A_\mu = (\varphi, \mathbf{A}) \xrightarrow{C} -(\varphi, \mathbf{A}) = -A_\mu \rightarrow U_C |\gamma\rangle = (-1) |\gamma\rangle$$

$$j^\mu A_\mu \xrightarrow{C} j^\mu A_\mu$$

$$U_C |n\gamma\rangle = (-1)^n |n\gamma\rangle$$

Boson-antiboson: CM frame

$$U_C |b\bar{b}\rangle = U_P |b\bar{b}\rangle = (-1)^l |b\bar{b}\rangle \rightarrow \eta_C = (-1)^l$$

Fermion-antifermion: CM frame

$U_C = U_P U_S$ Parity * Spin exchange

$$U_P |f \bar{f}\rangle = (-1)^{l+1} |f\bar{f}\rangle$$

$$U_S |\uparrow\downarrow - \downarrow\uparrow\rangle = |\downarrow\uparrow - \uparrow\downarrow\rangle = -|\uparrow\downarrow - \downarrow\uparrow\rangle, \quad U_S |\uparrow\downarrow + \downarrow\uparrow\rangle = |\downarrow\uparrow + \uparrow\downarrow\rangle = |\uparrow\downarrow + \downarrow\uparrow\rangle,$$

$$U_S |\uparrow\uparrow\rangle = |\uparrow\uparrow\rangle, \quad U_S |\downarrow\downarrow\rangle = |\downarrow\downarrow\rangle$$

$$\rightarrow U_C |f \bar{f}\rangle = (-1)^{l+s} |f\bar{f}\rangle$$

Applications - II

π^0, η, \dots : $\eta_C = (-1)^2 = +1$ 2 photons decays

States $f \bar{f}$ decaying to n photons:

$$C: (-1)^{l+s} \rightarrow S \text{ state} \begin{cases} \eta_C = +1 \text{ singlet} \\ \eta_C = -1 \text{ triplet} \end{cases}$$

$$n \text{ photons: } \eta_C = (-1)^l \rightarrow \begin{cases} 2 \text{ photons } \eta_C = +1 \\ 3 \text{ photons } \eta_C = -1 \end{cases}$$

$$\rightarrow \begin{cases} \text{singlet} \rightarrow 2 \text{ photons} \\ \text{triplet} \rightarrow 3 \text{ photons} \end{cases}$$

Meson: Fermion-Antifermion bound state

L	$S=0$	$S=1$
0	0^+	1^-
1	1^+	$0^{++}, 1^{++}, 2^{++}$
2	2^+	$1^-, 2^-, 3^-$

States decaying to 2 pions

J	$PC = ++$	$PC = +-, -+, --$
0	OK	KO
1	KO	KO, KO, OK
2	OK	KO

Time Reversal

Need to make clear at the outset:

These considerations are relevant to physical systems with a (very) small number of degrees of freedom

Do not apply to complex systems, whose time evolution is driven by the II Law of thermodynamics. For them, a *time arrow* can be defined with no ambiguities. Or so we believe... (A *very* hard subject)

Can state another ‘relativity principle’, about the choice of the positive direction of the time coordinate of physical events:

*The laws of physics are the same for all the observers,
irrespective of the chosen positive direction of their ‘time axis’*

Experimental fact:

All interactions, except weak interaction, are invariant wrt to time reversal

Classical Physics

General behavior of physical quantities wrt to time reversal:

$$t \rightarrow -t :$$

$$\mathbf{r} \rightarrow \mathbf{r}$$

$$\mathbf{v} = \frac{d\mathbf{r}}{dt} \rightarrow \frac{d\mathbf{r}}{d(-t)} = -\mathbf{v}, \mathbf{p} \rightarrow -\mathbf{p}$$

$$\mathbf{a} = \frac{d^2\mathbf{r}}{dt^2} \rightarrow \mathbf{a}, \mathbf{F} \rightarrow \mathbf{F}$$

$$\mathbf{L} = \mathbf{r} \times \mathbf{p} \rightarrow -\mathbf{L}$$

$$E \rightarrow E$$

$$\rho \rightarrow \rho$$

$$\mathbf{j} = \rho\mathbf{v} \rightarrow -\mathbf{j}$$

$$\varphi \rightarrow \varphi$$

$$\mathbf{A} \rightarrow -\mathbf{A}$$

$$\mathbf{E} \rightarrow \mathbf{E}$$

$$\mathbf{B} \rightarrow -\mathbf{B}$$

Observe: Ohm's law is *not* TR invariant

$$\mathbf{j} = \sigma\mathbf{E} \rightarrow -\mathbf{j} = \sigma\mathbf{E}, \text{ take } \sigma \text{ as TR invariant}$$

Example of a *macroscopic* law...

Schrodinger Equation

$$T : t \rightarrow t' = -t$$

First guess...

$$\langle t' | \psi' \rangle = \langle t | \psi \rangle = \langle -t' | \psi \rangle$$

$$\langle t | \psi' \rangle = \langle -t | \psi \rangle$$

$$\rightarrow \psi'(t) = \psi(-t)$$

$$U_T : |\psi\rangle \rightarrow |\psi'\rangle \equiv U_T |\psi\rangle$$

$$U_T : |t\rangle \rightarrow |t'\rangle \equiv U_T |t\rangle = \eta |-t\rangle \quad \text{Wrong!}$$

Indeed:

$$H |\psi(t)\rangle = i \frac{\partial |\psi(t)\rangle}{\partial t} \xrightarrow{U_T} H |\psi(-t)\rangle = -i \frac{\partial |\psi(-t)\rangle}{\partial t}$$

Redefine:

$$\left. \begin{array}{l} T : t \rightarrow -t \\ K : i \rightarrow -i \end{array} \right\}, U_T = KT$$

$$|\psi(t)\rangle \rightarrow U_T |\psi(t)\rangle = |\psi^T(t)\rangle = |\psi^*(-t)\rangle \quad \text{OK}$$

Time Reversal - I

Take a plane wave:

$$e^{-i(Et-\mathbf{p}\cdot\mathbf{r})} \xrightarrow{U_T} e^{+i(E(-t)-\mathbf{p}\cdot\mathbf{r})} = e^{-i(Et+\mathbf{p}\cdot\mathbf{r})} = e^{-i(Et-(-\mathbf{p})\cdot\mathbf{r})}$$

Quite natural...:

U_T sends a progressive plane wave
into a regressive one

Puzzled? Think of it in this way:

U_T inverts the positive direction of time for the 2nd observer. Therefore, the 2nd observer has a clock which is running *backwards*.

Then the same wave which is seen as *progressive* by the first observer, is seen as *regressive* by the second.

Rather simple for spinless particles..

Time Reversal - II

Take a particle with $s=1/2$:

$$\begin{pmatrix} \psi_+(t) \\ \psi_-(t) \end{pmatrix} \rightarrow \begin{pmatrix} \psi_+^T(t) \\ \psi_-^T(t) \end{pmatrix} = M \begin{pmatrix} \psi_+^*(-t) \\ \psi_-^*(-t) \end{pmatrix}$$

Now, must take into account that \mathbf{s} is *reversed* by T

$$\sigma_y = \begin{pmatrix} 0 & -i \\ +i & 0 \end{pmatrix} \rightarrow M = -i\sigma_y = \begin{pmatrix} 0 & -1 \\ +1 & 0 \end{pmatrix}$$

Spin-reversing operator

$$\begin{pmatrix} \psi_+(t) \\ \psi_-(t) \end{pmatrix} \xrightarrow{U_T} \begin{pmatrix} \psi_+^T(t) \\ \psi_-^T(t) \end{pmatrix} = -i\sigma_y \begin{pmatrix} \psi_+^*(-t) \\ \psi_-^*(-t) \end{pmatrix} = \begin{pmatrix} -\psi_-^*(-t) \\ +\psi_+^*(-t) \end{pmatrix} \quad \text{OK}$$

Apply U_T a second time:

$$\begin{pmatrix} \psi_+^T(t) \\ \psi_-^T(t) \end{pmatrix} \xrightarrow{U_T} -i\sigma_y \begin{pmatrix} -\psi_-^*(t) \\ +\psi_+^*(t) \end{pmatrix}^* = \begin{pmatrix} -\psi_+(t) \\ -\psi_-(t) \end{pmatrix} = - \begin{pmatrix} \psi_+(t) \\ \psi_-(t) \end{pmatrix} \quad \text{Similar to parity for spin 1/2}$$

Applications

Basic remark:

U_T is *not* a unitary operator

U_T is *not* a Hermitian operator

Therefore:

No real eigenvalues

No (conserved) observables

→ No 'time parity'

Nevertheless:

Quite useful to establish properties of fundamental interactions

Because is not linear, rather it is antilinear

$$U_T (a|\psi_1\rangle + b|\psi_2\rangle) = a^*U_T|\psi_1\rangle + b^*U_T|\psi_2\rangle$$

Complex conjugation

$$K : \alpha z \rightarrow (\alpha z)^* = \alpha^* z^*$$

is a non-linear operation.

Proof:

$$\alpha = ae^{i\varphi}, z = \rho e^{i\psi} \rightarrow \alpha^* = ae^{-i\varphi}, z^* = \rho e^{-i\psi}$$

$$\alpha z = ae^{i\varphi} \rho e^{i\psi} = a\rho e^{i(\varphi+\psi)} \rightarrow (\alpha z)^* = a\rho e^{-i(\varphi+\psi)} = \alpha^* z^*$$

But: $\alpha^* z^* \neq \alpha z^* \rightarrow$ Non linear whenever $\alpha \neq \alpha^* !!$

Indeed, a linear operator must satisfy:

$$A : (\alpha z) \rightarrow A(\alpha z) = \alpha A(z)$$

Kramers Degeneracy etc

Wave functions:

$$\psi_T(x) = \langle x | \psi_T \rangle = \langle x | U_T | \psi \rangle = \langle x | \psi \rangle^* = \psi^*(x) \quad \text{C.Conjugate}$$

If $H | \psi \rangle = E | \psi \rangle$, $[H, U_T] = 0$, $| \psi \rangle$ non-degenerate

$$\rightarrow H | \psi_T \rangle = H U_T | \psi \rangle = U_T H | \psi \rangle = E U_T | \psi \rangle = E | \psi_T \rangle \rightarrow | \psi_T \rangle = | \psi \rangle$$

$\rightarrow \psi(x)$ real

Degeneracy of fermion states:

$$U_T^2 | \psi \rangle = - | \psi \rangle$$

$$\rightarrow \langle \psi | \psi_T \rangle = \langle \psi_T | \psi \rangle^* = \langle \psi | U_T^\dagger | \psi \rangle = - \langle \psi | U_T^\dagger U_T^2 | \psi \rangle = - \langle \psi | U_T | \psi \rangle = - \langle \psi | \psi_T \rangle$$

$$\rightarrow \langle \psi | \psi_T \rangle = 0$$

$\rightarrow | \psi \rangle, | \psi_T \rangle$ orthogonal, independent

Example (trivial):

Spin $\frac{1}{2}$ particle in rest frame \rightarrow 2 independent states (up & down)

Effect on Scalar Products

$\{|i\rangle, i = 1, \dots\}$ complete set of states

$$\langle i|j\rangle = \delta_{ij}$$

$\rightarrow \langle U_T i|U_T j\rangle = \delta_{ij}$ norm is U_T - invariant

$$|\psi_1\rangle = \sum_i |i\rangle \langle i|\psi_1\rangle \rightarrow |U_T \psi_1\rangle = \sum_i |U_T i\rangle \langle i|\psi_1\rangle^*$$

$$\langle \psi_2| = \sum_j \langle \psi_2|j\rangle \langle j| \rightarrow \langle U_T \psi_2| = \sum_j \langle \psi_2|j\rangle^* \langle U_T j|$$

$$\rightarrow \langle U_T \psi_2|U_T \psi_1\rangle = \sum_j \langle \psi_2|j\rangle^* \langle U_T j|\sum_i |U_T i\rangle \langle i|\psi_1\rangle^*$$

$$= \sum_{i,j} \langle \psi_2|j\rangle^* \underbrace{\langle U_T j|U_T i\rangle}_{\delta_{ij}} \langle i|\psi_1\rangle^* = \sum_i \langle \psi_2|i\rangle^* \langle i|\psi_1\rangle^*$$

$$= \sum_i \langle i|\psi_2\rangle \langle \psi_1|i\rangle = \langle \psi_1|\psi_2\rangle \quad U_T \text{ swaps states in any scalar product}$$

Matrix Elements

Matrix element for a transition: Initial \leftrightarrow Final

H T -invariant \rightarrow Same matrix element for direct and reversed transition

$\underbrace{1+2}_i \rightarrow \underbrace{3+4}_f$ 2-body process

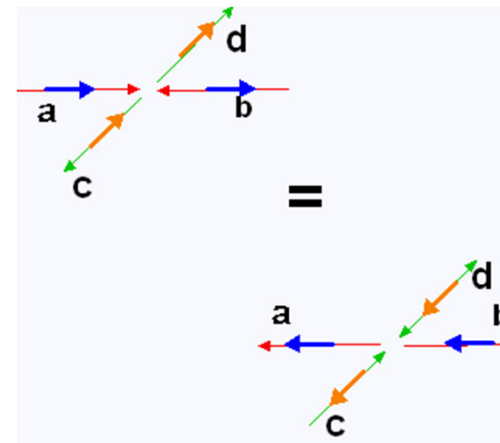
Reciprocity Theorem:

$$\langle \mathbf{p}_f, \mathbf{s}_f | S | \mathbf{p}_i, \mathbf{s}_i \rangle = \frac{\langle -\mathbf{p}_i, -\mathbf{s}_i | S | -\mathbf{p}_f, -\mathbf{s}_f \rangle^*}{T | \mathbf{p}_i, \mathbf{s}_i \rangle T \langle \mathbf{p}_f, \mathbf{s}_f |}$$

Detailed Balance Theorem:

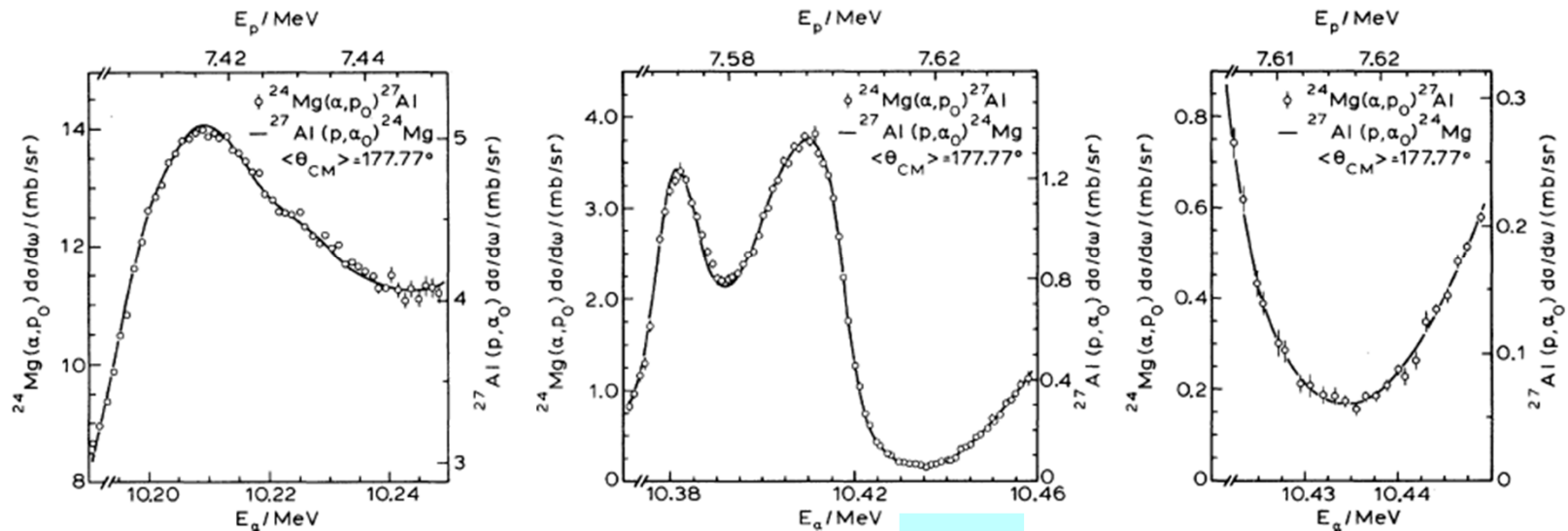
$$\frac{\frac{d\sigma_{if}}{d\Omega}}{\frac{d\sigma_{fi}}{d\Omega}} = \left(\frac{p_f}{p_i} \right)^2 \frac{(2s_3 + 1)(2s_4 + 1)}{(2s_1 + 1)(2s_2 + 1)}$$

OK for Strong, E.M., KO for Weak



Example: Test of Detailed Balance

- Nuclear reactions:
- (1) $\alpha + {}^{24}\text{Mg} \rightarrow p + {}^{27}\text{Al}$
 - (2) $p + {}^{27}\text{Al} \rightarrow \alpha + {}^{24}\text{Mg}$



@TBA

Full line: Reaction (1)
Circles: Reaction (2)

T-Violating Correlations

Example (many more exist):

$$\Lambda^0 \rightarrow p + \pi^-$$

$$\left. \begin{aligned} \frac{1^+}{2} &= \frac{1^+}{2} \oplus 0^- \oplus L^{(-1)^L} = \frac{1}{2} \oplus L \\ + &= + \cdot - \cdot (-1)^L = (-1)^{L+1} \end{aligned} \right\} \rightarrow L = 0, 1$$

Decay amplitude = Sum of S,P waves
S,P waves do interfere because of
 parity violation. So:
 Angular distribution has a term $\propto \cos \theta$

Most general form of angular distribution:

$$\frac{d\Gamma}{d\Omega_\pi} = 1 + A(\mathbf{J}_\Lambda + \mathbf{J}_p) \cdot \hat{\mathbf{p}}_\pi + B(\mathbf{J}_\Lambda \times \mathbf{J}_p) \cdot \hat{\mathbf{p}}_\pi + C(\mathbf{J}_\Lambda \cdot \mathbf{J}_p) + (1 - C)(\mathbf{J}_\Lambda \cdot \hat{\mathbf{p}}_\pi)(\mathbf{J}_p \cdot \hat{\mathbf{p}}_\pi)$$

$$(\mathbf{J}_\Lambda \times \mathbf{J}_p) \cdot \hat{\mathbf{p}}_\pi \xrightarrow{U_T} ((-\mathbf{J}_\Lambda) \times (-\mathbf{J}_p)) \cdot (-\hat{\mathbf{p}}_\pi) = -(\mathbf{J}_\Lambda \times \mathbf{J}_p) \cdot \hat{\mathbf{p}}_\pi \quad \text{T-Violating term}$$

→ Expect $B = 0$

Must correct for final state interaction: $p\pi$ scattering

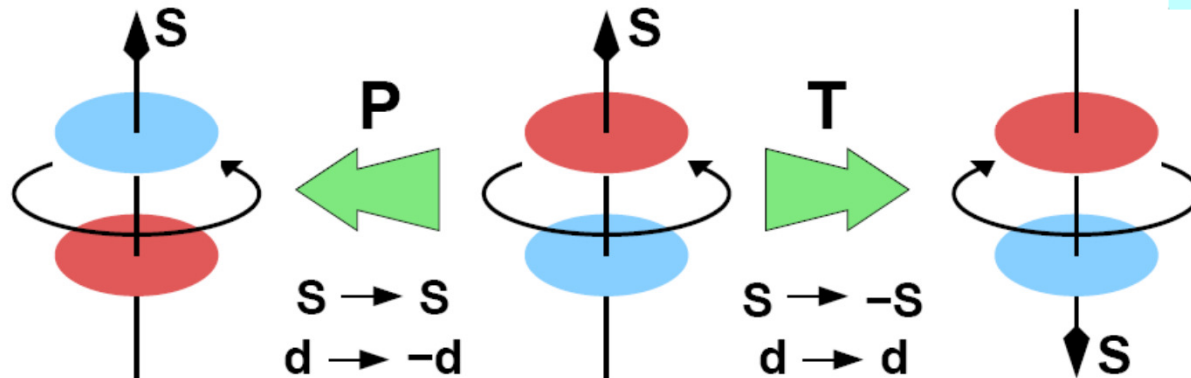
→ Get an upper limit of a few %

Electric Dipole Moments - I

Electric Dipole Moments

Elementary particle can only have a permanent electric dipole moment if both **parity** and **time reversal** symmetries are broken:

@TBA



$$\vec{\mu} = g \frac{e}{mc} \vec{S} \quad \vec{d} = \eta \frac{e}{2mc} \vec{S}$$

$d \parallel S$, otherwise there would be an extra degree of freedom which is not seen

Warning: True in the absence of a degenerate ground state

E.g., polar molecules like H_2O *do* have degenerate ground states...

Electric Dipole Moments - II

For a particle with both magnetic and electric dipole moments:

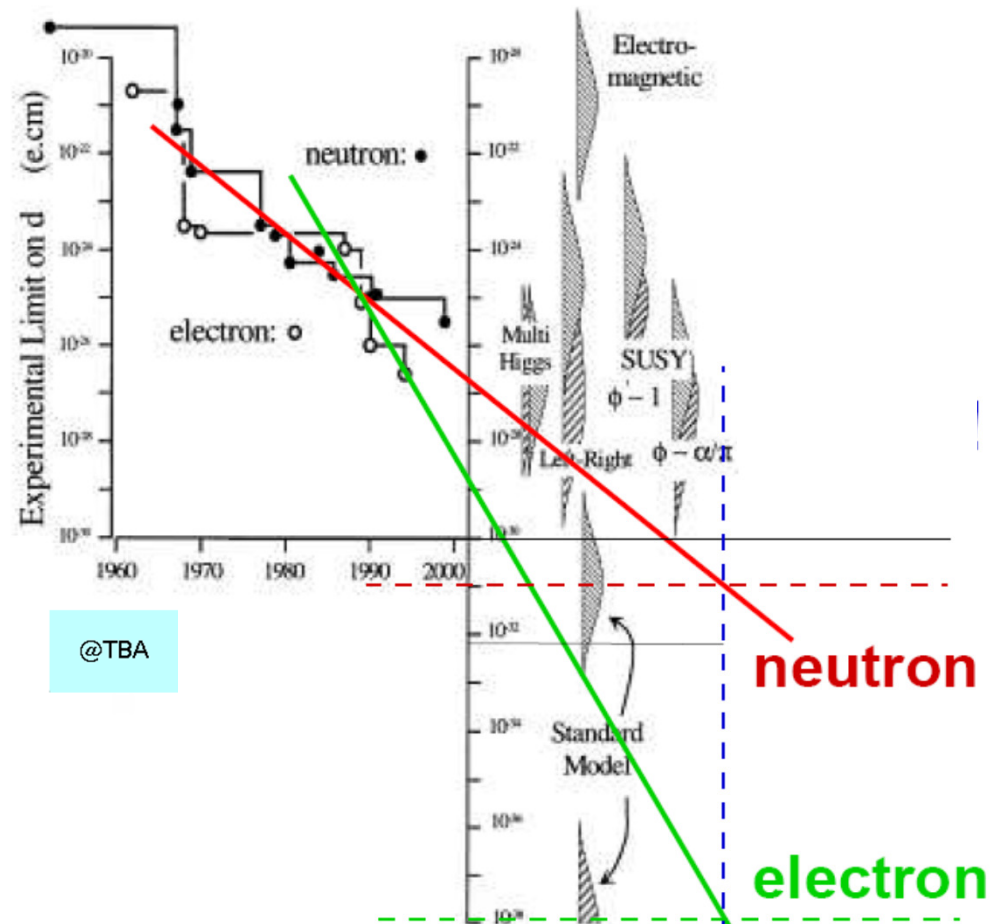
$$H' = -\boldsymbol{\mu} \cdot \mathbf{B} - \mathbf{d} \cdot \mathbf{E} \text{ Interaction energy}$$

Can observe in external \mathbf{E} :

Level shift

Spin precession

Status of the EDM



C, P, T in Quantum Field Theory

Just a few, oversimplified remarks:

Observables are (functions of) field operators

Just as in Quantum Mechanics, one finds different kinds of operators:

Scalars, Spinors, Vectors, ...

Fields of each kind have their own 'parity' built-in

States of a given field do inherit field 'parity', which is called *intrinsic parity*

C,P and T in QFT - I

1) Spin = 0

$$\phi(t, \mathbf{r}) = N \int d^3 \mathbf{p} [b(\mathbf{p}) e^{-ipx} + d^\dagger(\mathbf{p}) e^{+ipx}]$$

$$\phi^\dagger(t, \mathbf{r}) = N \int d^3 \mathbf{p} [d(\mathbf{p}) e^{-ipx} + b^\dagger(\mathbf{p}) e^{+ipx}]$$

$$\begin{cases} U_P \phi(t, \mathbf{r}) U_P^\dagger = \eta_P \phi(t, -\mathbf{r}) \\ U_P \phi^\dagger(t, \mathbf{r}) U_P^\dagger = \eta_P \phi^\dagger(t, -\mathbf{r}) \end{cases} \leftrightarrow \begin{cases} U_P \begin{matrix} b(\mathbf{p}) \\ d^\dagger(\mathbf{p}) \end{matrix} U_P^\dagger = \eta_P \begin{matrix} b(-\mathbf{p}) \\ d^\dagger(-\mathbf{p}) \end{matrix} \\ U_P \begin{matrix} b^\dagger(\mathbf{p}) \\ d(\mathbf{p}) \end{matrix} U_P^\dagger = \eta_P \begin{matrix} b^\dagger(\mathbf{p}) \\ d(\mathbf{p}) \end{matrix} \end{cases}, e^{\pm ipx} \rightarrow e^{\pm ipx}$$

$$\begin{cases} U_C \phi(t, \mathbf{r}) U_C^\dagger = \eta_C \phi^\dagger(t, \mathbf{r}) \\ U_C \phi^\dagger(t, \mathbf{r}) U_C^\dagger = \eta_C^* \phi(t, \mathbf{r}) \end{cases} \leftrightarrow \begin{cases} U_C \begin{matrix} b(\mathbf{p}) \\ d^\dagger(\mathbf{p}) \end{matrix} U_C^\dagger = \eta_C \begin{matrix} d(\mathbf{p}) \\ b^\dagger(\mathbf{p}) \end{matrix} \\ U_C \begin{matrix} b^\dagger(\mathbf{p}) \\ d(\mathbf{p}) \end{matrix} U_C^\dagger = \eta_C^* \begin{matrix} d^\dagger(\mathbf{p}) \\ b(\mathbf{p}) \end{matrix} \end{cases}, e^{\pm ipx} \rightarrow e^{\pm ipx}$$

$$\begin{cases} \bar{U}_T \phi(t, \mathbf{r}) \bar{U}_T^\dagger = \underbrace{\eta_T}_{=1} \phi(-t, \mathbf{r}) \\ \bar{U}_T \phi^\dagger(t, \mathbf{r}) \bar{U}_T^\dagger = \phi^\dagger(-t, \mathbf{r}) \end{cases} \leftrightarrow \begin{cases} \bar{U}_T \begin{matrix} b(\mathbf{p}) \\ d^\dagger(\mathbf{p}) \end{matrix} \bar{U}_T^\dagger = \begin{matrix} b(-\mathbf{p}) \\ d^\dagger(-\mathbf{p}) \end{matrix} \\ \bar{U}_T \begin{matrix} d(\mathbf{p}) \\ b^\dagger(\mathbf{p}) \end{matrix} \bar{U}_T^\dagger = \begin{matrix} d(-\mathbf{p}) \\ b^\dagger(-\mathbf{p}) \end{matrix} \end{cases}, e^{\pm ipx} \rightarrow e^{\mp ipx}$$

General:

Observables depend on $\phi \phi^\dagger$

Any observable yields a factor

$$\eta_P \eta_P^* = 1, \eta_C \eta_C^* = 1$$

$\rightarrow \eta_P, \eta_C$ unobservable

ϕ Hermitian:

$$\phi = \phi^\dagger$$

$$\rightarrow \eta_C = \eta_C^*, \eta_P = \pm 1$$

$$\rightarrow \begin{cases} \eta_P = \pm 1 \\ \eta_C = \pm 1 \end{cases} \text{ Observable}$$

η_T unobservable

C,P and T in QFT - II

2) Spin 1/2

$$\psi(t, \mathbf{r}) = \sum_s \int Nd^3\mathbf{p} [b(\mathbf{p}, s)u(\mathbf{p}, s)e^{+ipx} + d^\dagger(\mathbf{p}, s)v(\mathbf{p}, s)e^{-ipx}]$$

$$\psi^\dagger(t, \mathbf{r}) = \sum_s \int Nd^3\mathbf{p} [b^\dagger(\mathbf{p}, s)\bar{u}(\mathbf{p}, s)e^{-ipx} + d(\mathbf{p}, s)\bar{v}(\mathbf{p}, s)e^{+ipx}]$$

$$\begin{cases} U_P \psi(t, \mathbf{r}) U_P^\dagger = \eta_P \gamma^0 \psi(t, -\mathbf{r}) \\ U_P \bar{\psi}(t, \mathbf{r}) U_P^\dagger = \eta_P^* \bar{\psi}(t, -\mathbf{r}) \gamma^0 \end{cases} \leftrightarrow \begin{cases} U_P b(\mathbf{p}, s) U_P^\dagger = \eta_P b(-\mathbf{p}, s) \\ U_P d(\mathbf{p}, s) U_P^\dagger = \eta_P d(-\mathbf{p}, s) \\ U_P d^\dagger(\mathbf{p}, s) U_P^\dagger = \eta_P^* d^\dagger(-\mathbf{p}, s) \\ U_P b^\dagger(\mathbf{p}, s) U_P^\dagger = \eta_P^* b^\dagger(-\mathbf{p}, s) \end{cases}, e^{\pm ipx} \rightarrow e^{\pm ipx}$$

$$\begin{cases} U_C \psi(t, \mathbf{r}) U_C^\dagger = \eta_C i \gamma_2 \gamma_0 \bar{\psi}(t, \mathbf{r}) \\ U_C \bar{\psi}(t, \mathbf{r}) U_C^\dagger = \eta_C^* \psi(t, \mathbf{r}) i \gamma_2 \gamma_0 \end{cases} \leftrightarrow \begin{cases} U_C b(\mathbf{p}, s) U_C^\dagger = \eta_C d(\mathbf{p}, s) \\ U_C b^\dagger(\mathbf{p}, s) U_C^\dagger = \eta_C d^\dagger(\mathbf{p}, s) \\ U_C d(\mathbf{p}, s) U_C^\dagger = \eta_C^* b(\mathbf{p}, s) \\ U_C d^\dagger(\mathbf{p}, s) U_C^\dagger = \eta_C^* b^\dagger(\mathbf{p}, s) \end{cases}, e^{\pm ipx} \rightarrow e^{\pm ipx}$$

$$\begin{cases} \bar{U}_T \psi(t, \mathbf{r}) \bar{U}_T^\dagger = \underbrace{\eta_T}_{=1} \gamma_1 \gamma_3 \psi(-t, \mathbf{r}) \\ \bar{U}_T \bar{\psi}(t, \mathbf{r}) \bar{U}_T^\dagger = \bar{\psi}(-t, \mathbf{r}) (-\gamma_1 \gamma_3) \end{cases} \leftrightarrow \begin{cases} \bar{U}_T b(\mathbf{p}, s) \bar{U}_T^\dagger = b(-\mathbf{p}, -s) \\ \bar{U}_T d^\dagger(\mathbf{p}, s) \bar{U}_T^\dagger = d^\dagger(-\mathbf{p}, -s) \\ \bar{U}_T b^\dagger(\mathbf{p}, s) \bar{U}_T^\dagger = b^\dagger(-\mathbf{p}, -s) \\ \bar{U}_T d(\mathbf{p}, s) \bar{U}_T^\dagger = d(-\mathbf{p}, -s) \end{cases}, e^{\pm ipx} \rightarrow e^{\mp ipx}$$

CPT

Fundamental theorem applying to all field theories, including our beloved Standard Model:

Lorentz invariance

Micro-causality (whatever it means...)

Spin-statistics

→ *The product transformation CPT is a good symmetry*

Consequences:

$$m_{particle} = m_{antiparticle}$$

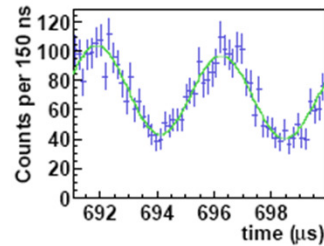
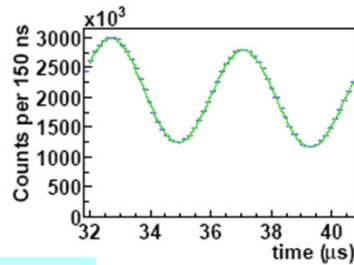
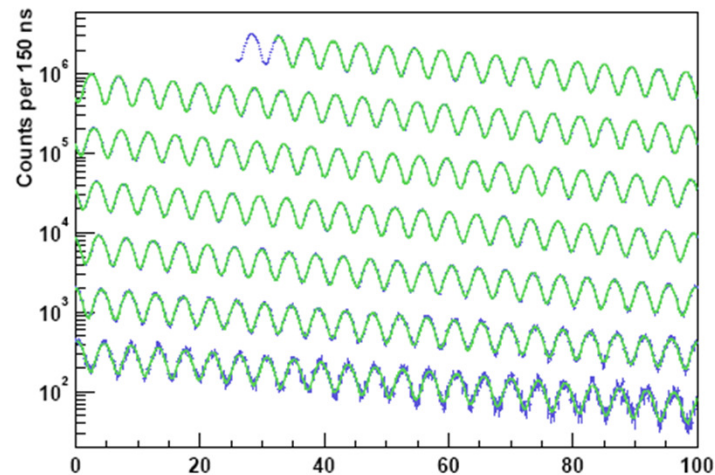
$$\tau_{particle} = \tau_{antiparticle}$$

$$|\boldsymbol{\mu}|_{particle} = |\boldsymbol{\mu}|_{antiparticle}$$

....more quantum numbers

QED Test: Muon Anomaly

a_μ : The Data



@TBA

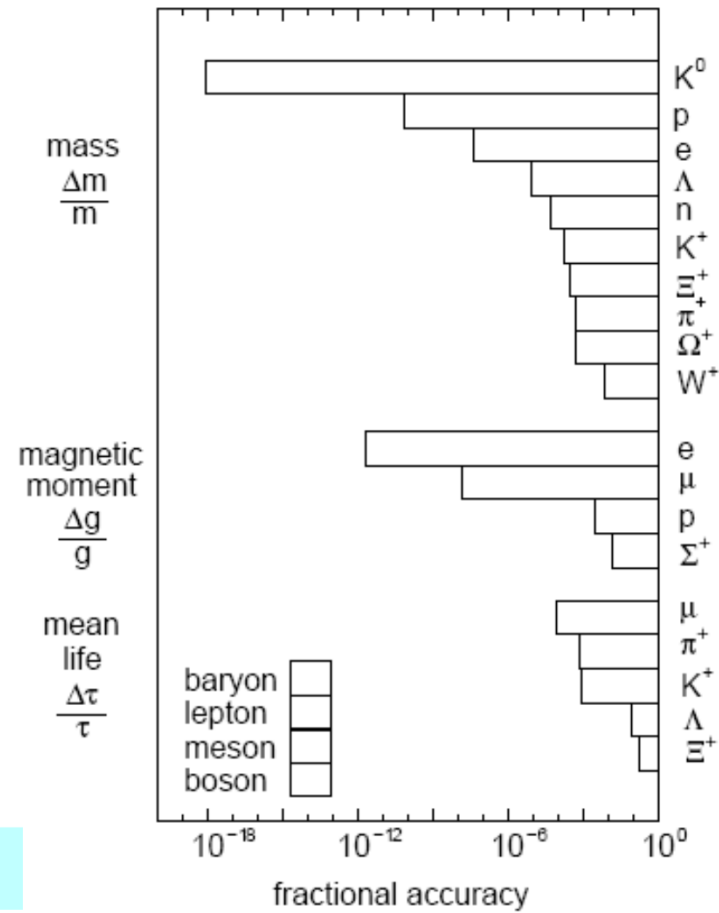
Also an excellent test
of CPT in EM interaction

$$\left. \begin{aligned} a_{\mu^+} &= (11659203 \pm 8) \times 10^{-10} \\ a_{\mu^-} &= (11659214 \pm 9) \times 10^{-10} \end{aligned} \right\} \Delta a_\mu = (11 \pm 12) \times 10^{-10}$$

CPT Tests

Just a flash on the general status of CPT tests...

...very comfortable



@TBA

Symmetries: Summary

Conserved quantity	Interaction		
	Strong	E.M.	Weak
4-momentum	OK	OK	OK
Charge	OK	OK	OK
Ang. Momentum, CM speed	OK	OK	OK
Baryonic number	OK	OK	OK
Leptonic numbers (3)	OK	OK	~OK
Parity	OK	OK	KO
Charge parity	OK	OK	KO
(Time reversal)	OK	OK	KO
CP	OK	OK	KO
(CPT)	OK	OK	OK
Flavor	OK	OK	KO