Elementary Particles I

2 – Electromagnetic Interaction

Form Factors, Structure Functions, Scaling, Partons

Leptons

Leptons

1st family	2nd family	3rd family
$\nu_e, \overline{\nu}_e$	$\nu_\mu, \overline{\nu}_\mu$	$\nu_\tau, \overline{\nu}_\tau$
e^-, e^+	μ^-, μ^+	τ^-, τ^+

Neutral, 'Massless'

Charged, Massive

"Pointlike", spin ½ Fermions
Electromagnetic and weak interactions

Lepton scattering by several targets as a powerful tool to probe constituents

- *Electromagnetic (and weak) coupling to leptons simple, well understood*

- *Small coupling constant→Perturbative expansion reliable*

Electromagnetic Interaction

Try to find transition amplitude for electromagnetic scattering

1st order perturbative contribution

$T_{fi} = -i\langle f | \int d^4x H' | i \rangle$ H': Interaction Hamiltonian density

$H' = j^\mu A_\mu$ Classical analogy, j_μ current 4-density

Reminder

For any system of charges and currents:

$u_E = \frac{1}{2}\rho\varphi$ Electrostatic potential energy density

$u_B = \frac{1}{2}\mathbf{j}\cdot\mathbf{A}$ Magnetostatic potential energy density

$j^\mu = (\rho, \mathbf{j})$ 4-current density

$A_\mu = (\varphi, \mathbf{A})$ 4-potential

An oversimplified example - I

First take a simple example:

Spinless, pointike “pion” scattering off a fixed, Coulomb potential

$$A_\mu = \left(\frac{eZ}{4\pi r}, \mathbf{0} \right)$$

$$j^\mu = (\rho, \mathbf{j}) = ie\left(\varphi^* \left(\frac{\partial\varphi}{\partial t} \right) - \left(\frac{\partial\varphi^*}{\partial t} \right)\varphi, \left(\left(\nabla\varphi^*\right)\varphi - \varphi^*\left(\nabla\varphi\right) \right) \right)$$

Usual definition of current density
ϕ: Stationary state

$$j^\mu = (\rho, \mathbf{j}) = ie\left(\left(\varphi'^* \left(\frac{\partial\varphi}{\partial t} \right) - \left(\frac{\partial\varphi'^*}{\partial t} \right)\varphi \right), \left(\left(\nabla\varphi'^*\right)\varphi - \varphi'^*\left(\nabla\varphi\right) \right) \right)$$

Generalize to a scattering state
ϕ,ϕ': Stationary states, plane waves

$$\rightarrow j^\mu = eNN'e^{-i\left((E-E')t-(\mathbf{p}-\mathbf{p}')\cdot\mathbf{r}\right)}\left((E+E'), (\mathbf{p}'+\mathbf{p}) \right)$$

$$\rightarrow j^\mu A_\mu = NN'(E+E')e^{-i\left((E-E')t-(\mathbf{p}-\mathbf{p}')\cdot\mathbf{r}\right)}\frac{e^2 Z}{4\pi r}$$

Integrate over time:

$$\int_{-\infty}^{+\infty} NN'e^{i(E-E')t}dt = NN'2\pi\delta(E-E')$$

Energy conservation; momentum *not conserved* by fixed Coulomb potential

An oversimplified example - II

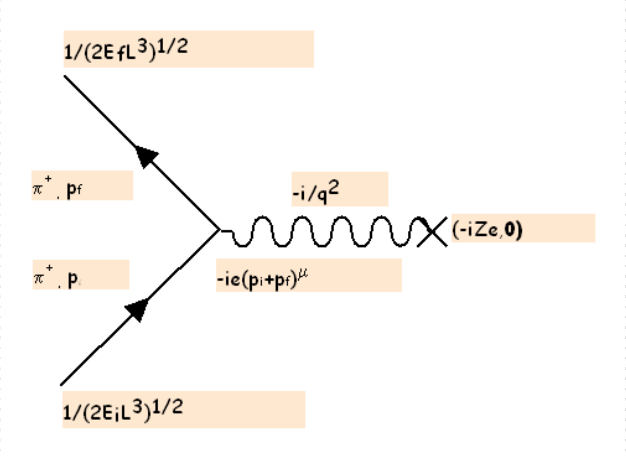
Integrate over space:

$$\int e^{+i((\mathbf{p}-\mathbf{p}')\cdot\mathbf{r})}\frac{e^2 Z}{4\pi r}d^3\mathbf{r} = \frac{Ze^2}{|\mathbf{q}|^2}, \mathbf{q} = \mathbf{p} - \mathbf{p}'$$

Matrix element:

$$T_{fi} = -i\langle f | \int d^4 x H' | i \rangle = -NN' 2\pi i \delta(E - E') \frac{Ze^2}{|\mathbf{q}|^2}$$

$$E = E' \rightarrow |\mathbf{q}|^2 = -q^2 \rightarrow T_{fi} = NN' 2\pi i \delta(E - E') \frac{Ze^2}{q^2}$$



Virtual photon:
Coupling fixed source to current

An oversimplified example - III

Evaluate transition probability

$$w = \frac{\left|T_{fi}\right|^2}{T}$$ Transition probability/Time

$$\left|\delta\left(E' - E\right)\right|^2 = \lim_{T\to\infty}\left|\frac{1}{2\pi}\int_{-T/2}^{+T/2} e^{i(E'-E)t}dt\right|^2 = \lim_{T\to\infty}\left|\frac{\sin\left[\left(E'-E\right)T/2\right]}{\pi\left(E'-E\right)}\right|^2 = \frac{T}{2\pi}\delta\left(E'-E\right)$$

$$w = N^2 N'^2 \frac{4\pi^2}{T}\left|\delta\left(E-E'\right)\right|^2 \frac{Z^2 e^4}{\left|\mathbf{q}\right|^4} = N^2 N'^2\, 2\pi\delta\left(E-E'\right)\frac{Z^2 e^4}{\left|\mathbf{q}\right|^4}$$

$$\to d\sigma = w\cdot\frac{\text{phase space}}{\text{incident flux}} = w\frac{\frac{Vd^3p'}{(2\pi)^3}}{\frac{p}{EV}} = w\frac{d^3p'}{(2\pi)^3}\frac{E}{p}V^2$$

$$d\sigma = N^2 N'^2\, 2\pi\delta\left(E-E'\right)\frac{Z^2 e^4}{\left|\mathbf{q}\right|^4}\frac{d^3p'}{(2\pi)^3}\frac{E}{p}V^2$$

$$N = N' = \frac{1}{\sqrt{V}} \to d\sigma = \frac{1}{V}\frac{1}{V}2\pi\delta\left(E-E'\right)\frac{Z^2e^4}{\left|\mathbf{q}\right|^4}\frac{d^3p'}{(2\pi)^3}\frac{E}{p}V^2 = 2\pi\delta\left(E-E'\right)\frac{Z^2e^4}{\left|\mathbf{q}\right|^4}\frac{d^3p'}{(2\pi)^3}\frac{E}{p}$$

An oversimplified example - IV

Calculate differential cross-section

$$\int d\sigma = \int 2\pi\delta(E-E')\frac{Z^2e^4}{|\mathbf{q}|^4}\frac{d^3p'}{(2\pi)^3}\frac{E}{p}$$

$$d^3p' = p'^2\,dp'\,d\Omega$$

$$p'dp' = E'dE' \to d^3p' = p'E'dE'd\Omega$$

$$\to \int 2\pi\delta(E-E')\frac{Z^2e^4}{|\mathbf{q}|^4}\frac{d^3p'}{(2\pi)^3}\frac{E}{p} = \int 2\pi\delta(E-E')\frac{Z^2e^4}{|\mathbf{q}|^4}\frac{dE'}{(2\pi)^3}\frac{E}{p}p'E'd\Omega$$

$$= \frac{1}{(2\pi)^2}\frac{Z^2e^4}{|\mathbf{q}|^4}\frac{E}{p}p'\int \delta(E-E')dE'E'd\Omega = \frac{1}{(2\pi)^2}\frac{Z^2e^4}{|\mathbf{q}|^4}E^2d\Omega$$

$$q = 2p\sin\frac{\theta}{2} \to q^4 = 16p^4\sin^4\frac{\theta}{2}$$

$$\to \frac{d\sigma}{d\Omega} = \frac{1}{(2\pi)^2}\frac{Z^2e^4}{16p^4\sin^4\theta/2}E^2 = \frac{Z^2e^4}{16(2\pi)^2}\frac{E^2}{p^4}\frac{1}{\sin^4\theta/2}$$

Useful to remember

$$E^2 = \mathbf{p}^2 + m^2$$

$$\to 2EdE = 2|\mathbf{p}|d|\mathbf{p}|$$

$$\to EdE = |\mathbf{p}|d|\mathbf{p}|$$

Compare to non-relativistic Rutherford cross-section:

$$\frac{d\sigma}{d\Omega} = \frac{1}{4}Z^2\alpha^2\frac{1}{T^2}\frac{1}{\sin^4\theta/2} = \frac{1}{4}\frac{Z^2e^4}{16\pi^2}\frac{1}{T^2}\frac{1}{\sin^4\theta/2}$$

Theme Variation: Spin ½ - I

Take now a spin ½ Dirac electron scattering off the same, static Coulomb potential

$$j^{\mu} = e\bar{\psi}\gamma^{\mu}\psi$$

$$\rightarrow \psi = \sqrt{\frac{m}{EV}}u(s,p)e^{ipx}, \bar{\psi} = \sqrt{\frac{m}{E'V}}\bar{u}(s',p')e^{-ip'x}$$

$$\rightarrow j^{\mu} = e\sqrt{\frac{m}{E'V}}\bar{u}(s',p')e^{-ip'x}\gamma^{\mu}\sqrt{\frac{m}{EV}}u(s,p)e^{ipx} = e\frac{m}{EV}e^{-i(p-p')x}\bar{u}\gamma^{\mu}u$$ Dirac transition current

$$j^{\mu}A_{\mu} = e\frac{m}{EV}e^{-i((E-E')t-(\mathbf{p}-\mathbf{p}')\cdot\mathbf{r})}\bar{u}(s',p')\gamma^{\mu}u(s,p)\left(\frac{eZ}{4\pi r},\mathbf{0}\right) = e\frac{m}{EV}e^{-i((E-E')t-(\mathbf{p}-\mathbf{p}')\cdot\mathbf{r})}\bar{u}\gamma^{0}u\frac{eZ}{4\pi r}$$

$$\underbrace{\bar{u}(s',p')}_{=u^{\dagger}\gamma^{0}}\gamma^{0}u(s,p) = u^{\dagger}(s',p')\overbrace{\gamma^{0}\gamma^{0}}^{=1}u(s,p) = u^{\dagger}(s',p')u(s,p)$$ Dirac matrices

$$\frac{d\sigma}{d\Omega} = \frac{m^2}{E^2}\frac{Z^2e^4}{16(2\pi)^2}\frac{E^2}{p^4}\frac{1}{\sin^4\theta/2}\left|u^{\dagger}(s',p')u(s,p)\right|^2 = \frac{Z^2e^4}{16(2\pi)^2}\frac{m^2}{p^4}\frac{1}{\sin^4\theta/2}\left|u^{\dagger}(s',p')u(s,p)\right|^2$$

Theme Variation: Spin ½ - II

Unpolarized cross-section:

Sum over final
Average over initial *Spin projections*

$$\frac{1}{2}\sum_{s=-1/2}^{+1/2}\sum_{s'=-1/2}^{+1/2}\left|u^{\dagger}(s',p')u(s,p)\right|^2 = 4\frac{E^2}{m^2}\left(1-\beta^2\sin^2\theta/2\right)$$

$$\rightarrow \frac{d\sigma}{d\Omega} = \frac{Z^2e^4}{16(2\pi)^2}\frac{m^2}{p^4}\frac{1}{\sin^4\theta/2}\frac{4E^2}{m^2}\left(1-\beta^2\sin^2\theta/2\right) = \frac{Z^2e^4}{(2\pi)^2}\frac{E^2}{4p^4}\frac{\left(1-\beta^2\sin^2\theta/2\right)}{\sin^4\theta/2}$$

$$\frac{d\sigma}{d\Omega} \underset{E\gg m}{\simeq} \frac{Z^2e^4}{(2\pi)^2}\frac{E^2}{4p^4}\frac{\cos^2\theta/2}{\sin^4\theta/2}$$

New factor, important at high speed
Reducing cross section at large angles (= *0* for $\theta\rightarrow\pi/2$)

Helicity Conservation

Dirac equation: High energy limit

$$E\psi = (\boldsymbol{\alpha}\cdot\mathbf{p} + \beta m)\psi$$

$u = \begin{pmatrix}\phi \\ \chi\end{pmatrix}$ Generic spinor; ϕ, χ 2-components spinors

$\boldsymbol{\alpha} = \begin{pmatrix}\boldsymbol{\sigma} & 0 \\ 0 & -\boldsymbol{\sigma}\end{pmatrix}, \beta = \begin{pmatrix}0 & 1 \\ 1 & 0\end{pmatrix}$ Dirac matrices, chiral representation, "2×2" block format

$$\begin{cases} E\phi = (\boldsymbol{\sigma}\cdot\mathbf{p})\phi + m\chi \\ E\chi = -(\boldsymbol{\sigma}\cdot\mathbf{p})\chi + m\phi \end{cases}$$

$$E \gg m \rightarrow \left.\begin{matrix} E \approx |\mathbf{p}| \\ m \approx 0 \end{matrix}\right\} \rightarrow \begin{cases} (\boldsymbol{\sigma}\cdot\mathbf{p})\phi \approx |\mathbf{p}|\phi \\ (\boldsymbol{\sigma}\cdot\mathbf{p})\chi \approx -|\mathbf{p}|\chi \end{cases} \rightarrow \begin{cases} \phi \simeq u_R \\ \chi \simeq u_L \end{cases} \rightarrow u \approx \begin{pmatrix} u_R \\ u_L \end{pmatrix} \rightarrow u^\dagger(s',p')u(s,p) \approx u'^\dagger_R u_R + u'^\dagger_L u_L$$

No mixed terms $\rightarrow$ *Helicity is conserved at high energy*

Explains the $(1-\beta^2\sin^2\theta/2)$ factor, cutting off the cross-section $\theta \rightarrow \pi$:
Solves conflicting helicity/angular momentum conservation

Always true for Dirac currents coupling to vector fields

Another Step into a Realistic Model

Take now a spin ½ Dirac electron scattering off a *distributed*, static source (like a *(A,Z)* nucleus)

$$A_\mu = (\varphi, \mathbf{0})$$

$$\varphi(r) = \frac{1}{4\pi}\int d^3\mathbf{r}'\frac{\rho(\mathbf{r}')}{|\mathbf{r}-\mathbf{r}'|}$$

$$\int \rho(\mathbf{r}')d^3\mathbf{r}' = Ze$$

Only change: the space integral

$$\int e^{+i((\mathbf{p}-\mathbf{p}')\cdot\mathbf{r})}\frac{e^2 Z}{4\pi r}d^3\mathbf{r} = \frac{Ze^2}{|\mathbf{q}|^2}, \mathbf{q} = \mathbf{p}-\mathbf{p}' \rightarrow \int d^3\mathbf{r}\int e^{+i((\mathbf{p}-\mathbf{p}')\cdot\mathbf{r})}\frac{\rho(\mathbf{r}')}{4\pi|\mathbf{r}'-\mathbf{r}|}d^3\mathbf{r}'$$

$$= \int d^3\mathbf{r}'\frac{\rho(\mathbf{r}')}{Ze}e^{+i(\mathbf{p}-\mathbf{p}')\cdot\mathbf{r}'}\int e^{+i((\mathbf{p}-\mathbf{p}')\cdot(\mathbf{r}-\mathbf{r}'))}\frac{Ze}{4\pi|\mathbf{r}'-\mathbf{r}|}d^3\mathbf{r}$$

$$F(\mathbf{q}) = \frac{\int \rho(\mathbf{r})e^{+i\mathbf{q}\cdot\mathbf{r}}d^3\mathbf{r}}{Ze}$$ Form factor of the charge distribution

$$\Rightarrow \frac{Z^2e^2}{|\mathbf{q}|^4} \rightarrow \frac{Z^2e^2}{|\mathbf{q}|^4}|F(\mathbf{q})|^2$$

The Form Factor

$$F(\mathbf{q}) = \frac{\int \rho(\mathbf{r}) e^{+i\mathbf{q}\cdot\mathbf{r}} d^3\mathbf{r}}{Ze} = \frac{1}{Ze}\int \rho(r) e^{+iqr\cos\theta} r^2 dr d\Omega$$

$$e^{+i|\mathbf{q}|r\cos\theta} = 1 + i|\mathbf{q}| r\cos\theta - \frac{1}{2}|\mathbf{q}|^2 r^2 \cos^2\theta ...$$

$$\rightarrow \frac{1}{Ze}\int \rho(r) e^{+i|\mathbf{q}|r\cos\theta} r^2 dr d\Omega \approx$$

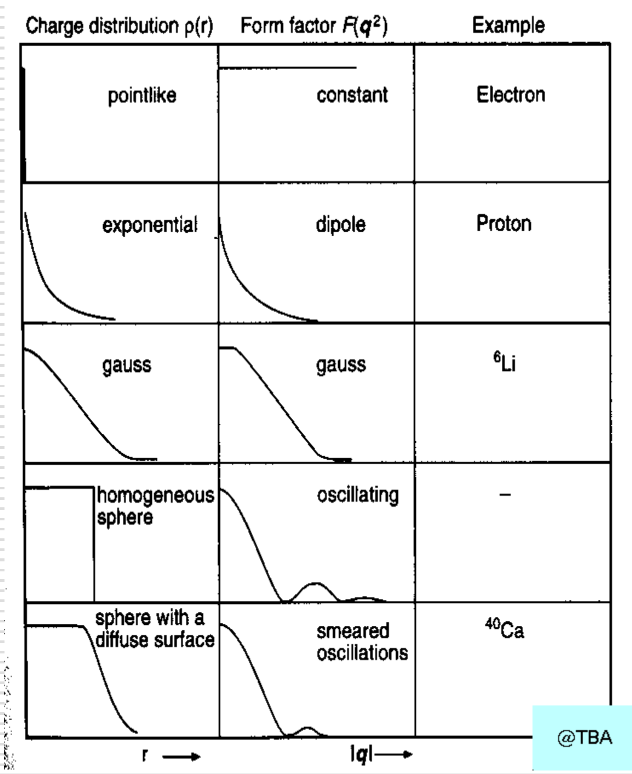
$$\approx \frac{1}{Ze}\left[\int \rho(r) r^2 dr d\Omega + i|\mathbf{q}|\int \rho(r) r^3 \cos\theta dr d\Omega - \frac{|\mathbf{q}|^2}{2}\int \rho(r) r^4 \cos^2\theta dr d\Omega\right]$$

$$\rightarrow F(\mathbf{q}) \approx 1 + \frac{i|\mathbf{q}|}{Ze}\int \rho(r) r^3 dr \underbrace{\int \cos\theta d\Omega}_{=\frac{2\pi}{2}\left(-\cos^2\theta\right)\Big|_0^\pi = 0} - \frac{|\mathbf{q}|^2}{2Ze}\underbrace{\int \rho(r) r^4 dr}_{=\frac{Ze\langle r^2\rangle}{4\pi}} \underbrace{\int \cos^2\theta d\Omega}_{=\frac{2\pi}{3}\left(-\cos^3\theta\right)\Big|_0^\pi = \frac{4\pi}{3}} = 1 - \frac{|\mathbf{q}|^2\langle r^2\rangle}{6}$$

$$\left.\begin{array}{l} F\left(|\mathbf{q}|^2\right) = F(0) + \dfrac{\partial F}{\partial |\mathbf{q}|^2}|\mathbf{q}|^2 + ... \\ F\left(|\mathbf{q}|^2\right) \approx 1 - \dfrac{1}{6}|\mathbf{q}|^2\langle r^2\rangle \end{array}\right\} \rightarrow\rightarrow \left\{\begin{array}{l} F(0) = 1 \\ \langle r^2\rangle = -6\left.\dfrac{\partial F}{\partial |\mathbf{q}|^2}\right|_{|\mathbf{q}|^2=0} \end{array}\right.$$

showing that measuring the form factor yields the rms charge radius

Form Factors: The Mathematical Game



$$\rho(r)=\frac{1}{8\pi a^3}e^{-ar}\rightarrow F\left(|\mathbf{q}|^2\right)=\left(\frac{1}{1+|\mathbf{q}|^2/a^2}\right)^2$$

$$\rho(r)=\begin{cases}\text{constant } r<R\\ 0 \quad\quad r>R\end{cases}\rightarrow F\left(|\mathbf{q}|^2\right)=\frac{3}{R^3|\mathbf{q}|^3}\left(\sin\left(R|\mathbf{q}|\right)-R|\mathbf{q}|\cos\left(R|\mathbf{q}|\right)\right)$$

Nuclear Form Factors

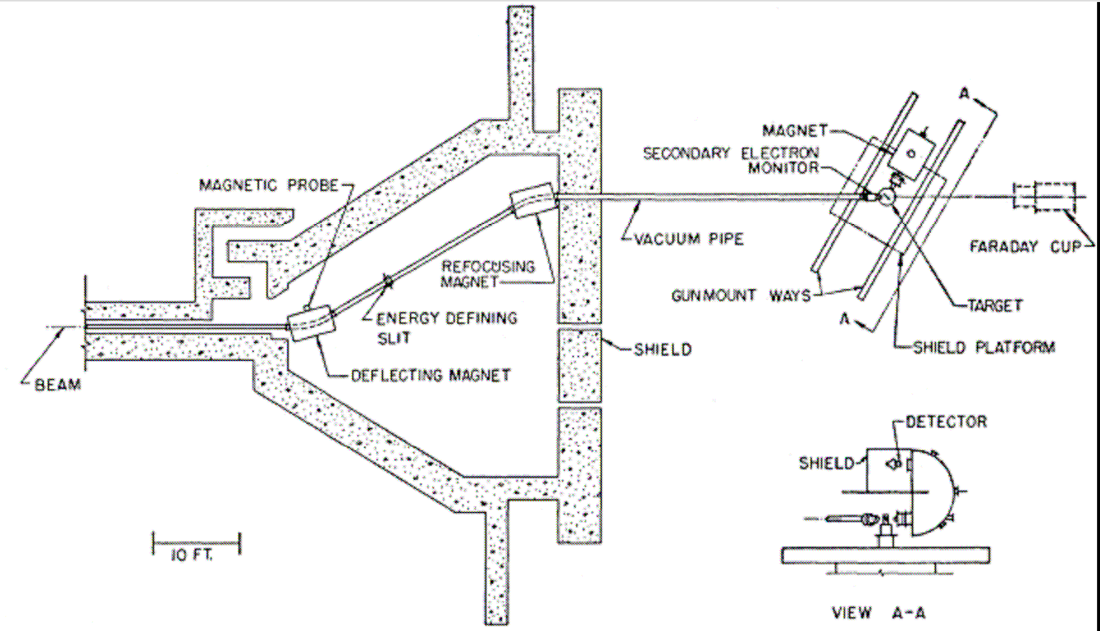


FIG. 18. The experimental installation of the 550-Mev spectrometer.

@TBA

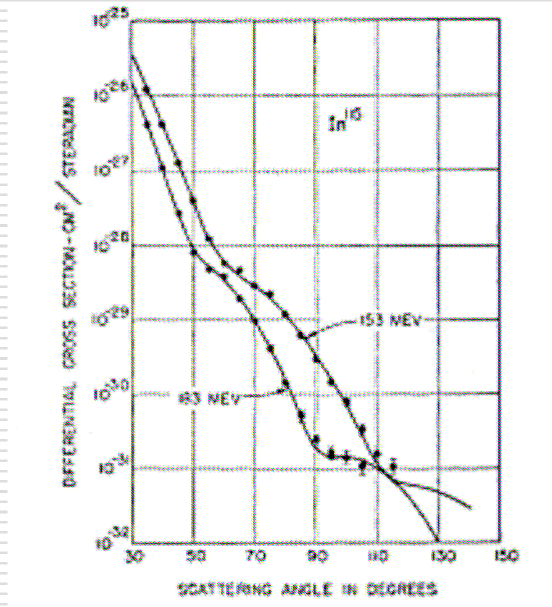


FIG. 46. Theoretical and experimental curves for In^{115} at two energies.

One of the Hofstadter's spectrometers at SLAC

Results for Indium

Particle-Particle Scattering

1st order Transition amplitude:

$$H' = j^\mu A_\mu \rightarrow H' = \left(j_1^\mu + j_2^\mu \right) A_\mu$$

$$\rightarrow M_{fi} = i(2\pi)^4 \delta\left(p_1 + p_2 - \left(p_1' + p_2'\right)\right) T_{fi} = i(2\pi)^4 \delta\left(p_1 + p_2 - \left(p_1' + p_2'\right)\right) j_\mu^{(1)} \frac{ig^{\mu\nu}}{q^2} j_\nu^{(2)}$$

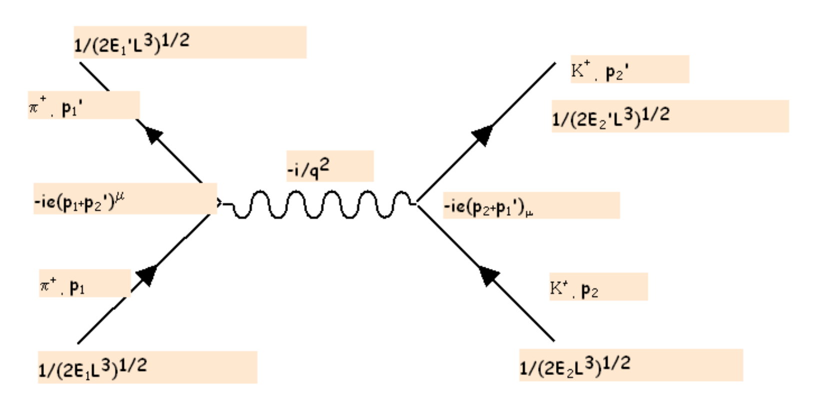
$q = p_1 - p_1' = p_2 - p_2'$ 4-momentum transfer

Transition currents:

$j^\mu = e\left(p_1 + p_3\right)^\mu$ scalar

$j^\mu = e\bar{u}_4 \gamma^\mu u_2$ fermion

....



Scattering Spin 0 – Spin ½

Just take the two currents as defined before:

$$T_{fi} = e\bar{u}'\gamma^{\mu}u\frac{g_{\mu\nu}}{q^2}e\left(p+p'\right)^{\nu} \rightarrow d\sigma = \frac{1}{4EE'\mathrm{v}}\left|T_{fi}\right|^2\left(2\pi\right)^4\delta^4\left(p_1^{'}+p_2^{'}-p_1-p_2\right)\frac{1}{\left(2\pi\right)^3}\frac{1}{\left(2\pi\right)^3}\frac{d^3\mathbf{p}_1^{'}}{2E_1^{'}}\frac{d^3\mathbf{p}_2^{'}}{2E_2^{'}}$$

$$\frac{1}{2}\sum_{s=-1/2}^{+1/2}\sum_{s'=-1/2}^{+1/2}\left|T_{fi}\right|^2 = \left(\frac{e^2}{q^2}\right)^2\frac{1}{2}\sum_{s,s'=-1/2}^{+1/2}\bar{u}\left(p_1',s'\right)\gamma^{\mu}u\left(p_1,s\right)\bar{u}\left(p_1,s\right)\gamma^{\nu}u\left(p_1',s'\right)\left(p_2+p_2'\right)_{\mu}\left(p_2+p_2'\right)_{\nu}$$

By defining…

$$T_{\mu\nu} = \left(p_2+p_2'\right)_{\mu}\left(p_2+p_2'\right)_{\nu}$$

$$L^{\mu\nu} = 2\left[p_1^{'\mu}p_1^{\nu}+p_1^{'\nu}p_1^{\mu}+\frac{q^2}{2}g^{\mu\nu}\right]$$

…it can be shown (!) that

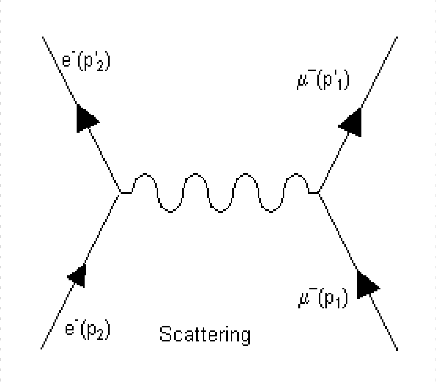
$$\frac{d\sigma}{dq^2} = \frac{2\alpha^2}{\left(p_1+p_2\right)^2q^4}\left[2\left(p_1\cdot p_2\right)\left(p_1\cdot p_2'\right)+\frac{q^2}{2}M^2\right]$$ Invariant cross-section

$$\left.\frac{d\sigma}{d\Omega}\right|_{LAB}\underset{E\gg m}{\simeq}\frac{\alpha^2}{4\left|\mathbf{p}_1\right|^2\sin^4\theta/2}\cos^2\frac{\theta}{2}\frac{\left|\mathbf{p}_1'\right|}{\left|\mathbf{p}_1\right|}$$ LAB = "2" rest frame

Another extension of Rutherford cross section
Includes *target recoil*

Scattering Spin ½ - Spin ½

Just to simplify things, take *different* spin ½ particles
(e.g. electron-muon scattering)



$$T_{fi}(s,s',r,r') = e\bar{u}'(p_2',s')\gamma^\mu u(p_2,s)\frac{g_{\mu\nu}}{q^2}\bar{u}'(p_1',r')\gamma^\nu u(p_1,r)$$

$$\frac{1}{4}\sum_{s,s',r,r'}\left|T_{fi}(s,s',r,r')\right|^2 = \frac{e^4}{q^4}L_{\mu\nu}M^{\mu\nu}$$

$$L^{\mu\nu} = 2\left[p_1'^{\mu}p_1^{\nu} + p_1'^{\nu}p_1^{\mu} + \frac{q^2}{2}g^{\mu\nu}\right]$$

$$M_{\mu\nu} = 2\left[p_{2\mu}'p_{2\nu} + p_{2\nu}'p_{2\mu} + \frac{q^2}{2}g_{\mu\nu}\right] \rightarrow \left.\frac{d\sigma}{d\Omega}\right|_{LAB} \underset{E\gg m}{\simeq} \frac{\alpha^2}{4|\mathbf{p}_1|^2\sin^4\theta/2}\cos^2\theta/2\frac{|\mathbf{p}_1'|}{|\mathbf{p}_1|}\left(1 - \frac{q^2\tan^2\theta/2}{2m_2^2}\right)$$

Yet another term...
Electron scattering off
the muon *magnetic moment*

Crossing Symmetry

Simple relationship between any pair of 2-body reactions

$a+b \to c+d$ Reaction A

$a+\underbrace{[c]}_{crossed} \to \underbrace{[b]}_{crossed}+d$ Reaction B

Define: Crossed particle ≡ Antiparticle

By changing the 4-momentum sign of the crossed particle. the two amplitudes are identical

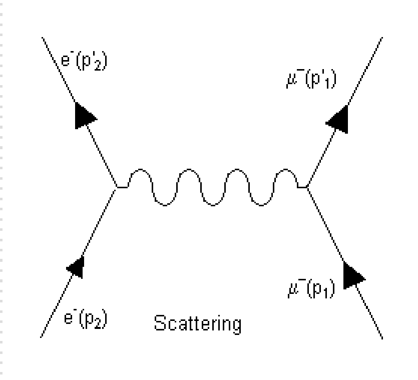
$$A\left[a(p_A)+b(p_B)\to c(p_C)+d(p_D)\right]=A\left[a(p_A)+\bar{c}(-p_C)\to \bar{b}(-p_B)+d(p_D)\right]$$

Annihilation

Apply crossing symmetry to electron-muon scattering

$$e^- + \mu^- \to e^- + \mu^- \qquad \text{A: Scattering}$$

$$e^- + \underset{crossed}{\left[e^-\right]} \to \underset{crossed}{\left[\mu^-\right]} + \mu^- \equiv e^- + e^+ \to \mu^+ + \mu^- \qquad \text{B: Annihilation}$$

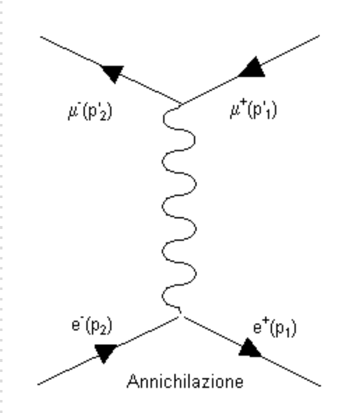


Amplitude for scattering:

$$T_{fi}\left(s,s',r,r'\right) = (-e)\bar{u}_{(\mu)}\left(p_2',s'\right)\gamma^{\mu}u_{(\mu)}\left(p_2,s\right)\frac{-ig_{\mu\nu}}{q^2}(-e)\bar{u}_{(e)}\left(p_1',r'\right)\gamma^{\nu}u_{(e)}\left(p_1,r\right)$$

$$q = p_1 - p_1' \to q^2 = \left(p_1 - p_1'\right)^2 = p_1^2 + p_1'^2 - 2p_1 \cdot p_1'$$ q=4-momentum transfer

$$q^2 = 2m_e^2 - 2\left(E_1E_1' - \mathbf{p}_1 \cdot \mathbf{p}_1'\right) \underset{E \gg m}{\simeq} -2\left(E_1E_1' - \mathbf{p}_1 \cdot \mathbf{p}_1'\right) < 0$$



Amplitude for annihilation:

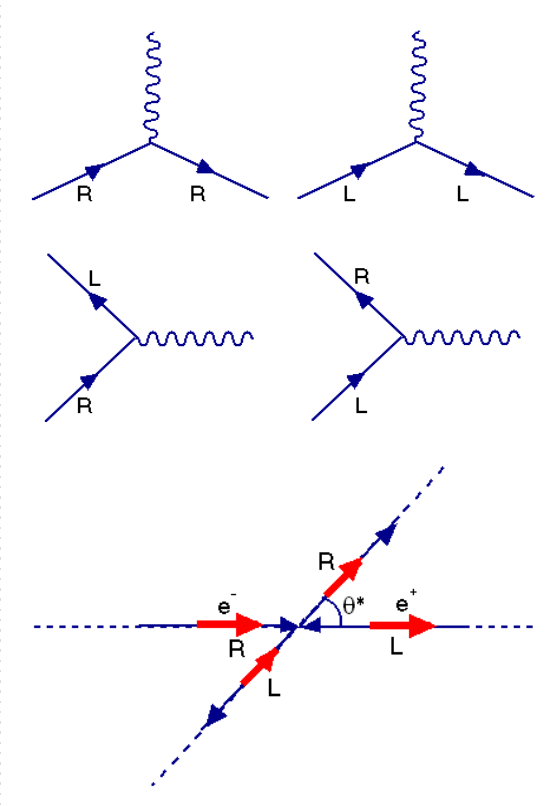
$$T_{fi}\left(s,s',r,r'\right) = (-e)\bar{u}_{(\mu)}\left(p_2',s'\right)\gamma^{\mu}v_{(\mu)}\left(p_1',r'\right)\frac{-ig_{\mu\nu}}{q^2}(-e)\bar{v}_{(e)}\left(p_1,s\right)\gamma^{\nu}u_{(e)}\left(p_2,r\right)$$

$$q = p_1 + p_2 \to q^2 = \left(p_1 + p_2\right)^2 = p_1^2 + p_2^2 + 2p_1 \cdot p_2$$ q=total 4-momentum

$$q^2 = 2m_e^2 + 2\left(E_1E_2 - \mathbf{p}_1 \cdot \mathbf{p}_2\right) \underset{E \gg m}{\simeq} 2\left(E_1E_2 - \mathbf{p}_1 \cdot \mathbf{p}_2\right) > 0$$

The Annihilation Cross-Section - I

Helicity conservation at high energy:
Consequence of electromagnetic field being a *vector*



Scattering: $R \to R$, $L \to L$

Annihilation: $R+L$, $L+R$

For both initial and final state:
Particle and antiparticle must have *opposite* helicity
Can decompose differential cross-section into 4 pieces:

$$\frac{d\sigma}{d\Omega^*} = \frac{1}{4}\cdot 2\left[\left.\frac{d\sigma}{d\Omega^*}\right|_{LR\to LR} + \left.\frac{d\sigma}{d\Omega^*}\right|_{RL\to RL} + \left.\frac{d\sigma}{d\Omega^*}\right|_{LR\to RL} + \left.\frac{d\sigma}{d\Omega^*}\right|_{RL\to LR} \right]$$

Average over initial states
Sum over final states

The Annihilation Cross-Section - II

Transition amplitude = Amplitude to find final particles at angle θ wrt to initial direction*

Phase space, incident flux and normalization factors just cancel out at high energy
Matrix element:

$$T_{fi} = \frac{\alpha}{q^2 \equiv s} \cdot Amplitude\ to\ find\ J=1\ state\ rotated\ by\ \theta^*$$

Use rotation matrices for a *J=1* state: Take *y*-axis ⊥ reaction plane

$$e^{-i\theta^* J_2}|J,m\rangle = \sum_{m'} d^J_{m,m'}(\theta^*)|J,m'\rangle, \quad d^J_{m,m'}(\theta^*) = \langle J,m|e^{-i\theta^* J_2}|J,m'\rangle$$

$$d^1_{+1,+1}(\theta^*) = d^1_{-1,-1}(\theta^*) = \frac{1}{2}(1+\cos\theta^*)$$

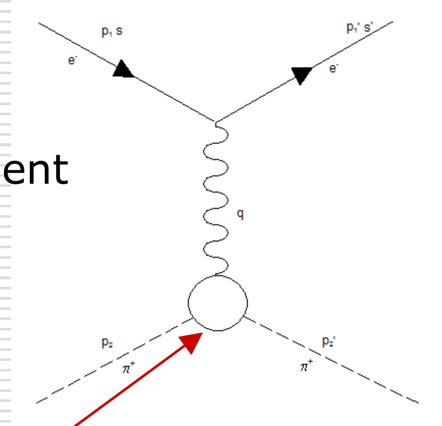
$$d^1_{+1,-1}(\theta^*) = d^1_{-1,+1}(\theta^*) = \frac{1}{2}(1-\cos\theta^*)$$

$$\left.\begin{aligned} \left.\frac{d\sigma}{d\Omega^*}\right|_{LR\to LR} &= \left.\frac{d\sigma}{d\Omega^*}\right|_{RL\to RL} = \frac{\alpha^2}{s}\left(\frac{1}{2}\right)^2(1+\cos\theta^*)^2 \\ \left.\frac{d\sigma}{d\Omega^*}\right|_{LR\to RL} &= \left.\frac{d\sigma}{d\Omega^*}\right|_{RL\to LR} = \frac{\alpha^2}{s}\left(\frac{1}{2}\right)^2(1-\cos\theta^*)^2 \end{aligned}\right\} \to \frac{d\sigma}{d\Omega^*} = \boxed{\frac{\alpha^2}{4s}(1+\cos^2\theta^*)} \qquad \sigma = \int_{4\pi} \frac{\alpha^2}{4s}(1+\cos^2\theta^*)d\Omega^* = \boxed{\frac{4\pi\alpha^2}{3s}}$$

The π Form Factor - I

Consider electron-pion scattering: The π is not a point-like object…
What are we to take for the pion current?
Must build a 4-vector operator
Some guesswork:
1) *Lorentz invariance*

$$\left.\begin{array}{ll} p_2, p_2', q & \text{Three 4-momentum vectors} \\ p_2' = p_2 + q & \text{Constraint} \end{array}\right\} \rightarrow \text{ 2 independent}$$



Blob indicating a non-QED vertex:
The pion is an extended object

Choose:

$$\left.\begin{array}{l} p_2' + p \\ p_2' - p = q \end{array}\right\} \text{ Both can contribute to the current}$$

Only one independent 4-scalar:

E.g. $(p_2')^2 = (p_2)^2 = m^2 \rightarrow p_2 \cdot p_2'$

Choose instead q^2

$$\rightarrow j^{\mu}_{(\pi)} = e\left[F(q^2)(p'+p)^{\mu} + G(q^2)q^{\mu}\right]e^{-iq\cdot x}$$

The π Form Factor - II

2) *Gauge Invariance*

Charge conservation $\leftrightarrow$ Current must be *divergenceless*

$$\partial_\mu j^\mu = 0 \to \partial_\mu j_\pi^\mu = e\partial_\mu \left[F\left(q^2\right)\left(p'+p\right)^\mu + G\left(q^2\right)q^\mu \right] e^{-iq\cdot x}$$

$$= -iq_\mu e \left[F\left(q^2\right)\left(p'+p\right)^\mu + G\left(q^2\right)q^\mu \right] e^{-iq\cdot x} = 0$$

$$\to \partial_\mu j^\mu = 0 \Rightarrow q_\mu j^\mu = 0$$

$$q_\mu \left[F\left(q^2\right)\left(p_2 + p_2'\right)^\mu + G\left(q^2\right)q^\mu \right] = 0$$

$$\left.\begin{array}{l} q_\mu \left(p_2 + p_2'\right)^\mu = \left(p_2 - p_2'\right)_\mu \left(p_2 + p_2'\right)^\mu = 0 \\ q_\mu q^\mu \neq 0 \end{array}\right\} \to G\left(q^2\right) = 0$$

$$\to j^\mu = e\left(p_2 + p_2'\right)^\mu F\left(q^2\right)$$

Just *one* form factor for a scalar particle like the π

The π Form Factor - III

What is $F(q^2)$?

In the CM frame:

$$q^2 = (E' - E, \mathbf{p}' - \mathbf{p})^2 = (E' - E)^2 - (\mathbf{p}' - \mathbf{p})^2 = 0 - \mathbf{q}^2 = -|\mathbf{q}|^2$$

$$\rightarrow F_{scatt}(q^2) = F_{scatt}(|\mathbf{q}|^2)$$

Again, Fourier transform of the charge distribution

If crossing is good, can extend to the reaction

$$e^+ + e^- \rightarrow \pi^+ + \pi^-$$

$$q^2 = (E_1 + E_2, \boldsymbol{p}_1 + \boldsymbol{p}_2)^2 = (E_1 + E_2)^2 - (\boldsymbol{p}_1 + \boldsymbol{p}_2)^2$$

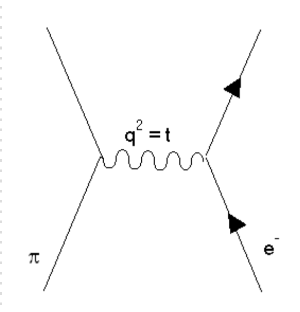
$$q^2 = E_{CM}^2$$

$$\rightarrow F_{annihil}(q^2) = F_{annihil}(E_{CM}^2)$$

$$\rightarrow F(q^2) = \begin{cases} F_{scatt}(q^2), q^2 < 0 \\ F_{annihil}(q^2), q^2 > 0 \end{cases}$$

Experiments – Space-like

π scattering off electrons

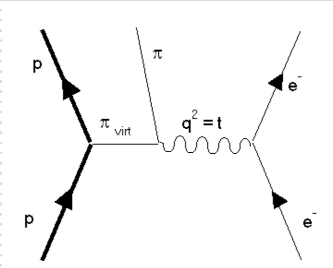


$$p^* = \frac{p_\pi m_e}{\sqrt{s}} \to t_{\max} = -4\frac{p_\pi^2 m_e^2}{s} = -4\frac{p_\pi^2 m_e^2}{m_\pi^2 + m_e^2 + 2E_\pi m_e} \approx -4\frac{p_\pi^2 m_e^2}{m_\pi^2 + 2E_\pi m_e}$$

$$\to t_{\max} \underset{p_\pi \to \infty}{\sim} -4\frac{p_\pi m_e^2}{2m_e} = -2p_\pi m_e$$

Unappealing, t_{max} too small

Electroproduction of one π



OK

Flux measurement

Pion Momentum

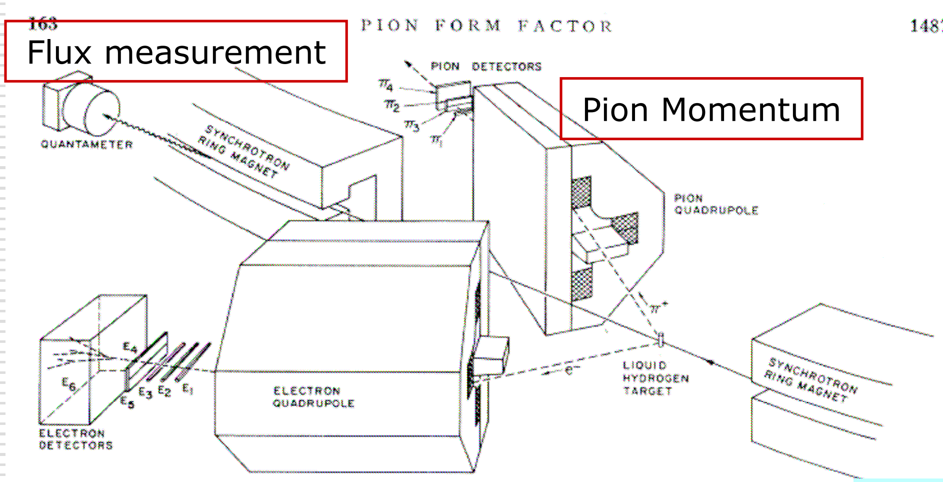
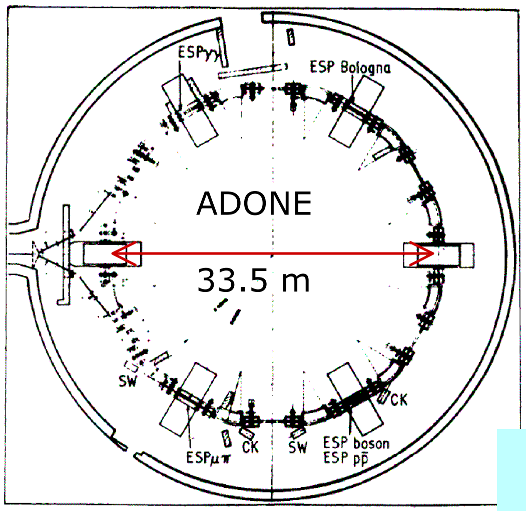


Fig. 5. Schematic view of the experimental layout.

Electron Identification & Momentum

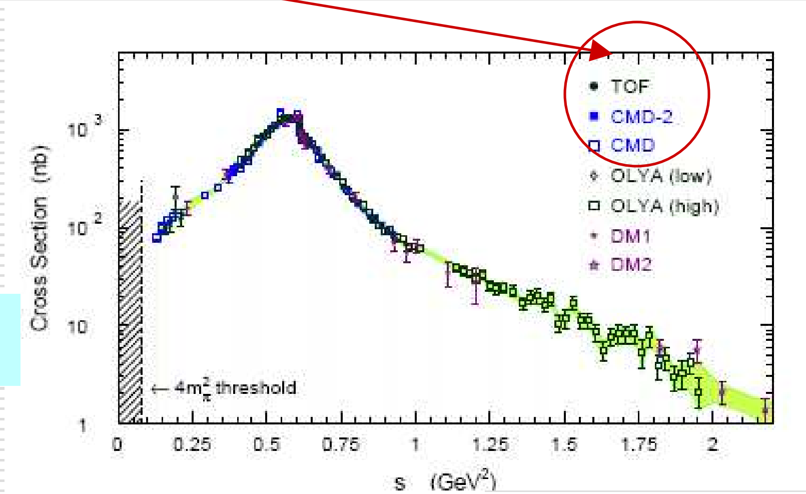
@TBA

Experiments – Time-like



@TBA

First e^+ - e^- colliding beams
ADONE – Frascati, 1967 etc.



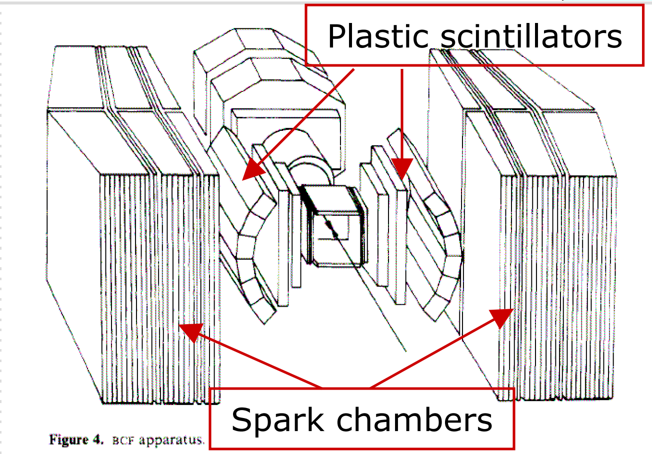
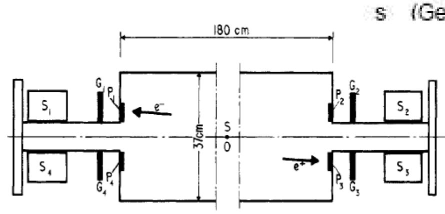


Figure 4. BCF apparatus.



Luminosity monitor measures Bhabha scattering rate at small angles

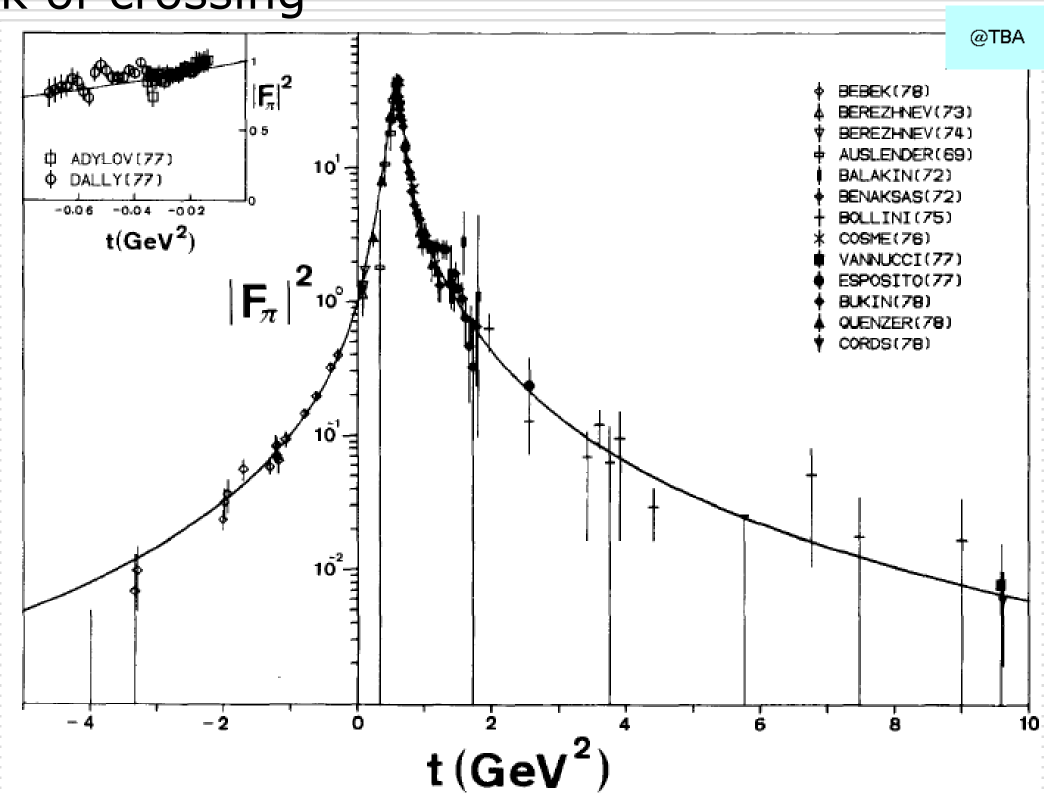
$Rate_{Bhabha} = \sigma_{Bhabha} \cdot Luminosity$

$e^+ + e^- \rightarrow e^+ + e^-$ pure QED at low energy, small angle

$\rightarrow$ accurate, reliable σ_{Bhabha} prediction $\rightarrow Luminosity = Rate_{Bhabha} / \sigma_{Bhabha}$

The π Form Factor at Large

Is there a unique function $F(q^2)$?? Yes!
Good check of crossing



Electron Form Factors - I

$j^\mu = e\bar{\psi}\gamma^\mu\psi$ Dirac current

$\bar{u}(p')\gamma^\mu u(p) = \frac{1}{2m}\left[(p+p')^\mu + i\sigma^{\mu\nu}(p-p')_\nu\right]$ Gordon's identity

$$\begin{cases} \frac{e}{2m}u(p')(p+p')^\mu u(p) & \text{charge, like a scalar particle} \\ \frac{ie}{2m}\bar{u}(p')\sigma^{\mu\nu}\underbrace{(p-p')_\nu}_{=q_\nu}u(p) & \text{extra term} \end{cases}$$

Extra term due to *magnetic dipole current.* Indeed, it contributes the interaction energy:

$$\frac{ie}{2m}\bar{u}(p')\sigma^{\mu\nu}u(p)q_\nu A_\mu \underset{\text{low speed}}{\longrightarrow} -\frac{e}{2m}\phi'^\dagger\boldsymbol{\sigma}\cdot(\mathbf{q}\times\mathbf{A})\phi$$

Magnetic dipole interaction energy

$$\frac{e}{2m}\phi'^\dagger\boldsymbol{\sigma}\cdot(\nabla\times\mathbf{A})\phi \equiv \frac{e}{2m}\phi'^\dagger(\boldsymbol{\sigma}\cdot\mathbf{B})\phi \Rightarrow \frac{ie}{2m}\bar{u}(p')\sigma^{\mu\nu}u(p)q_\nu A_\mu \underset{\text{low speed}}{\longrightarrow} -\frac{e}{2m}\phi'^\dagger(\boldsymbol{\sigma}\cdot\mathbf{B})\phi$$

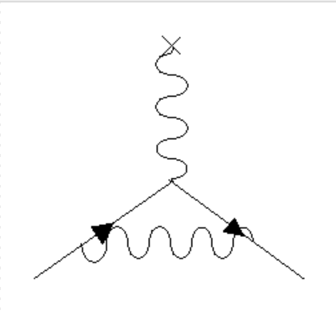
$\mu \approx \frac{e\hbar}{2mc}$ Magnetic moment , $j = \frac{1}{2}\hbar$ Spin, $\gamma \equiv \frac{\mu}{j}$ Gyromagnetic ratio

$\gamma \approx \frac{e\hbar}{2mc}\frac{2}{\hbar} \underset{\text{natural units}}{=} \frac{e}{2m}\cdot 2,$ Define $\gamma \equiv g\frac{e}{2m} \rightarrow g \approx 2$ Dirac g-factor

Electron Form Factors - II

Now: g-factor not exactly 2, as predicted by Dirac equation
Reason: *Radiative corrections*
Largest correction: Anomalous magnetic moment

$$\mu_{Dirac} = \frac{e}{2m} \rightarrow \mu = \frac{e}{2m}(1+\kappa_e)$$



$$j^\mu = \frac{e}{2m}\bar{u}(p')\Big[(p+p')^\mu + i\sigma^{\mu\nu}(1+\kappa_e)q_\nu\Big]u(p)$$

$$\bar{u}(p')\gamma^\mu u(p) = \frac{1}{2m}\bar{u}(p')\Big[(p+p')^\mu + i\sigma^{\mu\nu}q_\nu\Big]u(p)$$

$$\rightarrow \bar{u}(p')(p+p')^\mu u(p) = \bar{u}(p')\left(\gamma^\mu - i\sigma^{\mu\nu}q_\nu\right)u(p)$$

$$\rightarrow j^\mu = \frac{e}{2m}\bar{u}(p')\left[\gamma^\mu - i\sigma^{\mu\nu}q_\nu + i\sigma^{\mu\nu}(1+\kappa_e)q_\nu\right]u(p)$$

$$\rightarrow j^\mu = \frac{e}{2m}\bar{u}(p')\left[\gamma^\mu + i\kappa_e\sigma^{\mu\nu}q_\nu\right]u(p)$$

In spite of the electron *being* a pointlike fermion, radiative corrections make it behaving like an extended object

Further radiative corrections lumped into 2 *form factors*

$$j^\mu = \frac{e}{2m}\bar{u}(p')\left[f(q^2)\gamma^\mu + g(q^2)i\kappa_e\sigma^{\mu\nu}q_\nu\right]u(p)$$ Most general form (it can be shown..)

g-2 - Theory

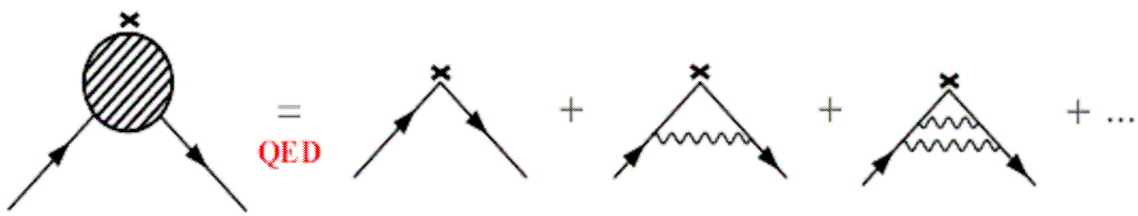
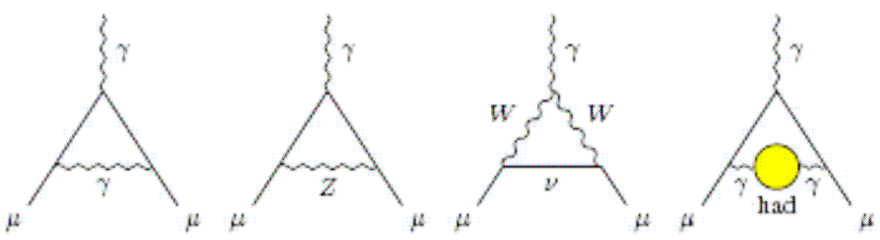


Figure 1: The perturbative expansion of $\Gamma^{\rho}(p', p)$ in single flavour QED. The tree graph gives $F_1 = 1$, $F_2 = F_3 = 0$. The one loop vertex correction graph gives the coefficient A_1 in Eq. (2.21). The cross denotes the insertion of the external field.

QED only...

@TBA



Contributions
rom new physics ?

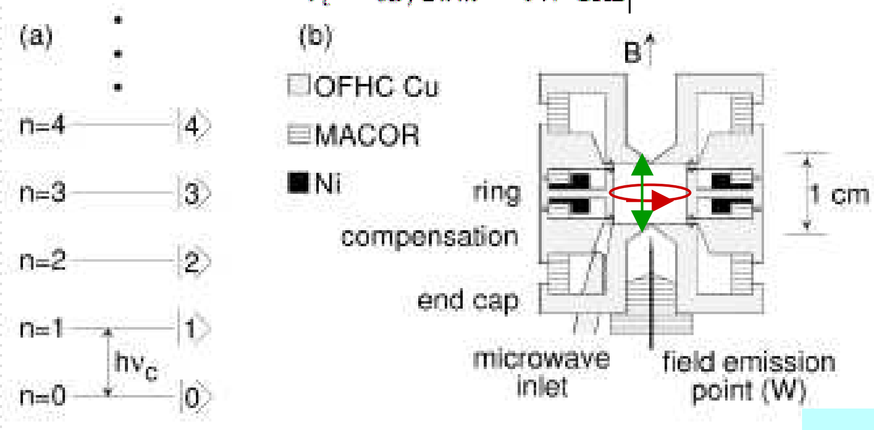
..and more

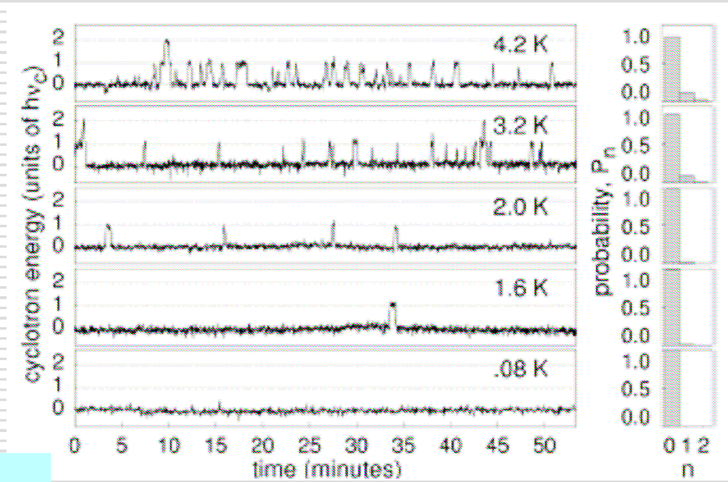
QED Electroweak QCD ???

Electron *g-2*: Quantum Cyclotron

Clever use of magnetic and electric fields at low temperature: The *Penning Trap*

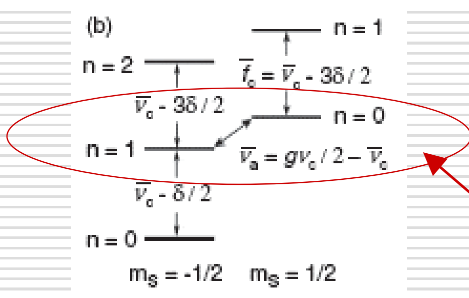
$$\nu_c = eB/2\pi m = 147\ \mathrm{GHz}$$

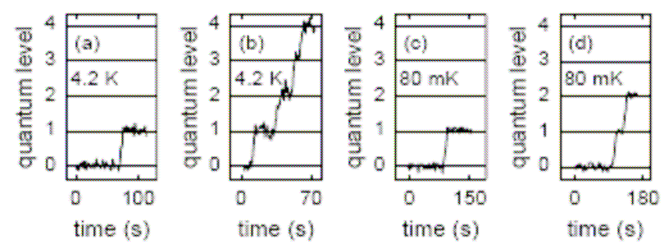




$$E(n, m_s) = \frac{g}{2} h\nu_c m_s + \left(n + \frac{1}{2}\right) h\bar{\nu}_c - \frac{1}{2} h\delta \left(n + \frac{1}{2} + m_s\right)^2$$

@TBA





$$g/2 = 1.001\,159\,652\,180\,85\,(76)\ [0.76\ \mathrm{ppt}]$$

Muon *g-2:* Experimental Method

$\mu \approx \frac{e\hbar}{2mc}$ Magnetic moment , $j = \frac{1}{2}\hbar$ Spin, $\gamma \equiv \frac{\mu}{j}$ Gyromagnetic ratio

$\gamma \approx \frac{e\hbar}{2mc}\frac{2}{\hbar} \underset{\text{natural units}}{=} \frac{e}{2m}\cdot 2 \equiv g\frac{e}{2m} \rightarrow g \approx 2$

$a \equiv \frac{g-2}{2} \ll 1$ Anomaly

For a charged particle moving in a uniform magnetic field:

$\frac{d\boldsymbol{p}}{dt} = e\left(\boldsymbol{p}\times\boldsymbol{B}\right) \rightarrow \omega_c = \frac{eB}{m}$ Cyclotron frequency

For a particle carrying a magnetic moment $\boldsymbol{\mu}$:

$\left.\begin{array}{l}\boldsymbol{\mu} = \gamma\boldsymbol{J}\\ \boldsymbol{\tau} = \boldsymbol{\mu}\times\boldsymbol{B}\\ \boldsymbol{\tau} = d\boldsymbol{J}/dt\end{array}\right\} \rightarrow \frac{d\boldsymbol{\mu}}{dt} = \gamma\left(\boldsymbol{\mu}\times\boldsymbol{B}\right)$ Precession of $\boldsymbol{\mu}$ around $\boldsymbol{B}$ in the particle rest frame

$\omega_s = \frac{2\mu B}{\hbar} = g\frac{eB}{2m} = (1+a)\frac{eB}{m}$ Precession frequency, unchanged by Lorentz transformation

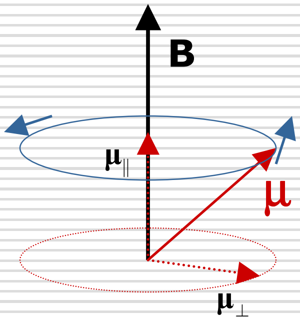
Get *a* by measuring the *beat frequency* ω_s-ω_c

Larmor Precession

Just a short reminder of the classical result:

$$\frac{d\boldsymbol{\mu}}{dt}=\gamma\left(\boldsymbol{\mu}\times\boldsymbol{B}\right)\rightarrow\boldsymbol{B}\cdot\frac{d\boldsymbol{\mu}}{dt}=\gamma\boldsymbol{B}\cdot\left(\boldsymbol{\mu}\times\boldsymbol{B}\right)=0\rightarrow\boldsymbol{\mu}_{\parallel}=\text{const}$$

Say $\boldsymbol{B}=B\hat{\boldsymbol{k}}\rightarrow\ \boldsymbol{\mu}_{\perp}=\mu_x\hat{\boldsymbol{i}}+\mu_y\hat{\boldsymbol{j}}$



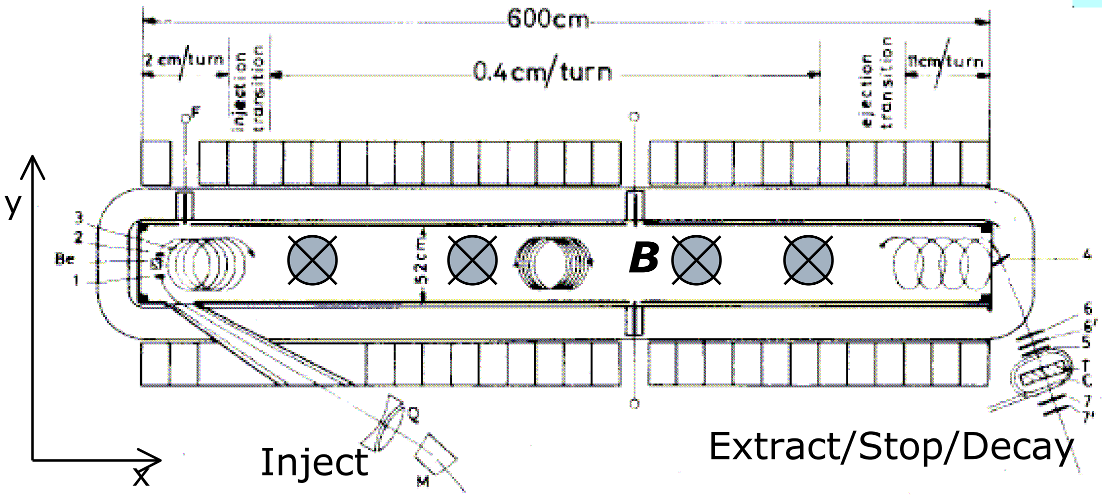
$$\rightarrow\frac{d\boldsymbol{\mu}_{\perp}}{dt}=\gamma\left(\boldsymbol{\mu}_{\perp}\times\boldsymbol{B}\right)\rightarrow\begin{cases}\dfrac{d\mu_x}{dt}=\gamma\left(\boldsymbol{\mu}_{\perp}\times\boldsymbol{B}\right)_x=+\gamma\mu_y B\\ \dfrac{d\mu_y}{dt}=\gamma\left(\boldsymbol{\mu}_{\perp}\times\boldsymbol{B}\right)_y=-\gamma\mu_x B\end{cases}$$

$$\tilde{\mu}\equiv\mu_x+i\mu_y\rightarrow\begin{cases}\dfrac{d\mu_x}{dt}=+\gamma\mu_y B\\ i\dfrac{d\mu_y}{dt}=-i\gamma\mu_x B\end{cases}\rightarrow\frac{d\mu_x}{dt}+i\frac{d\mu_y}{dt}=+\gamma B\left(\mu_y-i\mu_x\right)=-i\gamma B\left(\mu_x+i\mu_y\right)$$

$$\rightarrow\frac{d\tilde{\mu}}{dt}=-i\gamma B\tilde{\mu}\rightarrow\tilde{\mu}\left(t\right)=\tilde{\mu}_0 e^{-i\gamma Bt}\rightarrow\begin{cases}\mu_x\left(t\right)=\mu_{x0}\cos\gamma Bt\\ \mu_y\left(t\right)=-\mu_{y0}\sin\gamma Bt\end{cases}$$

The First Muon *g-2* at CERN

Small ***B*** *y*-gradient allowing for orbit drift along *x*



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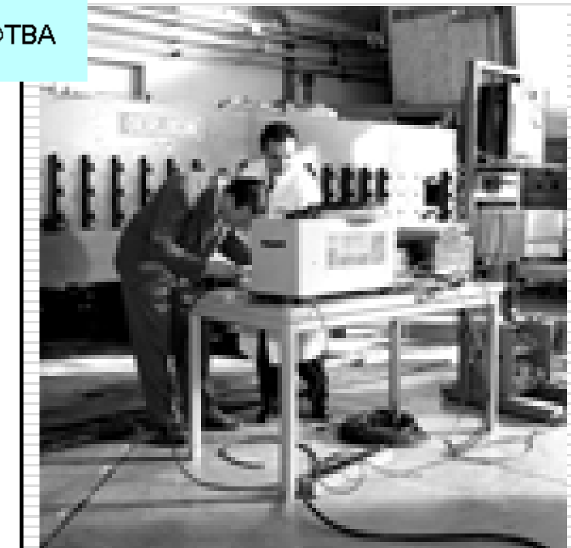
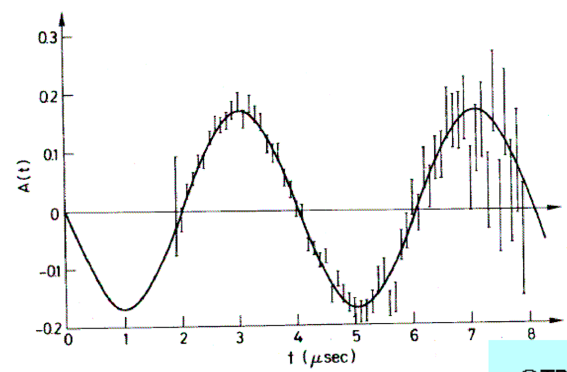


Stopping μ^+ weakly decays to $\overline{\nu}_\mu e^+ \nu_e$ with τ = 2.2 μs
Parity violation enhances decays where positron is emitt
along $\boldsymbol{s}_\mu$ →Polarization direction can be measured
By counting

$$\frac{N_{e^+}^{forward} - N_{e^+}^{backward}}{N_{e^+}^{forward} + N_{e^+}^{backward}} \propto \text{spin rotation angle } = (1+a)\frac{eB}{mc}\gamma t$$



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An Improved *g-2* at CERN

@TBA

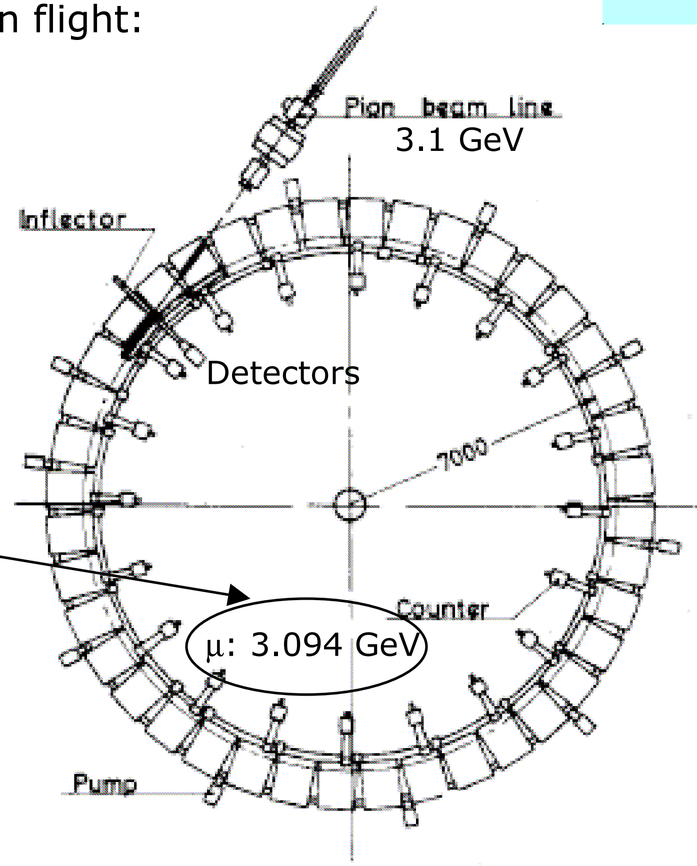
Detect high energy electrons from μ's decaying in flight:
Forward-emitted in the μ rest frame

$$N(t) = N_0 e^{-t/\tau} \{1 - A \sin(2\pi f_a t + \phi)\}$$

$$\omega_a = a \frac{eB}{mc} \gamma$$

Why this momentum?

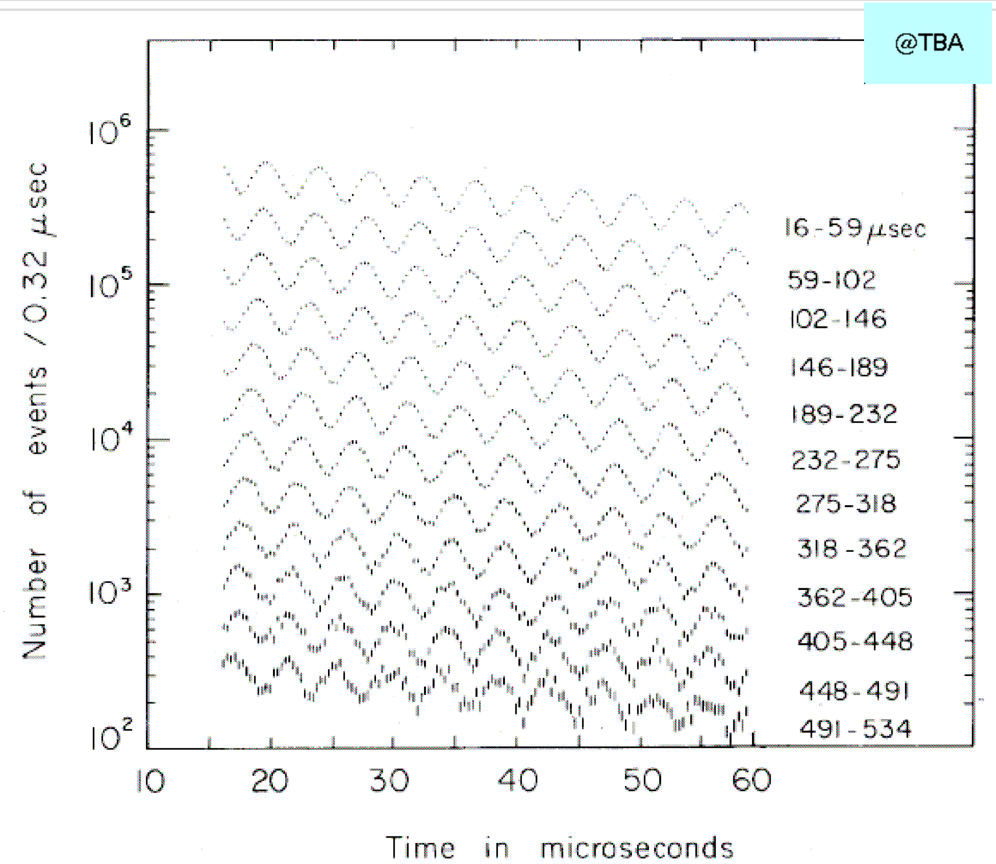
'Magic' cancellation of spurious effects
(*E*-field appearing in the μ rest frame
from Lorentz-transformed *B*-field in the
LAB frame)



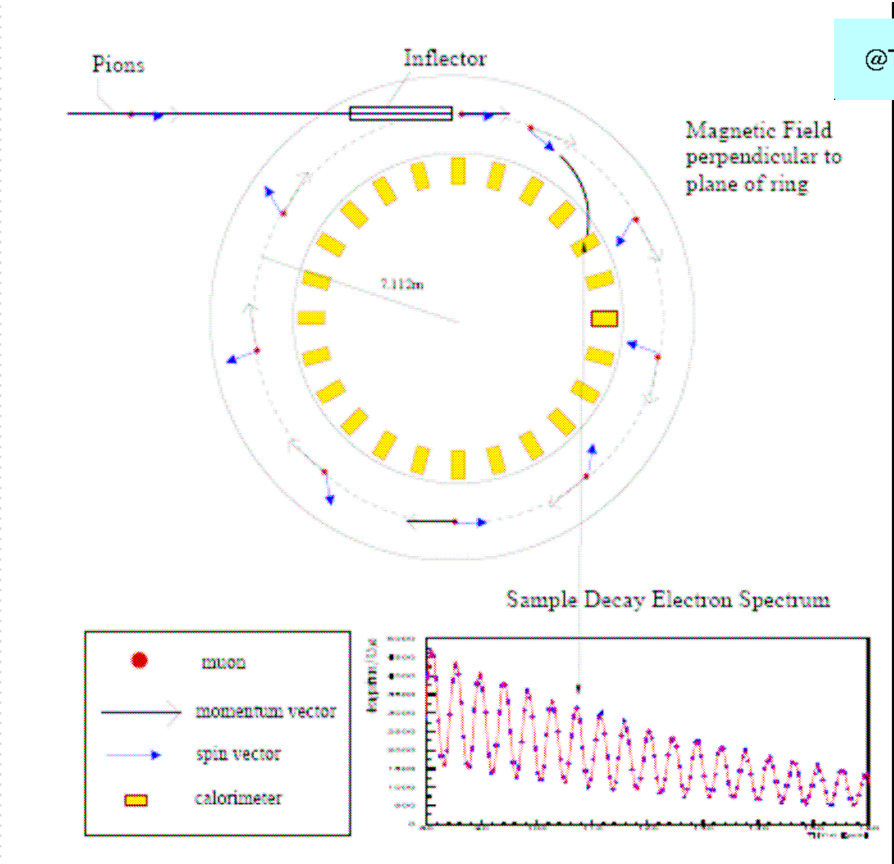
Astonishing Precision…

$$a_{\mu^+} = 1\ 165\ 911\ (11) \times 10^{-9} \quad (10\ \text{ppm})$$

$$a_{\mu^-} = 1\ 165\ 937\ (12) \times 10^{-9} \quad (10\ \text{ppm})$$



The Latest Muon *g-2* at BNL



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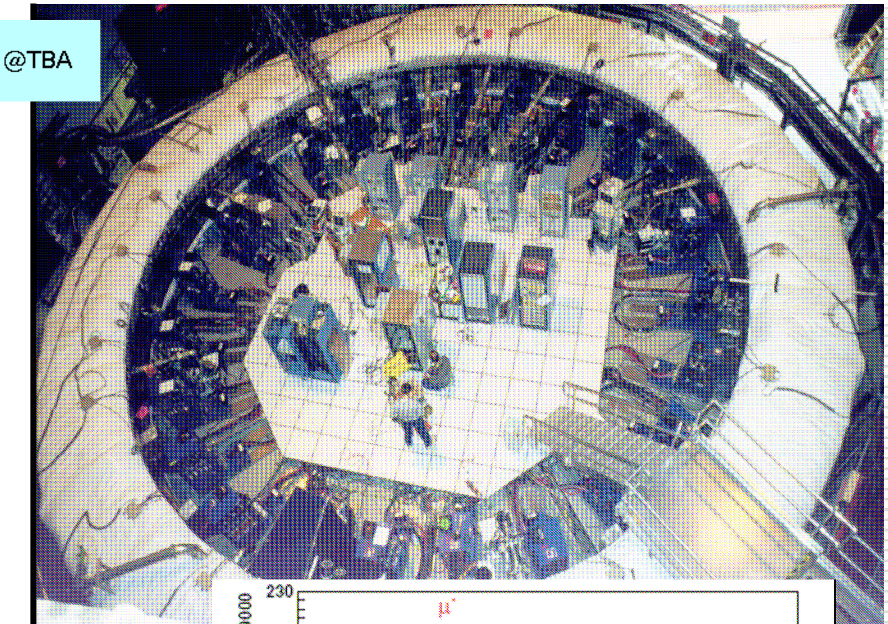
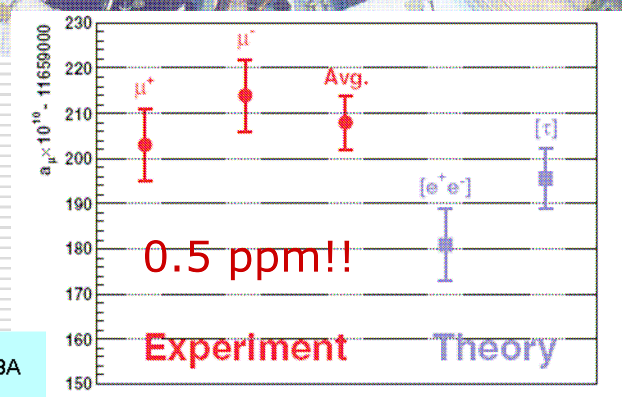




@TBA

Nucleon Form Factors

Take the same current for the nucleon

$$j_p^\mu = e\bar{u}(p')\left(F(q^2)\gamma^\mu + G(q^2)\, i\kappa_p \frac{\sigma^{\mu\nu}}{2m} q_\nu\right)u(p)$$

Anomalous magnetic moment well measured, not understood

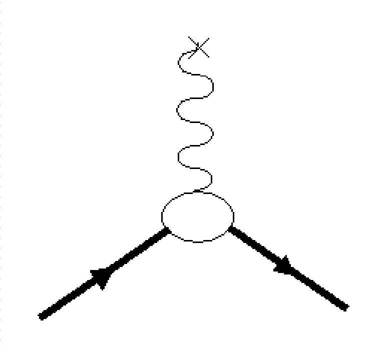
$$\kappa_p = ?$$

Anomaly originating here from the extended shape of the proton, rather than radiative corrections

$$F_1(q^2) = F(q^2)$$

$$F_2(q^2) = 2MG(q^2)$$

$$\rightarrow j_p^\mu = e\bar{u}(p')\left(F_1(q^2)\gamma^\mu + \frac{i\kappa_p F_2(q^2)}{2M}\sigma^{\mu\nu} q_\nu\right)u(p)$$



Blob indicates a non-QED vertex

Redefine:

$$G_E(q^2) = F_1 + \frac{\kappa_P q^2}{4M^2} F_2$$ Electric form factor

$$G_M(q^2) = F_1 + \kappa_P F_2$$ Magnetic form factor

Nucleon Magnetic Moments

Electron-Proton comparison

$$\mu_B = \frac{e\hbar}{2m_e} = 5.78\times10^{-5} \quad eV/T$$

$$\mu = g\,s\,\mu_B$$

μ_e/μ_B = 1.001 159 652 187 ± 0.000 000 000 004 $\quad \frac{g}{2}$: Electron

$$\mu_N = \frac{e\hbar}{2m_p} = 3.15\times10^{-8} \quad eV/T$$

$$\mu_p = g\,s\,\mu_N$$

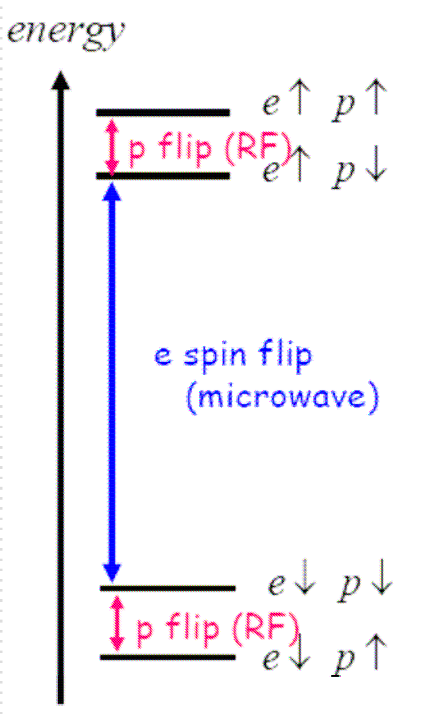
μ_p/μ_N = 2.792847351 ± 0.000000028 $\quad \frac{g}{2}$: Proton

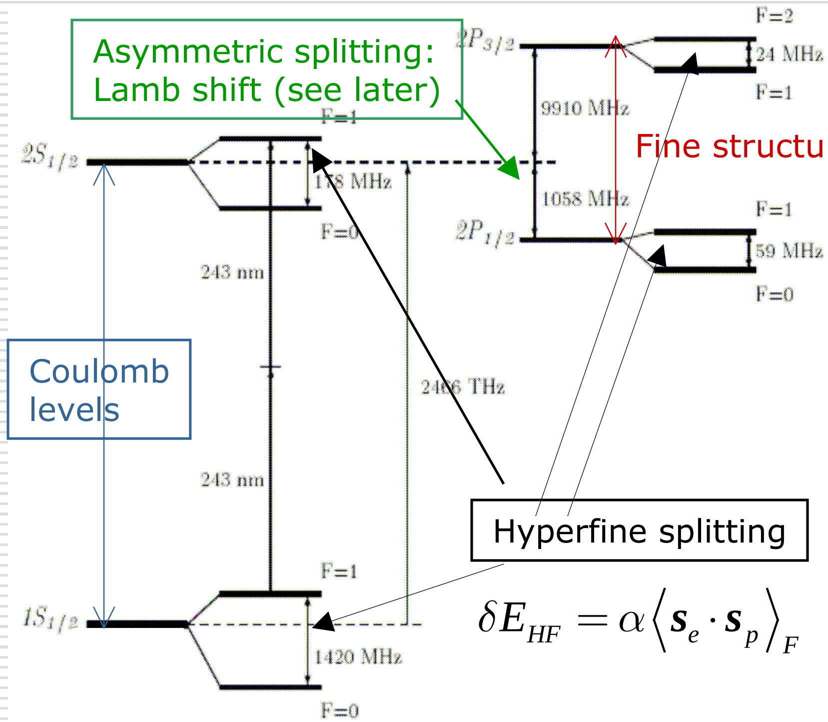
Reminder: For a free Dirac particle *g=2*

Strong indication: *The nucleon is not a point-like particle*

Proton Magnetic Moment - I

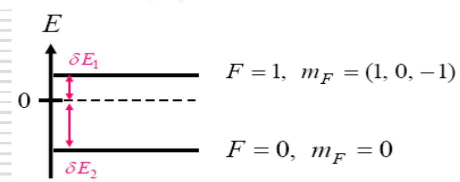
Modern method:
RF+Microwave atomic spectroscopy





$$\delta E_{HF} = \alpha \left\langle \mathbf{s}_e \cdot \mathbf{s}_p \right\rangle_F$$

@TBA



$$\delta E_1 + \delta E_2 = 5.8 \times 10^{-6} \;\; eV$$

Proton Magnetic Moment - II

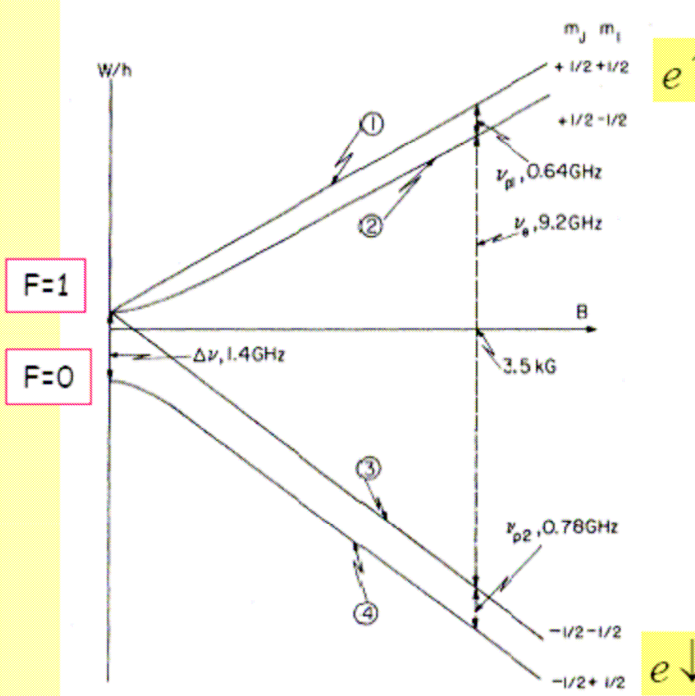


FIG. 1. Energy levels of hydrogen in the ground electronic state in an applied magnetic field.

$e\uparrow$ $p\uparrow,\ m_F=1\ \therefore\ F=1$

$p\downarrow,\ m_F=0$

The electron's magnetic moment is almost 2000x larger than that of the proton and of opposite sign. The main splitting between states at large B is due to the electron.

The hyperfine interaction between the electron and proton spins still causes the F = 1 state to have higher energy. This explains the order of states in the diagram.

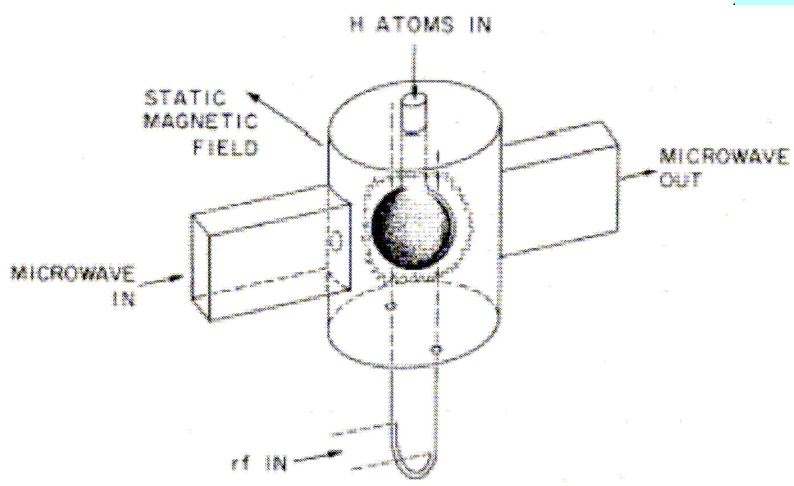
$e\downarrow$ $p\downarrow,\ m_F=-1\ \therefore\ F=1$

$p\uparrow,\ m_F=0$

@TBA

Proton Magnetic Moment - III

@TBA



$$\frac{\mu_{e^-}(\mathrm{H})}{\mu_p(\mathrm{H})} = -658.210\,7058(66) \quad [1.0 \times 10^{-8}],$$

Correct for atomic effects

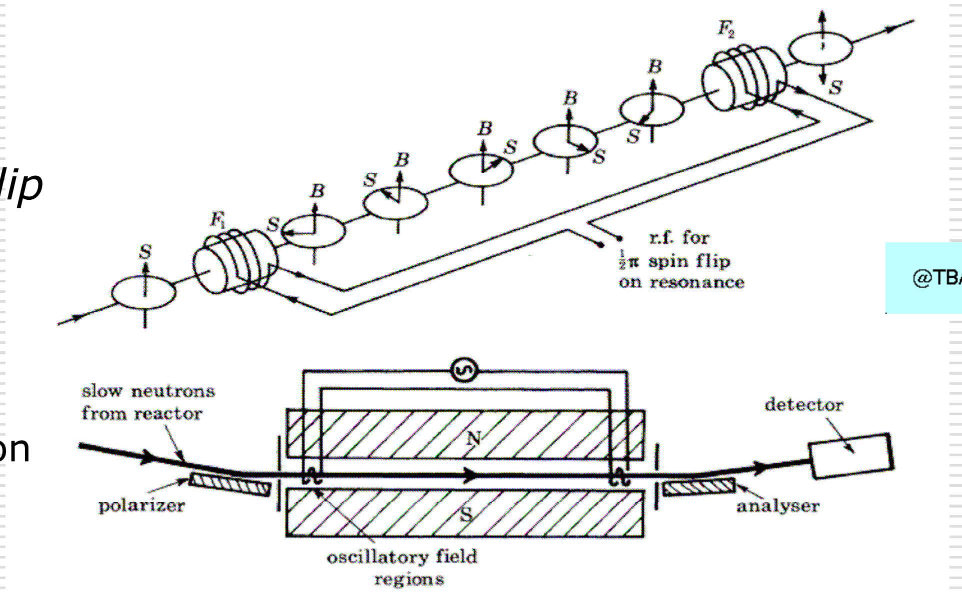
$$\frac{\mu_{e^-}}{\mu_p} = \frac{g_p(\mathrm{H})}{g_p}\left(\frac{g_{e^-}(\mathrm{H})}{g_{e^-}}\right)^{-1}\frac{\mu_{e^-}(\mathrm{H})}{\mu_p(\mathrm{H})}$$
$$= -658.210\,6860(66) \quad [1.0 \times 10^{-8}],$$

$$\mu_p/\mu_N = \underline{2.7928}47351 \pm 0.000000028$$

Neutron Magnetic Moment - I

Modern measurements: Thermal neutrons from a reactor

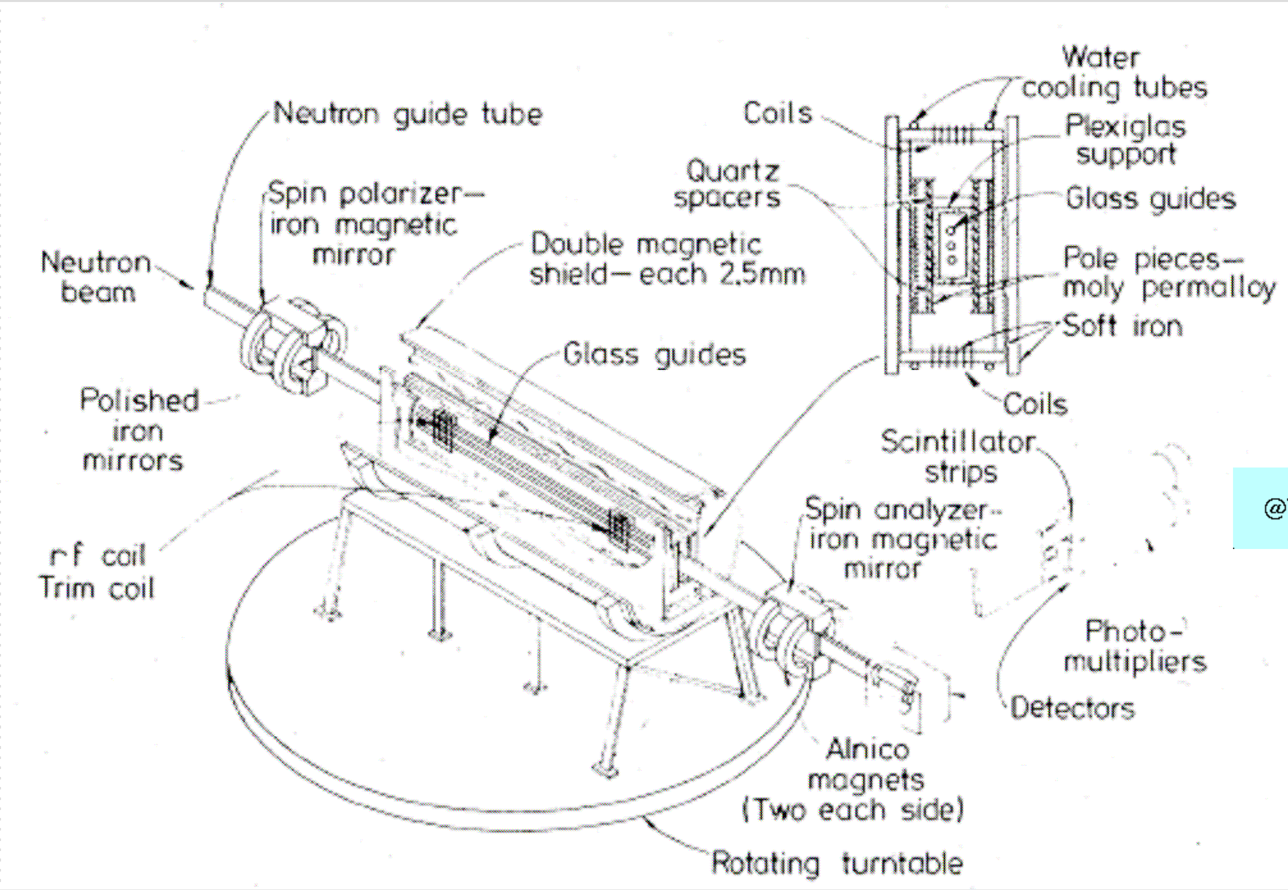
RF pulses in *F1,F2*
Periodic spin rotation in *F1,F2* : Get ***s*** ⊥ ***B***
Time of flight between F_1,F_2
Spin precession in ***B*** field
Count analyzed neutrons
'Resonance':
Count rate vs. frequency of spin flip



@TBA

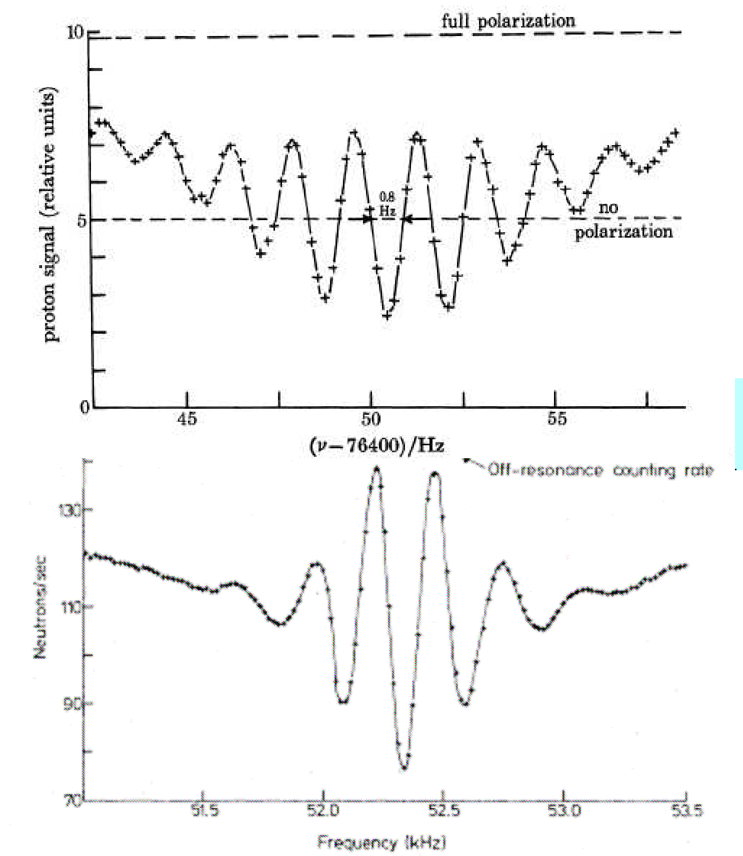
Polarizer,Analyzer:
Neutron Scattering from magnetized iron crystals, sensitive to spin orientation

Neutron Magnetic Moment - II



@TBA

Neutron Magnetic Moment - III



Proton resonance (Calibration)

@TBA

Neutron resonance

$$\mu_n/\mu_N = -1.913\,041\,84\,(88)\ (0.45\ \text{ppm})\,.$$

The Rosenbluth Formula

Consider elastic electron-nucleon scattering
Going through the same steps as for electron-muon scattering

$$\begin{cases} A(q^2) = F_1^2(q^2) - \kappa_p^2 \dfrac{q^2}{4M^2} F_2^2(q^2) \\ B(q^2) = -\dfrac{q^2}{2M^2}\left(F_1(q^2) + \kappa_p F_2(q^2)\right)^2 \end{cases}$$

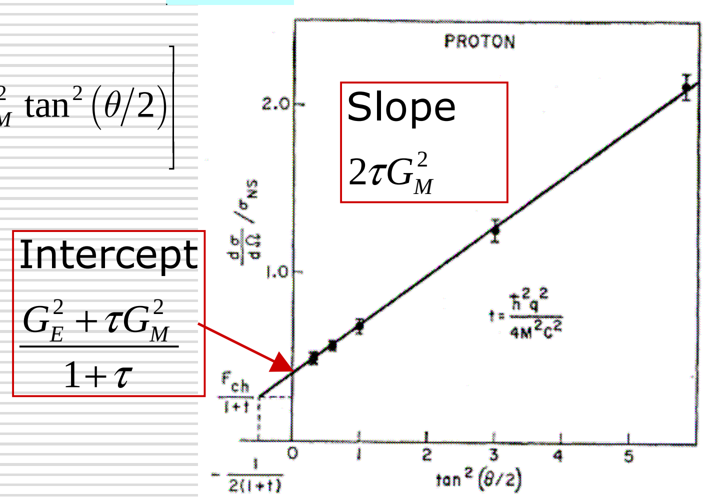
$$\left.\frac{d\sigma}{d\Omega}\right|_{LAB} = \frac{\alpha^2}{4|\mathbf{p}_1|^2 \sin^4\theta/2} \cos^2\theta/2 \frac{|\mathbf{p}_1'|}{|\mathbf{p}_1|}\left(A(q^2) + B(q^2)\tan^2\theta/2\right)$$

$$\left.\frac{d\sigma}{d\Omega}\right|_{LAB} = \frac{\alpha^2}{4|\mathbf{p}_1|^2 \sin^4\theta/2} \cos^2\theta/2 \frac{|\mathbf{p}_1'|}{|\mathbf{p}_1|}\left[\frac{G_E^2 - (q^2/4m^2)G_M^2}{1 - q^2/4m^2} - \frac{q^2}{m^2} G_M^2 \tan^2(\theta/2)\right]$$

$$\left.\frac{d\sigma}{d\Omega}\right|_{LAB} = \left(\frac{d\sigma}{d\Omega}\right)_{Mott}\left[\frac{G_E^2 + \tau G_M^2}{1+\tau} + 4\tau G_M^2 \tan^2\theta/2\right], \tau = -\frac{q^2}{4m^2}$$

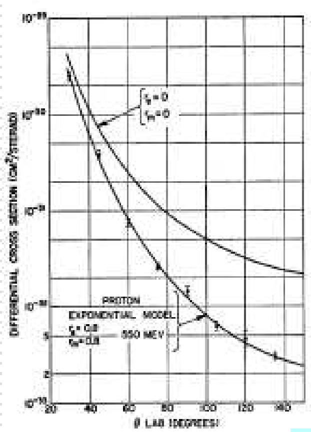
@TBA

Fixed q^2



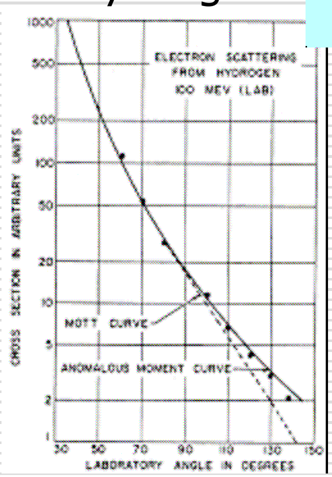
'Rosenbluth separation' gives G_E, G_M

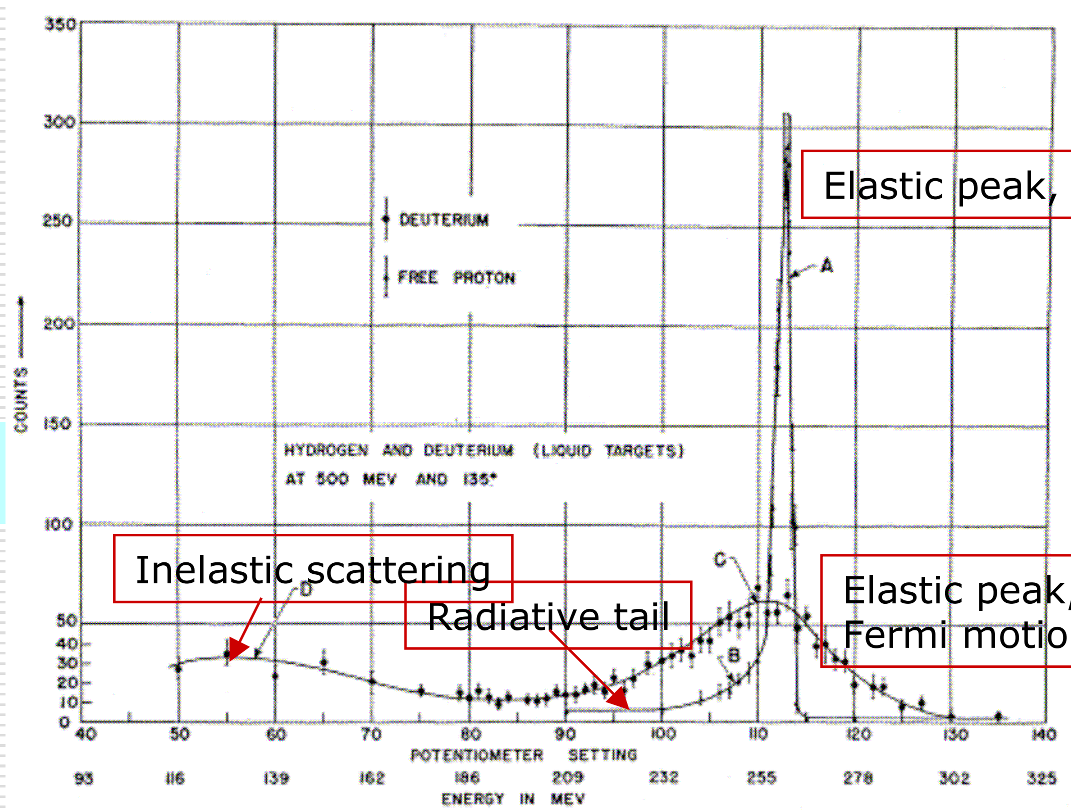
Measurements



Hydrogen

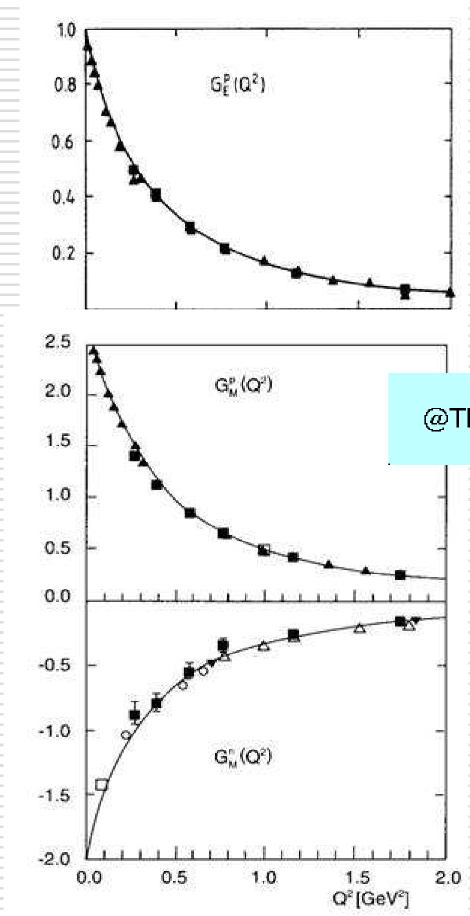
@TBA



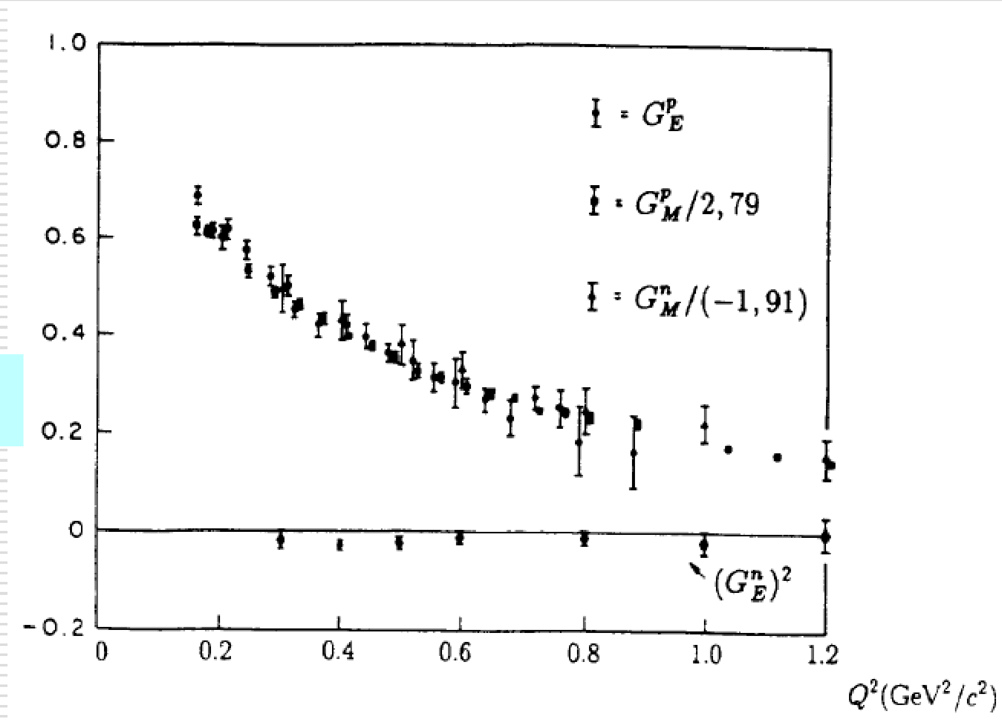


Electron energy at fixed angle for H_2, D_2

Experimental Results



@TBA



Same shape for the 3 non-zero..
Probing the same structure

Time-Like Form Factors

$e^+ + e^- \rightarrow p + \bar{p}$

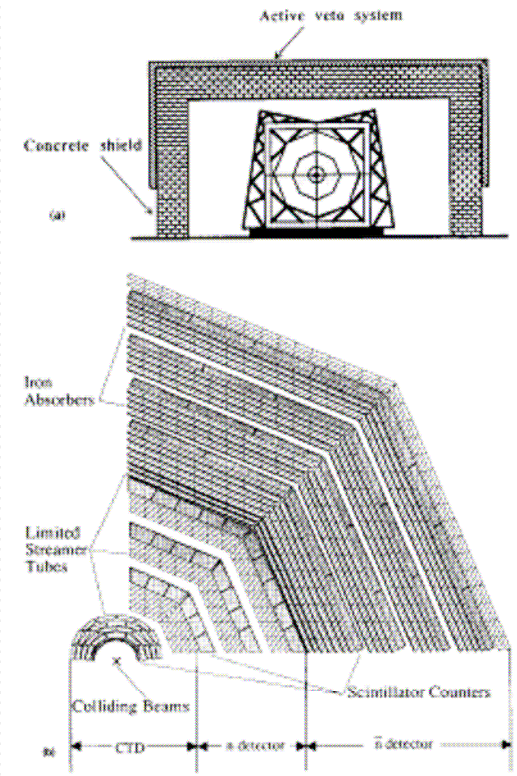
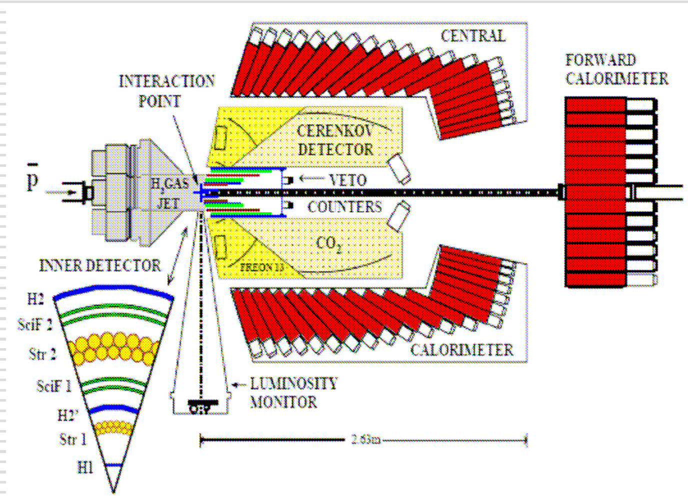


Fig. 1. (a) Schematic view of the FENICE detector: the concrete shield is also shown. (b) Detailed view of the FENICE detector showing two contiguous octant.

FENICE (Frascati)

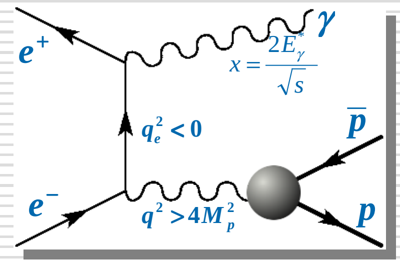
$\bar{p} + p \rightarrow e^+ + e^-$

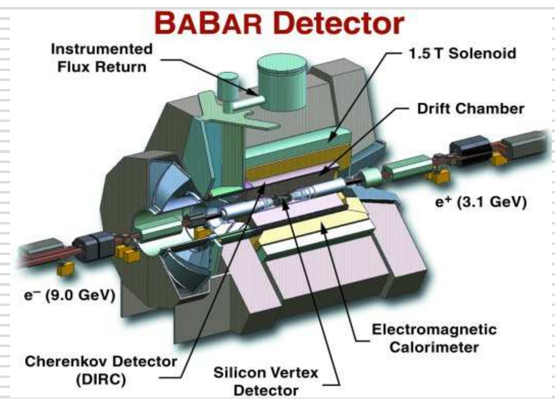


E760/E835 (Fermilab)

@TBA

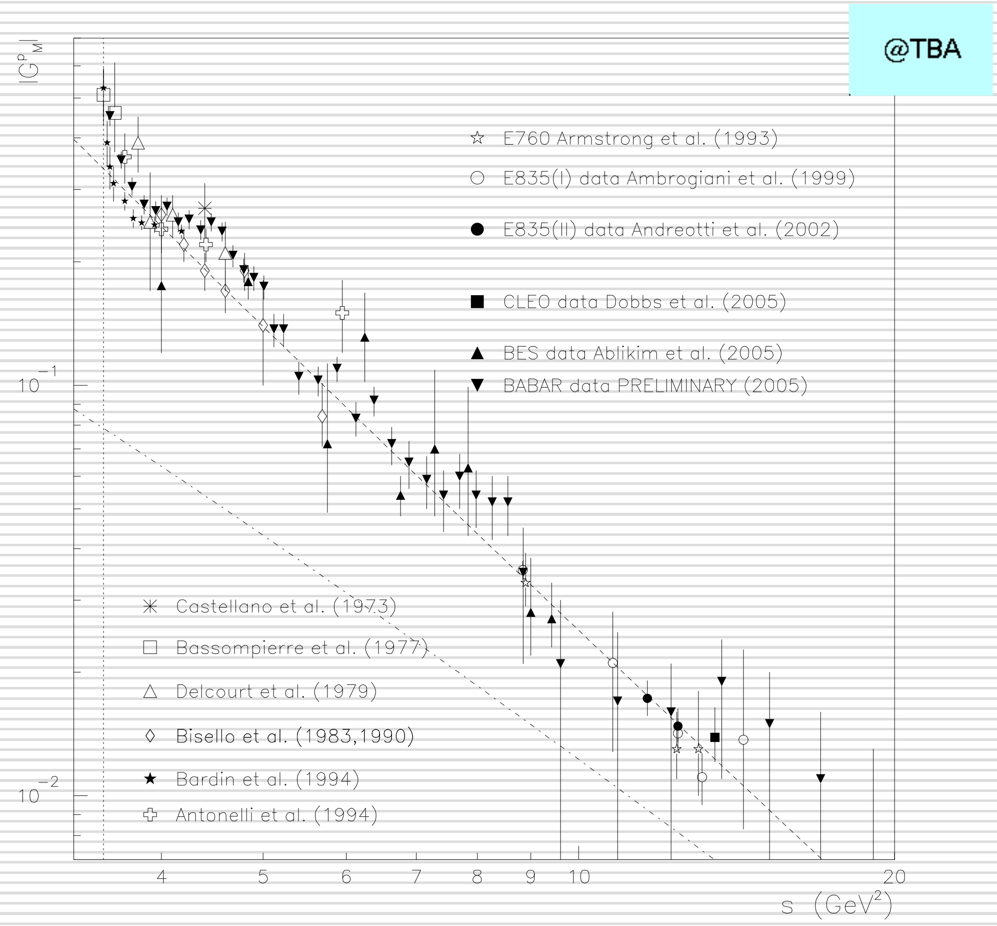
$e^+ + e^- \rightarrow p + \bar{p} + \gamma$





BaBar (SLAC)

Experimental Results



Elastic Scattering Kinematics

4-momentum conservation at the nucleon vertex

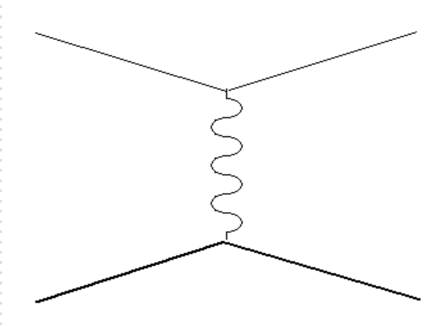
$$p + q = p'$$
$$\rightarrow (p+q)^2 = p'^2 \rightarrow p^2 + q^2 + 2p \cdot q = p'^2$$
$$p^2 = p'^2 = M^2$$
$$\rightarrow 2p \cdot q = -q^2$$

Rewrite in the LAB frame, take massless lepton

$$P^\mu = (M, \mathbf{0})$$
$$p^\mu = (E, \mathbf{p}) \approx (|\mathbf{p}|, \mathbf{p})$$
$$p'^\mu \approx (|\mathbf{p}'|, \mathbf{p}')$$
$$\rightarrow q^2 \approx (|\mathbf{p}| - |\mathbf{p}'|, \mathbf{p} - \mathbf{p}')^2 = |\mathbf{p}|^2 + |\mathbf{p}'|^2 - 2|\mathbf{p}||\mathbf{p}'| - \left(|\mathbf{p}|^2 + |\mathbf{p}'|^2 - 2\mathbf{p} \cdot \mathbf{p}'\right)$$
$$\rightarrow q^2 \approx (|\mathbf{p}| - |\mathbf{p}'|, \mathbf{p} - \mathbf{p}')^2 = -2|\mathbf{p}||\mathbf{p}'| + 2\mathbf{p} \cdot \mathbf{p}' = -2|\mathbf{p}||\mathbf{p}'|(1 - \cos\theta) = -4|\mathbf{p}||\mathbf{p}'|\sin^2\theta/2$$
$$Q^2 \equiv -q^2$$
$$p \cdot q \approx (M, \mathbf{0}) \cdot (|\mathbf{p}| - |\mathbf{p}'|, \mathbf{p} - \mathbf{p}') = M(|\mathbf{p}| - |\mathbf{p}'|)$$
$$\nu \equiv |\mathbf{p}| - |\mathbf{p}'| \rightarrow p \cdot q \approx M\nu$$

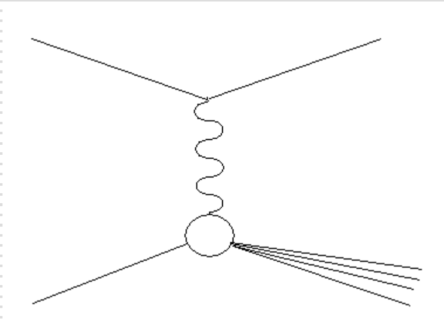
Inelastic Scattering

Elastic:

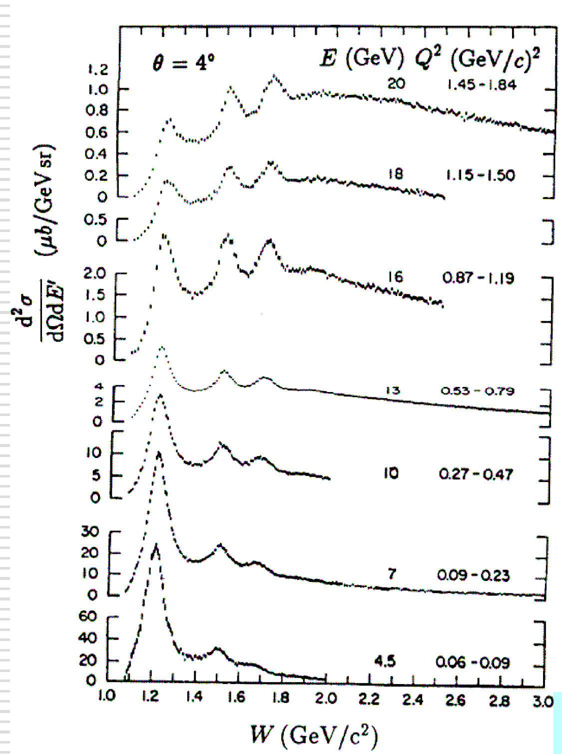


Invariant mass = M

Generalise to inelastic reactions:



Invariant mass = W>M

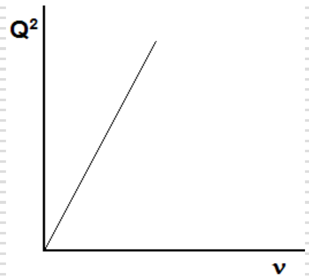


@TBA

Copious production of *resonances* (nucleon excited states), when q^2 not too big

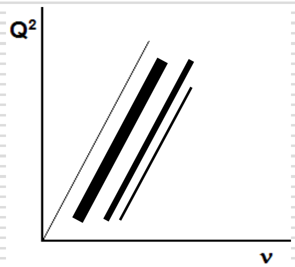
Inelastic Scattering Kinematics

$$\nu = \frac{Q^2}{2M} \rightarrow 2M\nu = Q^2$$



Elastic

$$2M\nu = Q^2 + M'^2 - M^2$$



Inelastic

Scattered electron:

2-body kinematics,
E′, θ totally correlated

>2-body kinematics
E′, θ uncorrelated

Measure *E′*, $\theta \rightarrow$ Get q^2, ν

Generalise Rosenbluth cross-section to account for variable *W*:
Introduce *structure functions* W_1, W_2 to replace form factors F_1, F_2

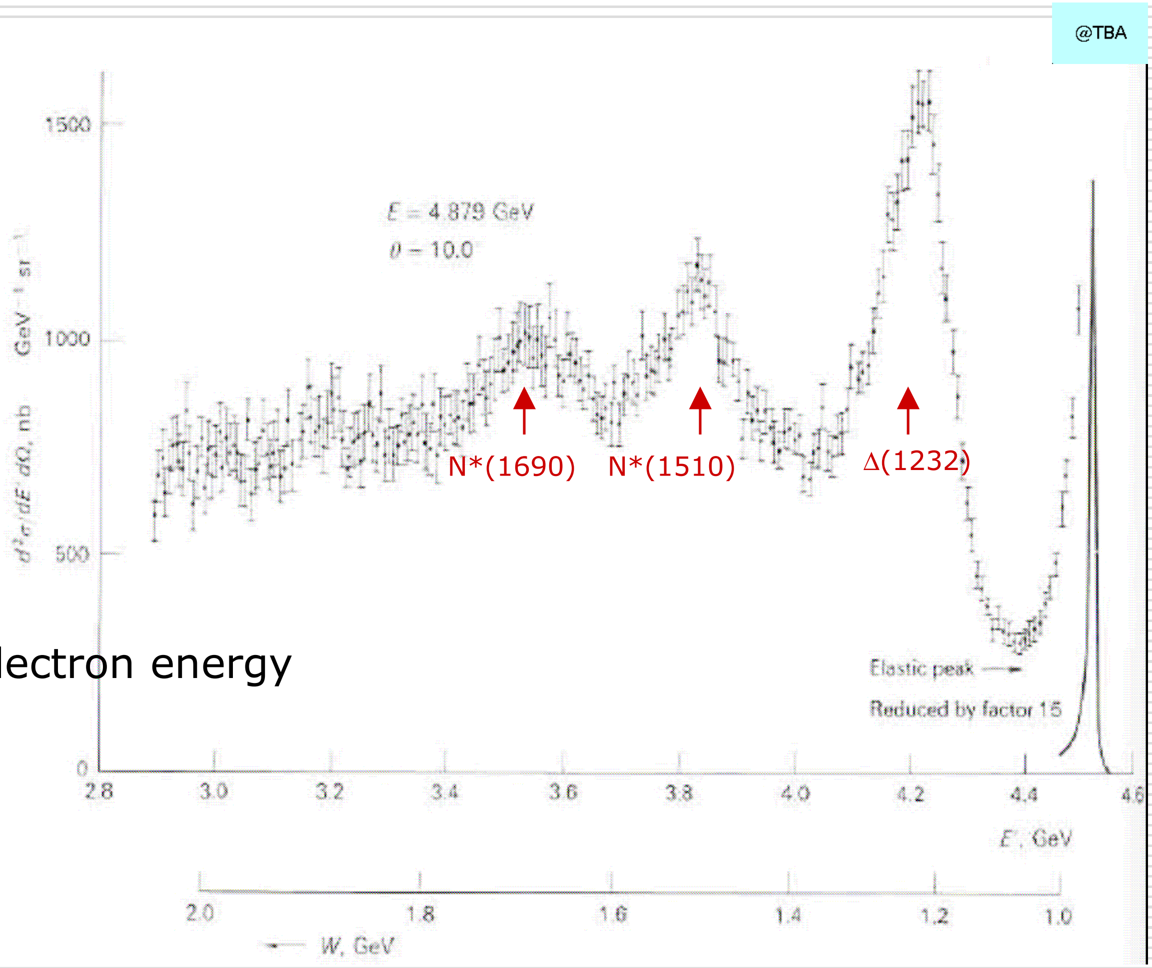
$$\left.\frac{d\sigma}{d\Omega dE'}\right|_{LAB} = \frac{\alpha^2}{4|\mathbf{p}_1|^2 \sin^4\theta/2} \cos^2\theta/2 \left(W_2\left(\nu, q^2\right) + W_1\left(\nu, q^2\right)\tan^2\theta/2\right)$$

$W_{1,2}$ depending on q^2 *and* ν

Resonance Excitation at DESY

First DESY machine

Electron synchrotron
E_{max} = 6 GeV

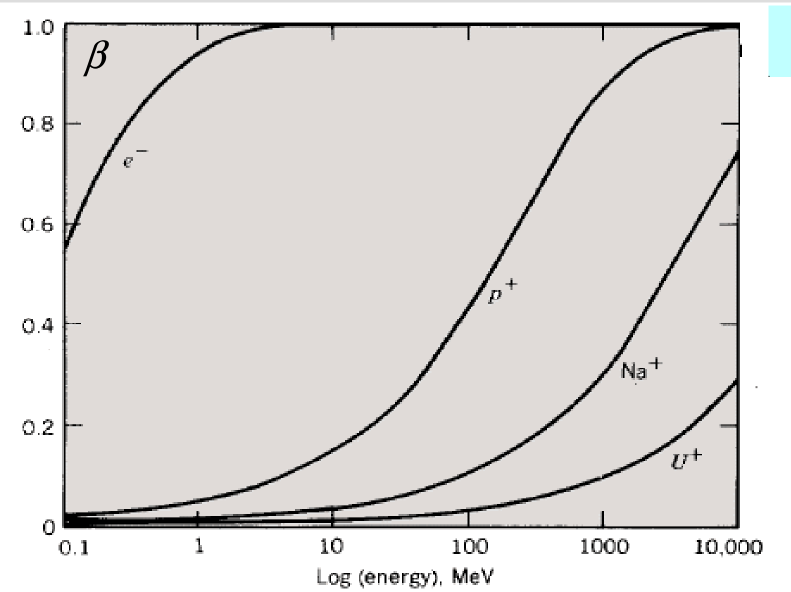


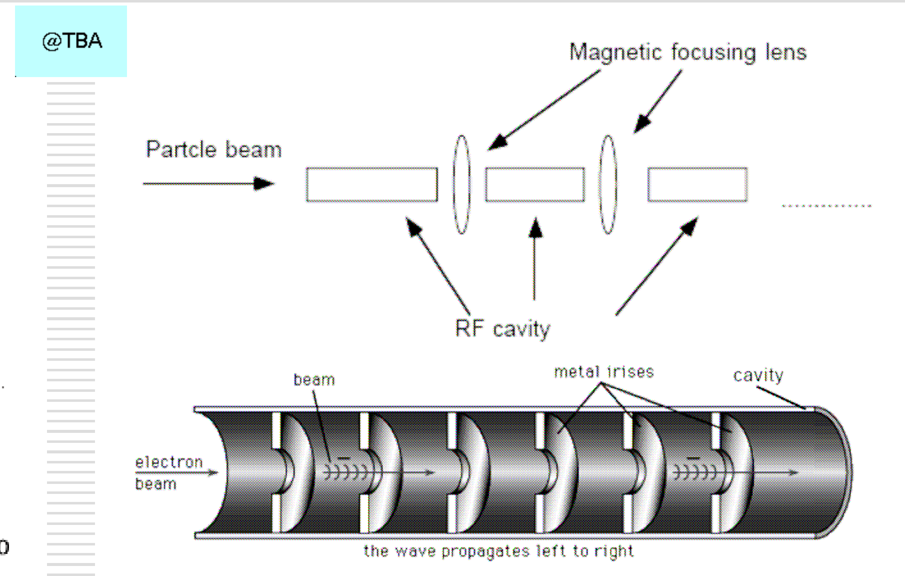
$e+p \rightarrow e+X$

E, E': Incident, scattered electron energy

$W = M_X$

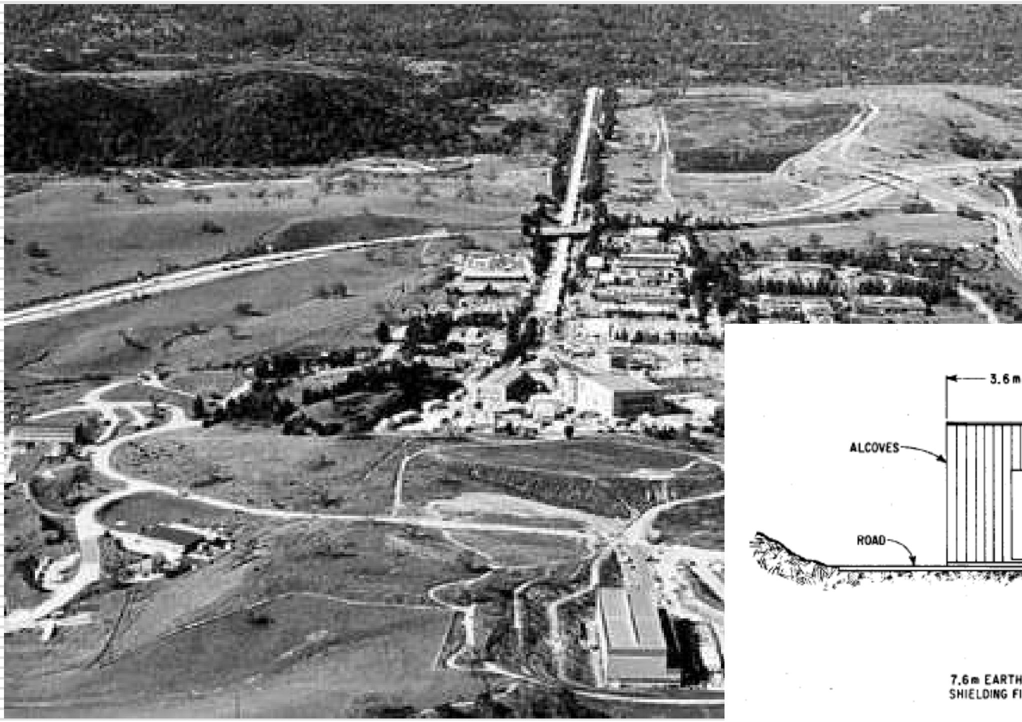
The Electron LINAC

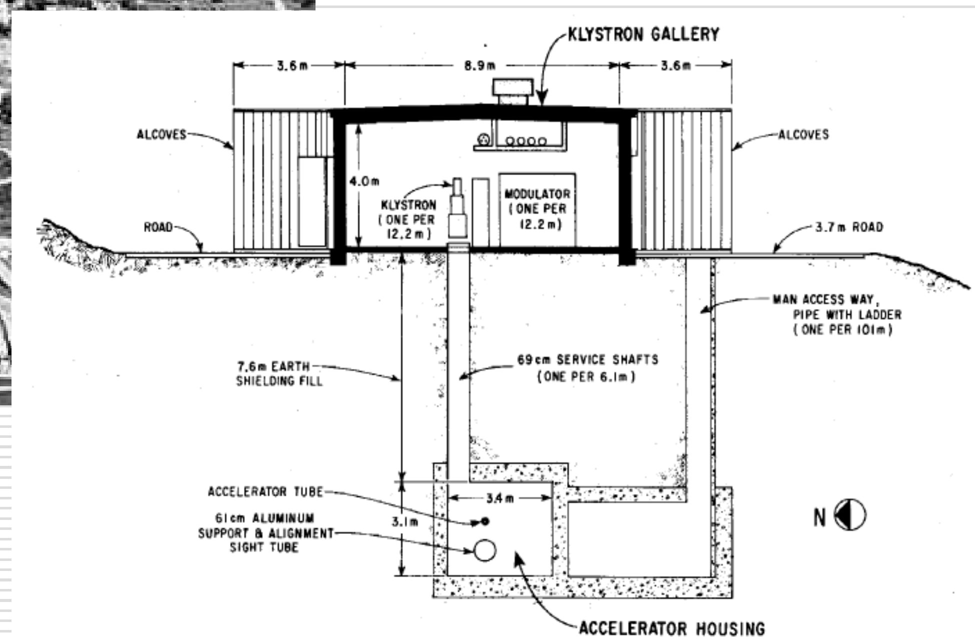




Traveling wave linear accelerator:
Electrons riding the traveling EM wave at constant phase

SLAC - I

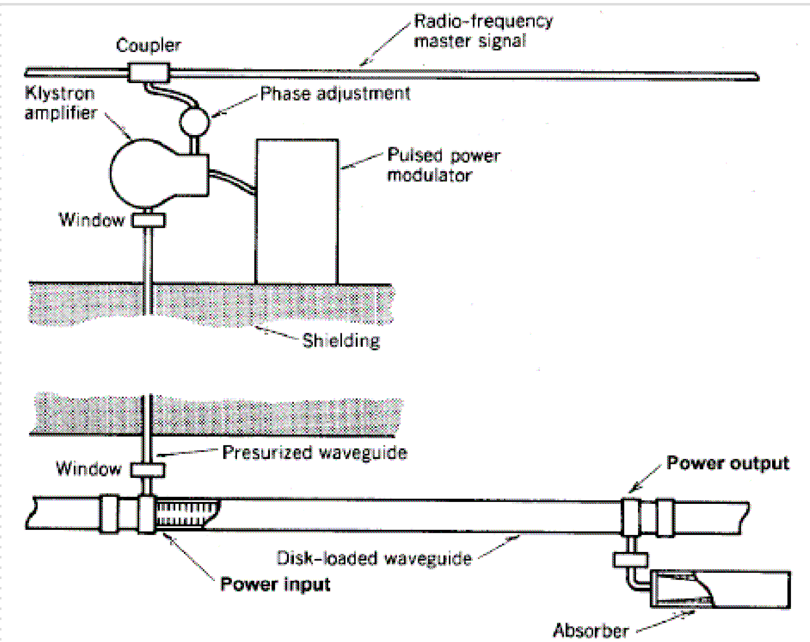



KLYSTRON GALLERY
3.6m
8.9m
3.6m
ALCOVES
ALCOVES
4.0m
KLYSTRON
(ONE PER
12.2m)
MODULATOR
(ONE PER
12.2m)
ROAD
3.7m ROAD
MAN ACCESS WAY,
PIPE WITH LADDER
(ONE PER 101m)
7.6m EARTH
SHIELDING FILL
69cm SERVICE SHAFTS
(ONE PER 6.1m)
ACCELERATOR TUBE
3.4m
61cm ALUMINUM
SUPPORT & ALIGNMENT
SIGHT TUBE
3.1m
N
ACCELERATOR HOUSING

SLAC - II

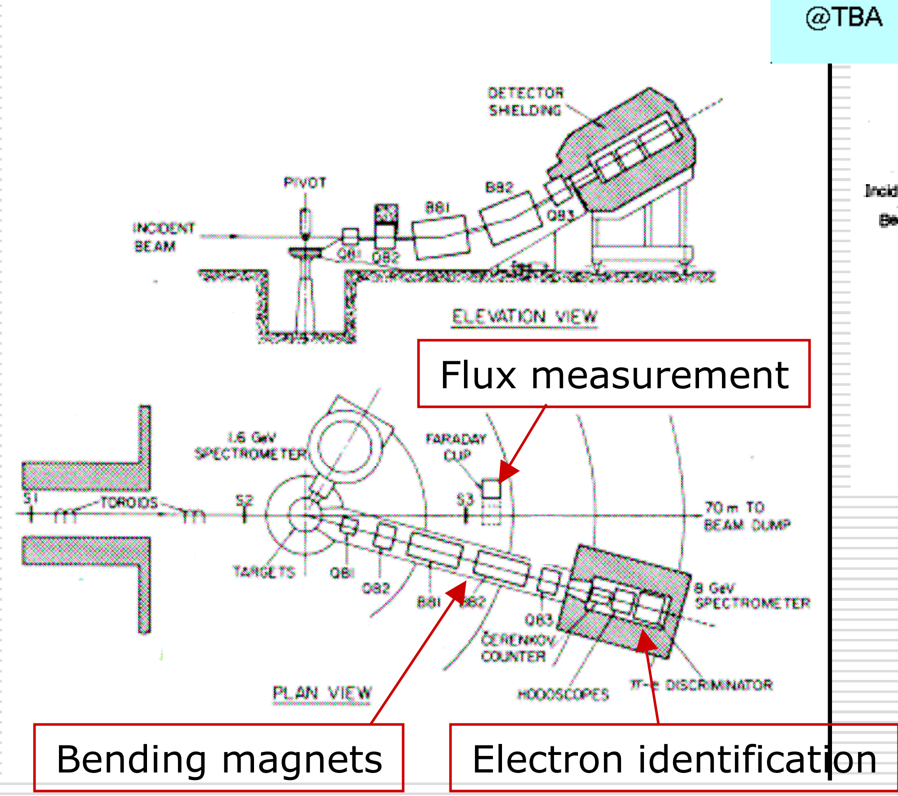
Accelerator length	3100 m
Length between power feeds	3.1 m
Number of accelerator sections	960
Number of klystrons	245
Peak power per klystron	6–24 MW
Beam pulse repetition rate	1–360 pulses/s
Radio-frequency pulse length	2.5 μs
Filling time	0.83 μs
Shunt impedance	53 MΩ/m
Electron energy (unloaded)	11.1–22.2 GeV
Electron energy (loaded)	10–20 GeV
Electron beam peak current	25–50 mA
Electron beam average current	15–30 μA
Average electron beam power	0.15–0.6 MW
Efficiency	4.3%
Positron energy	7.4–14.8 GeV
Positron average beam current	0.45 μA
Operating frequency	2.856 GHz
Accelerating structure	Iris-loaded waveguide
Waveguide outer diameter	10.5 cm
Aperture diameter	1.9 cm

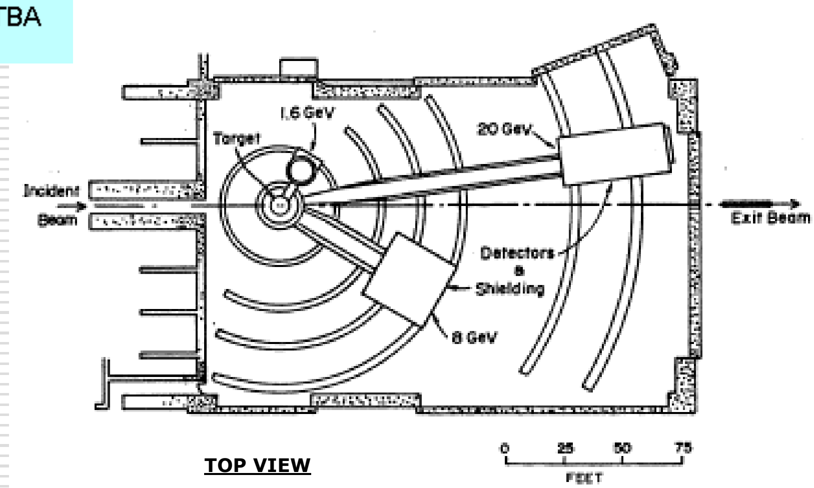
@TBA



The SLAC Experiments

@TBA

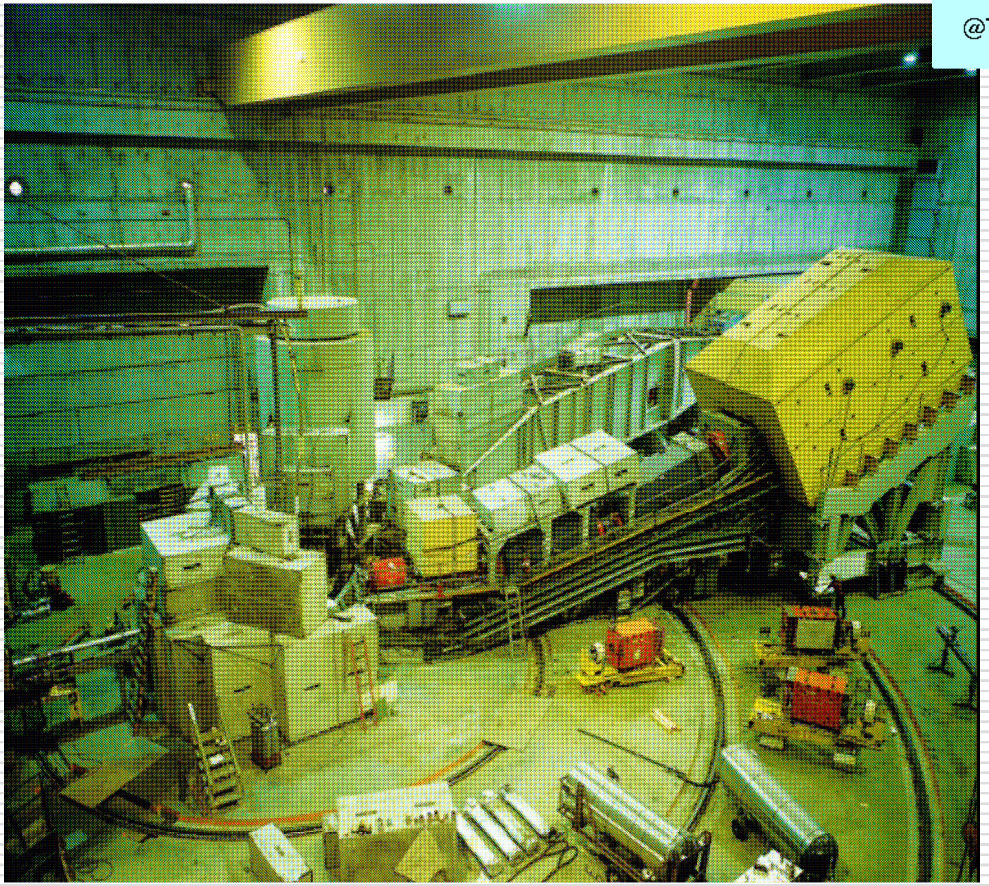




Measure E', θ of the scattered electron
$\rightarrow$Get q^2, ν

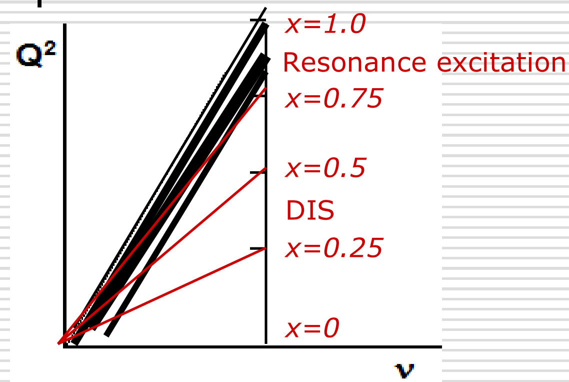
$$\frac{d^2\sigma}{dE'd\Omega} \rightarrow \frac{d^2\sigma}{dq^2 d\nu}$$

SLAC End Station A

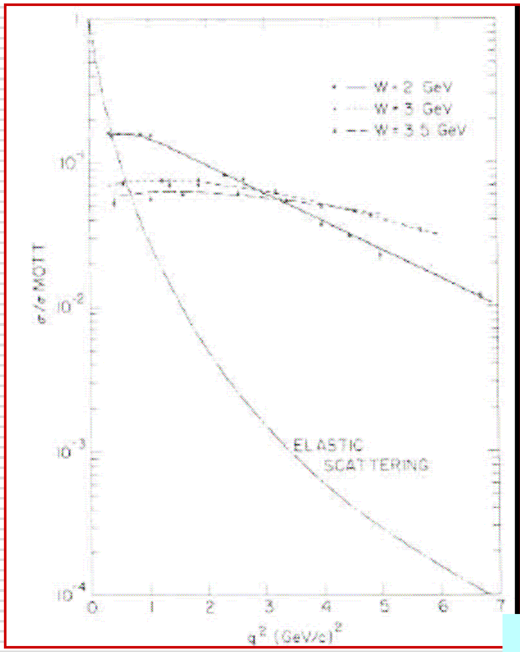


Deep Inelastic Scattering - I

Details of structure functions in the resonance region difficult to explain
But: Beyond small q^2, ν things are surprisingly simple!



Striking elastic/deep inelastic comparison:
no q^2 dependence in DIS



@TBA

$$\frac{d^2\sigma}{dE'd\Omega} \to \frac{d^2\sigma}{dq^2 d\nu}$$

$$\begin{cases} x = \dfrac{Q^2}{2M\nu} \\ y = \dfrac{\nu}{E_1} \end{cases} \to \frac{d^2\sigma}{dxdy} = \frac{4\pi\alpha^2 ME_1}{Q^4}\left[2xF_1\left(\frac{1+(1-y)^2}{2}\right)+(1-y)(F_2-2xF_1)-\frac{M^2 xyF_2}{s-M^2}\right]$$

Bjorken scaling hypothesis:

$$\frac{d^2\sigma}{dQ^2 d\nu} \underset{\substack{q^2,\nu\to\infty \\ \frac{q^2}{\nu}\ \text{finite}}}{\longrightarrow} f(x), \quad x = \frac{Q^2}{2M\nu}$$

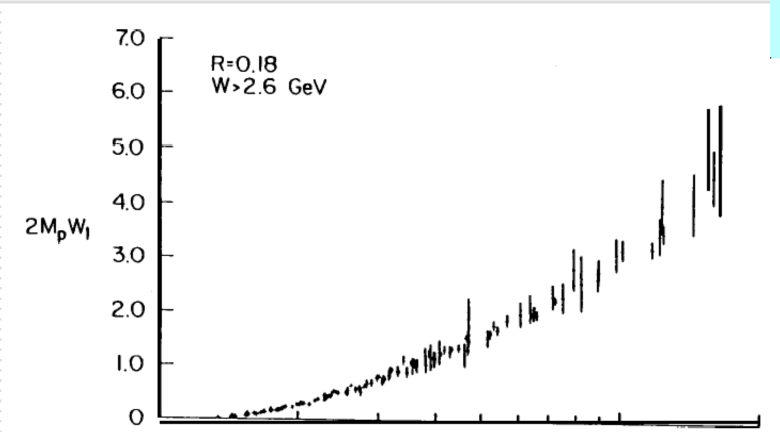
Universal function of x
Q^2 independent

Deep Inelastic Scattering - II

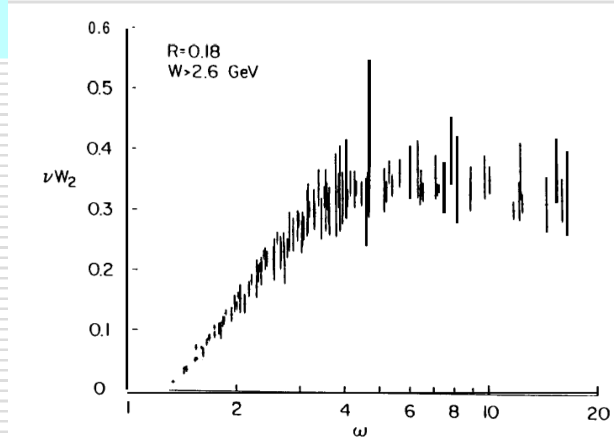
Expect:

$$\begin{cases} F_1\left(x,Q^2\right) \to F_1\left(x\right) \\ F_2\left(x,Q^2\right) \to F_2\left(x\right) \end{cases}$$ if scaling is good

Observe ($\omega = 1/x$):



@TBA

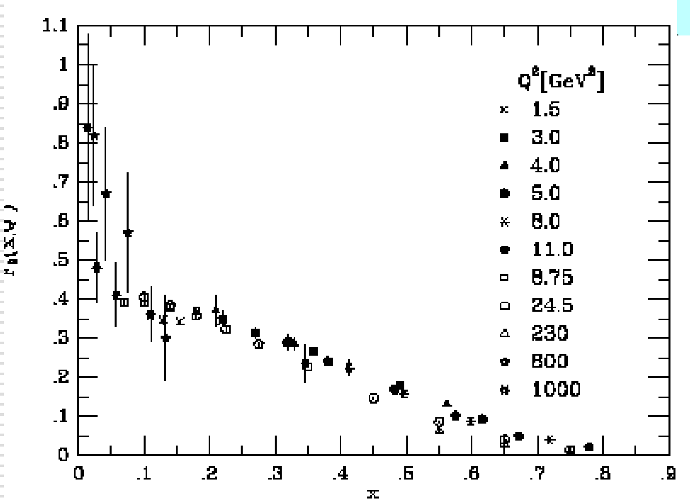


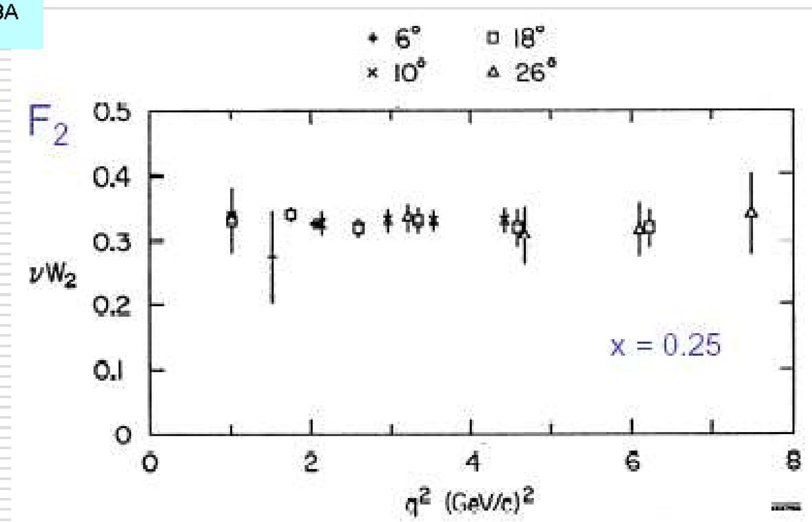
Scaling at high energy is indeed well verified!

Deep Inelastic Scattering - III

W_2: Universal function of x…………………….q^2-independent!

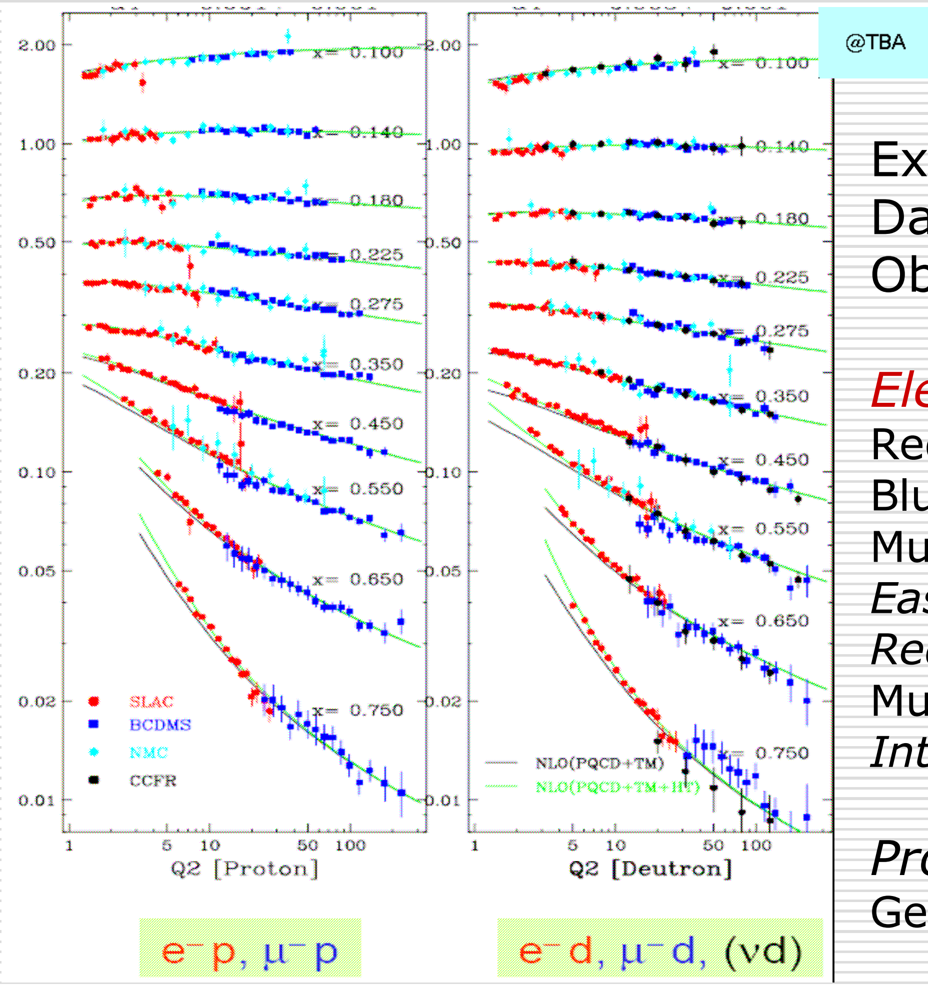
@TBA





As compared to fast varying elastic f.f., just astonishing…

Scaling Violations



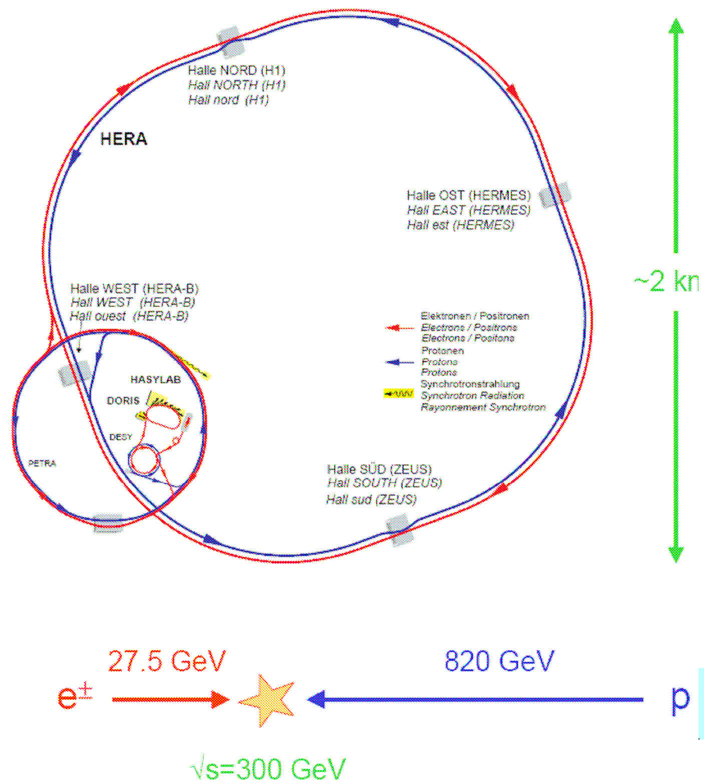
Extensive compilation
Data from fixed target experiments
Observe:

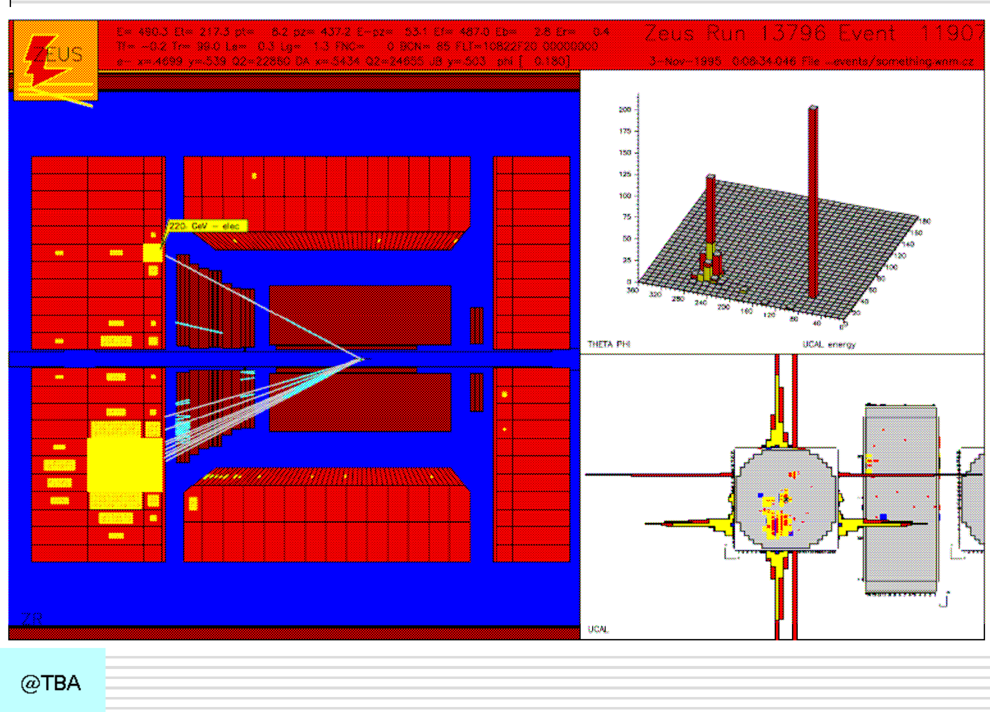
Electron vs. *Muon*
Red points are from electron DIS
Blue points are from muon DIS
Muon merits:
Easier to get to high energy
Reduced radiative corrections
Muon drawbacks:
Intensity

Proton (L) vs. Deuteron (R)
Get *neutron* structure function

HERA

First example of *asymmetric* collider





A Recent Compilation

Results from several experiments:

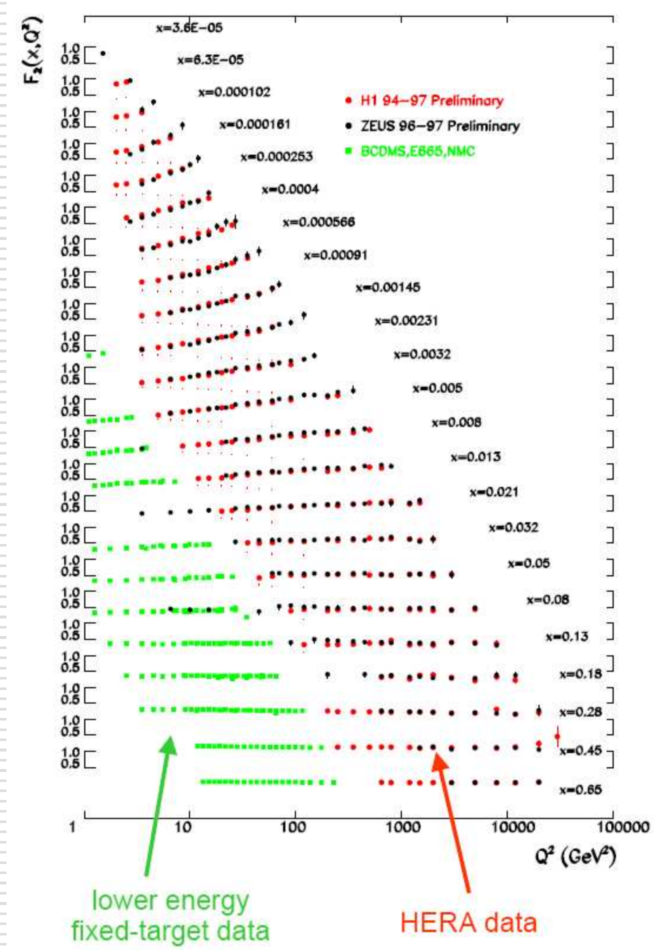
muon DIS NMC, BCDMS, E665
CERN FNAL

electron DIS at HERA collider
DESY

Huge q^2, x range

Small, measurable scaling violation
Interesting features at small x

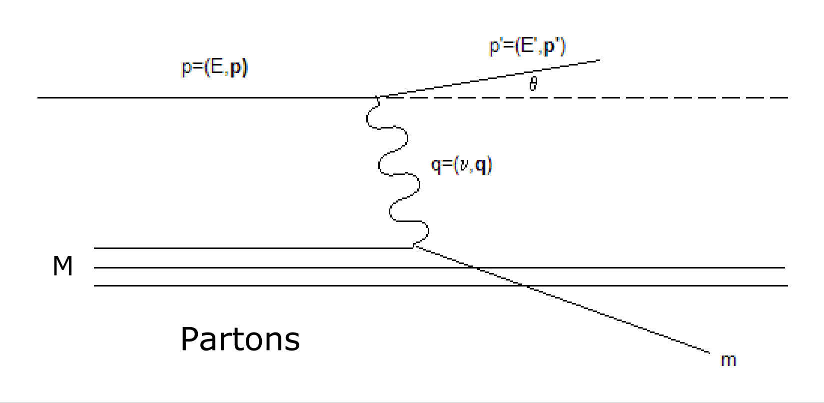
$\rightarrow$ QCD !



Parton Model - I

Structure functions generalise form factors
Form factor of a point source = *constant*
Feynman suggestion:

Maybe deep <u>inelastic</u> scaling just indicates <u>elastic</u> scattering off free, pointlike constituents



Differential cross-section for elastic scattering off a free, pointlike constituent of mass m

$$\left.\frac{d\sigma}{d\Omega}\right|_{LAB} = \frac{\alpha^2}{4|\mathbf{p}|^2 \sin^4\theta/2}\cos^2\theta/2\,\frac{|\mathbf{p}'|}{|\mathbf{p}|}\left(1-\frac{q^2\tan^2\theta/2}{2m^2}\right)$$

$$= \frac{\alpha^2}{4|\mathbf{p}|^2 \sin^4\theta/2}\frac{|\mathbf{p}'|}{|\mathbf{p}|}\left(\cos^2\theta/2-\frac{q^2}{2m^2}\sin^2\theta/2\right)$$

$$m^2 = (m+\nu,\mathbf{q})^2 = m^2 + 2m\nu + \underbrace{\nu^2 - |\mathbf{q}|^2}_{=q^2}$$

$$\rightarrow \nu + \frac{q^2}{2m} = 0$$

Parton Model - II

In the elastic scattering off a parton, energy and angle of the scattered electron are fully correlated

Formally write the differential cross section as

Full E', θ correlation

z: Parton charge, units e

$$\frac{d\sigma}{d\Omega} = \int dE' \frac{d^2\sigma}{dE'd\Omega} = \int dE' \frac{\alpha^2 z^2}{4E^2 \sin^4 \theta/2} \left(\cos^2 \theta/2 - \frac{q^2}{2m^2} \sin^2 \theta/2 \right) \delta\left(\nu + \frac{q^2}{2m} \right)$$

$$\begin{cases} \nu = E - E' \\ q^2 = -4EE' \sin^2 \theta/2 \end{cases} \rightarrow E - E' = \frac{4EE' \sin^2 \theta/2}{2m} \rightarrow E' \left(1 + \frac{4E}{2m} \sin^2 \theta/2 \right) = E$$

$$\rightarrow E' = \frac{E}{1 + \frac{4E}{2m} \sin^2 \theta/2}$$

$$\nu + \frac{q^2}{2m} = 0, x = -\frac{q^2}{2M\nu} \rightarrow x = \frac{m}{M}$$

$$\frac{d^2\sigma}{dE'd\Omega} = \frac{\alpha^2 z^2}{4E^2 \sin^2 \theta/2} \left(\cos^2 \theta/2 - \frac{q^2}{2M^2 x^2} \sin^2 \theta/2 \right) \delta\left(\nu + \frac{q^2}{2Mx} \right)$$

Parton Model - III

Summing over all types of partons

$$\frac{d^2\sigma}{dE'd\Omega} = \frac{\alpha^2}{4E^2\sin^2\theta/2}\left(\cos^2\theta/2 - \frac{q^2}{2M^2x^2}\sin^2\theta/2\right)\delta\left(\nu + \frac{q^2}{2Mx}\right)\left(\sum_i z_i^2 n_i\right)$$

Compare to inelastic cross-section

$$\left.\frac{d\sigma}{d\Omega dE'}\right|_{LAB} = \frac{\alpha^2}{4E^2\sin^4\theta/2}\left(W_2\left(\nu,q^2\right)\cos^2\theta/2 + W_1\left(\nu,q^2\right)\sin^2\theta/2\right)$$

Then for structure functions:

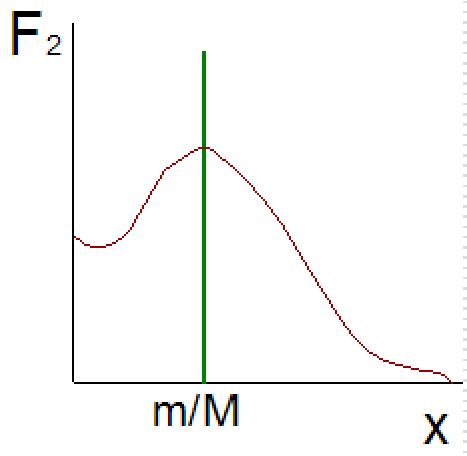
$$\rightarrow \begin{cases} W_2 = \dfrac{F_2}{\nu} = \left(\sum_i z_i^2 n_i\right)\delta\left(\nu + \dfrac{q^2}{2Mx}\right) \\ W_1 = \dfrac{2F_1}{M} = \left(\sum_i z_i^2 n_i\right)\left(\dfrac{-q^2}{2M^2x^2}\right)\delta\left(\nu + \dfrac{q^2}{2Mx}\right) \end{cases}$$

$$\rightarrow F_2 = \nu\left(\sum_i z_i^2 n_i\right)\delta\left(\nu + \frac{q^2}{2Mx}\right) = \left(\sum_i z_i^2 n_i\right)\delta\left(1 + \frac{q^2}{2Mx\nu}\right)$$

$$= \left(\sum_i z_i^2 n_i\right)x\delta\left(x + \frac{q^2}{2M\nu}\right) = \left(\sum_i z_i^2 n_i\right)x\delta\left(x - \frac{m}{M}\right)$$

Parton model prediction

Actual shape



Parton Model - IV

The true meaning of *x*

P, q: proton, virtual photon 4-momenta

$$x = \frac{Q^2}{2M\nu} = \frac{Q^2}{2P\cdot q} \quad \text{invariant}$$

$$x = \frac{m}{M}$$

$$m^2 = x^2 M^2 = x^2\left(E^2 - \mathbf{p}^2\right)$$

$$m^2 = \left(E_{parton}, \mathbf{p}_{parton}\right)^2 = E_{parton}{}^2 - \mathbf{p}_{parton}{}^2$$

$$\left(E_{parton}, \mathbf{p}_{parton}\right) = x\cdot\left(E, \mathbf{P}\right)$$

$$\rightarrow \mathbf{p}_{parton} \approx x\cdot\mathbf{P} \quad \text{when } m \ll |\mathbf{p}|$$

$$P = \left(E_p \underset{comp.trasversa}{\underline{\mathbf{0}}}, |\mathbf{P}|\right), \quad E_p = \sqrt{M^2 + |\mathbf{P}|^2} \approx |\mathbf{P}|$$

$$q = \left(E_\gamma, \mathbf{q_T}, 0\right)$$

$$\rightarrow P\cdot q \approx |\mathbf{P}| E_\gamma$$

$$\rightarrow E_\gamma \approx \frac{P\cdot q}{|\mathbf{P}|} = \frac{Q^2}{2x|\mathbf{P}|}$$

Lot of insight in this limit (Feynman):
(Proton) Infinite Momentum Frame

$\beta \rightarrow 1 \Rightarrow \gamma \rightarrow \infty$ Large time dilation

Time constants of internal motions:

$\tau \rightarrow \infty$ in the IMF

Constituents seen as *still* by the DIS virtual photon

Use time-energy indeterminacy relation:

$$\tau_0 \sim \frac{1}{E_\gamma} \approx \frac{2x|\mathbf{P}|}{Q^2} \quad \text{DIS time scale}$$

$$\tau \sim \frac{1}{\Delta E} \approx \frac{2x|\mathbf{P}|}{p_T^2} \quad \text{Constituents motion time scale}$$

$$\rightarrow \frac{\tau_0}{\tau} \approx \frac{p_T^2}{Q^2} \sim 0$$

$\rightarrow$ *No binding effects*
$\rightarrow$ *Free constituents OK*

Therefore:
x is the momentum fraction carried by the struck parton

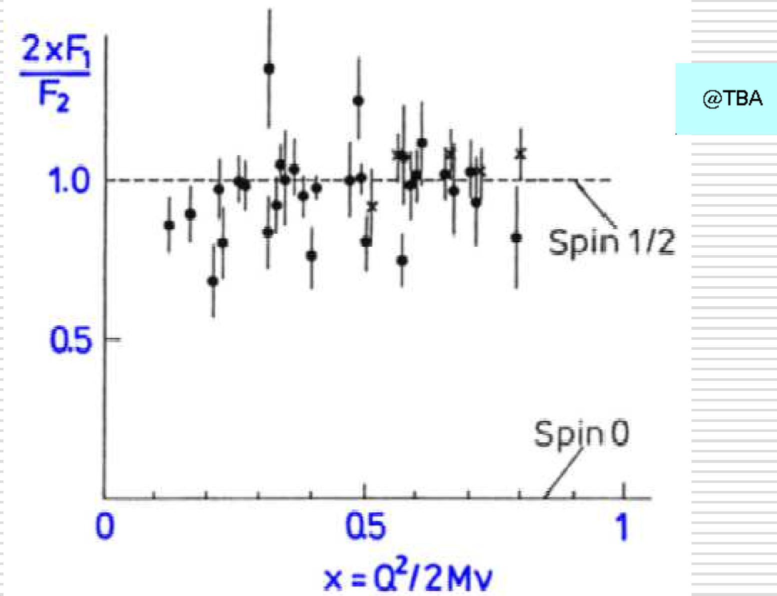
Parton Model - V

Callan-Gross relation for spin 1/2 partons

$$\begin{cases} \dfrac{F_2}{\nu} = \left(\sum_i z_i^2 n_i\right) \delta\left(\nu + \dfrac{q^2}{2Mx}\right) \\ \dfrac{2F_1}{M} = \left(\sum_i z_i^2 n_i\right)\left(\dfrac{-q^2}{2M^2 x^2}\right) \delta\left(\nu + \dfrac{q^2}{2Mx}\right) \end{cases}$$

$$\rightarrow \frac{F_2}{\nu} = \frac{2F_1}{M}\frac{2M^2x^2}{-q^2} = \frac{2F_1}{M}\frac{2M^2x^2}{2M\nu x} = \frac{2F_1 x}{\nu}$$

$$\rightarrow F_2 = 2F_1 x$$



Parton Model - VI

Several unanswered questions…

Most important issues:

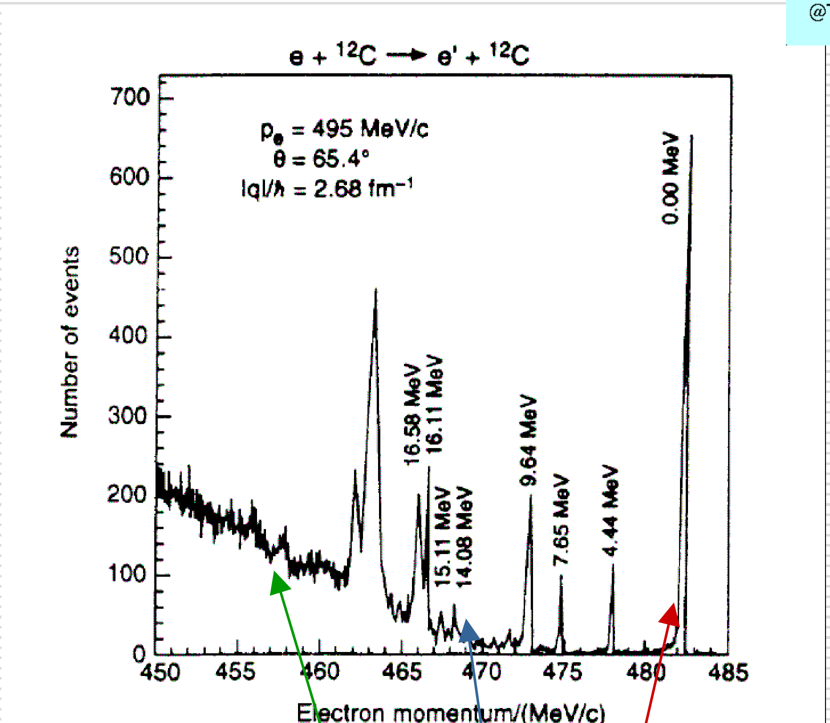
One does not observe any free constituent out of the collision

Constituents seem to be essentially free (as partons)
and tightly bound (as never observed free outside the nucleon)
at the same time

For some time, this point was believed to rule out any constituent model

The ‘Nuclear Parton Model’

@TBA

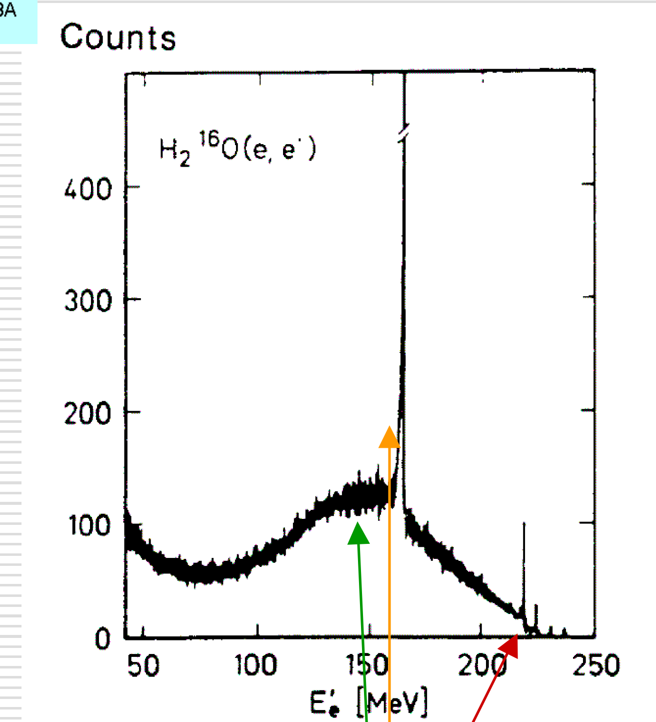


Scattering off ^{12}C

Elastic scattering off the *whole nucleus*

Nuclear levels excitation (inelastic)

Beginning of elastic scattering off *protons*



Scattering off water

Elastic scattering off the *whole ^{16}O nucleus*

Nuclear levels excitation (inelastic)

Elastic scattering off ^{16}O *protons*

Elastic scattering off *free protons*

The Quark Parton Model

Write down F_2 in terms of PDFs

$$F_2 = \left(\sum_i z_i^2 n_i\right) x\delta\left(x - \frac{m}{M}\right)$$

$$F_2(x) = x\left(\sum_i z_i^2 q_i(x)\right)$$

$$F_2^p(x) = x\left[\left(\frac{2}{3}\right)^2 u_p(x) + \left(-\frac{1}{3}\right)^2 d_p(x)\right];$$ p = uud

$$\rightarrow F_2^p(x) = x\left[\frac{4}{9}u_p(x) + \frac{1}{9}d_p(x)\right]$$

$$F_2^n(x) = x\left[\left(-\frac{1}{3}\right)^2 d_n(x) + \left(\frac{2}{3}\right)^2 u_n(x)\right];$$ n = ddu

From isospin symmetry:

$$F_2^n(x) = x\left[\left(-\frac{1}{3}\right)^2 u_p(x) + \left(\frac{2}{3}\right)^2 d_p(x)\right]$$

$$\rightarrow F_2^n(x) = x\left[\frac{1}{9}u_p(x) + \frac{4}{9}d_p(x)\right]$$

Consider the deuteron structure function:

$$F_2^d(x) = \frac{1}{2}\left(F_2^p + F_2^n\right) = \frac{5}{9}\frac{x}{2}\left[u_p(x) + d_p(x)\right]$$

$$\rightarrow F_2^n(x) = F_2^d(x) - F_2^p(x)$$

$$= \frac{5}{18}x\left[u_p(x) + d_p(x)\right] - \frac{1}{9}x\left[u_p(x) - 4d_p(x)\right]$$

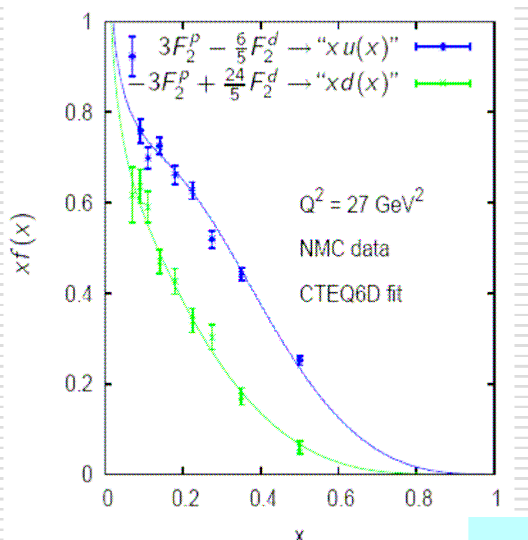
$$= \frac{3}{18}x\left[u_p(x) - d_p(x)\right]$$

Finally extract PDFs from measured F_2

$$xu_p(x) = xd_n(x) = 3F_2^p(x) - \frac{6}{5}F_2^d(x)$$

$$xu_n(x) = xd_p(x) = 3F_2^p(x) + \frac{24}{5}F_2^d(x)$$

The Parton Distribution Functions



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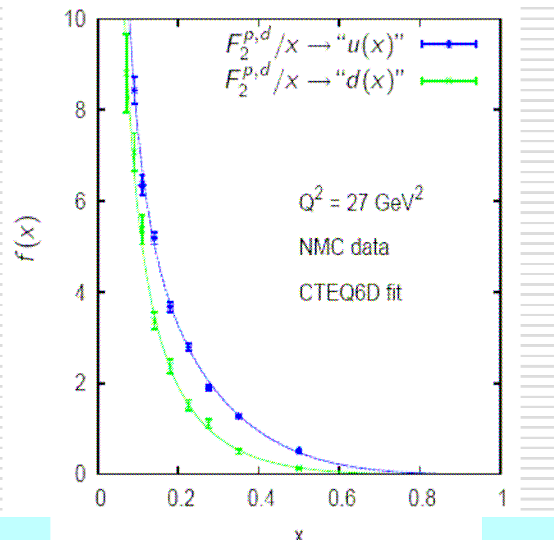

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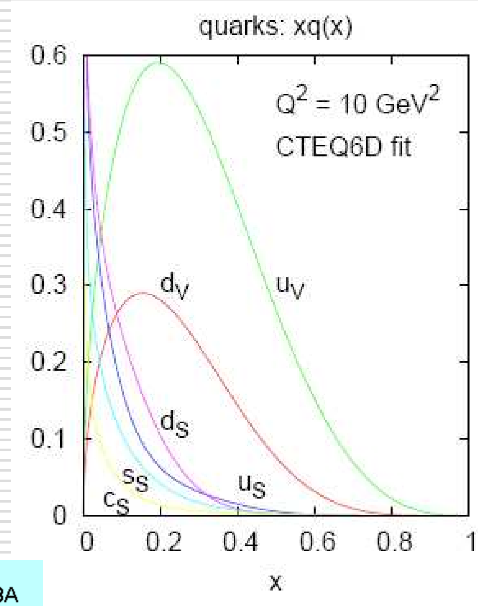


Parton model predictions: *Sum Rules* (= Integral relations) for PDFs
Examples: Proton quark content is *uud*

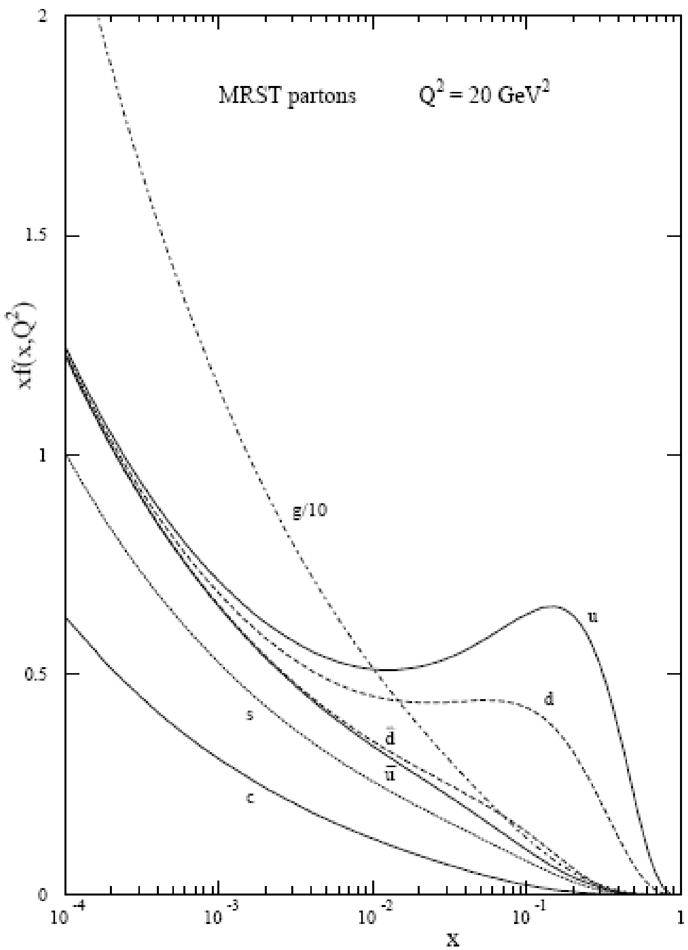
$$\int \left[u_p(x) - \bar{u}_p(x)\right] dx = 2$$

$$\int \left[d_p(x) - \bar{d}_p(x)\right] dx = 1$$

$$\int \left[s_p(x) - \bar{s}_p(x)\right] dx = 0$$

What's the origin of antiquarks in the nucleon?
QCD! See later..

The PDFs at Low *x*

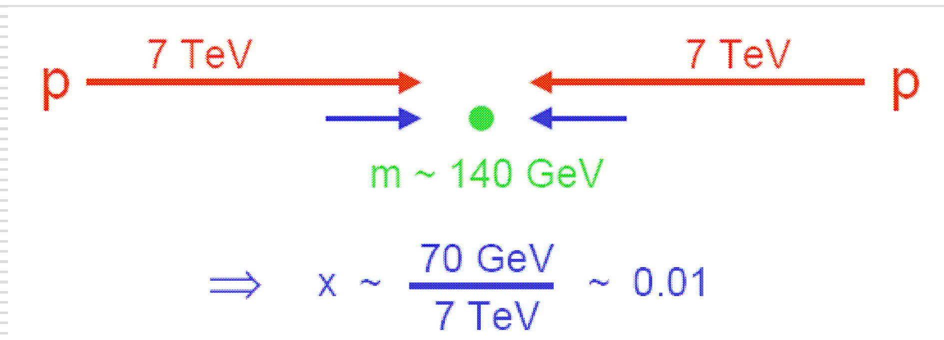


Data-based calculation

Low-x region very important at LHC

Example:

Production of a Higgs with m_H = *140 GeV*



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