

Elementary Particles I

2 – Electromagnetic Interaction

Form Factors, Structure Functions, Scaling, Partons

Leptons

Leptons

1st family	2nd family	3rd family
$\nu_e, \bar{\nu}_e$	$\nu_\mu, \bar{\nu}_\mu$	$\nu_\tau, \bar{\nu}_\tau$
e^-, e^+	μ^-, μ^+	τ^-, τ^+

Neutral, 'Massless'

Charged, Massive

"Pointlike", spin 1/2 Fermions

Electromagnetic and weak interactions

Lepton scattering by several targets as a powerful tool to probe constituents

- *Electromagnetic (and weak) coupling to leptons simple, well understood*
- *Small coupling constant $\rightarrow$ Perturbative expansion reliable*

Electromagnetic Interaction

Try to find transition amplitude for electromagnetic scattering
1st order perturbative contribution

$$T_{fi} = -i \langle f | \int d^4x H' | i \rangle \quad H': \text{Interaction Hamiltonian density}$$

$$H' = j^\mu A_\mu \quad \text{Classical analogy, } j_\mu \text{ current 4-density}$$

Reminder

For any system of charges and currents:

$$u_E = \frac{1}{2} \rho \varphi \quad \text{Electrostatic potential energy density}$$

$$u_B = \frac{1}{2} \mathbf{j} \cdot \mathbf{A} \quad \text{Magnetostatic potential energy density}$$

$$j^\mu = (\rho, \mathbf{j}) \quad \text{4-current density}$$

$$A_\mu = (\varphi, \mathbf{A}) \quad \text{4-potential}$$

An oversimplified example - I

First take a simple example:

Spinless, pointlike “pion” scattering off a fixed, Coulomb potential

$$A_\mu = \left(\frac{eZ}{4\pi r}, \mathbf{0} \right)$$

$$j^\mu = (\rho, \mathbf{j}) = ie \left(\varphi^* \left(\frac{\partial \varphi}{\partial t} \right) - \left(\frac{\partial \varphi^*}{\partial t} \right) \varphi, ((\nabla \varphi^*) \varphi - \varphi^* (\nabla \varphi)) \right)$$

Usual definition of current density
 ϕ : Stationary state

$$j^\mu = (\rho, \mathbf{j}) = ie \left(\left(\varphi'^* \left(\frac{\partial \varphi}{\partial t} \right) - \left(\frac{\partial \varphi'^*}{\partial t} \right) \varphi \right), ((\nabla \varphi'^*) \varphi - \varphi'^* (\nabla \varphi)) \right)$$

Generalize to a scattering state
 ϕ, ϕ' : Stationary states, plane waves

$$\rightarrow j^\mu = eNN' e^{-i((E-E')t - (\mathbf{p}-\mathbf{p}')\cdot\mathbf{r})} ((E+E'), (\mathbf{p}'+\mathbf{p}))$$

$$\rightarrow j^\mu A_\mu = NN' (E+E') e^{-i((E-E')t - (\mathbf{p}-\mathbf{p}')\cdot\mathbf{r})} \frac{e^2 Z}{4\pi r}$$

Integrate over time:

$$\int_{-\infty}^{+\infty} NN' e^{i(E-E')t} dt = NN' 2\pi \delta(E-E')$$

Energy conservation; momentum
not conserved by fixed Coulomb
 potential

An oversimplified example - II

Integrate over space:

$$\int e^{+i((\mathbf{p}-\mathbf{p}')\cdot\mathbf{r})} \frac{e^2 Z}{4\pi r} d^3\mathbf{r} = \frac{Ze^2}{|\mathbf{q}|^2}, \mathbf{q} = \mathbf{p} - \mathbf{p}'$$

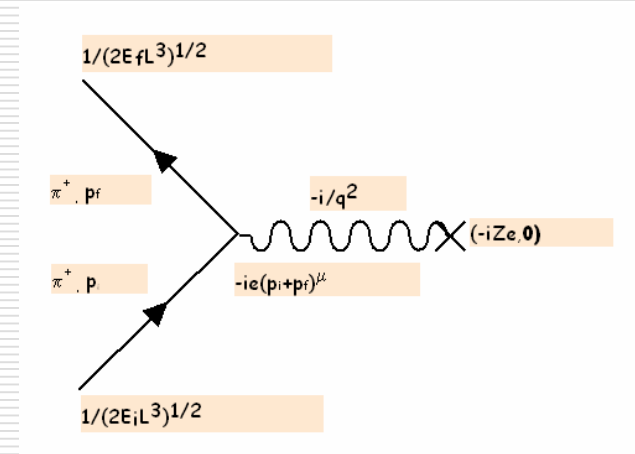
Matrix element:

$$T_{fi} = -i \langle f | \int d^4x H' | i \rangle = -NN' 2\pi i \delta(E - E') \frac{Ze^2}{|\mathbf{q}|^2}$$

$$E = E' \rightarrow |\mathbf{q}|^2 = -q^2 \rightarrow T_{fi} = NN' 2\pi i \delta(E - E') \frac{Ze^2}{q^2}$$

Virtual photon:

Coupling fixed source to current



An oversimplified example - III

Evaluate transition probability

$$w = \frac{|T_{fi}|^2}{T} \quad \text{Transition probability/Time}$$

$$|\delta(E' - E)|^2 = \lim_{T \rightarrow \infty} \left| \frac{1}{2\pi} \int_{-T/2}^{+T/2} e^{i(E' - E)t} dt \right|^2 = \lim_{T \rightarrow \infty} \left| \frac{\sin[(E' - E)T/2]}{\pi(E' - E)} \right|^2 = \frac{T}{2\pi} \delta(E' - E)$$

$$w = N^2 N'^2 \frac{4\pi^2}{T} |\delta(E - E')|^2 \frac{Z^2 e^4}{|\mathbf{q}|^4} = N^2 N'^2 2\pi \delta(E - E') \frac{Z^2 e^4}{|\mathbf{q}|^4}$$

$$\rightarrow d\sigma = w \cdot \frac{\text{phase space}}{\text{incident flux}} = w \cdot \frac{\frac{V d^3 p'}{(2\pi)^3}}{\frac{p}{EV}} = w \frac{d^3 p'}{(2\pi)^3} \frac{E}{p} V^2$$

$$d\sigma = N^2 N'^2 2\pi \delta(E - E') \frac{Z^2 e^4}{|\mathbf{q}|^4} \frac{d^3 p'}{(2\pi)^3} \frac{E}{p} V^2$$

$$N = N' = \frac{1}{\sqrt{V}} \rightarrow d\sigma = \frac{1}{V} \frac{1}{V} 2\pi \delta(E - E') \frac{Z^2 e^4}{|\mathbf{q}|^4} \frac{d^3 p'}{(2\pi)^3} \frac{E}{p} V^2 = 2\pi \delta(E - E') \frac{Z^2 e^4}{|\mathbf{q}|^4} \frac{d^3 p'}{(2\pi)^3} \frac{E}{p}$$

An oversimplified example - IV

Calculate differential cross-section

$$\int d\sigma = \int 2\pi\delta(E - E') \frac{Z^2 e^4}{|\mathbf{q}|^4} \frac{d^3 p'}{(2\pi)^3} \frac{E}{p}$$

$$d^3 p' = p'^2 dp' d\Omega$$

$$p' dp' = E' dE' \rightarrow d^3 p' = p' E' dE' d\Omega$$

$$\rightarrow \int 2\pi\delta(E - E') \frac{Z^2 e^4}{|\mathbf{q}|^4} \frac{d^3 p'}{(2\pi)^3} \frac{E}{p} = \int 2\pi\delta(E - E') \frac{Z^2 e^4}{|\mathbf{q}|^4} \frac{dE'}{(2\pi)^3} \frac{E}{p} p' E' d\Omega$$

$$= \frac{1}{(2\pi)^2} \frac{Z^2 e^4}{|\mathbf{q}|^4} \frac{E}{p} p' \int \delta(E - E') dE' E' d\Omega = \frac{1}{(2\pi)^2} \frac{Z^2 e^4}{|\mathbf{q}|^4} E^2 d\Omega$$

$$q = 2p \sin \frac{\theta}{2} \rightarrow q^4 = 16p^4 \sin^4 \frac{\theta}{2}$$

$$\rightarrow \frac{d\sigma}{d\Omega} = \frac{1}{(2\pi)^2} \frac{Z^2 e^4}{16p^4 \sin^4 \theta/2} E^2 = \frac{Z^2 e^4}{16(2\pi)^2} \frac{E^2}{p^4} \frac{1}{\sin^4 \theta/2}$$

Useful to remember

$$E^2 = \mathbf{p}^2 + m^2$$

$$\rightarrow 2EdE = 2|\mathbf{p}|d|\mathbf{p}|$$

$$\rightarrow EdE = |\mathbf{p}|d|\mathbf{p}|$$

Compare to non-relativistic
Rutherford cross-section:

$$\frac{d\sigma}{d\Omega} = \frac{1}{4} Z^2 \alpha^2 \frac{1}{T^2} \frac{1}{\sin^4 \theta/2} = \frac{1}{4} \frac{Z^2 e^4}{16\pi^2} \frac{1}{T^2} \frac{1}{\sin^4 \theta/2}$$

Theme Variation: Spin 1/2 - I

Take now a spin 1/2 Dirac electron scattering off the same, static Coulomb potential

$$j^\mu = e\bar{\psi}\gamma^\mu\psi$$

$$\rightarrow \psi = \sqrt{\frac{m}{EV}}u(s, p)e^{ipx}, \bar{\psi} = \sqrt{\frac{m}{E'V}}\bar{u}(s', p')e^{-ip'x}$$

$$\rightarrow j^\mu = e\sqrt{\frac{m}{E'V}}\bar{u}(s', p')e^{-ip'x}\gamma^\mu\sqrt{\frac{m}{EV}}u(s, p)e^{ipx} = e\frac{m}{EV}e^{-i(p-p')x}\bar{u}\gamma^\mu u \quad \text{Dirac transition current}$$

$$j^\mu A_\mu = e\frac{m}{EV}e^{-i((E-E')t-(\mathbf{p}-\mathbf{p}')\cdot\mathbf{r})}\bar{u}(s', p')\gamma^\mu u(s, p)\left(\frac{eZ}{4\pi r}, \mathbf{0}\right) = e\frac{m}{EV}e^{-i((E-E')t-(\mathbf{p}-\mathbf{p}')\cdot\mathbf{r})}\bar{u}\gamma^0 u \frac{eZ}{4\pi r}$$

$$\underbrace{\bar{u}(s', p')\gamma^0 u(s, p)}_{=\bar{u}^\dagger\gamma^0} = u^\dagger(s', p')\overbrace{\gamma^0\gamma^0}^{=1}u(s, p) = u^\dagger(s', p')u(s, p) \quad \text{Dirac matrices}$$

$$\frac{d\sigma}{d\Omega} = \frac{m^2}{E^2} \frac{Z^2 e^4}{16(2\pi)^2} \frac{E^2}{p^4} \frac{1}{\sin^4 \theta/2} |u^\dagger(s', p')u(s, p)|^2 = \frac{Z^2 e^4}{16(2\pi)^2} \frac{m^2}{p^4} \frac{1}{\sin^4 \theta/2} |u^\dagger(s', p')u(s, p)|^2$$

Theme Variation: Spin 1/2 - II

Unpolarized cross-section:

Sum over final *Spin projections*
Average over initial

$$\frac{1}{2} \sum_{s=-1/2}^{+1/2} \sum_{s'=-1/2}^{+1/2} |u^\dagger(s', p') u(s, p)|^2 = 4 \frac{E^2}{m^2} (1 - \beta^2 \sin^2 \theta/2)$$
$$\rightarrow \frac{d\sigma}{d\Omega} = \frac{Z^2 e^4}{16(2\pi)^2} \frac{m^2}{p^4} \frac{1}{\sin^4 \theta/2} \frac{4E^2}{m^2} (1 - \beta^2 \sin^2 \theta/2) = \frac{Z^2 e^4}{(2\pi)^2} \frac{E^2}{4p^4} \frac{(1 - \beta^2 \sin^2 \theta/2)}{\sin^4 \theta/2}$$
$$\frac{d\sigma}{d\Omega} \underset{E \gg m}{\simeq} \frac{Z^2 e^4}{(2\pi)^2} \frac{E^2}{4p^4} \frac{\cos^2 \theta/2}{\sin^4 \theta/2}$$

New factor, important at high speed

Reducing cross section at large angles (= 0 for $\theta \rightarrow \pi/2$)

Helicity Conservation

Dirac equation: High energy limit

$$E\psi = (\boldsymbol{\alpha} \cdot \mathbf{p} + \beta m)\psi$$

$$u = \begin{pmatrix} \phi \\ \chi \end{pmatrix} \quad \text{Generic spinor; } \phi, \chi \text{ 2-components spinors}$$

$$\boldsymbol{\alpha} = \begin{pmatrix} \boldsymbol{\sigma} & 0 \\ 0 & -\boldsymbol{\sigma} \end{pmatrix}, \beta = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \quad \text{Dirac matrices, chiral representation, "2x2" block format}$$

$$\begin{cases} E\phi = (\boldsymbol{\sigma} \cdot \mathbf{p})\phi + m\chi \\ E\chi = -(\boldsymbol{\sigma} \cdot \mathbf{p})\chi + m\phi \end{cases}$$

$$E \gg m \rightarrow \begin{cases} E \approx |\mathbf{p}| \\ m \approx 0 \end{cases} \rightarrow \begin{cases} (\boldsymbol{\sigma} \cdot \mathbf{p})\phi \approx |\mathbf{p}|\phi \\ (\boldsymbol{\sigma} \cdot \mathbf{p})\chi \approx -|\mathbf{p}|\chi \end{cases} \rightarrow \begin{cases} \phi \simeq u_R \\ \chi \simeq u_L \end{cases} \rightarrow u \approx \begin{pmatrix} u_R \\ u_L \end{pmatrix} \rightarrow u^\dagger(s', p') u(s, p) \approx u_R^\dagger u_R + u_L^\dagger u_L$$

No mixed terms $\rightarrow$ *Helicity is conserved at high energy*

Explains the $(1 - \beta^2 \sin^2 \theta/2)$ factor, cutting off the cross-section $\theta \rightarrow \pi$:
Solves conflicting helicity/angular momentum conservation

Always true for Dirac currents coupling to vector fields

Another Step into a Realistic Model

Take now a spin $\frac{1}{2}$ Dirac electron scattering off a *distributed*, static source (like a (A,Z) nucleus)

$$A_\mu = (\varphi, \mathbf{0})$$

$$\varphi(r) = \frac{1}{4\pi} \int d^3\mathbf{r}' \frac{\rho(\mathbf{r}')}{|\mathbf{r} - \mathbf{r}'|}$$

$$\int \rho(\mathbf{r}') d^3\mathbf{r}' = Ze$$

Only change: the space integral

$$\int e^{+i(\mathbf{p}-\mathbf{p}')\cdot\mathbf{r}} \frac{e^2 Z}{4\pi r} d^3\mathbf{r} = \frac{Ze^2}{|\mathbf{q}|^2}, \mathbf{q} = \mathbf{p} - \mathbf{p}' \rightarrow \int d^3\mathbf{r} \int e^{+i(\mathbf{p}-\mathbf{p}')\cdot\mathbf{r}} \frac{\rho(\mathbf{r}')}{4\pi |\mathbf{r}' - \mathbf{r}|} d^3\mathbf{r}'$$

$$= \int d^3\mathbf{r}' \frac{\rho(\mathbf{r}')}{Ze} e^{+i(\mathbf{p}-\mathbf{p}')\cdot\mathbf{r}'} \int e^{+i((\mathbf{p}-\mathbf{p}')\cdot(\mathbf{r}-\mathbf{r}'))} \frac{Ze}{4\pi |\mathbf{r}' - \mathbf{r}|} d^3\mathbf{r}$$

$$F(\mathbf{q}) = \frac{\int \rho(\mathbf{r}) e^{+i\mathbf{q}\cdot\mathbf{r}} d^3\mathbf{r}}{Ze} \quad \text{Form factor of the charge distribution}$$

$$\Rightarrow \frac{Z^2 e^2}{|\mathbf{q}|^4} \rightarrow \frac{Z^2 e^2}{|\mathbf{q}|^4} |F(\mathbf{q})|^2$$

The Form Factor

$$F(\mathbf{q}) = \frac{\int \rho(\mathbf{r}) e^{+i\mathbf{q}\cdot\mathbf{r}} d^3\mathbf{r}}{Ze} = \frac{1}{Ze} \int \rho(r) e^{+iqr\cos\theta} r^2 dr d\Omega$$

$$e^{+i|\mathbf{q}|r\cos\theta} = 1 + i|\mathbf{q}|r\cos\theta - \frac{1}{2}|\mathbf{q}|^2 r^2 \cos^2\theta \dots$$

$$\rightarrow \frac{1}{Ze} \int \rho(r) e^{+i|\mathbf{q}|r\cos\theta} r^2 dr d\Omega \approx$$

$$\approx \frac{1}{Ze} \left[\int \rho(r) r^2 dr d\Omega + i|\mathbf{q}| \int \rho(r) r^3 \cos\theta dr d\Omega - \frac{|\mathbf{q}|^2}{2} \int \rho(r) r^4 \cos^2\theta dr d\Omega \right]$$

$$\rightarrow F(\mathbf{q}) \approx 1 + \frac{i|\mathbf{q}|}{Ze} \underbrace{\int \rho(r) r^3 dr \int \cos\theta d\Omega}_{=\frac{2\pi}{2}(-\cos^2\theta)\Big|_0^\pi=0} - \frac{|\mathbf{q}|^2}{2Ze} \underbrace{\int \rho(r) r^4 dr}_{=\frac{Ze\langle r^2 \rangle}{4\pi}} \underbrace{\int \cos^2\theta d\Omega}_{=\frac{2\pi}{3}(-\cos^3\theta)\Big|_0^\pi=\frac{4\pi}{3}} = 1 - \frac{|\mathbf{q}|^2 \langle r^2 \rangle}{6}$$

$$\left. \begin{aligned} F(|\mathbf{q}|^2) &= F(0) + \frac{\partial F}{\partial |\mathbf{q}|^2} |\mathbf{q}|^2 + \dots \\ F(|\mathbf{q}|^2) &\approx 1 - \frac{1}{6} |\mathbf{q}|^2 \langle r^2 \rangle \end{aligned} \right\} \rightarrow \rightarrow \left\{ \begin{aligned} F(0) &= 1 \\ \langle r^2 \rangle &= -6 \frac{\partial F}{\partial |\mathbf{q}|^2} \Big|_{|\mathbf{q}|^2=0} \end{aligned} \right.$$

showing that measuring the form factor yields the rms charge radius

Form Factors: The Mathematical Game

Charge distribution $\rho(r)$	Form factor $F(q^2)$	Example
pointlike	constant	Electron
exponential	dipole	Proton
gauss	gauss	${}^6\text{Li}$
homogeneous sphere	oscillating	—
sphere with a diffuse surface	smeared oscillations	${}^{40}\text{Ca}$

$r \rightarrow$ $|q| \rightarrow$

@TBA

$$\rho(r) = \frac{1}{8\pi a^3} e^{-ar} \rightarrow F(|\mathbf{q}|^2) = \left(\frac{1}{1 + |\mathbf{q}|^2/a^2} \right)^2$$

$$\rho(r) = \begin{cases} \text{constant} & r < R \\ 0 & r > R \end{cases} \rightarrow F(|\mathbf{q}|^2) = \frac{3}{R^3 |\mathbf{q}|^3} (\sin(R|\mathbf{q}|) - R|\mathbf{q}| \cos(R|\mathbf{q}|))$$

Nuclear Form Factors

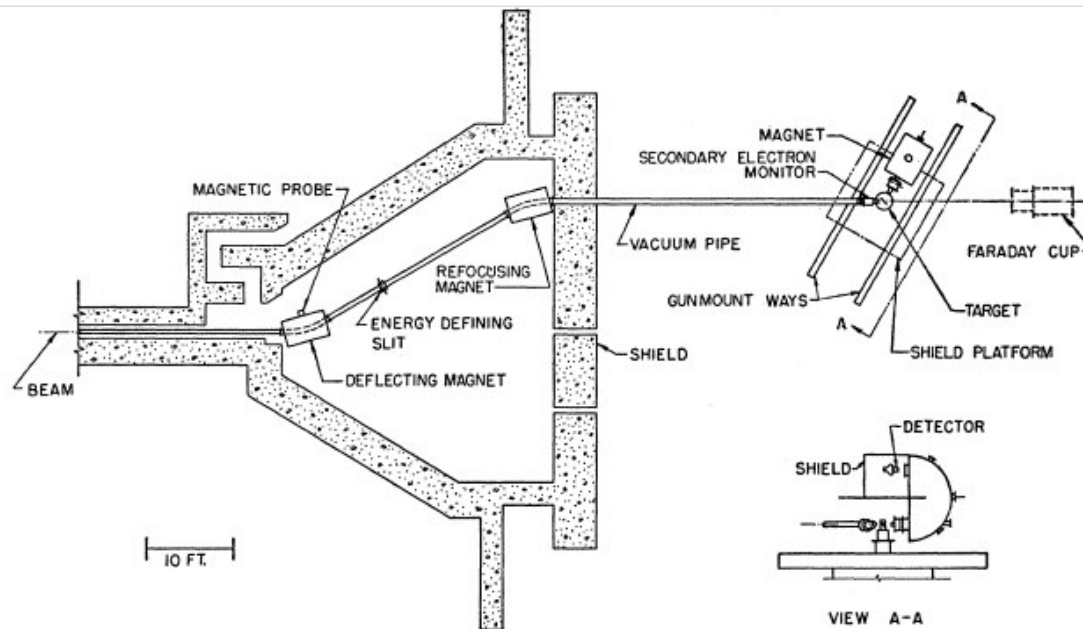


FIG. 18. The experimental installation of the 550-Mev spectrometer.

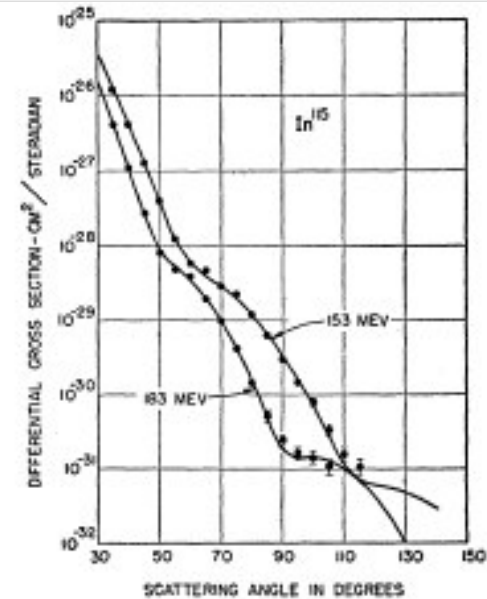


FIG. 46. Theoretical and experimental curves for In^{115} at two energies.

@TBA

One of the Hofstadter's spectrometers at SLAC

Results for Indium

Particle-Particle Scattering

1st order Transition amplitude:

$$H' = j^\mu A_\mu \rightarrow H' = (j_1^\mu + j_2^\mu) A_\mu$$

$$\rightarrow M_{fi} = i(2\pi)^4 \delta(p_1 + p_2 - (p_1' + p_2')) T_{fi} = i(2\pi)^4 \delta(p_1 + p_2 - (p_1' + p_2')) j_\mu^{(1)} \frac{ig^{\mu\nu}}{q^2} j_\nu^{(2)}$$

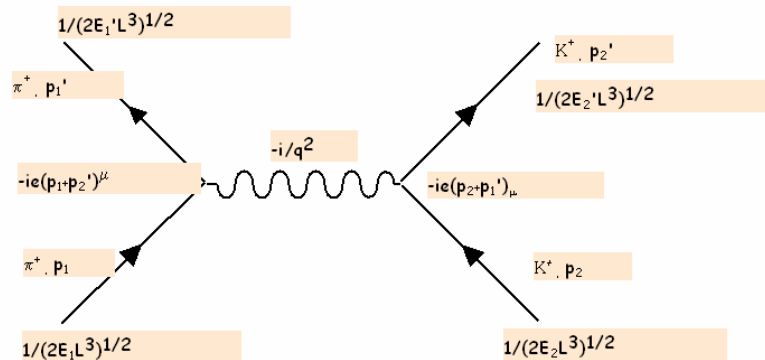
$$q = p_1 - p_1' = p_2 - p_2' \quad \text{4-momentum transfer}$$

Transition currents:

$$j^\mu = e(p_1 + p_3)^\mu \quad \text{scalar}$$

$$j^\mu = e\bar{u}_4\gamma^\mu u_2 \quad \text{fermion}$$

.....



Scattering Spin 0 – Spin 1/2

Just take the two currents as defined before:

$$T_{fi} = e \bar{u}' \gamma^\mu u \frac{g_{\mu\nu}}{q^2} e (p + p')^\nu \rightarrow d\sigma = \frac{1}{4EE'_{\text{V}}} |T_{fi}|^2 (2\pi)^4 \delta^4(p_1' + p_2' - p_1 - p_2) \frac{1}{(2\pi)^3} \frac{1}{(2\pi)^3} \frac{d^3\mathbf{p}_1'}{2E_1'} \frac{d^3\mathbf{p}_2'}{2E_2'}$$

$$\frac{1}{2} \sum_{s=-1/2}^{+1/2} \sum_{s'=-1/2}^{+1/2} |T_{fi}|^2 = \left(\frac{e^2}{q^2} \right)^2 \frac{1}{2} \sum_{s,s'=-1/2}^{+1/2} \bar{u}(p_1', s') \gamma^\mu u(p_1, s) \bar{u}(p_1, s) \gamma^\nu u(p_1', s') (p_2 + p_2')_\mu (p_2 + p_2')_\nu$$

By defining...

$$T_{\mu\nu} = (p_2 + p_2')_\mu (p_2 + p_2')_\nu$$

$$L^{\mu\nu} = 2 \left[p_1'^\mu p_1^\nu + p_1'^\nu p_1^\mu + \frac{q^2}{2} g^{\mu\nu} \right]$$

...it can be shown (!) that

$$\frac{d\sigma}{dq^2} = \frac{2\alpha^2}{(p_1 + p_2)^2 q^4} \left[2(p_1 \cdot p_2)(p_1 \cdot p_2') + \frac{q^2}{2} M^2 \right] \quad \text{Invariant cross-section}$$

$$\left. \frac{d\sigma}{d\Omega} \right|_{\text{LAB}} \underset{E \gg m}{\simeq} \frac{\alpha^2}{4|\mathbf{p}_1|^2 \sin^4 \theta/2} \cos^2 \frac{\theta}{2} \frac{|\mathbf{p}_1'|}{|\mathbf{p}_1|} \quad \text{LAB = "2" rest frame}$$

Another extension of
Rutherford cross section
Includes *target recoil*

Scattering Spin 1/2 - Spin 1/2

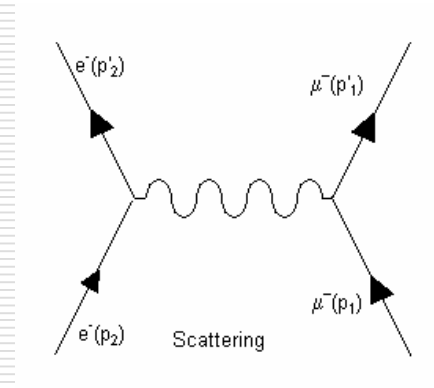
Just to simplify things, take *different* spin 1/2 particles
(e.g. electron-muon scattering)

$$T_{fi}(s, s', r, r') = e \bar{u}'(p_2', s') \gamma^\mu u(p_2, s) \frac{g_{\mu\nu}}{q^2} \bar{u}'(p_1', r') \gamma^\nu u(p_1, r)$$

$$\frac{1}{4} \sum_{s, s', r, r'} |T_{fi}(s, s', r, r')|^2 = \frac{e^4}{q^4} L_{\mu\nu} M^{\mu\nu}$$

$$L^{\mu\nu} = 2 \left[p_1'^\mu p_1^\nu + p_1'^\nu p_1^\mu + \frac{q^2}{2} g^{\mu\nu} \right]$$

$$M_{\mu\nu} = 2 \left[p_{2\mu} p_{2\nu} + p_{2\nu} p_{2\mu} + \frac{q^2}{2} g_{\mu\nu} \right] \rightarrow \frac{d\sigma}{d\Omega} \Big|_{LAB} \stackrel{E \gg m}{\approx} \frac{\alpha^2}{4 |\mathbf{p}_1|^2 \sin^4 \theta/2} \cos^2 \theta/2 \frac{|\mathbf{p}_1'|}{|\mathbf{p}_1|} \left(1 - \frac{q^2 \tan^2 \theta/2}{2m_2^2} \right)$$



Yet another term...
Electron scattering off
the muon *magnetic moment*

Crossing Symmetry

Simple relationship between any pair of 2-body reactions

$$a + b \rightarrow c + d$$

Reaction A

$$a + \underset{\text{crossed}}{[c]} \rightarrow \underset{\text{crossed}}{[b]} + d$$

Reaction B

Define: Crossed particle $\equiv$ Antiparticle

By changing the 4-momentum sign of the crossed particle. the two amplitudes are identical

$$A[a(p_A) + b(p_B) \rightarrow c(p_C) + d(p_D)] = A[a(p_A) + \bar{c}(-p_C) \rightarrow \bar{b}(-p_B) + d(p_D)]$$

Annihilation

Apply crossing symmetry to electron-muon scattering

$$e^- + \mu^- \rightarrow e^- + \mu^-$$

A: Scattering

$$e^- + \left[e^- \right]_{\text{crossed}} \rightarrow \left[\mu^- \right]_{\text{crossed}} + \mu^- \equiv e^- + e^+ \rightarrow \mu^+ + \mu^-$$

B: Annihilation

Amplitude for scattering:

$$T_{fi}(s, s', r, r') = (-e) \bar{u}_{(\mu)}(p_2', s') \gamma^\mu u_{(\mu)}(p_2, s) \frac{-ig_{\mu\nu}}{q^2} (-e) \bar{u}_{(e)}(p_1', r') \gamma^\nu u_{(e)}(p_1, r)$$

$$q = p_1 - p_1' \rightarrow q^2 = (p_1 - p_1')^2 = p_1^2 + p_1'^2 - 2p_1 \cdot p_1' \quad q=4\text{-momentum transfer}$$

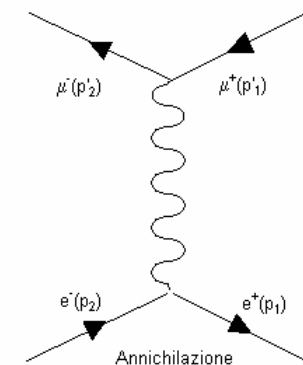
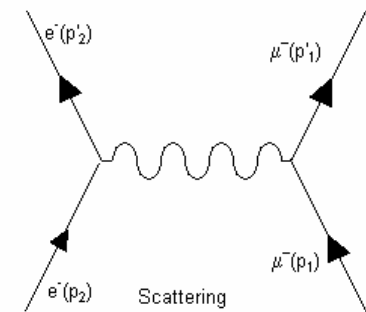
$$q^2 = 2m_e^2 - 2(E_1 E_1' - \mathbf{p}_1 \cdot \mathbf{p}_1') \underset{E \gg m}{\simeq} -2(E_1 E_1' - \mathbf{p}_1 \cdot \mathbf{p}_1') < 0$$

Amplitude for annihilation:

$$T_{fi}(s, s', r, r') = (-e) \bar{u}_{(\mu)}(p_2', s') \gamma^\mu v_{(\mu)}(p_1', r') \frac{-ig_{\mu\nu}}{q^2} (-e) \bar{v}_{(e)}(p_1, s) \gamma^\nu u_{(e)}(p_2, r)$$

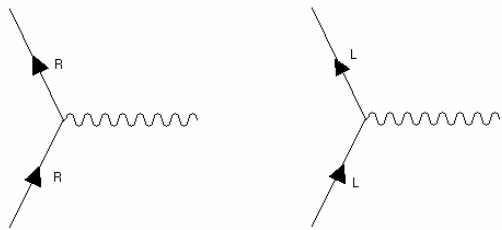
$$q = p_1 + p_2 \rightarrow q^2 = (p_1 + p_2)^2 = p_1^2 + p_2^2 + 2p_1 \cdot p_2 \quad q=\text{total 4-momentum}$$

$$q^2 = 2m_e^2 + 2(E_1 E_2 - \mathbf{p}_1 \cdot \mathbf{p}_2) \underset{E \gg m}{\simeq} 2(E_1 E_2 - \mathbf{p}_1 \cdot \mathbf{p}_2) > 0$$

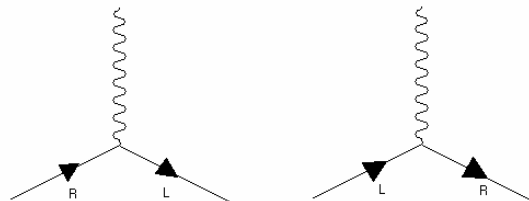


The Annihilation Cross-Section - I

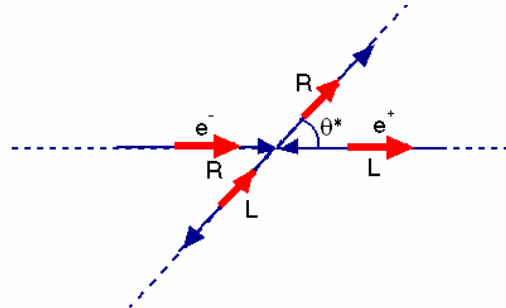
Helicity conservation at high energy:
Consequence of electromagnetic field being a *vector*



Scattering: $R \rightarrow R, L \rightarrow L$



Annihilation: $R+L, L+R$



For both initial and final state:
Particle and antiparticle must have *opposite* helicity
Can decompose differential cross-section into 4 pieces:

$$\frac{d\sigma}{d\Omega^*} = \frac{1}{4} \cdot 2 \left[\frac{d\sigma}{d\Omega^*} \Big|_{LR \rightarrow LR} + \frac{d\sigma}{d\Omega^*} \Big|_{RL \rightarrow RL} + \frac{d\sigma}{d\Omega^*} \Big|_{LR \rightarrow RL} + \frac{d\sigma}{d\Omega^*} \Big|_{RL \rightarrow LR} \right]$$

Average over initial states
Sum over final states

The Annihilation Cross-Section - II

Transition amplitude = Amplitude to find final particles at angle θ^ wrt to initial direction*

Phase space, incident flux and normalization factors just cancel out at high energy
Matrix element:

$$T_{fi} = \frac{\alpha}{q^2 \equiv s} \cdot \text{Amplitude to find } J=1 \text{ state rotated by } \theta^*$$

Use rotation matrices for a $J=1$ state: Take y-axis $\perp$ reaction plane

$$e^{-i\theta^* J_2} |J, m\rangle = \sum_{m'} d_{m, m'}^J(\theta^*) |J, m'\rangle, \quad d_{m, m'}^J(\theta^*) = \langle J, m | e^{-i\theta^* J_2} | J, m'\rangle$$

$$d_{+1, +1}^1(\theta^*) = d_{-1, -1}^1(\theta^*) = \frac{1}{2}(1 + \cos \theta^*)$$

$$d_{+1, -1}^1(\theta^*) = d_{-1, +1}^1(\theta^*) = \frac{1}{2}(1 - \cos \theta^*)$$

$$\left. \begin{aligned} \frac{d\sigma}{d\Omega^*} \Big|_{LR \rightarrow LR} &= \frac{d\sigma}{d\Omega^*} \Big|_{RL \rightarrow RL} = \frac{\alpha^2}{s} \left(\frac{1}{2} \right)^2 (1 + \cos \theta^*)^2 \\ \frac{d\sigma}{d\Omega^*} \Big|_{LR \rightarrow RL} &= \frac{d\sigma}{d\Omega^*} \Big|_{RL \rightarrow LR} = \frac{\alpha^2}{s} \left(\frac{1}{2} \right)^2 (1 - \cos \theta^*)^2 \end{aligned} \right\} \rightarrow \frac{d\sigma}{d\Omega^*} = \frac{\alpha^2}{4s} (1 + \cos^2 \theta^*) \quad \sigma = \int_{4\pi} \frac{\alpha^2}{4s} (1 + \cos^2 \theta^*) d\Omega^* = \frac{4\pi\alpha^2}{3s}$$

The π Form Factor - I

Consider electron-pion scattering: The π is not a point-like object...
What are we to take for the pion current?

Must build a 4-vector operator

Some guesswork:

1) *Lorentz invariance*

p_2, p_2', q Three 4-momentum vectors
 $p_2' = p_2 + q$ Constraint

} $\rightarrow$ 2 independent

Choose:

$p_2' + p$
 $p_2' - p = q$

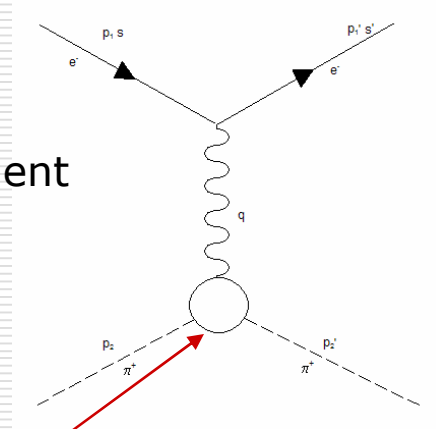
} Both can contribute to the current

Only one independent 4-scalar:

E.g. $(p_2')^2 = (p_2)^2 = m^2 \rightarrow p_2 \cdot p_2'$

Choose instead q^2

$$\rightarrow j_{(\pi)}^\mu = e \left[F(q^2) (p_2' + p_2)^\mu + G(q^2) q^\mu \right] e^{-iq \cdot x}$$



Blob indicating a non-QED vertex:
The pion is an extended object

The π Form Factor - II

2) Gauge Invariance

Charge conservation $\leftrightarrow$ Current must be *divergenceless*

$$\partial_\mu j^\mu = 0 \rightarrow \partial_\mu j^\mu_\pi = e \partial_\mu \left[F(q^2) (p' + p)^\mu + G(q^2) q^\mu \right] e^{-iq \cdot x}$$

$$= -iq_\mu e \left[F(q^2) (p' + p)^\mu + G(q^2) q^\mu \right] e^{-iq \cdot x} = 0$$

$$\rightarrow \partial_\mu j^\mu = 0 \Rightarrow q_\mu j^\mu = 0$$

$$q_\mu \left[F(q^2) (p_2 + p_2')^\mu + G(q^2) q^\mu \right] = 0$$

$$\left. \begin{aligned} q_\mu (p_2 + p_2')^\mu &= (p_2 - p_2')_\mu (p_2 + p_2')^\mu = 0 \\ q_\mu q^\mu &\neq 0 \end{aligned} \right\} \rightarrow G(q^2) = 0$$

$$\rightarrow j^\mu = e (p_2 + p_2')^\mu F(q^2)$$

Just *one* form factor for a scalar particle like the π

The π Form Factor - III

What is $F(q^2)$?

In the CM frame:

$$q^2 = (E' - E, \mathbf{p}' - \mathbf{p})^2 = (E' - E)^2 - (\mathbf{p}' - \mathbf{p})^2 = 0 - \mathbf{q}^2 = -|\mathbf{q}|^2$$

$$\rightarrow F_{scatt}(q^2) = F_{scatt}(|\mathbf{q}|^2)$$

Again, Fourier transform of the charge distribution

If crossing is good, can extend to the reaction

$$e^+ + e^- \rightarrow \pi^+ + \pi^-$$

$$q^2 = (E_1 + E_2, \mathbf{p}_1 + \mathbf{p}_2)^2 = (E_1 + E_2)^2 - (\mathbf{p}_1 + \mathbf{p}_2)^2$$

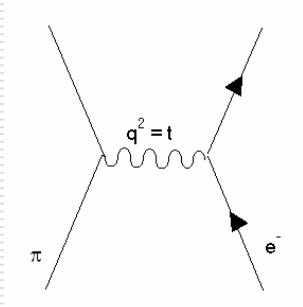
$$q^2 = E_{CM}^2$$

$$\rightarrow F_{annihil}(q^2) = F_{annihil}(E_{CM}^2)$$

$$\rightarrow F(q^2) = \begin{cases} F_{scatt}(q^2), & q^2 < 0 \\ F_{annihil}(q^2), & q^2 > 0 \end{cases}$$

Experiments – Space-like

π scattering off electrons

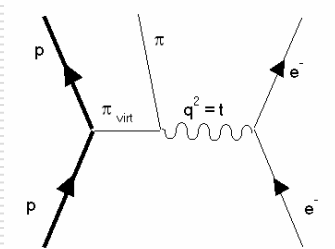


$$p^* = \frac{p_\pi m_e}{\sqrt{s}} \rightarrow t_{\max} = -4 \frac{p_\pi^2 m_e^2}{s} = -4 \frac{p_\pi^2 m_e^2}{m_\pi^2 + m_e^2 + 2E_\pi m_e} \approx -4 \frac{p_\pi^2 m_e^2}{m_\pi^2 + 2E_\pi m_e}$$

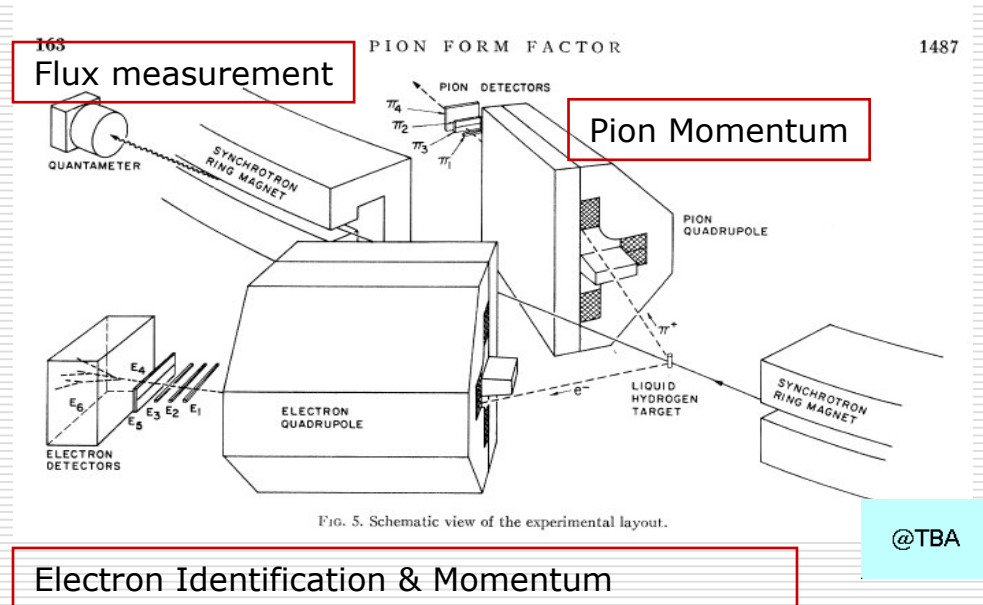
$$\rightarrow t_{\max} \underset{p_\pi \rightarrow \infty}{\sim} -4 \frac{p_\pi m_e^2}{2m_e} = -2p_\pi m_e$$

Unappealing, $t_{\max}$ too small

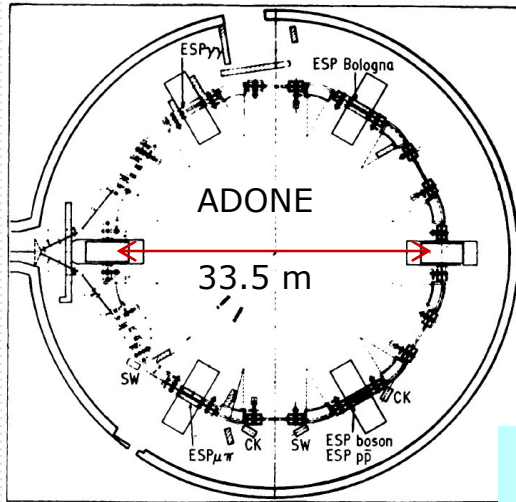
Electroproduction of one π



OK



Experiments – Time-like



First $e^+ - e^-$ colliding beams
ADONE – Frascati, 1967 etc.

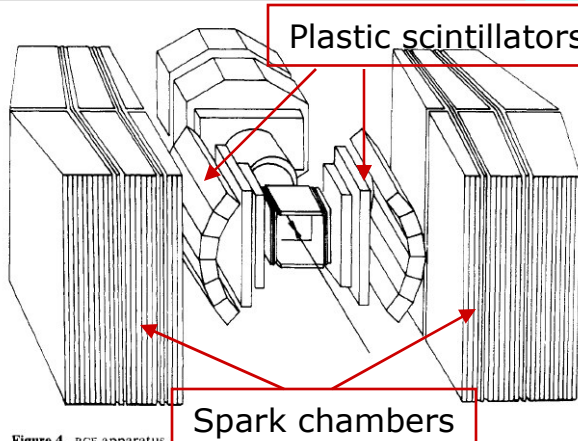
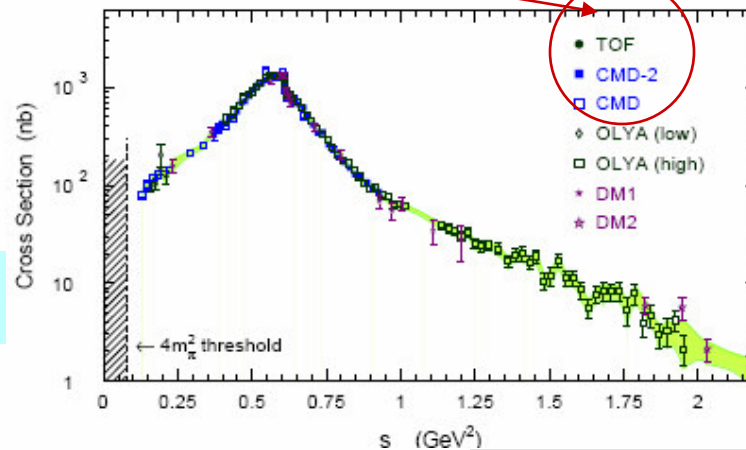
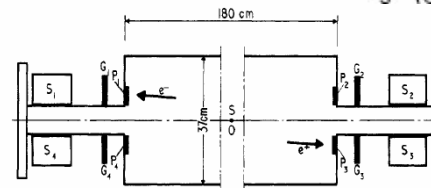


Figure 4. ACF apparatus.



Luminosity monitor
measures Bhabha
scattering rate
at small angles

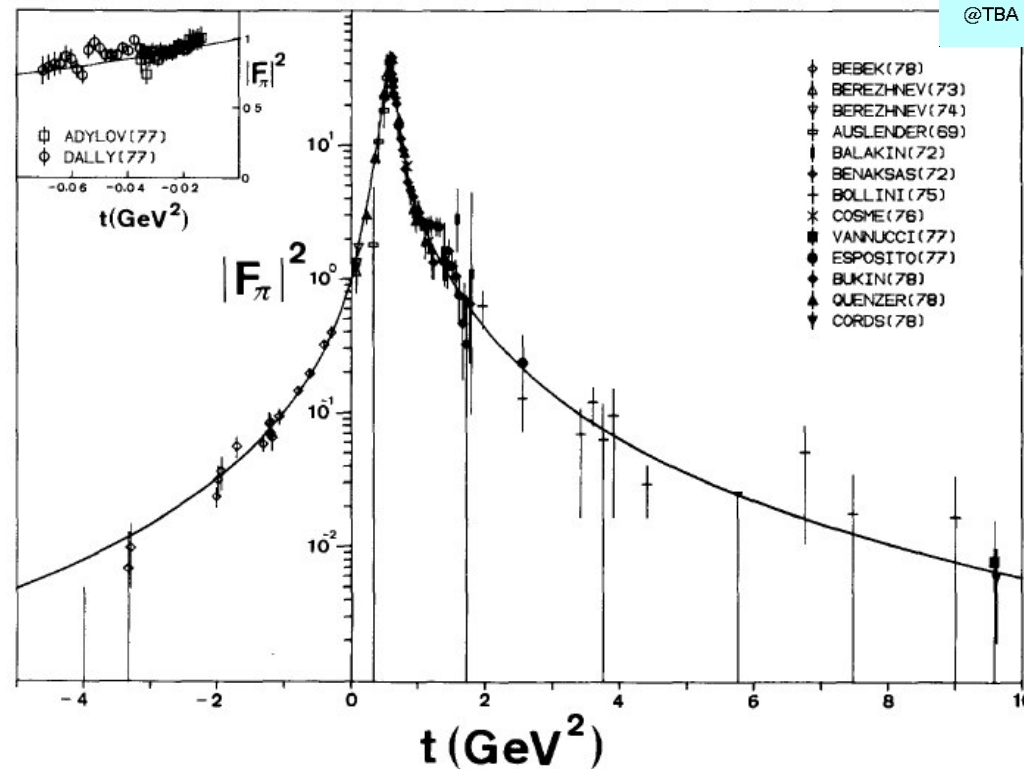
$$Rate_{Bhabha} = \sigma_{Bhabha} \cdot Luminosity$$

$e^+ + e^- \rightarrow e^+ + e^-$ pure QED at low energy, small angle

$\rightarrow$ accurate, reliable σ_{Bhabha} prediction $\rightarrow Luminosity = Rate_{Bhabha} / \sigma_{Bhabha}$

The π Form Factor at Large

Is there a unique function $F(q^2)$?? Yes!
Good check of crossing



Electron Form Factors - I

$$j^\mu = e\bar{\psi}\gamma^\mu\psi \quad \text{Dirac current}$$

$$\bar{u}(p')\gamma^\mu u(p) = \frac{1}{2m} \left[(p+p')^\mu + i\sigma^{\mu\nu}(p-p')_\nu \right] \quad \text{Gordon's identity}$$

$$\left\{ \begin{array}{l} \frac{e}{2m} u(p')(p+p')^\mu u(p) \quad \text{charge, like a scalar particle} \\ \frac{ie}{2m} \bar{u}(p') \underbrace{\sigma^{\mu\nu}(p-p')_\nu}_{=q_\nu} u(p) \quad \text{extra term} \end{array} \right.$$

Extra term due to *magnetic dipole current*. Indeed, it contributes the interaction energy:

$$\frac{ie}{2m} \bar{u}(p')\sigma^{\mu\nu}u(p)q_\nu A_\mu \xrightarrow{\text{low speed}} -\frac{e}{2m} \phi'^\dagger \boldsymbol{\sigma} \cdot (\mathbf{q} \times \mathbf{A}) \phi$$

Magnetic dipole interaction energy

$$\frac{e}{2m} \phi'^\dagger \boldsymbol{\sigma} \cdot (\nabla \times \mathbf{A}) \phi \equiv \frac{e}{2m} \phi'^\dagger (\boldsymbol{\sigma} \cdot \mathbf{B}) \phi \Rightarrow \frac{ie}{2m} \bar{u}(p')\sigma^{\mu\nu}u(p)q_\nu A_\mu \xrightarrow{\text{low speed}} -\frac{e}{2m} \phi'^\dagger (\boldsymbol{\sigma} \cdot \mathbf{B}) \phi$$

$$\mu \approx \frac{e\hbar}{2mc} \quad \text{Magnetic moment, } j = \frac{1}{2}\hbar \text{ Spin, } \gamma \equiv \frac{\mu}{j} \quad \text{Gyromagnetic ratio}$$

$$\gamma \approx \frac{e\hbar}{2mc} \frac{2}{\hbar} = \frac{e}{2m} \cdot 2, \quad \text{Define } \gamma \equiv g \frac{e}{2m} \rightarrow g \approx 2 \quad \text{Dirac } g\text{-factor}$$

Electron Form Factors - II

Now: g -factor not exactly 2, as predicted by Dirac equation

Reason: *Radiative corrections*

Largest correction: Anomalous magnetic moment

$$\mu_{Dirac} = \frac{e}{2m} \rightarrow \mu = \frac{e}{2m}(1 + \kappa_e)$$

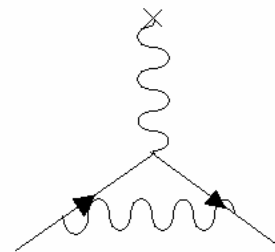
$$j^\mu = \frac{e}{2m} \bar{u}(p') \left[(p + p')^\mu + i\sigma^{\mu\nu} (1 + \kappa_e) q_\nu \right] u(p)$$

$$\bar{u}(p') \gamma^\mu u(p) = \frac{1}{2m} \bar{u}(p') \left[(p + p')^\mu + i\sigma^{\mu\nu} q_\nu \right] u(p)$$

$$\rightarrow \bar{u}(p') (p + p')^\mu u(p) = \bar{u}(p') (\gamma^\mu - i\sigma^{\mu\nu} q_\nu) u(p)$$

$$\rightarrow j^\mu = \frac{e}{2m} \bar{u}(p') \left[\gamma^\mu - i\sigma^{\mu\nu} q_\nu + i\sigma^{\mu\nu} (1 + \kappa_e) q_\nu \right] u(p)$$

$$\rightarrow j^\mu = \frac{e}{2m} \bar{u}(p') \left[\gamma^\mu + i\kappa_e \sigma^{\mu\nu} q_\nu \right] u(p)$$



In spite of the electron *being* a pointlike fermion, radiative corrections make it behaving like an extended object

Further radiative corrections lumped into 2 *form factors*

$$j^\mu = \frac{e}{2m} \bar{u}(p') \left[f(q^2) \gamma^\mu + g(q^2) i\kappa_e \sigma^{\mu\nu} q_\nu \right] u(p) \quad \text{Most general form (it can be shown..)}$$

$g-2$ - Theory

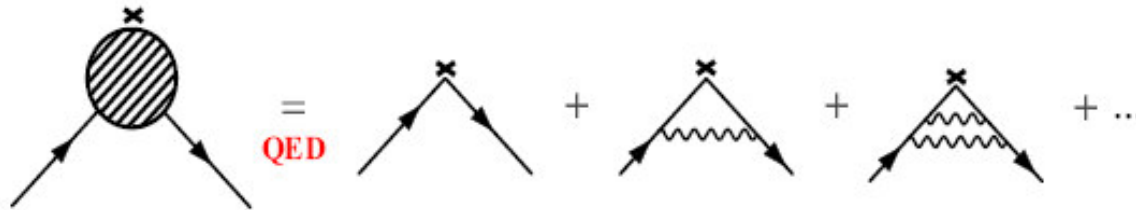
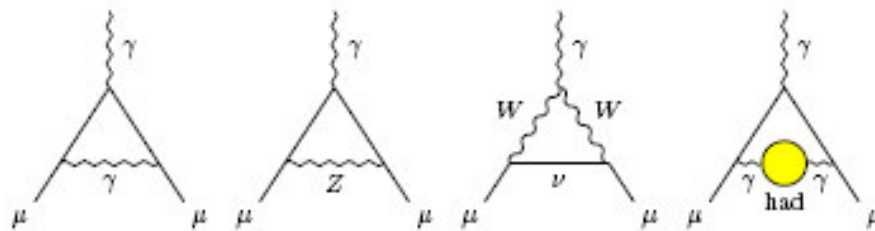


Figure 1: The perturbative expansion of $\Gamma^\rho(p', p)$ in single flavour QED. The tree graph gives $F_1 = 1$, $F_2 = F_3 = 0$. The one loop vertex correction graph gives the coefficient A_1 in Eq. (2.21). The cross denotes the insertion of the external field.

QED only...

@TBA



QED

Electroweak

QCD

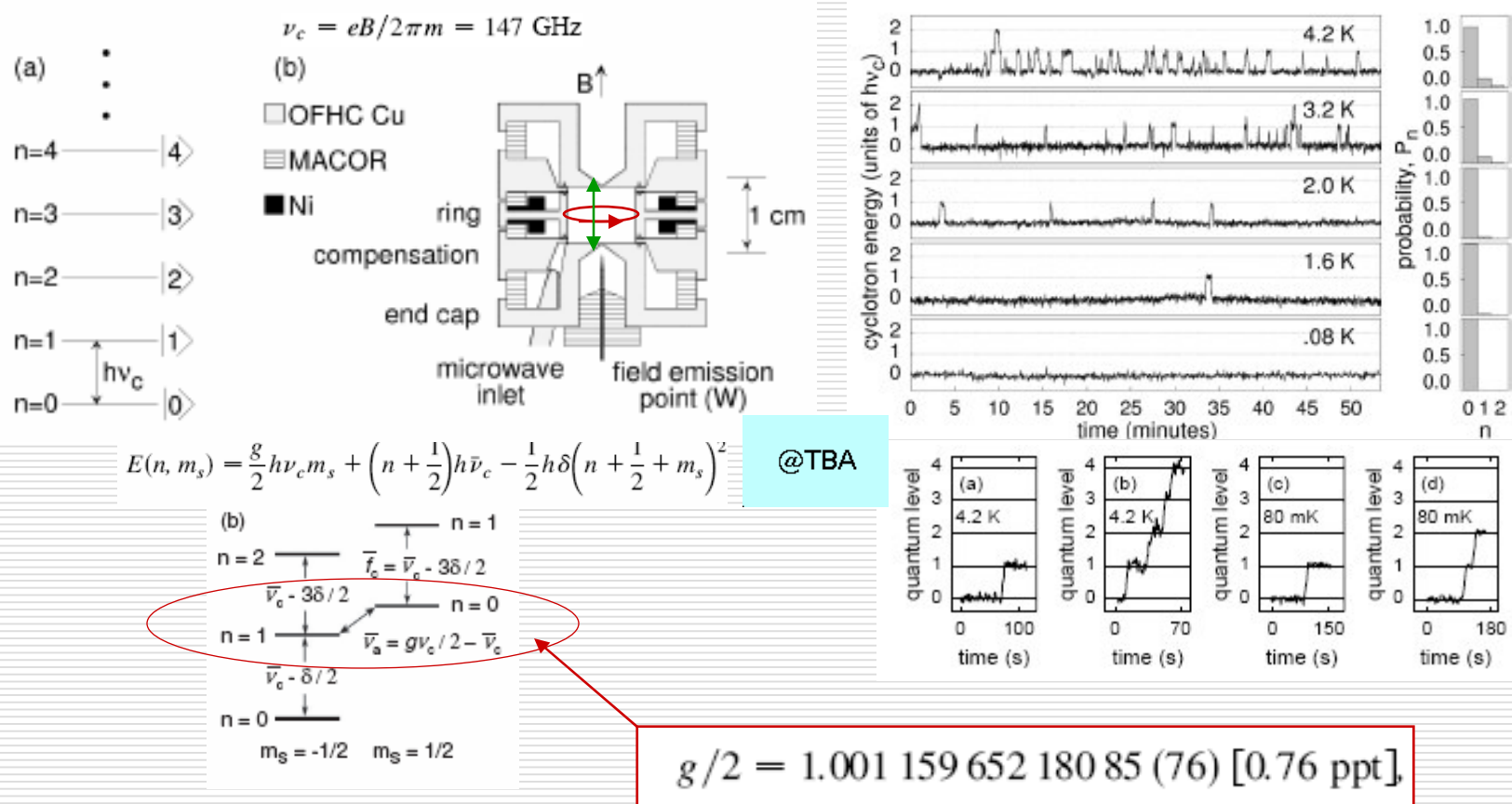
???

Contributions
from new physics ?

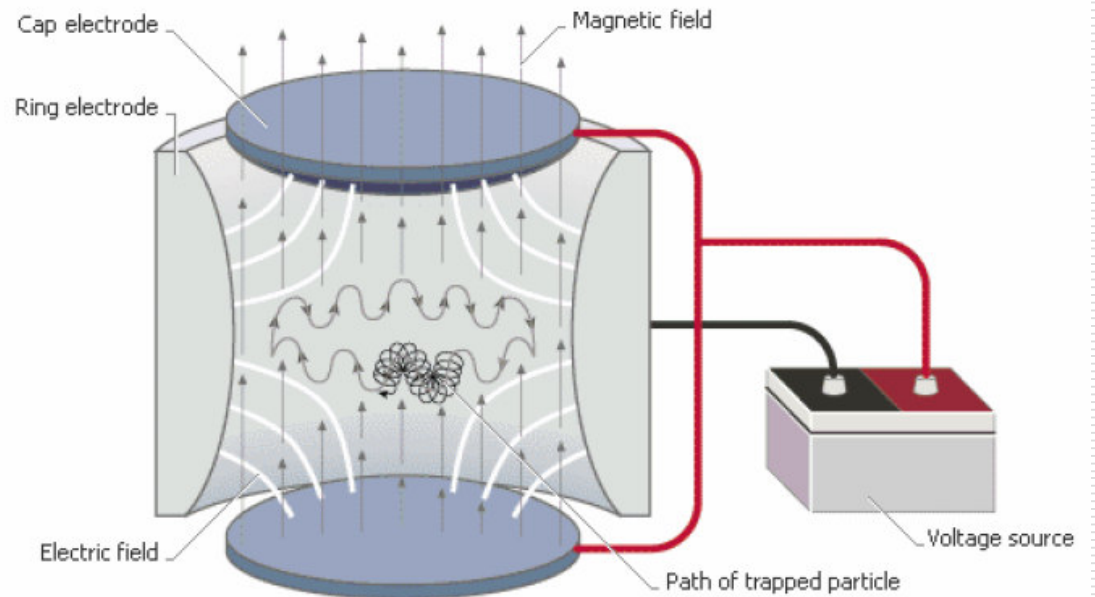
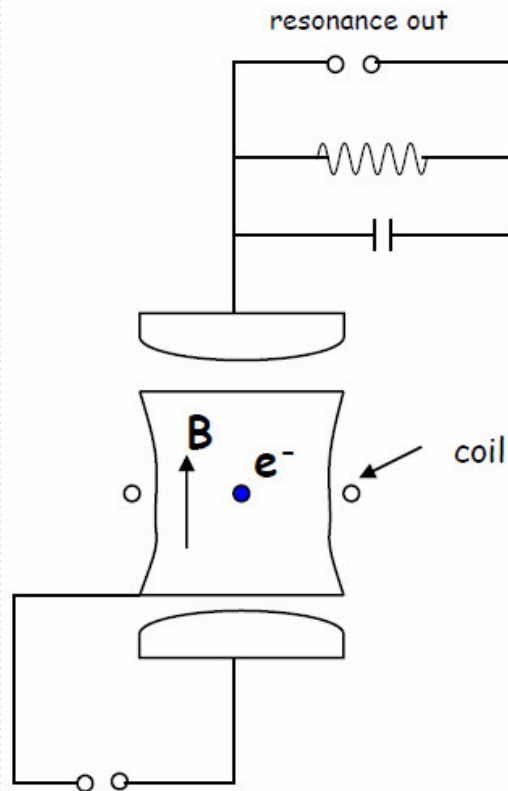
..and more

Electron g -2: Quantum Cyclotron

Clever use of magnetic and electric fields at low temperature: The *Penning Trap*



The Penning Trap



Muon $g-2$: Experimental Method

$$\mu \approx \frac{e\hbar}{2mc} \text{ Magnetic moment, } j = \frac{1}{2}\hbar \text{ Spin, } \gamma \equiv \frac{\mu}{j} \text{ Gyromagnetic ratio}$$

$$\gamma \approx \frac{e\hbar}{2mc} \frac{2}{\hbar} = \frac{e}{2m} \cdot 2 \equiv g \frac{e}{2m} \rightarrow g \approx 2$$

$$a \equiv \frac{g-2}{2} \ll 1 \text{ Anomaly}$$

For a charged particle moving in a uniform magnetic field:

$$\frac{d\mathbf{p}}{dt} = e(\mathbf{p} \times \mathbf{B}) \rightarrow \omega_c = \frac{eB}{m} \text{ Cyclotron frequency}$$

For a particle carrying a magnetic moment μ :

$$\left. \begin{array}{l} \mu = \gamma \mathbf{J} \\ \boldsymbol{\tau} = \mu \times \mathbf{B} \\ \boldsymbol{\tau} = d\mathbf{J}/dt \end{array} \right\} \rightarrow \frac{d\mu}{dt} = \gamma(\mu \times \mathbf{B}) \text{ Precession of } \mu \text{ around } \mathbf{B} \text{ in the particle rest frame}$$

$$\omega_s = \frac{2\mu B}{\hbar} = g \frac{eB}{2m} = (1+a) \frac{eB}{m} \text{ Precession frequency, unchanged by Lorentz transformation}$$

Get a by measuring the *beat frequency*
 $\omega_s - \omega_c$

Larmor Precession

Just a short reminder of the classical result:

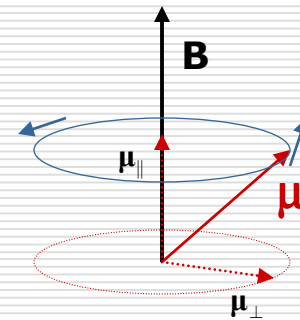
$$\frac{d\boldsymbol{\mu}}{dt} = \gamma(\boldsymbol{\mu} \times \mathbf{B}) \rightarrow \mathbf{B} \cdot \frac{d\boldsymbol{\mu}}{dt} = \gamma \mathbf{B} \cdot (\boldsymbol{\mu} \times \mathbf{B}) = 0 \rightarrow \mu_{\parallel} = \text{const}$$

Say $\mathbf{B} = B\hat{\mathbf{k}} \rightarrow \boldsymbol{\mu}_{\perp} = \mu_x \hat{\mathbf{i}} + \mu_y \hat{\mathbf{j}}$

$$\rightarrow \frac{d\boldsymbol{\mu}_{\perp}}{dt} = \gamma(\boldsymbol{\mu}_{\perp} \times \mathbf{B}) \rightarrow \begin{cases} \frac{d\mu_x}{dt} = \gamma(\boldsymbol{\mu}_{\perp} \times \mathbf{B})_x = +\gamma\mu_y B \\ \frac{d\mu_y}{dt} = \gamma(\boldsymbol{\mu}_{\perp} \times \mathbf{B})_y = -\gamma\mu_x B \end{cases}$$

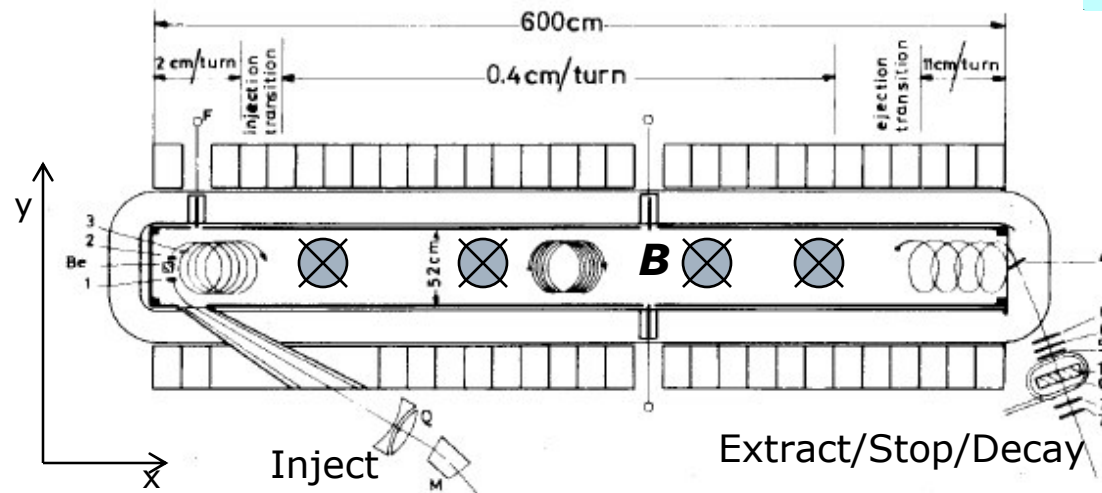
$$\tilde{\mu} \equiv \mu_x + i\mu_y \rightarrow \begin{cases} \frac{d\mu_x}{dt} = +\gamma\mu_y B \\ i \frac{d\mu_y}{dt} = -i\gamma\mu_x B \end{cases} \rightarrow \frac{d\mu_x}{dt} + i \frac{d\mu_y}{dt} = +\gamma B(\mu_y - i\mu_x) = -i\gamma B(\mu_x + i\mu_y)$$

$$\rightarrow \frac{d\tilde{\mu}}{dt} = -i\gamma B \tilde{\mu} \rightarrow \tilde{\mu}(t) = \tilde{\mu}_0 e^{-i\gamma B t} \rightarrow \begin{cases} \mu_x(t) = \mu_{x0} \cos \gamma B t \\ \mu_y(t) = -\mu_{y0} \sin \gamma B t \end{cases}$$

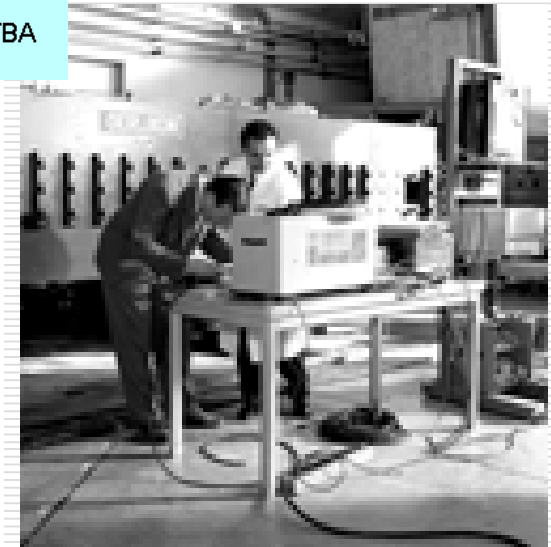


The First Muon $g-2$ at CERN

Small B y -gradient allowing for orbit drift along x

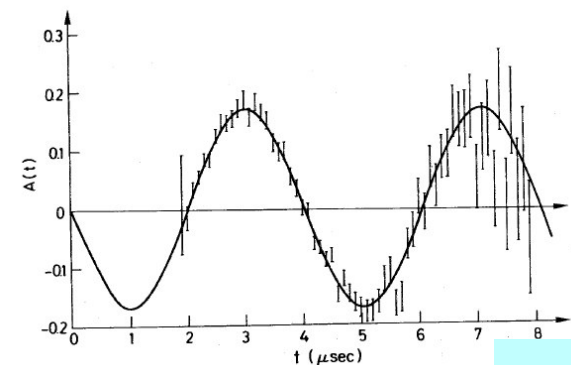


@TBA



Stopping μ^+ weakly decays to $\bar{\nu}_\mu e^+ \nu_e$ with $\tau = 2.2 \mu s$
 Parity violation enhances decays where positron is emitted along $\mathbf{s}_\mu \rightarrow$ Polarization direction can be measured
 By counting

$$\frac{N_{e^+}^{forward} - N_{e^+}^{backward}}{N_{e^+}^{forward} + N_{e^+}^{backward}} \propto \text{spin rotation angle} = (1+a) \frac{eB}{mc} \gamma t$$



@TBA

An Improved $g-2$ at CERN

@TBA

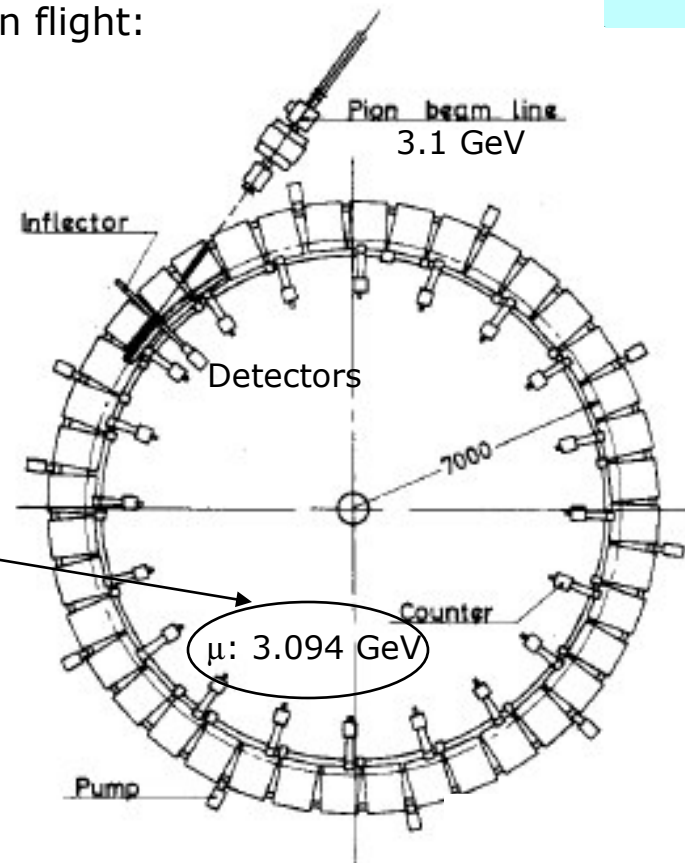
Detect high energy electrons from μ 's decaying in flight:
Forward-emitted in the μ rest frame

$$N(t) = N_0 e^{-t/\tau} \{1 - A \sin(2\pi f_a t + \phi)\}$$

$$\omega_a = a \frac{eB}{mc} \gamma$$

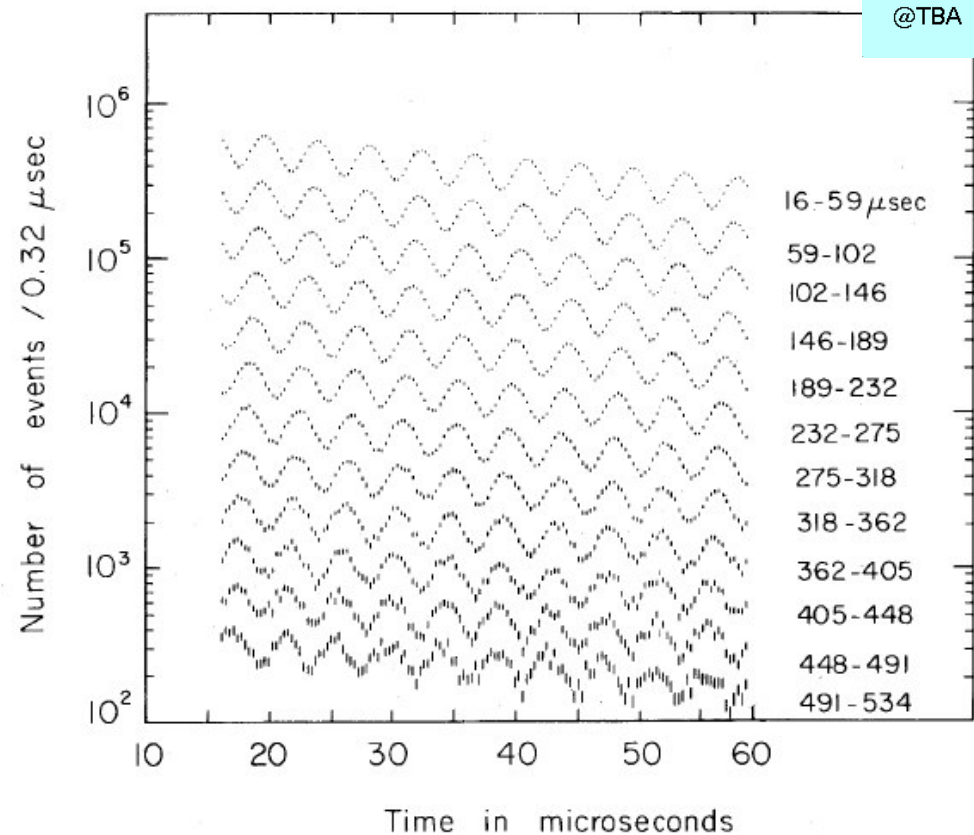
Why this momentum?

'Magic' cancellation of spurious effects
(E -field appearing in the μ rest frame
from Lorentz-transformed B -field in the
LAB frame)



Astonishing Precision...

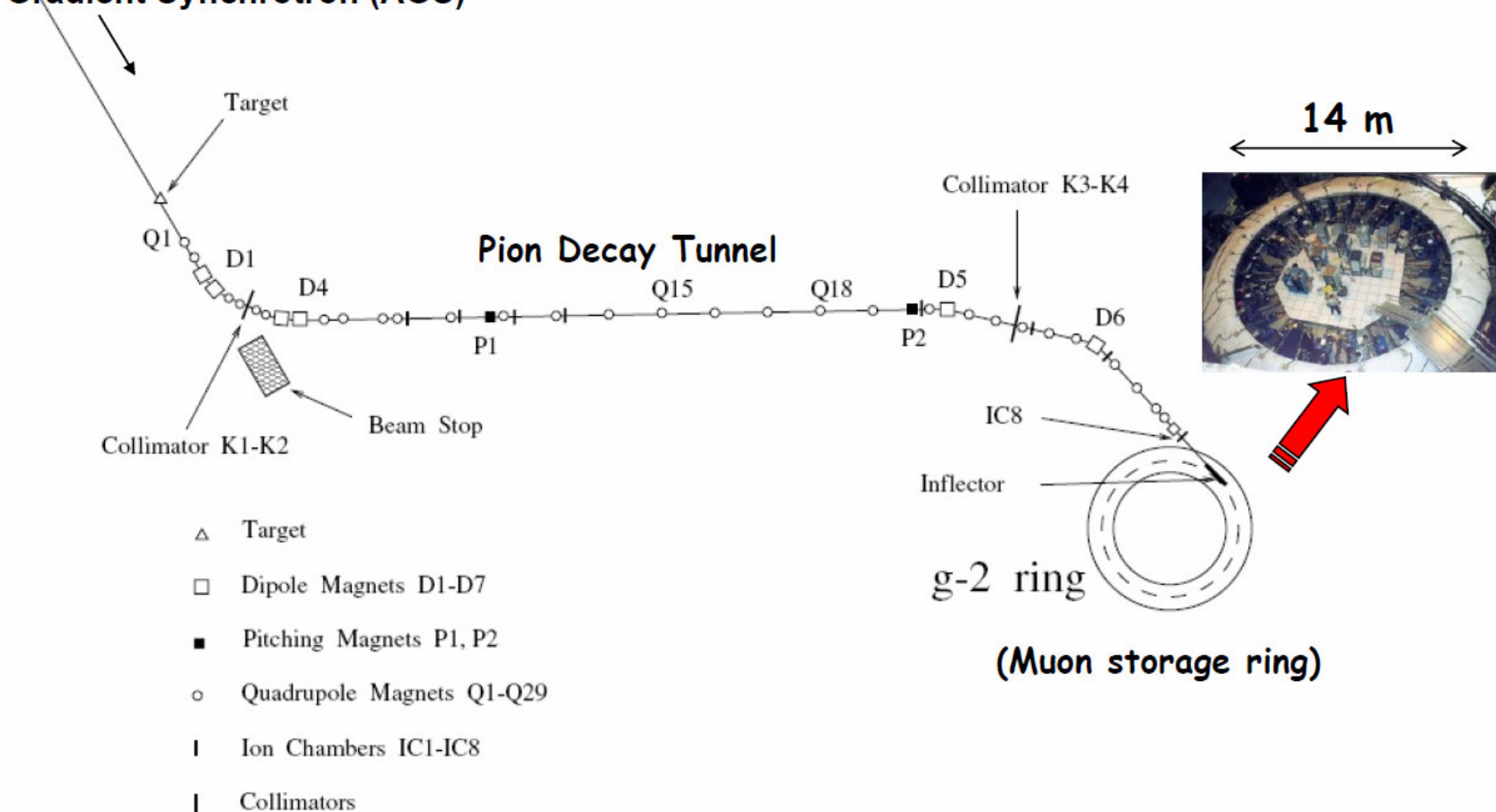
$$a_{\mu^+} = 1\,165\,911\,(11) \times 10^{-9} \quad (10 \text{ ppm})$$
$$a_{\mu^-} = 1\,165\,937\,(12) \times 10^{-9} \quad (10 \text{ ppm})$$



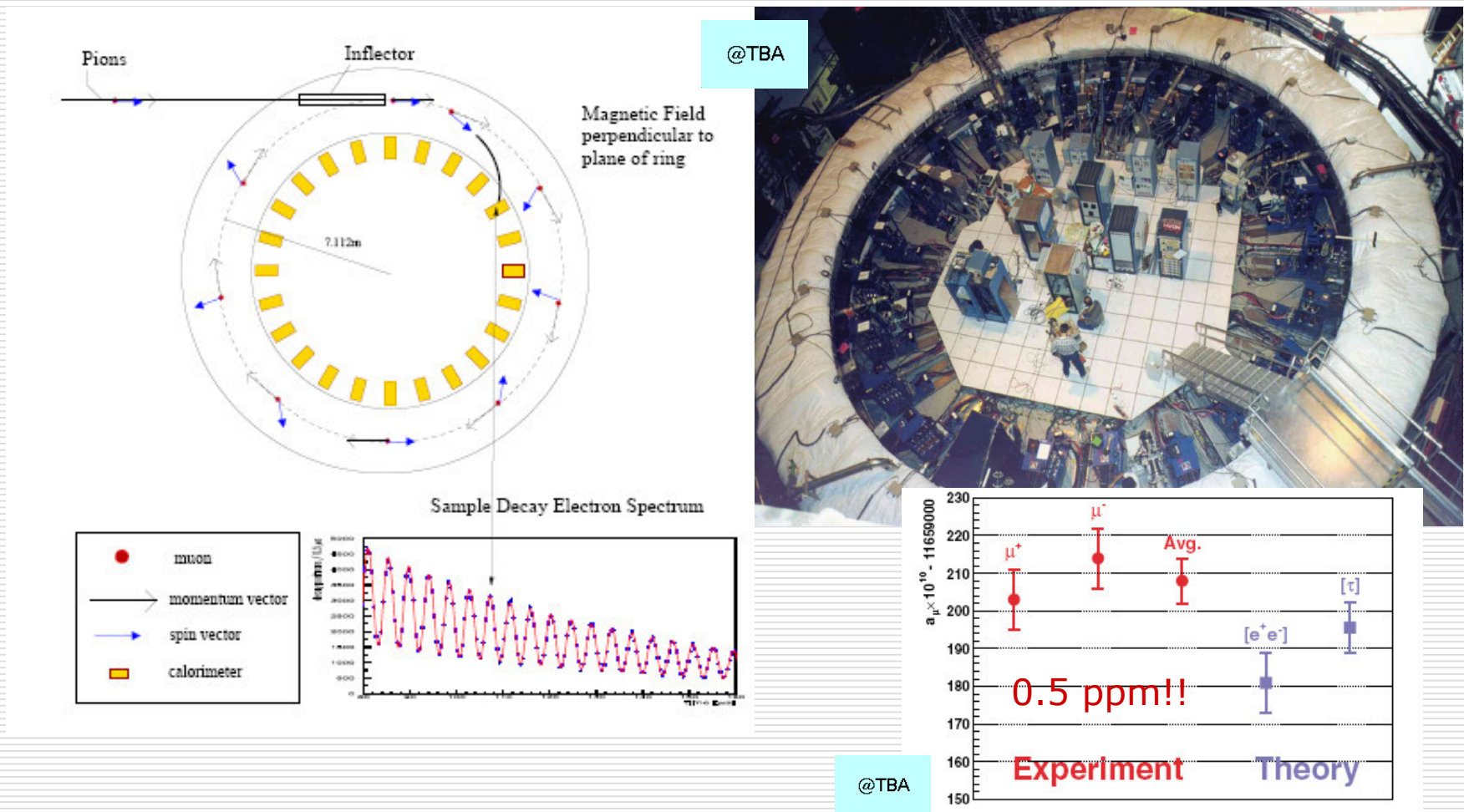
The Latest Muon $g-2$ at BNL - I

24 GeV Protons from the Alternating Gradient Synchrotron (AGS)

Brookhaven National Laboratory



The Latest Muon $g-2$ at BNL - II



Nucleon Form Factors

Take the same current for the nucleon

$$j_p^\mu = e \bar{u}(p') \left(F(q^2) \gamma^\mu + G(q^2) i \kappa_p \frac{\sigma^{\mu\nu}}{2m} q_\nu \right) u(p)$$

Anomalous magnetic moment
well measured, not understood

$$\kappa_p = ?$$

Anomaly originating here from the extended shape of the proton,
rather than radiative corrections

$$F_1(q^2) = F(q^2)$$

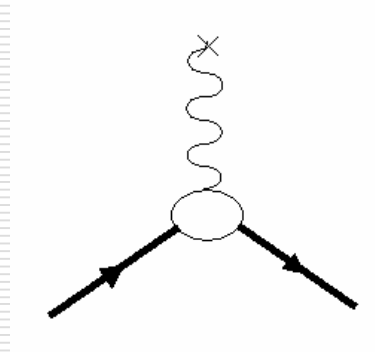
$$F_2(q^2) = 2MG(q^2)$$

$$\rightarrow j_p^\mu = e \bar{u}(p') \left(F_1(q^2) \gamma^\mu + \frac{i \kappa_p F_2(q^2)}{2M} \sigma^{\mu\nu} q_\nu \right) u(p)$$

Redefine:

$$G_E(q^2) = F_1 + \frac{\kappa_p q^2}{4M^2} F_2 \quad \text{Electric form factor}$$

$$G_M(q^2) = F_1 + \kappa_p F_2 \quad \text{Magnetic form factor}$$



Blob indicates a non-QED vertex

Nucleon Magnetic Moments

Electron-Proton comparison

$$\mu_B = \frac{e\hbar}{2m_e} = 5.78 \times 10^{-5} \text{ eV/T}$$

$$\mu = g s \mu_B$$

$$\mu_e/\mu_B = \underline{1.001\,159\,652\,187} \pm 0.000\,000\,000\,004 \quad \frac{g}{2}: \text{ Electron}$$

$$\mu_N = \frac{e\hbar}{2m_p} = 3.15 \times 10^{-8} \text{ eV/T}$$

$$\mu_p = g s \mu_N$$

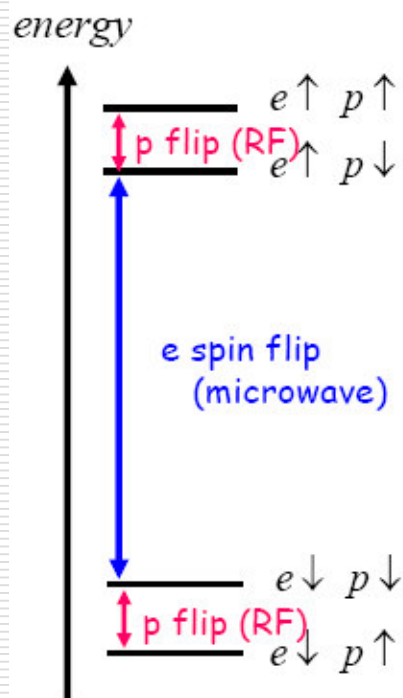
$$\mu_p/\mu_N = \underline{2.792847351} \pm 0.0000000028 \quad \frac{g}{2}: \text{ Proton}$$

Reminder: For a free Dirac particle $g=2$

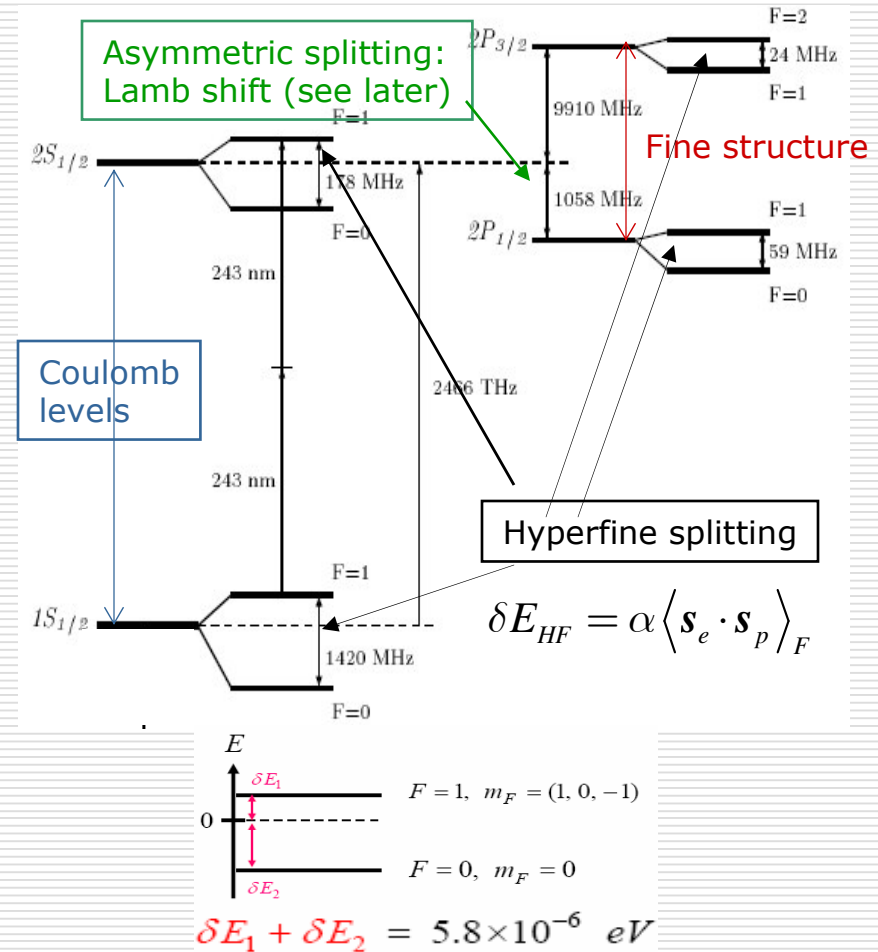
Strong indication: *The nucleon is not a point-like particle*

Proton Magnetic Moment - I

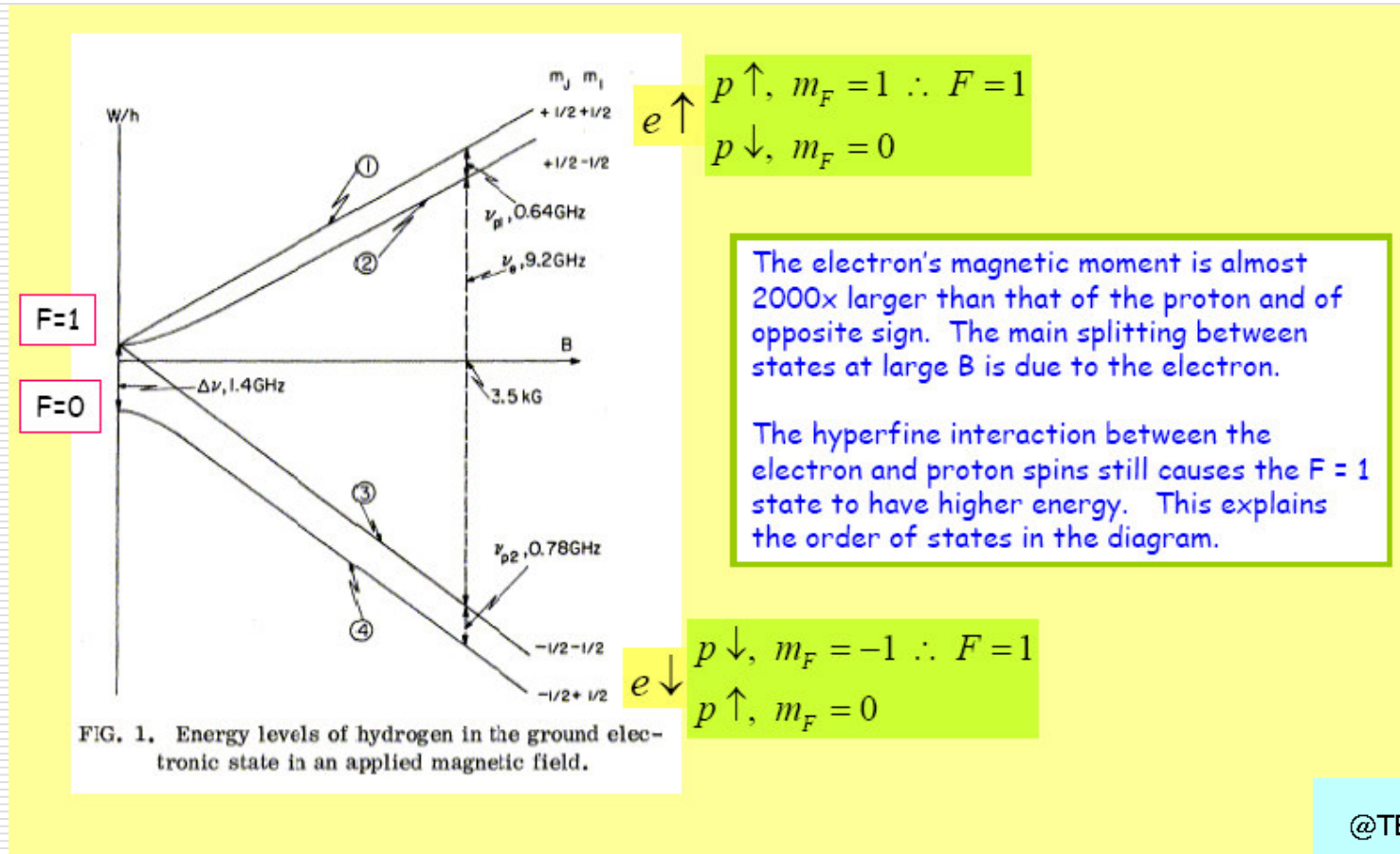
Modern method:
RF+Microwave atomic spectroscopy



@TBA

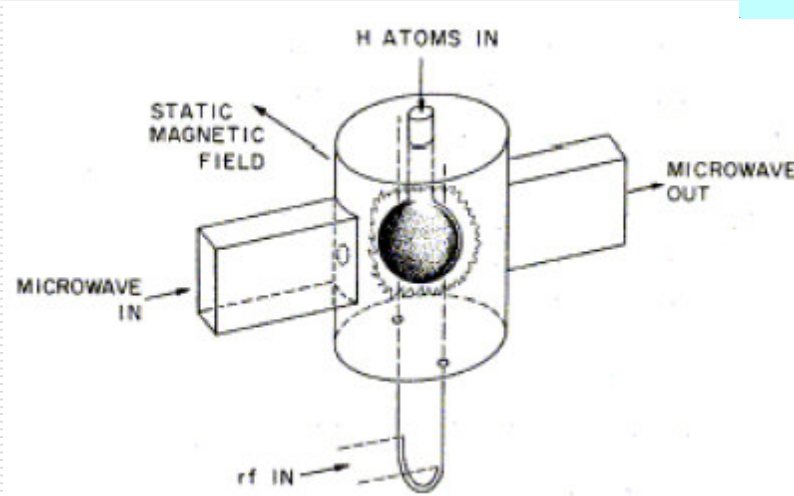


Proton Magnetic Moment - II



Proton Magnetic Moment - III

@TBA



$$\frac{\mu_{e^-}(\text{H})}{\mu_p(\text{H})} = -658.210\,7058(66) \quad [1.0 \times 10^{-8}],$$

Correct for atomic effects

$$\begin{aligned} \frac{\mu_{e^-}}{\mu_p} &= \frac{g_p(\text{H})}{g_p} \left(\frac{g_{e^-}(\text{H})}{g_{e^-}} \right)^{-1} \frac{\mu_{e^-}(\text{H})}{\mu_p(\text{H})} \\ &= -658.210\,6860(66) \quad [1.0 \times 10^{-8}], \end{aligned}$$

$$\mu_p/\mu_N = \underline{2.792847351} \pm 0.000000028$$

Neutron Magnetic Moment - I

Modern measurements: Thermal neutrons from a reactor

RF pulses in F_1, F_2

Periodic spin rotation in F_1, F_2 : Get $\mathbf{s} \perp \mathbf{B}$

Time of flight between F_1, F_2

Spin precession in $\mathbf{B}$ field

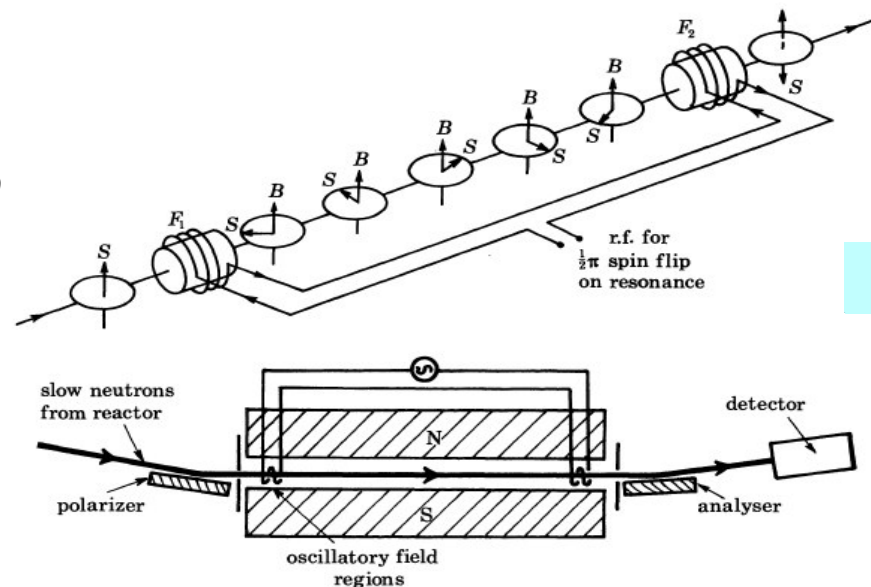
Count analyzed neutrons

'Resonance':

Count rate vs. frequency of spin flip

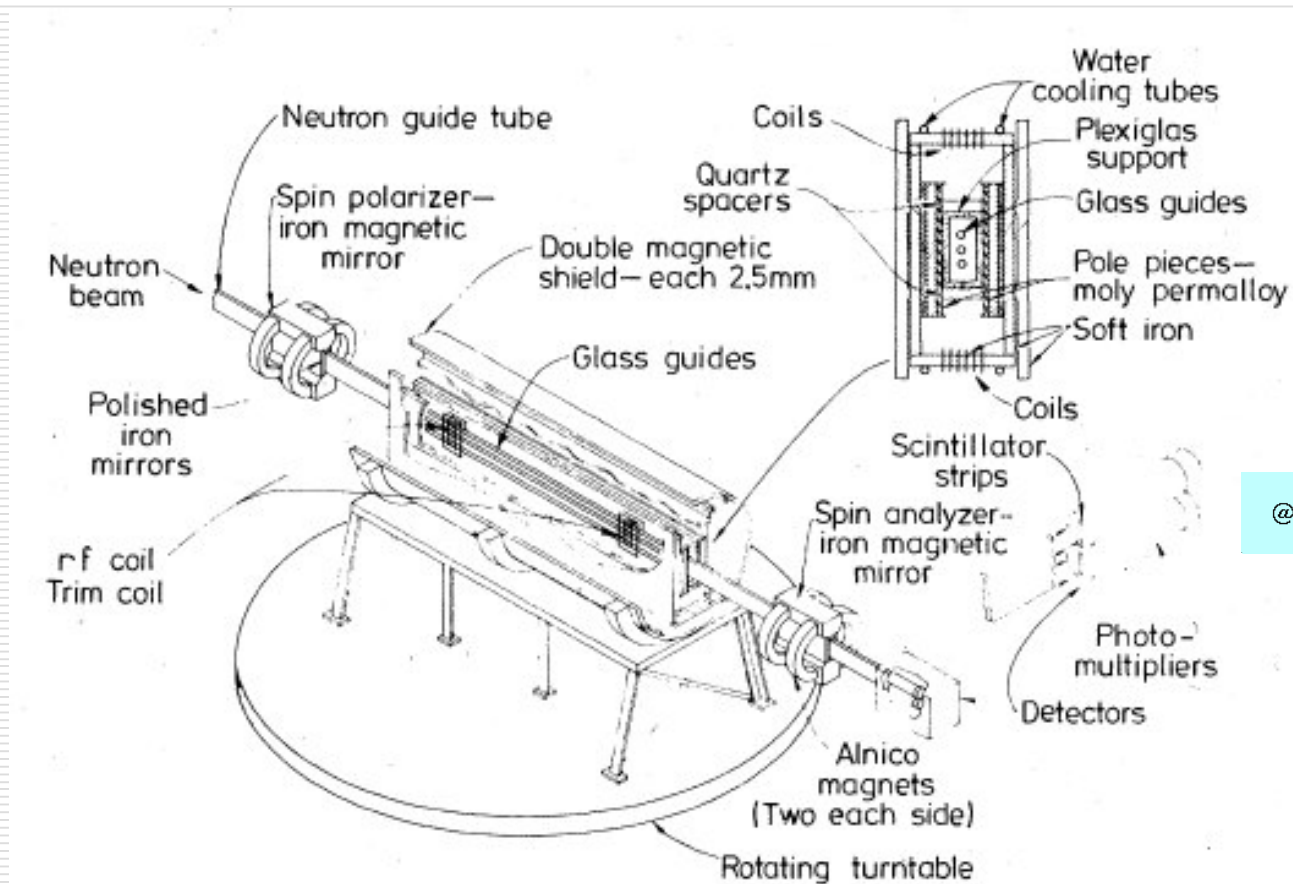
Polarizer, Analyzer:

Neutron Scattering from magnetized iron crystals, sensitive to spin orientation



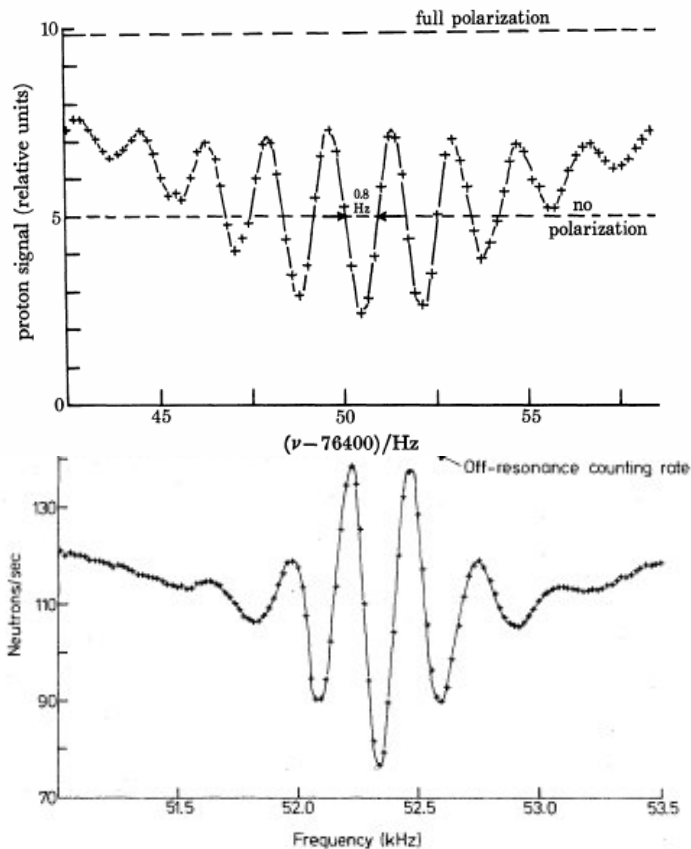
@TBA

Neutron Magnetic Moment - II



@TBA

Neutron Magnetic Moment - III



Proton resonance (Calibration)

@TBA

Neutron resonance

$$\mu_n / \mu_N = -1.913\,041\,84\,(88)\,(0.45\,\text{ppm}) .$$

The Rosenbluth Formula

Consider elastic electron-nucleon scattering
Going through the same steps as for electron-muon scattering

$$\begin{cases} A(q^2) = F_1^2(q^2) - \kappa_p^2 \frac{q^2}{4M^2} F_2^2(q^2) \\ B(q^2) = -\frac{q^2}{2M^2} (F_1(q^2) + \kappa_p F_2(q^2))^2 \end{cases}$$

$$\left. \frac{d\sigma}{d\Omega} \right|_{LAB} = \frac{\alpha^2}{4|\mathbf{p}_1|^2 \sin^4 \theta/2} \cos^2 \theta/2 \frac{|\mathbf{p}_1'|}{|\mathbf{p}_1|} (A(q^2) + B(q^2) \tan^2 \theta/2)$$

$$\left. \frac{d\sigma}{d\Omega} \right|_{LAB} = \frac{\alpha^2}{4|\mathbf{p}_1|^2 \sin^4 \theta/2} \cos^2 \theta/2 \frac{|\mathbf{p}_1'|}{|\mathbf{p}_1|} \left[\frac{G_E^2 - (q^2/4m^2) G_M^2}{1 - q^2/4m^2} - \frac{q^2}{m^2} G_M^2 \tan^2(\theta/2) \right]$$

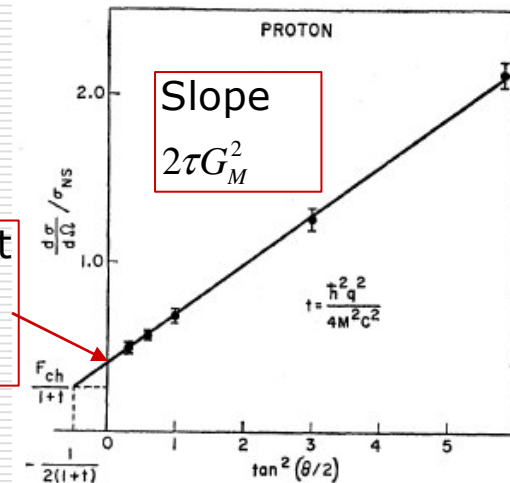
$$\left. \frac{d\sigma}{d\Omega} \right|_{LAB} = \left(\frac{d\sigma}{d\Omega} \right)_{Mott} \left[\frac{G_E^2 + \tau G_M^2}{1 + \tau} + 4\tau G_M^2 \tan^2 \theta/2 \right], \tau = -\frac{q^2}{4m^2}$$

'Rosenbluth separation' gives G_E, G_M

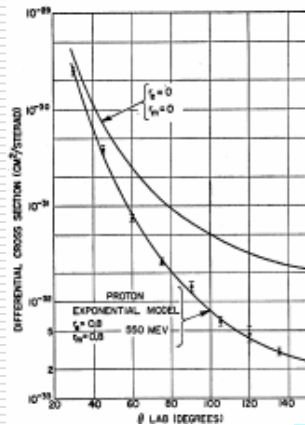
@TBA

Fixed q^2

Intercept
 $\frac{G_E^2 + \tau G_M^2}{1 + \tau}$

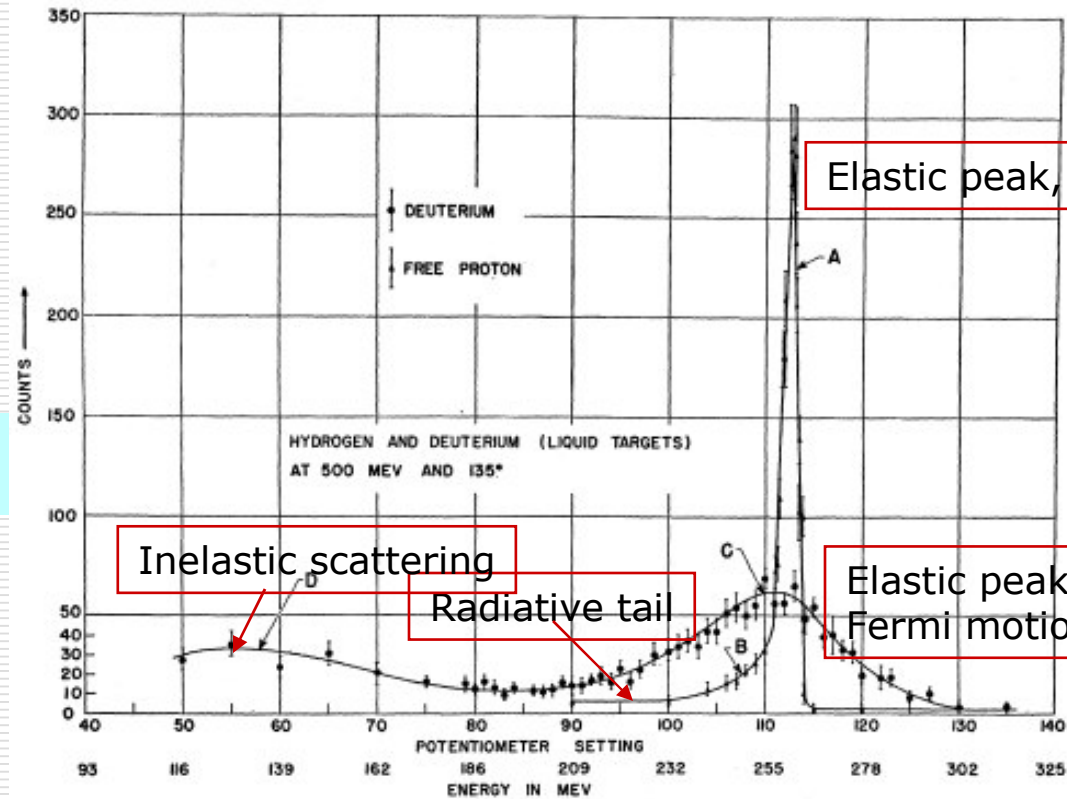
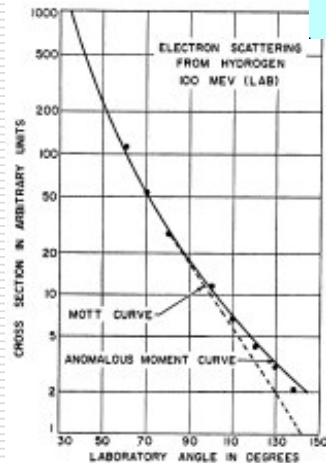


Measurements



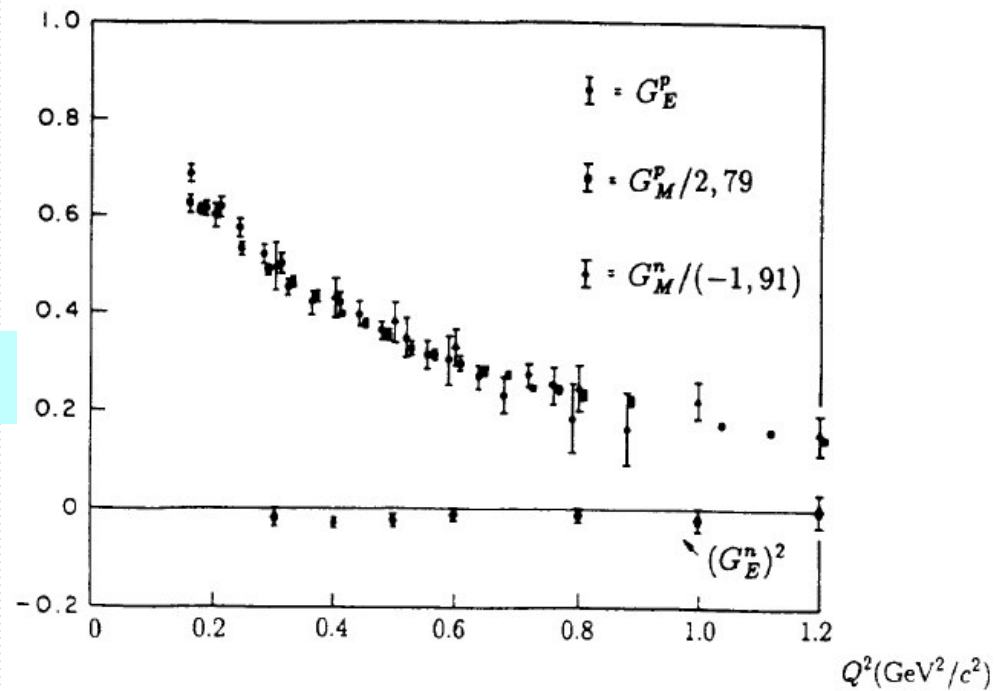
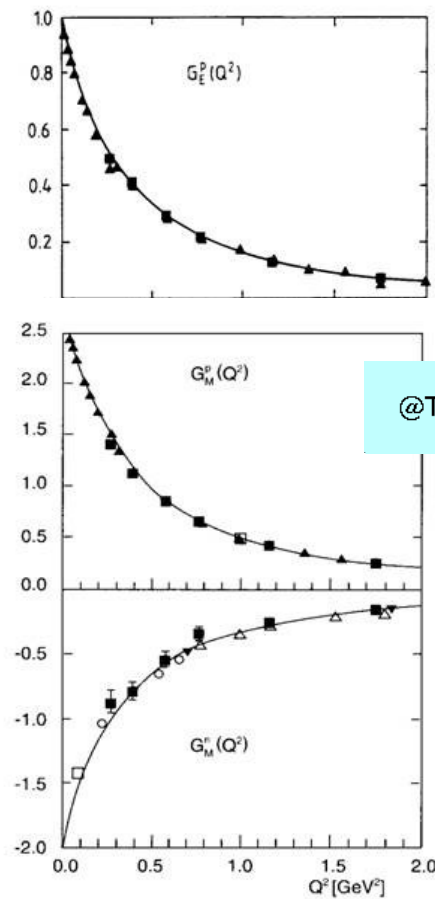
Hydrogen

@TBA



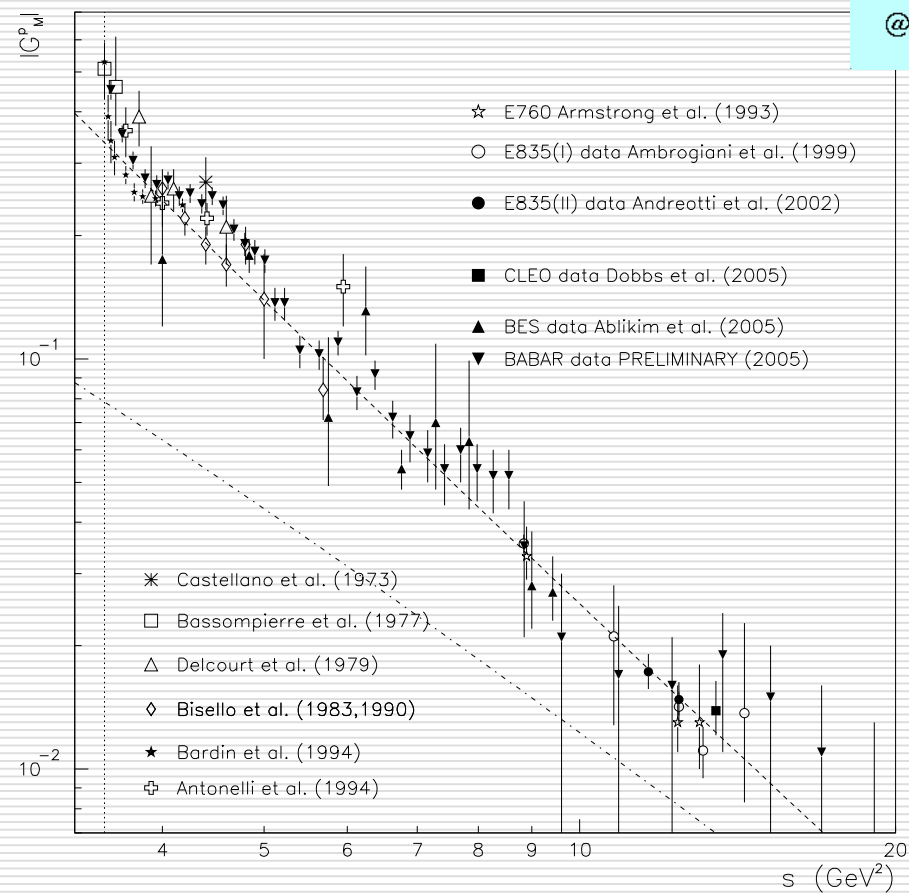
Electron energy at fixed angle for H_2 , D_2

Experimental Results



Same shape for the 3 non-zero..
Probing the same structure

Experimental Results



@TBA

Elastic Scattering Kinematics

4-momentum conservation at the nucleon vertex

$$p + q = p'$$

$$\rightarrow (p + q)^2 = p'^2 \rightarrow p^2 + q^2 + 2p \cdot q = p'^2$$

$$p^2 = p'^2 = M^2$$

$$\rightarrow 2p \cdot q = -q^2$$

Rewrite in the LAB frame, take massless lepton

$$P^\mu = (M, \mathbf{0})$$

$$p^\mu = (E, \mathbf{p}) \approx (|\mathbf{p}|, \mathbf{p})$$

$$p'^\mu \approx (|\mathbf{p}'|, \mathbf{p}')$$

$$\rightarrow q^2 \approx (|\mathbf{p}| - |\mathbf{p}'|, \mathbf{p} - \mathbf{p}')^2 = |\mathbf{p}|^2 + |\mathbf{p}'|^2 - 2|\mathbf{p}||\mathbf{p}'| - (|\mathbf{p}|^2 + |\mathbf{p}'|^2 - 2\mathbf{p} \cdot \mathbf{p}')$$

$$\rightarrow q^2 \approx (|\mathbf{p}| - |\mathbf{p}'|, \mathbf{p} - \mathbf{p}')^2 = -2|\mathbf{p}||\mathbf{p}'| + 2\mathbf{p} \cdot \mathbf{p}' = -2|\mathbf{p}||\mathbf{p}'|(1 - \cos \theta) = -4|\mathbf{p}||\mathbf{p}'|\sin^2 \theta/2$$

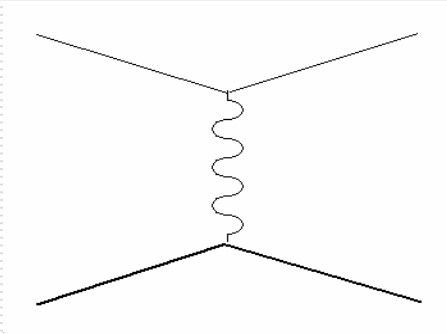
$$Q^2 \equiv -q^2$$

$$p \cdot q \approx (M, \mathbf{0}) \cdot (|\mathbf{p}| - |\mathbf{p}'|, \mathbf{p} - \mathbf{p}') = M(|\mathbf{p}| - |\mathbf{p}'|)$$

$$\nu \equiv |\mathbf{p}| - |\mathbf{p}'| \rightarrow p \cdot q \approx M\nu$$

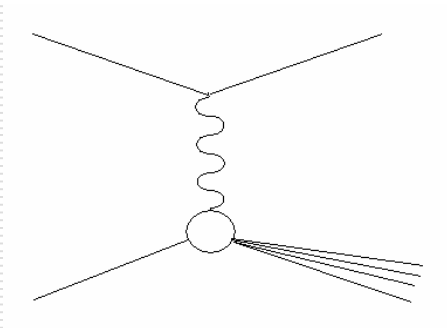
Inelastic Scattering

Elastic:

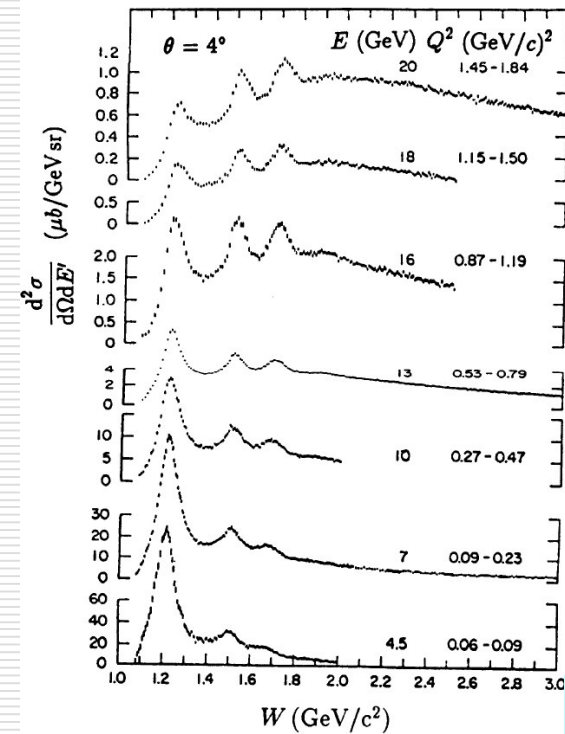


Invariant mass = M

Generalise to inelastic reactions:



Invariant mass = $W > M$

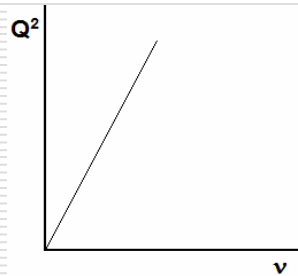


@TBA

Copious production of *resonances* (nucleon excited states), when q^2 not too big

Inelastic Scattering Kinematics

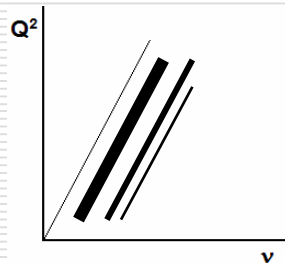
$$\nu = \frac{Q^2}{2M} \rightarrow 2M\nu = Q^2$$



Elastic

Scattered electron:
2-body kinematics,
 E', θ totally correlated

$$2M\nu = Q^2 + M'^2 - M^2$$



Inelastic

>2-body kinematics
 E', θ uncorrelated

Measure $E', \theta \rightarrow$ Get q^2, ν

Generalise Rosenbluth cross-section to account for variable W :
Introduce *structure functions* W_1, W_2 to replace form factors F_1, F_2

$$\left. \frac{d\sigma}{d\Omega dE'} \right|_{LAB} = \frac{\alpha^2}{4|\mathbf{p}_1|^2 \sin^4 \theta/2} \cos^2 \theta/2 \left(W_2(\nu, q^2) + W_1(\nu, q^2) \tan^2 \theta/2 \right)$$

$W_{1,2}$ depending on q^2 and ν

Resonance Excitation at DESY

First DESY machine

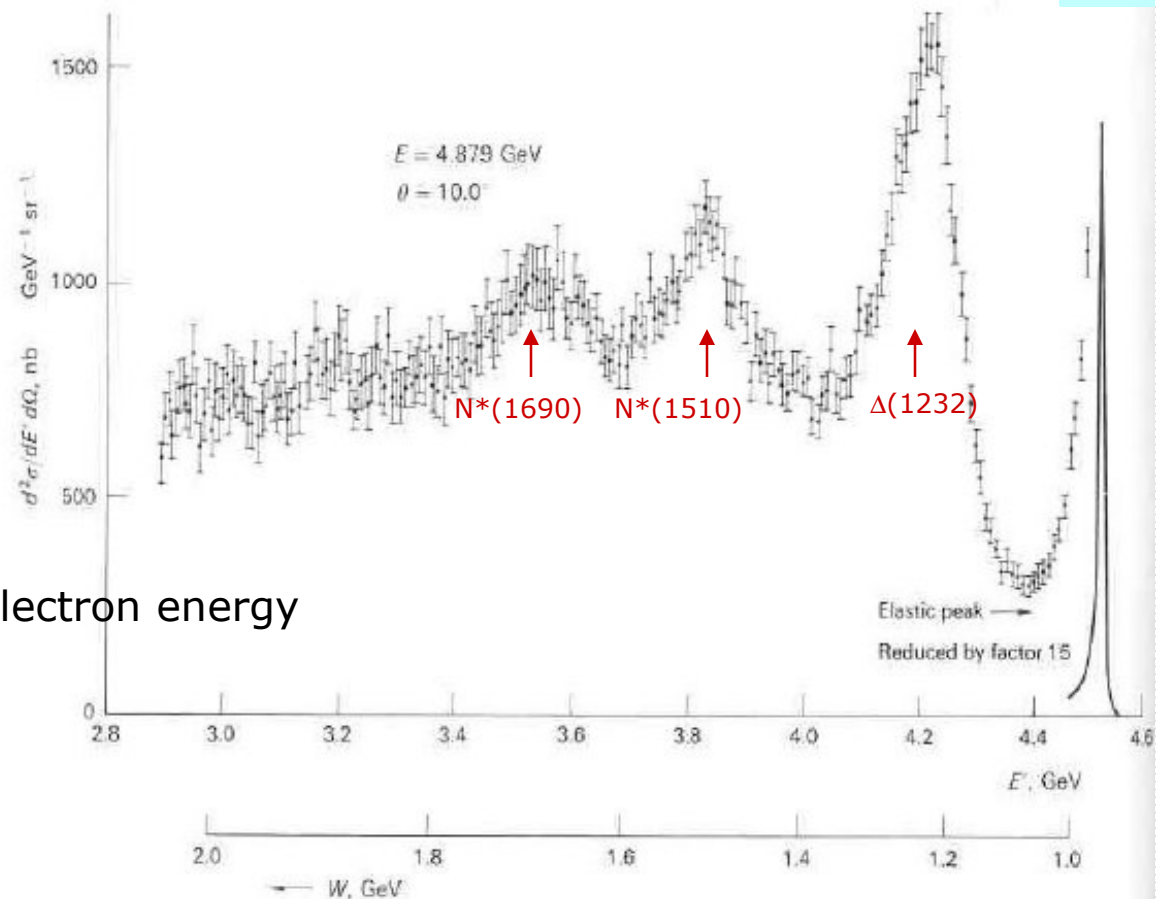
Electron synchrotron

$E_{max} = 6 \text{ GeV}$

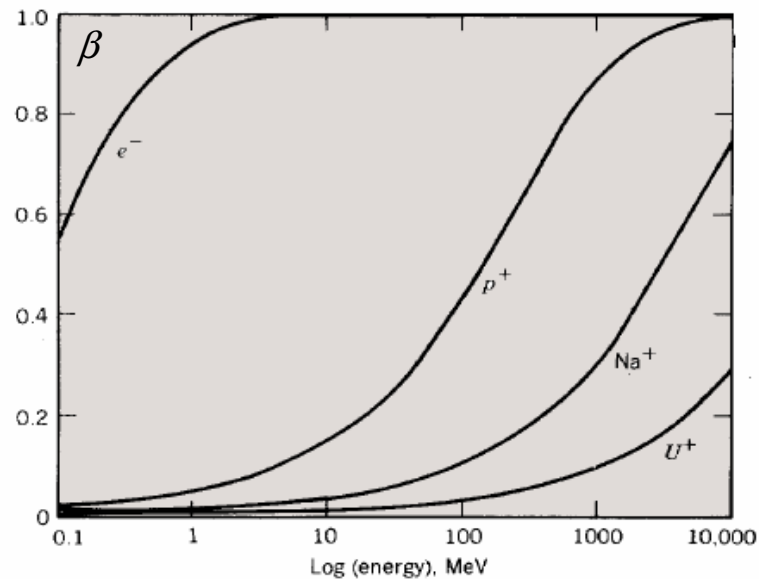
$e + p \rightarrow e + X$

E, E' : Incident, scattered electron energy

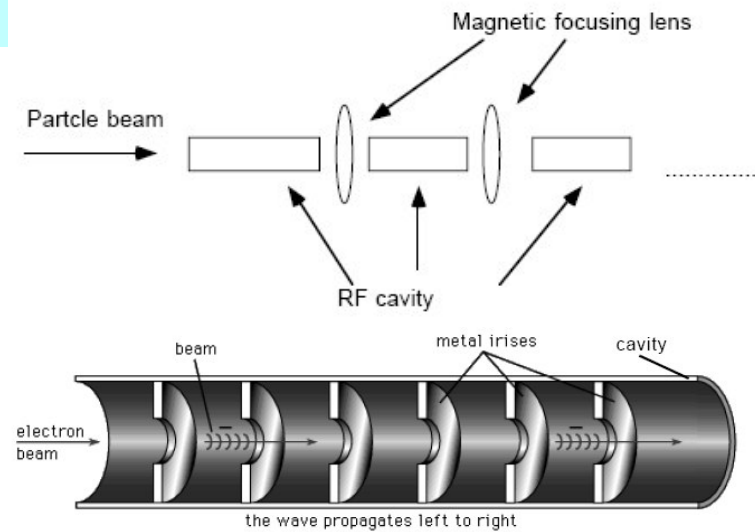
$W = M_X$



The Electron LINAC

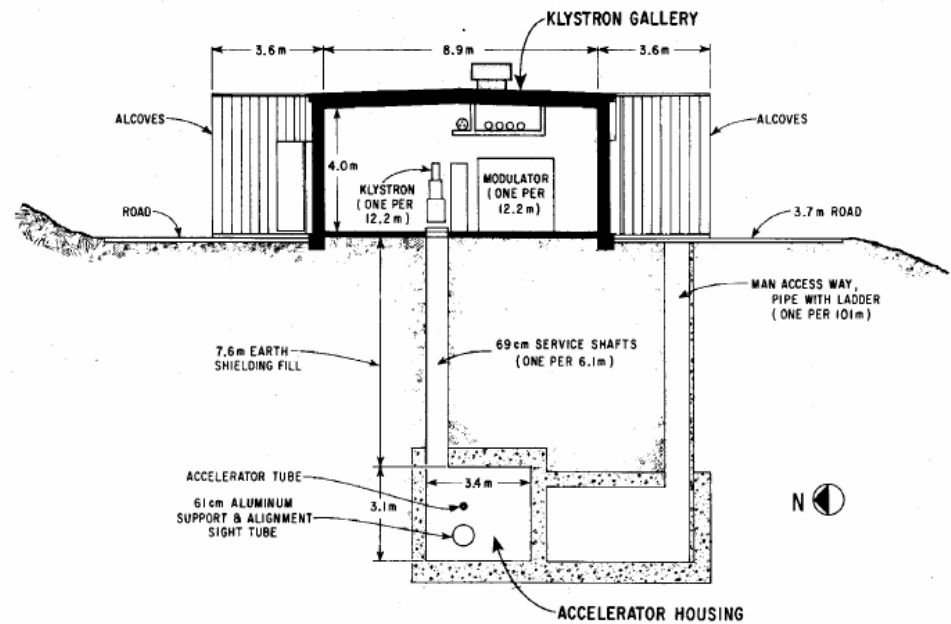
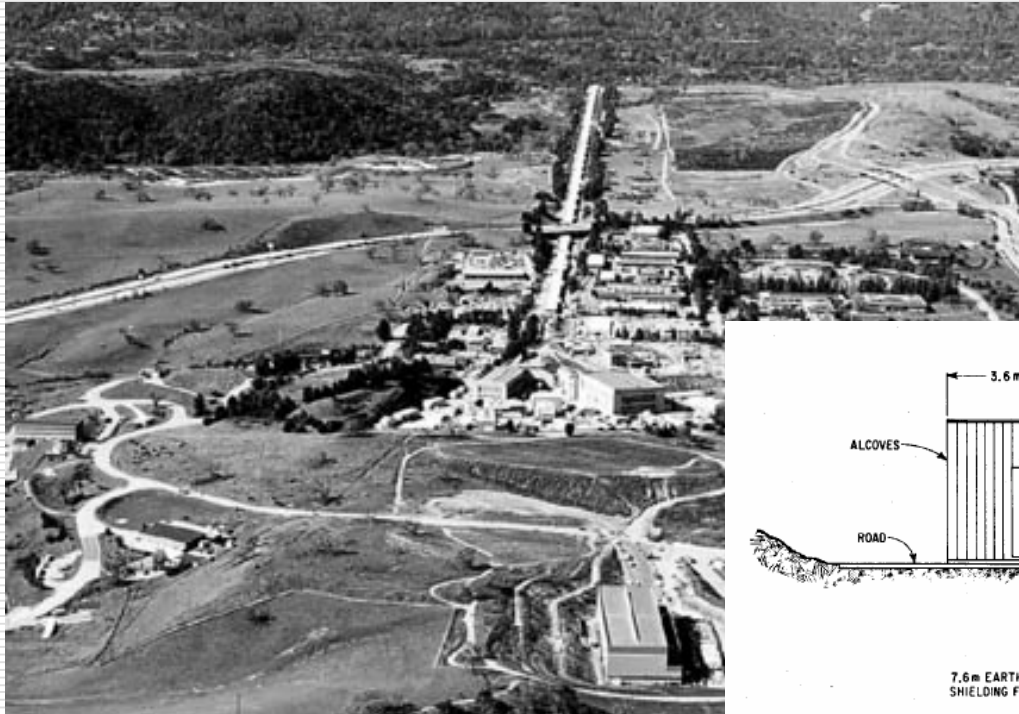


@TBA



Traveling wave linear accelerator:
Electrons riding the traveling EM wave at constant phase

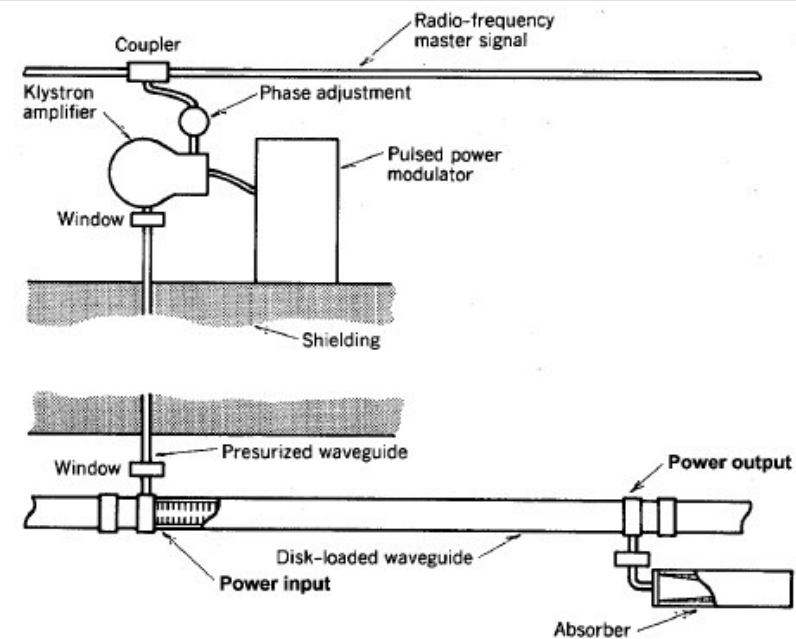
SLAC - I



SLAC - II

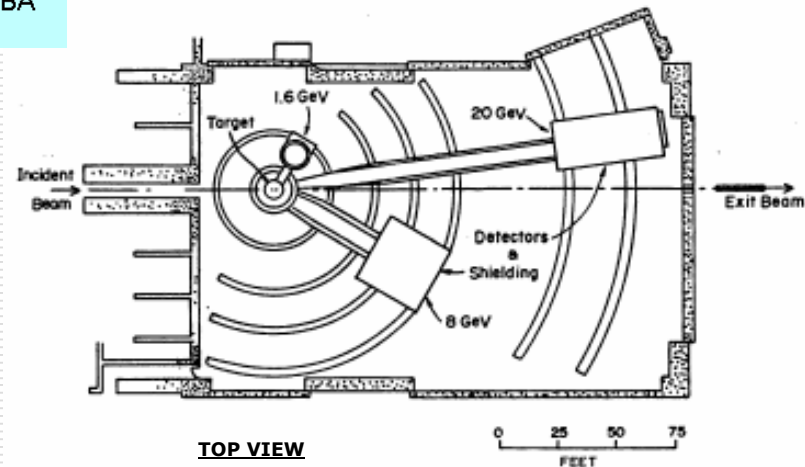
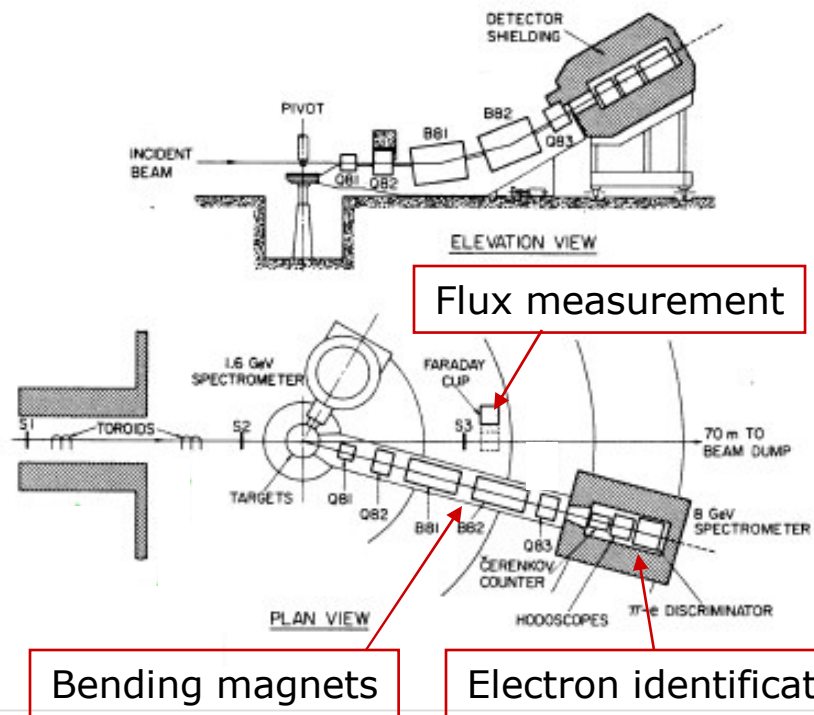
Accelerator length	3100 m
Length between power feeds	3.1 m
Number of accelerator sections	960
Number of klystrons	245
Peak power per klystron	6–24 MW
Beam pulse repetition rate	1–360 pulses/s
Radio-frequency pulse length	2.5 μ s
Filling time	0.83 μ s
Shunt impedance	53 M Ω /m
Electron energy (unloaded)	11.1–22.2 GeV
Electron energy (loaded)	10–20 GeV
Electron beam peak current	25–50 mA
Electron beam average current	15–30 μ A
Average electron beam power	0.15–0.6 MW
Efficiency	4.3%
Positron energy	7.4–14.8 GeV
Positron average beam current	0.45 μ A
Operating frequency	2.856 GHz
Accelerating structure	Iris-loaded waveguide
Waveguide outer diameter	10.5 cm
Aperture diameter	1.9 cm

@TBA



The SLAC Experiments

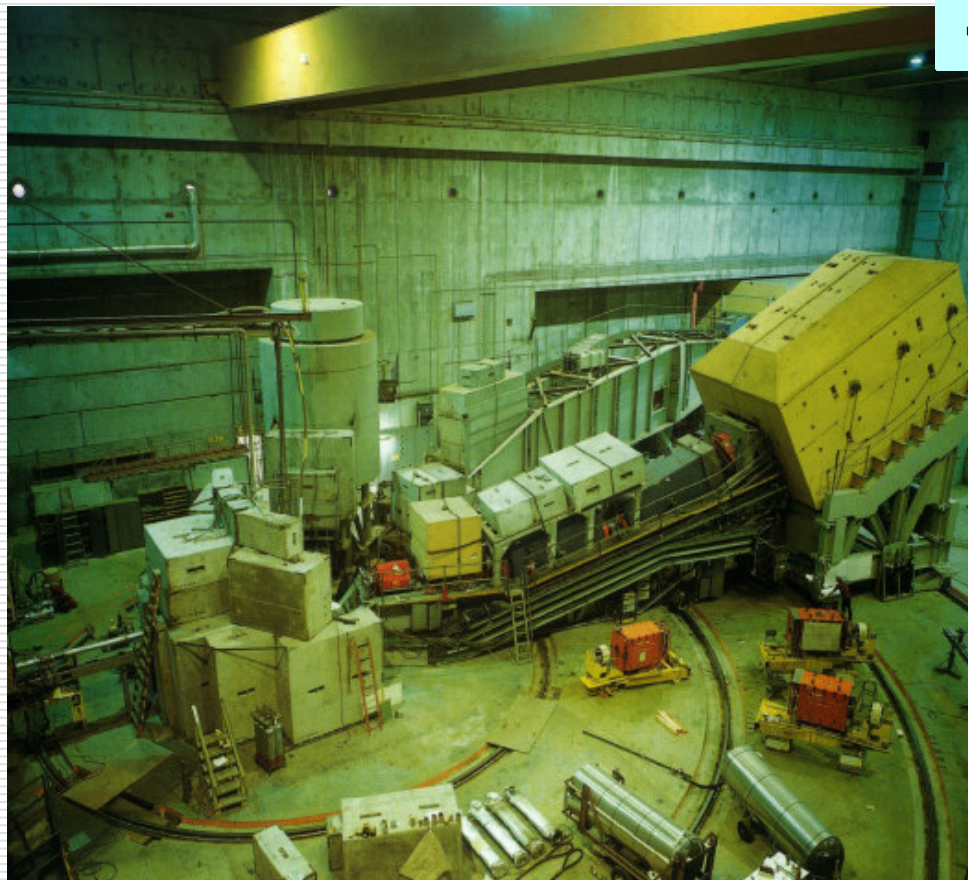
@TBA



Measure E' , θ of the scattered electron
 $\rightarrow$ Get q^2 , ν

$$\frac{d^2\sigma}{dE' d\Omega} \rightarrow \frac{d^2\sigma}{dq^2 d\nu}$$

SLAC End Station A

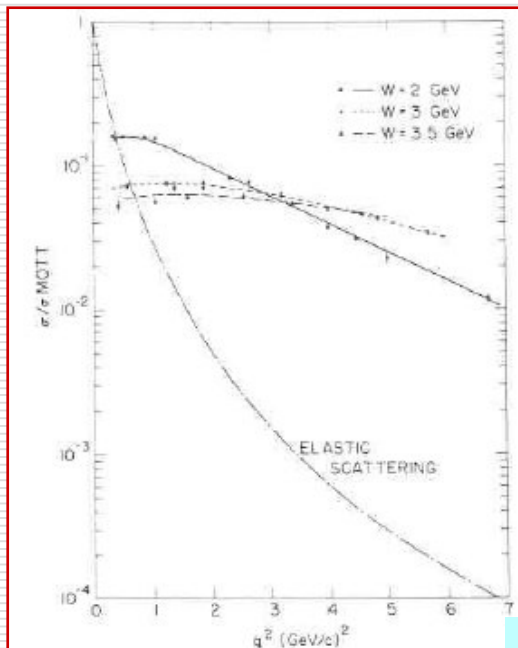


@TBA

Deep Inelastic Scattering - I

Details of structure functions in the resonance region difficult to explain
But: Beyond small q^2, ν things are surprisingly simple!

Striking elastic/deep inelastic comparison:
no q^2 dependence in DIS



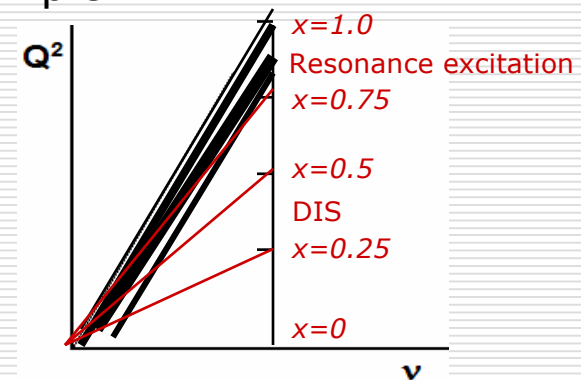
$$\frac{d^2\sigma}{dE'd\Omega} \rightarrow \frac{d^2\sigma}{dq^2 d\nu}$$

$$\begin{cases} x = \frac{Q^2}{2M\nu} \\ y = \frac{\nu}{E_1} \end{cases} \rightarrow \frac{d^2\sigma}{dxdy} = \frac{4\pi\alpha^2 ME_1}{Q^4} \left[2xF_1 \left(\frac{1+(1-y)^2}{2} \right) + (1-y)(F_2 - 2xF_1) - \frac{M^2 xy F_2}{s-M^2} \right]$$

Bjorken scaling hypothesis:

$$\frac{d^2\sigma}{dQ^2 d\nu} \xrightarrow[\frac{q^2}{\nu} \text{ finite}]{q^2, \nu \rightarrow \infty} f(x), \quad x = \frac{Q^2}{2M\nu}$$

Universal function of x
 Q^2 independent

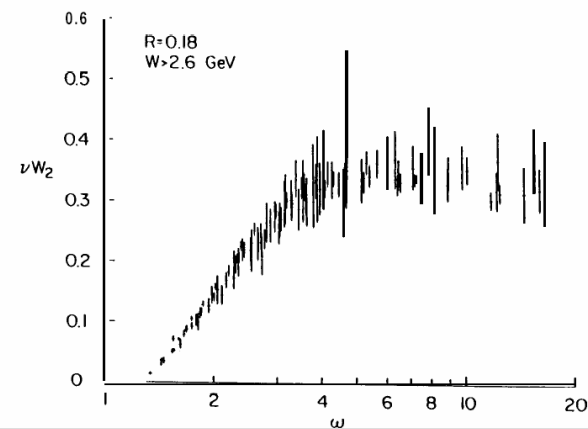
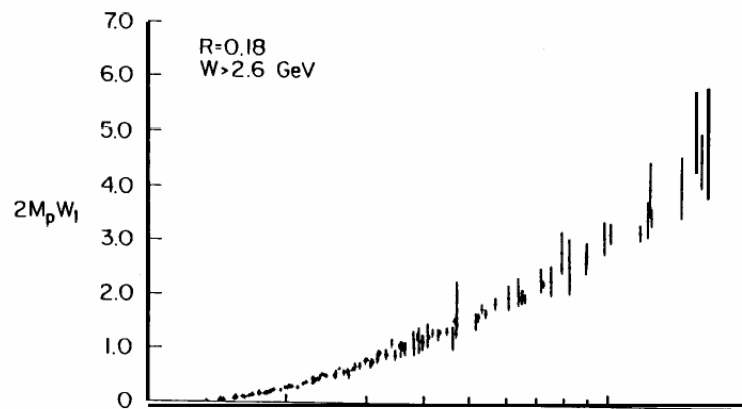


Deep Inelastic Scattering - II

Expect:

$$\begin{cases} F_1(x, Q^2) \rightarrow F_1(x) \\ F_2(x, Q^2) \rightarrow F_2(x) \end{cases} \text{ if scaling is good}$$

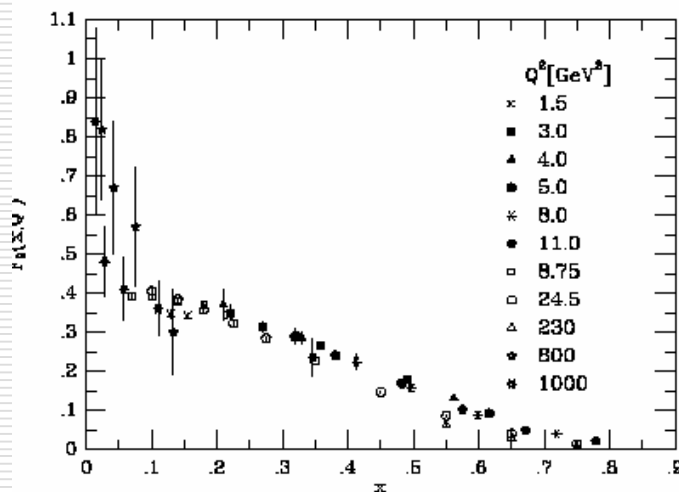
Observe ($\omega = 1/x$):



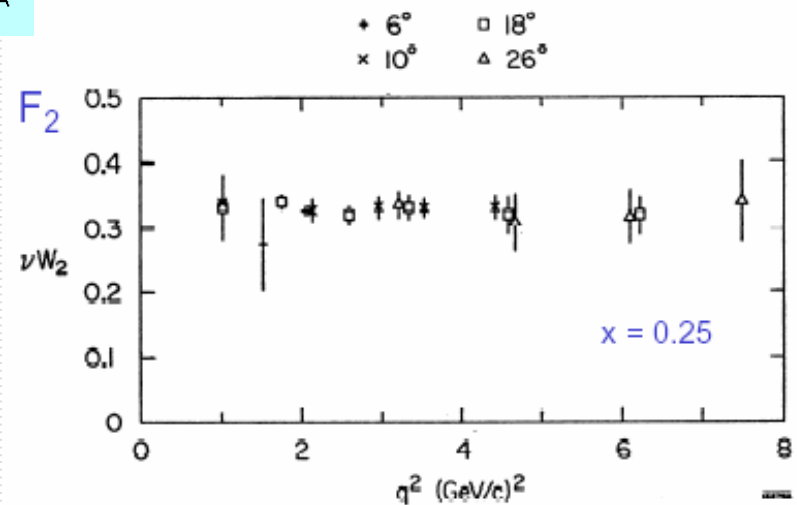
Scaling at high energy is indeed well verified!

Deep Inelastic Scattering - III

W_2 : Universal function of x q^2 -independent!

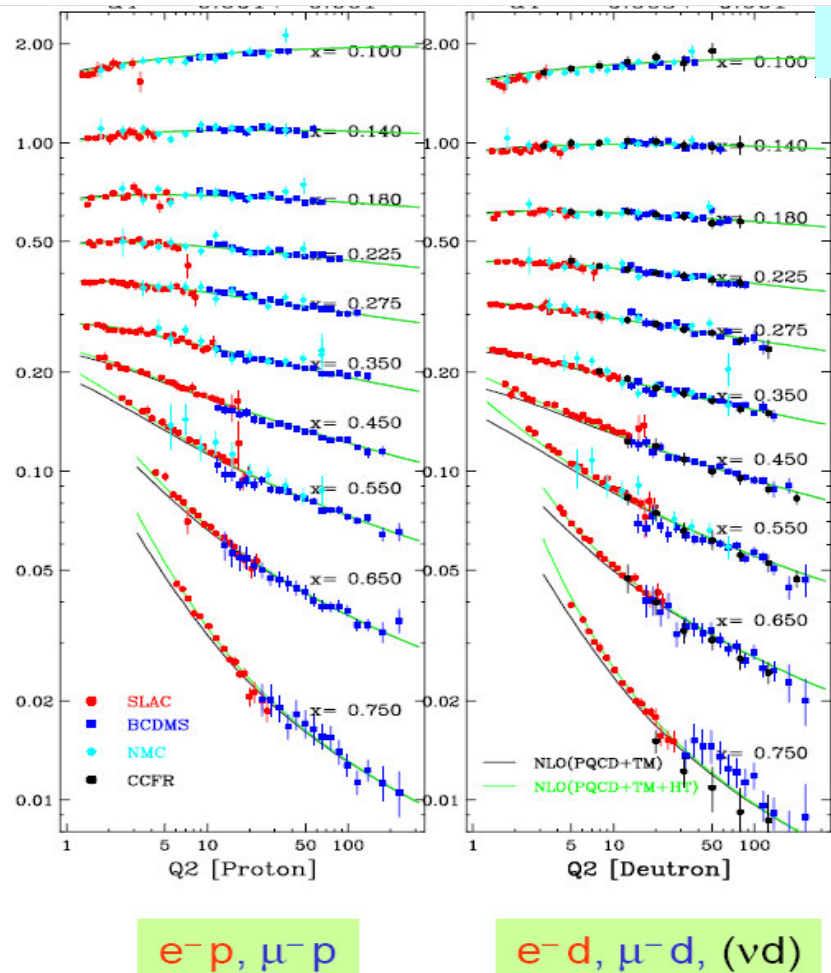


@TBA



As compared to fast varying elastic f.f., just astonishing...

Scaling Violations



@TBA

Extensive compilation
Data from fixed target experiments
Observe:

Electron vs. *Muon*

Red points are from electron DIS

Blue points are from muon DIS

Muon merits:

Easier to get to high energy

Reduced radiative corrections

Muon drawbacks:

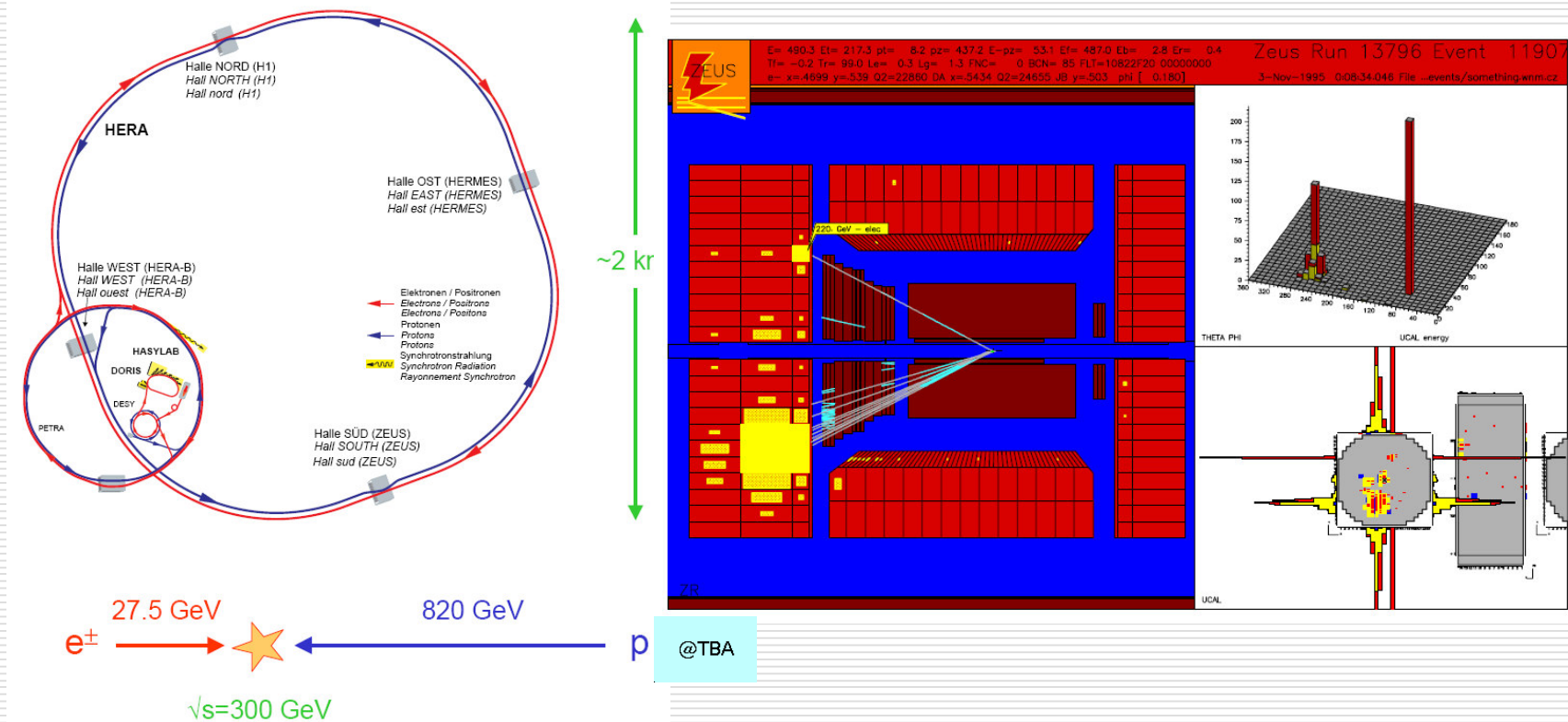
Intensity

Proton (L) vs. Deuteron (R)

Get *neutron* structure function

HERA

First example of *asymmetric* collider



A Recent Compilation

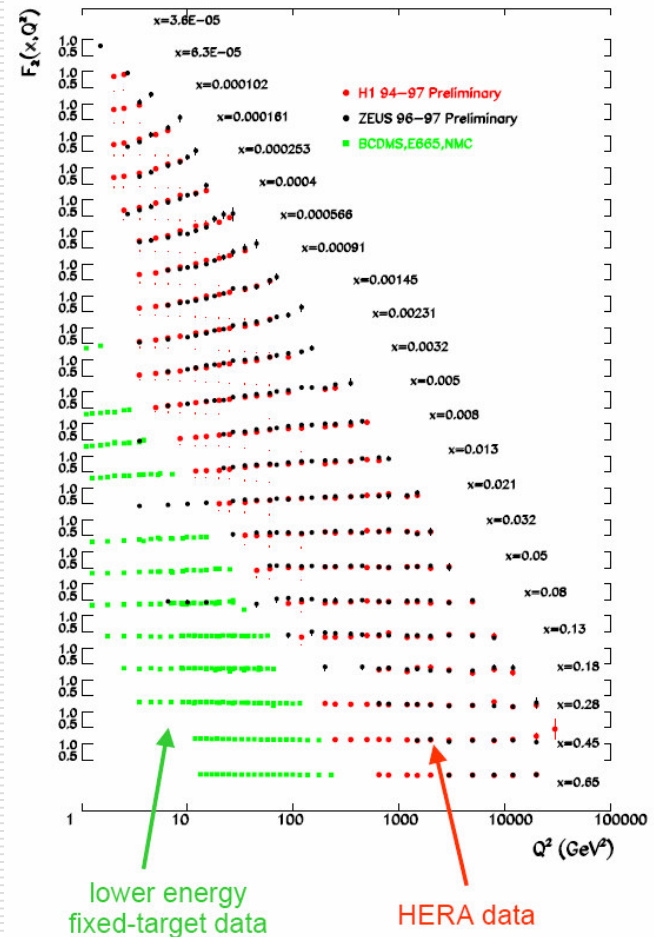
Results from several experiments:

muon DIS NMC, BCDMS, E665
CERN FNAL
electron DIS at HERA collider
DESY

Huge q^2 , x range

Small, measurable scaling violation
Interesting features at small x

→ QCD !



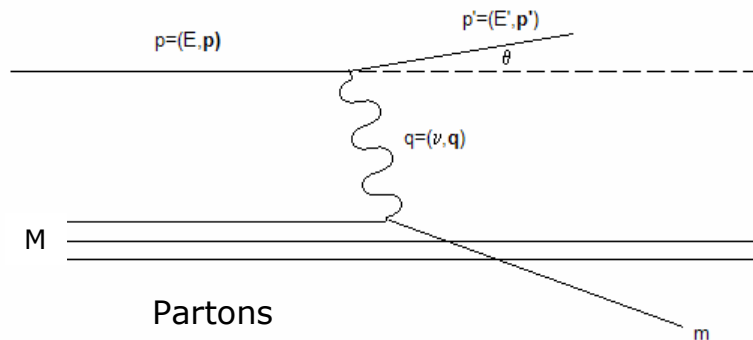
Parton Model - I

Structure functions generalise form factors

Form factor of a point source = *constant*

Feynman suggestion:

Maybe deep inelastic scaling just indicates elastic scattering off free, pointlike constituents



Differential cross-section for elastic scattering off a free, pointlike constituent of mass m

$$\left. \frac{d\sigma}{d\Omega} \right|_{LAB} = \frac{\alpha^2}{4|\mathbf{p}|^2 \sin^4 \theta/2} \cos^2 \theta/2 \frac{|\mathbf{p}'|}{|\mathbf{p}|} \left(1 - \frac{q^2 \tan^2 \theta/2}{2m^2} \right)$$

$$= \frac{\alpha^2}{4|\mathbf{p}|^2 \sin^4 \theta/2} \frac{|\mathbf{p}'|}{|\mathbf{p}|} \left(\cos^2 \theta/2 - \frac{q^2}{2m^2} \sin^2 \theta/2 \right)$$

$$m^2 = (m + \nu, \mathbf{q})^2 = m^2 + 2m\nu + \underbrace{\nu^2 - |\mathbf{q}|^2}_{=-q^2}$$

$$\rightarrow \nu + \frac{q^2}{2m} = 0$$

Parton Model - II

In the elastic scattering off a parton, energy and angle of the scattered electron are fully correlated

Formally write the differential cross section as

Full E' , θ correlation

z : Parton charge, units e

$$\frac{d\sigma}{d\Omega} = \int dE' \frac{d^2\sigma}{dE' d\Omega} = \int dE' \frac{\alpha^2 z^2}{4E^2 \sin^4 \theta/2} \left(\cos^2 \theta/2 - \frac{q^2}{2m^2} \sin^2 \theta/2 \right) \delta \left(\nu + \frac{q^2}{2m} \right)$$

$$\begin{cases} \nu = E - E' \\ q^2 = -4EE' \sin^2 \theta/2 \end{cases} \rightarrow E - E' = \frac{4EE' \sin^2 \theta/2}{2m} \rightarrow E' \left(1 + \frac{4E}{2m} \sin^2 \theta/2 \right) = E$$

$$\rightarrow E' = \frac{E}{1 + \frac{4E}{2m} \sin^2 \theta/2}$$

$$\nu + \frac{q^2}{2m} = 0, x = -\frac{q^2}{2M\nu} \rightarrow x = \frac{m}{M}$$

$$\frac{d^2\sigma}{dE' d\Omega} = \frac{\alpha^2 z^2}{4E^2 \sin^2 \theta/2} \left(\cos^2 \theta/2 - \frac{q^2}{2M^2 x^2} \sin^2 \theta/2 \right) \delta \left(\nu + \frac{q^2}{2Mx} \right)$$

Parton Model - III

Summing over all types of partons

$$\frac{d^2\sigma}{dE'd\Omega} = \frac{\alpha^2}{4E^2 \sin^2 \theta/2} \left(\cos^2 \theta/2 - \frac{q^2}{2M^2 x^2} \sin^2 \theta/2 \right) \delta \left(\nu + \frac{q^2}{2Mx} \right) \left(\sum_i z_i^2 n_i \right)$$

Compare to inelastic cross-section

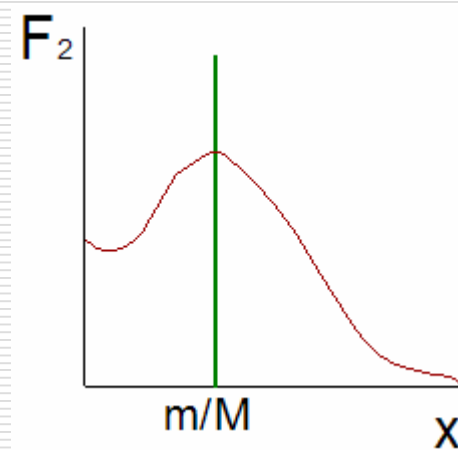
$$\left. \frac{d\sigma}{d\Omega dE'} \right|_{LAB} = \frac{\alpha^2}{4E^2 \sin^4 \theta/2} \left(W_2(\nu, q^2) \cos^2 \theta/2 + W_1(\nu, q^2) \sin^2 \theta/2 \right)$$

Then for structure functions:

$$\rightarrow \begin{cases} W_2 = \frac{F_2}{\nu} = \left(\sum_i z_i^2 n_i \right) \delta \left(\nu + \frac{q^2}{2Mx} \right) \\ W_1 = \frac{2F_1}{M} = \left(\sum_i z_i^2 n_i \right) \left(\frac{-q^2}{2M^2 x^2} \right) \delta \left(\nu + \frac{q^2}{2Mx} \right) \end{cases}$$

$$\begin{aligned} \rightarrow F_2 &= \nu \left(\sum_i z_i^2 n_i \right) \delta \left(\nu + \frac{q^2}{2Mx} \right) = \left(\sum_i z_i^2 n_i \right) \delta \left(1 + \frac{q^2}{2Mx\nu} \right) \\ &= \left(\sum_i z_i^2 n_i \right) x \delta \left(x + \frac{q^2}{2M\nu} \right) = \left(\sum_i z_i^2 n_i \right) x \delta \left(x - \frac{m}{M} \right) \end{aligned}$$

Parton model prediction
Actual shape



Parton Model - IV

The true meaning of x

P, q : proton, virtual photon 4-momenta

$$x = \frac{Q^2}{2M\nu} = \frac{Q^2}{2P \cdot q} \quad \text{invariant}$$

$$x = \frac{m}{M}$$

$$m^2 = x^2 M^2 = x^2 (E^2 - \mathbf{p}^2)$$

$$m^2 = (E_{parton}, \mathbf{p}_{parton})^2 = E_{parton}^2 - \mathbf{p}_{parton}^2$$

$$(E_{parton}, \mathbf{p}_{parton}) = x \cdot (E, \mathbf{P})$$

$$\rightarrow \mathbf{p}_{parton} \approx x \cdot \mathbf{P} \quad \text{when } m \ll |\mathbf{p}|$$

$$P = \left(E_p, \underset{\text{comp.transversa}}{\mathbf{0}}, |\mathbf{P}| \right), \quad E_p = \sqrt{M^2 + |\mathbf{P}|^2} \approx |\mathbf{P}|$$

$$q = (E_\gamma, \mathbf{q}_T, 0)$$

$$\rightarrow P \cdot q \approx |\mathbf{P}| E_\gamma$$

$$\rightarrow E_\gamma \approx \frac{P \cdot q}{|\mathbf{P}|} = \frac{Q^2}{2x|\mathbf{P}|}$$

Lot of insight in this limit (Feynman):
(Proton) Infinite Momentum Frame

$\beta \rightarrow 1 \Rightarrow \gamma \rightarrow \infty$ Large time dilation

Time constants of internal motions:

$\tau \rightarrow \infty$ in the IMF

Constituents seen as *still* by the DIS virtual photon

Use time-energy indeterminacy relation:

$$\tau_0 \sim \frac{1}{E_\gamma} \approx \frac{2x|\mathbf{P}|}{Q^2} \quad \text{DIS time scale}$$

$$\tau \sim \frac{1}{\Delta E} \approx \frac{2x|\mathbf{P}|}{p_T^2} \quad \text{Constituents motion time scale}$$

$$\rightarrow \frac{\tau_0}{\tau} \approx \frac{p_T^2}{Q^2} \sim 0$$

$\rightarrow$ No binding effects

$\rightarrow$ Free constituents OK

Therefore:

x is the momentum fraction carried by the struck parton

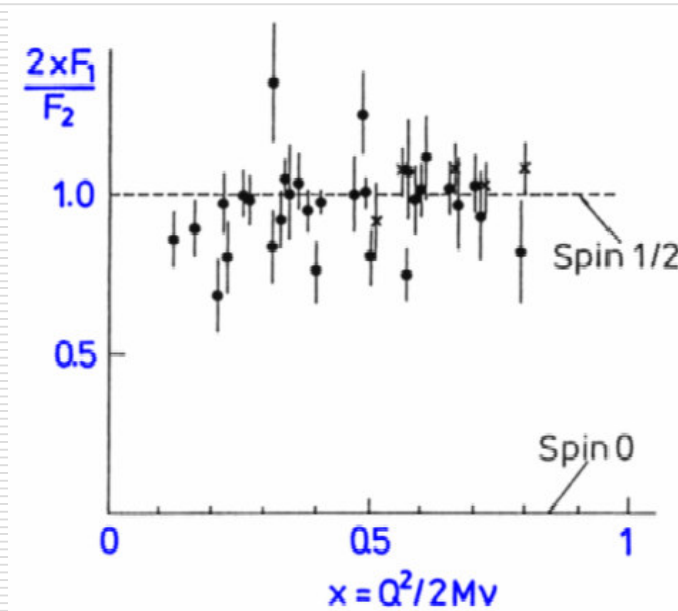
Parton Model - V

Callan-Gross relation for spin 1/2 partons

$$\left\{ \begin{aligned} \frac{F_2}{\nu} &= \left(\sum_i z_i^2 n_i \right) \delta \left(\nu + \frac{q^2}{2Mx} \right) \\ \frac{2F_1}{M} &= \left(\sum_i z_i^2 n_i \right) \left(\frac{-q^2}{2M^2 x^2} \right) \delta \left(\nu + \frac{q^2}{2Mx} \right) \end{aligned} \right.$$

$$\rightarrow \frac{F_2}{\nu} = \frac{2F_1}{M} \frac{2M^2 x^2}{-q^2} = \frac{2F_1}{M} \frac{2M^2 x^2}{2M \nu x} = \frac{2F_1 x}{\nu}$$

$$\rightarrow F_2 = 2F_1 x$$



Parton Model - VI

Several unanswered questions...

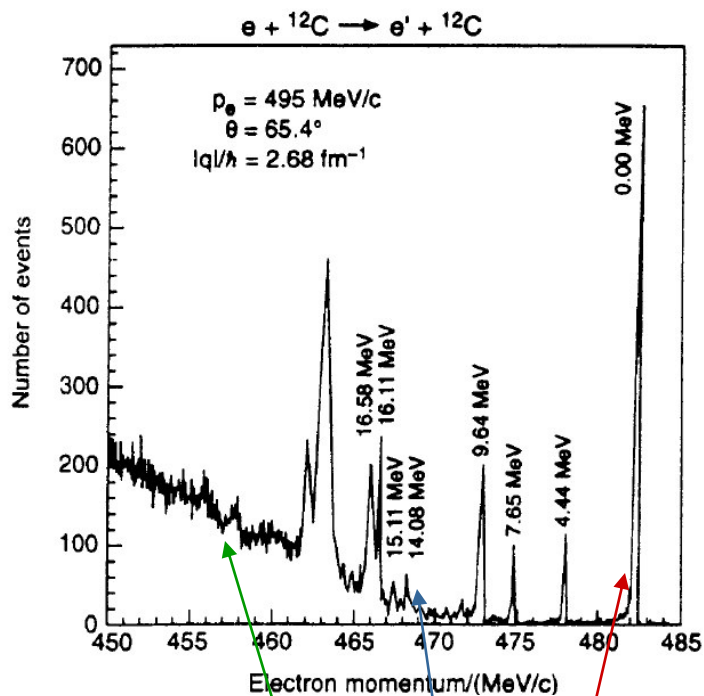
Most important issues:

One does not observe any free constituent out of the collision

*Constituents seem to be essentially free (as partons)
and tightly bound (as never observed free outside the nucleon)
at the same time*

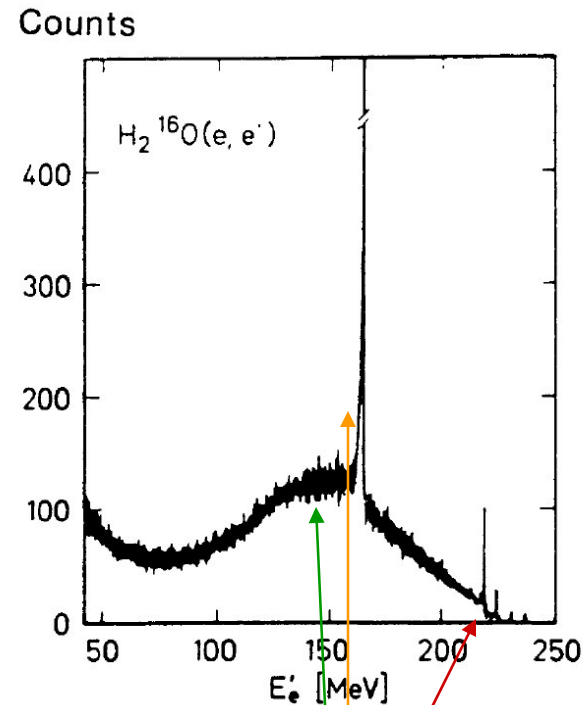
For some time, this point was believed to rule out any constituent model

The 'Nuclear Parton Model'



Scattering off ${}^{12}\text{C}$

Elastic scattering off the *whole nucleus*
 Nuclear levels excitation (inelastic)
 Beginning of elastic scattering off *protons*



Scattering off water

Elastic scattering off the *whole ${}^{16}\text{O}$ nucleus*
 Nuclear levels excitation (inelastic)
 Elastic scattering off ${}^{16}\text{O}$ *protons*
 Elastic scattering off *free protons*

The Quark Parton Model

Write down F_2 in terms of PDFs

$$F_2 = \left(\sum_i z_i^2 n_i \right) x \delta \left(x - \frac{m}{M} \right)$$

$$F_2(x) = x \left(\sum_i z_i^2 q_i(x) \right)$$

$$F_2^p(x) = x \left[\left(\frac{2}{3} \right)^2 u_p(x) + \left(-\frac{1}{3} \right)^2 d_p(x) \right]; \quad p = uud$$

$$\rightarrow F_2^p(x) = x \left[\frac{4}{9} u_p(x) + \frac{1}{9} d_p(x) \right]$$

$$F_2^n(x) = x \left[\left(-\frac{1}{3} \right)^2 d_n(x) + \left(\frac{2}{3} \right)^2 u_n(x) \right]; \quad n = ddu$$

From isospin symmetry:

$$F_2^n(x) = x \left[\left(-\frac{1}{3} \right)^2 u_p(x) + \left(\frac{2}{3} \right)^2 d_p(x) \right]$$

$$\rightarrow F_2^n(x) = x \left[\frac{1}{9} u_p(x) + \frac{4}{9} d_p(x) \right]$$

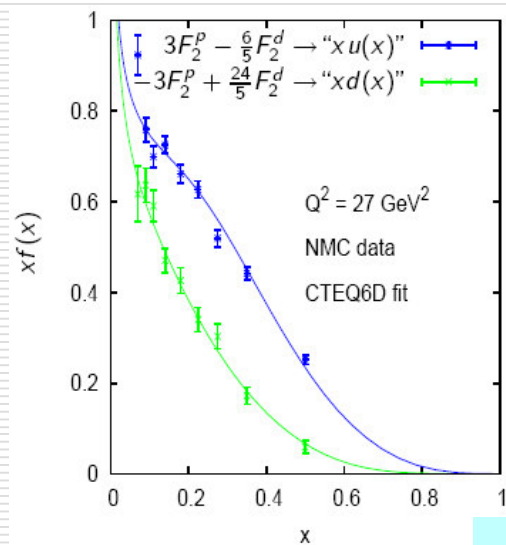
Consider the deuteron structure function:

$$\begin{aligned} F_2^d(x) &= \frac{1}{2} (F_2^p + F_2^n) = \frac{5}{9} x [u_p(x) + d_p(x)] \\ \rightarrow F_2^n(x) &= F_2^d(x) - F_2^p(x) \\ &= \frac{5}{18} x [u_p(x) + d_p(x)] - \frac{1}{9} x [u_p(x) + 4d_p(x)] \\ &= \frac{3}{18} x [u_p(x) - d_p(x)] \end{aligned}$$

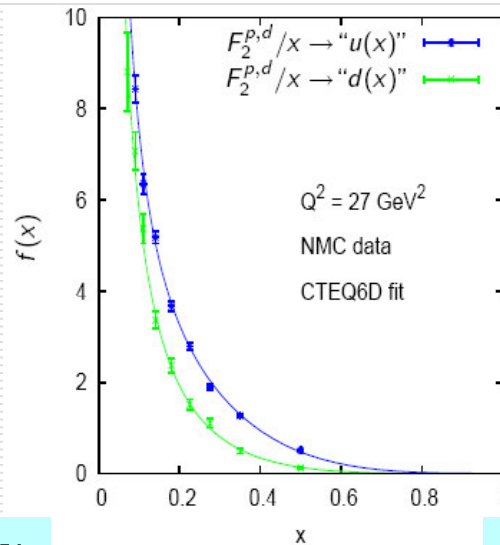
Finally extract PDFs from measured F_2

$$\begin{aligned} xu_p(x) &= xd_n(x) = 3F_2^p(x) - \frac{6}{5} F_2^d(x) \\ xu_n(x) &= xd_p(x) = 3F_2^n(x) + \frac{24}{5} F_2^d(x) \end{aligned}$$

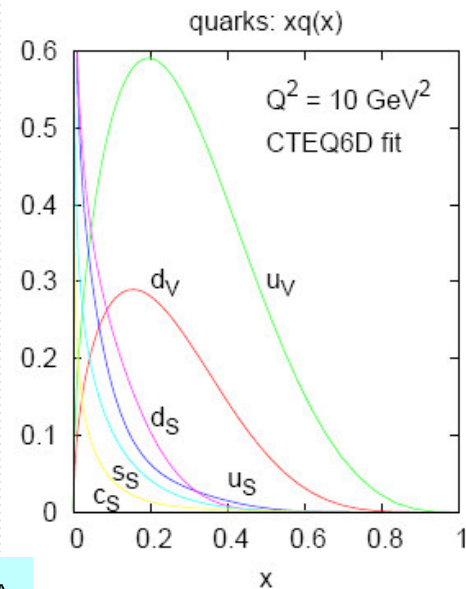
The Parton Distribution Functions



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Parton model predictions: *Sum Rules* (= Integral relations) for PDFs

Examples: Proton quark content is uud

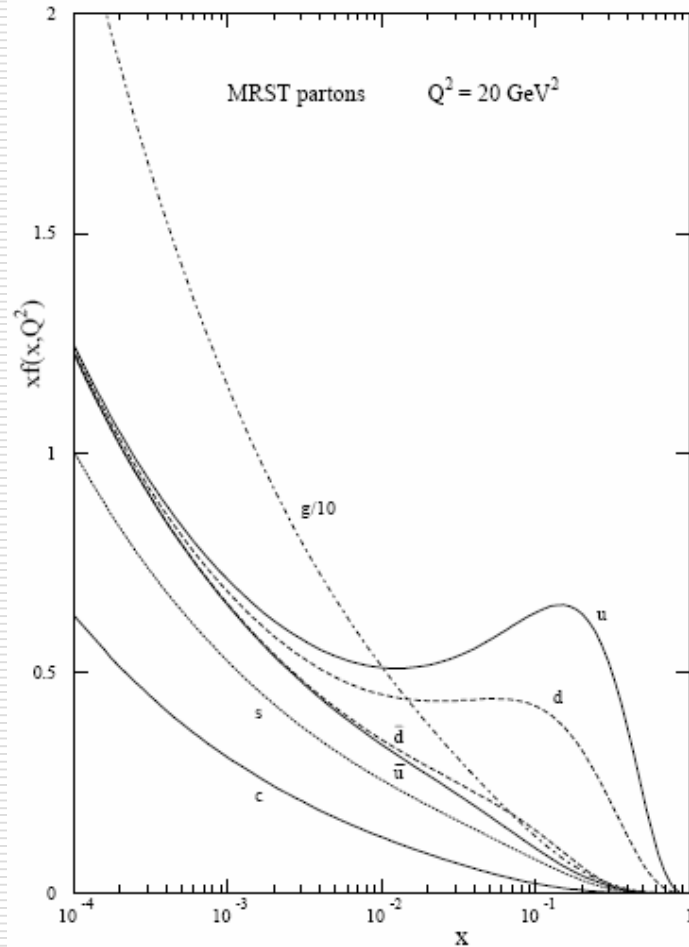
$$\int [u_p(x) - \bar{u}_p(x)] dx = 2$$

$$\int [d_p(x) - \bar{d}_p(x)] dx = 1$$

$$\int [s_p(x) - \bar{s}_p(x)] dx = 0$$

What's the origin of antiquarks in the nucleon?
QCD! See later..

The PDFs at Low x

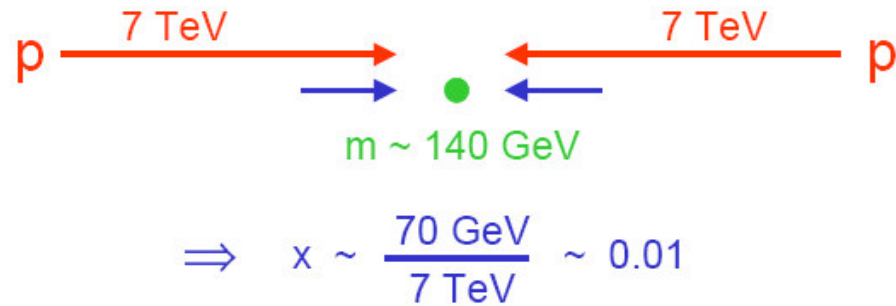


Data-based calculation

Low- x region very important at LHC

Example:

Production of a Higgs with $m_H = 140 \text{ GeV}$



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