Elementary Particles I

3 – Structure II: Quarks

Strong Interaction

Originally pictured as an attractive force between nucleons, required to overcome repulsive Coulomb interaction between protons in nuclei

Main features:

Strength
Short range
Charge independence

Several, rather complicated features (repulsive core, many body effects,…)
For a long time, difficult to understand: lot of guesswork, many models

Today, believed to be a *residual force* between ‘color neutral’ particles (*hadrons*), a remnant of color interaction between colored quarks and gluons

Somewhat similar to Van der Waals/Covalent bond between ‘neutral’ molecules, coming from electromagnetic interaction between charged electrons and nuclei

Yukawa Theory

First attempt to model strong interaction after the electromagnetic:
Exchange of mediator particles $\rightarrow$ Prediction of *pion*

Mass > 0 Limited range

Spin $\neq 1$ Vector particle would yield repulsive forces between identical particle

Charged,Neutral Same force for $pp,\ nn,\ pn$

Electromagnetism

$\frac{\partial^2\varphi}{\partial t^2} - \nabla^2\varphi = -\rho$ Wave equation - Scalar potential

$\nabla^2\varphi = \rho$ Static case

$\rho_G(\mathbf{r}) = e\delta(\mathbf{r})$ Point source at the origin

$\rightarrow \varphi_G(\mathbf{r}) = \frac{e}{r}$ Green's function $\equiv$ Coulomb potential

Yukawa

$\frac{\partial^2\varphi}{\partial t^2} - \nabla^2\varphi - m^2 = -\rho$ Wave equation - Pion field

$\nabla^2\varphi + m^2 = \rho$ Static case

$\rho_G(\mathbf{r}) = g\delta(\mathbf{r})$ Point source at the origin

$\rightarrow \varphi_G(\mathbf{r}) = \frac{g\ e^{-mr}}{r}$ Green's function $\equiv$ Yukawa potential

Pions

Discovered after the II World War (Cosmic Rays, Accelerators)
Properties

Mass	135 MeV	Neutral
	139 MeV	Charged
Spin	0	
Parity	-	
Charge parity	+	
Lifetime	$25\ 10^{-9}$ s	Charged
	10^{-16} s	Neutral
Decay modes (Dominant)	$\mu\nu$	Charged
	$\gamma\gamma$	Neutral

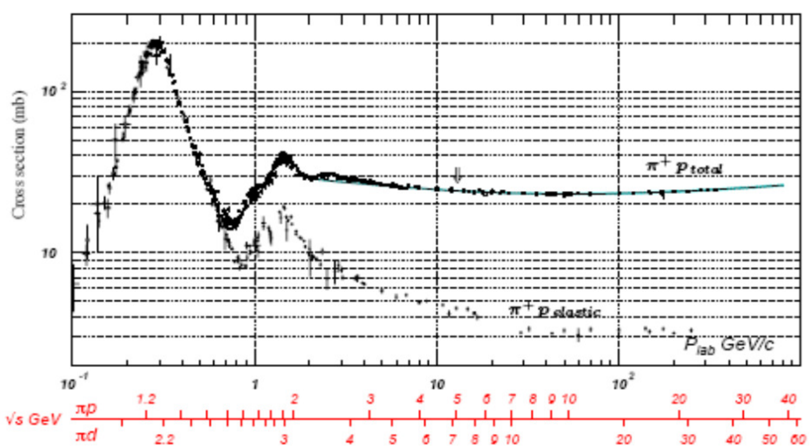
Stable vs. strong decays, as the *lightest hadron*
Copiously produced at first accelerators (synchrocylotrons)
Charged pions easily focused into collimated, high energy beams

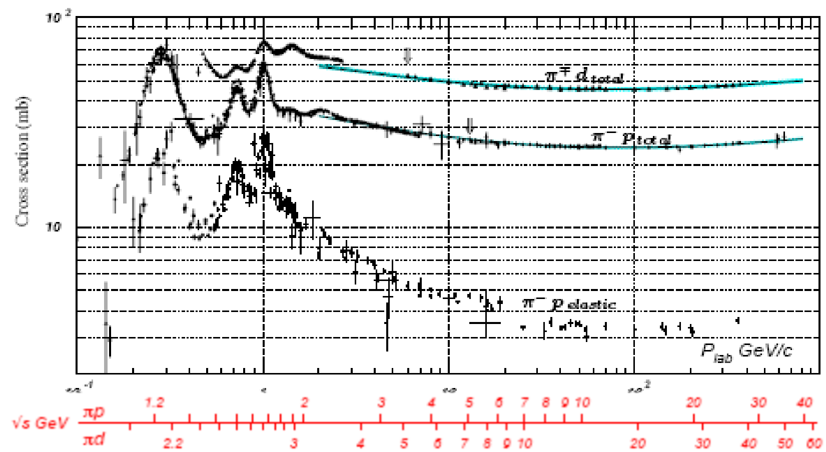
Scattering

As for electromagnetic, strong interaction can be investigated by scattering experiments
Perform experiments like

$$p+p, \quad p+n, \quad \pi^{\pm}+p, \quad \pi^{\pm}+n$$

Pion: Spinless $\rightarrow$ Understanding πN scattering easier than NN



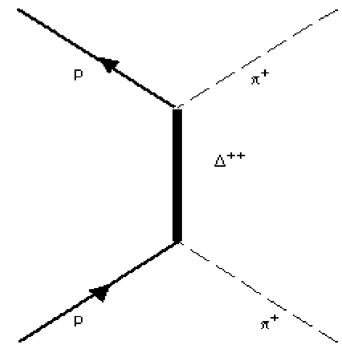


Total cross section plots - Observe lot of structure

Δ-Resonance: Formation

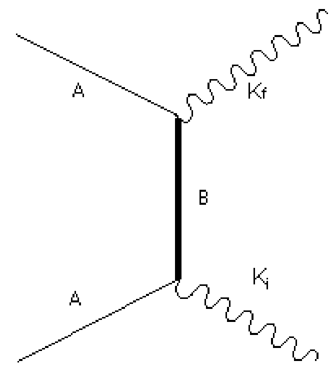
First observed by Fermi and collaborators in πN scattering (1951)

$$\pi^+ + p \to \Delta^{++} \to \pi^+ + p$$



With some caveats, can be considered as a kind of excited nucleon state (But: Different spin, quark content)

Also observed in other charge states Δ^+, Δ^-, Δ^0 and in many different processes (strong, e.m. and weak)



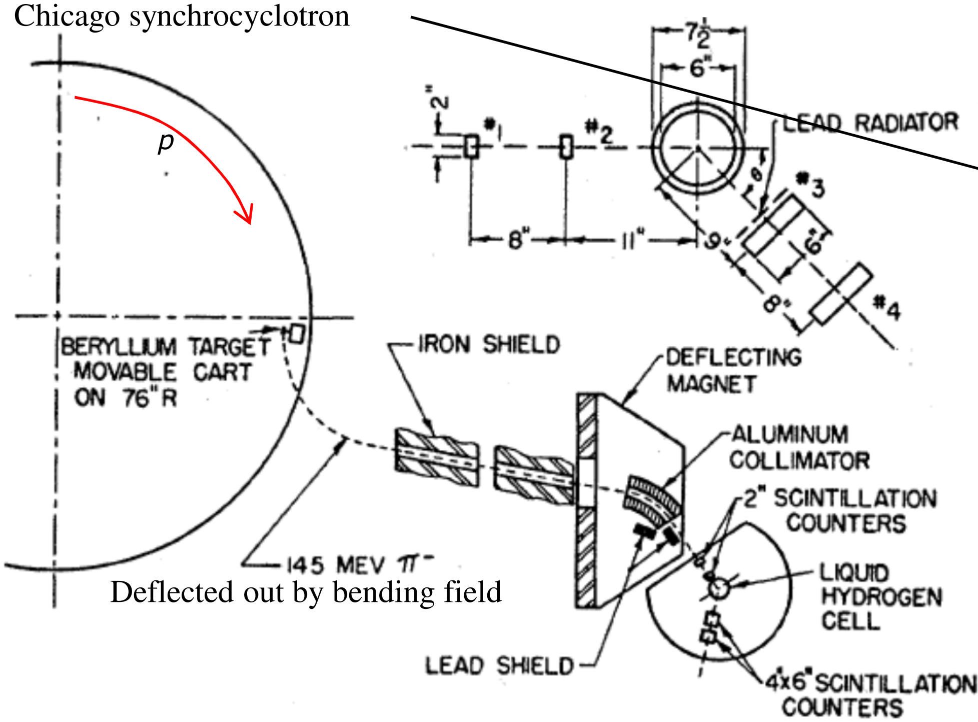
Some analogy with photon excitation of atomic levels

$$\gamma + A \to B \to \gamma + A, \quad \text{A ground state, B excited level}$$

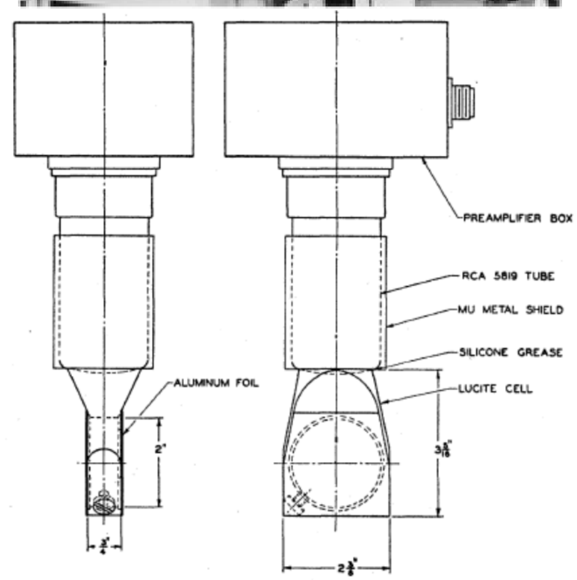
Good indication that the nucleon is a *composite* object

Discovery of Δ - 1951

Chicago synchrocyclotron



Deflected out by bending field



Collect first data on 2-body reactions:

$$\pi^+ + p \to \pi^+ + p$$

$$\pi^- + p \to \pi^- + p$$

$$\pi^- + p \to \pi^0 + n$$

Plastic scintillators

Δ^{++} Resonance

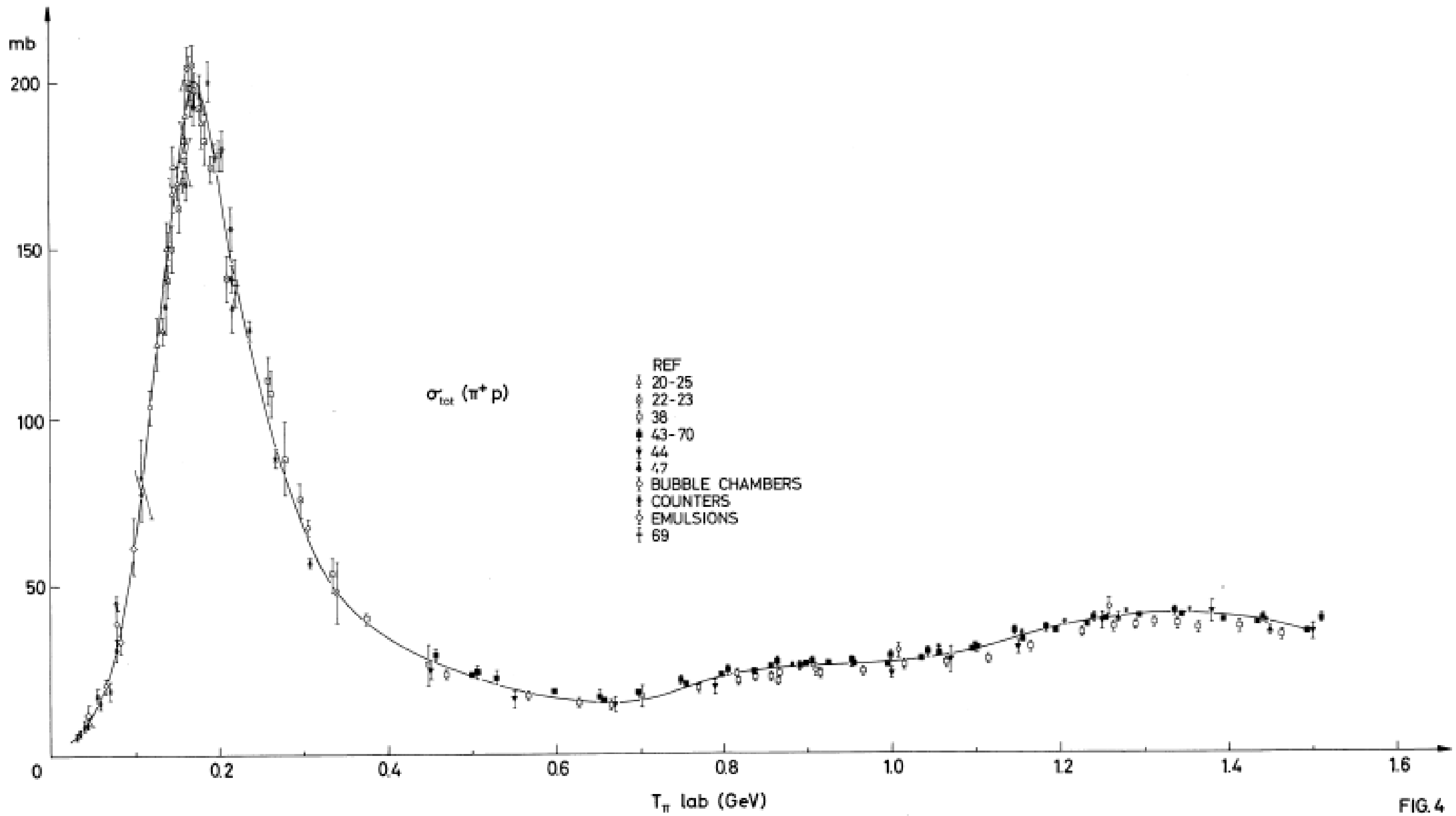


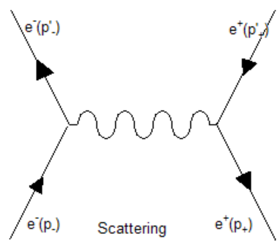
FIG. 4

Propagators

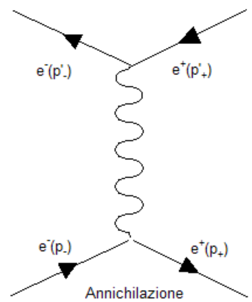
Take first a QED example: Bhabha scattering at $\sqrt{s} \ll M_{Z^0}$

$$e^- + e^+ \to e^- + e^+$$

Two one-photon diagrams



t-channel



s-channel

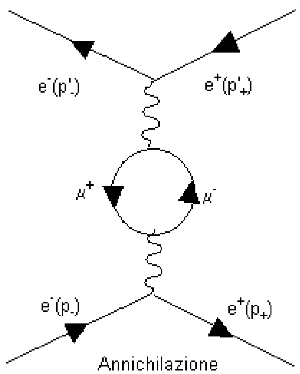
t-channel: Virtual photon has $q^2 < 0$ space-like
s-channel: Virtual photon has $q^2 > 0$ time-like

In both cases : Virtual photon propagator $= \dfrac{1}{q^2}$

Propagators in the *s*-channel - I

Taking radiative corrections to one loop:

$$\text{Virtual photon propagator} = \frac{1}{q^2\left(1-\overline{\Pi}_\gamma^{(2)}\left(q^2\right)\right)}$$



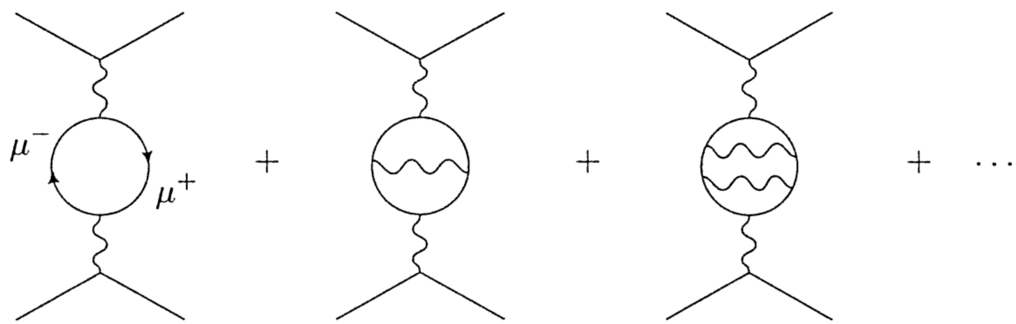
Correction resulting from fermion e.m. currents circulating in the loop, after renormalization
In principle: All fermion loops, leptons & quarks, should be included

$q^2 > 4m_f^2 \to \overline{\Pi}_\gamma^{(2)}\left(q^2\right)$ becomes *complex*

Nonzero amplitude for the virtual photon to materialize as a $f\,\bar{f}$ pair *on-shell*

Propagators in the *s*-channel - II

Among all fermion circulating in the loop, take a muon pair
Taking further perturbative expansion :



Higher order diagrams: Usually negligible

When $\sqrt{s} = E_{CM} \sim M_{bound} \equiv M$, Coulomb attractive force between muons very strong
→ Higher order diagrams large

Naive understanding:

A $\mu^+\mu^-$ pair has bound states, like a hydrogen atom
When $E_{CM} \approx M$: large amplitude for the scattering process to yield a $\mu^+\mu^-$ bound state

Propagators in the *s*-channel - III

Imaginary part tied to bound state being *unstable*:
Unlike the *H* atom, muonic atom annihilates into various channels

$$q^2 \sim M^2 \quad \to \overline{\Pi}_\gamma^{(2)}\left(q^2\right) \approx \frac{M^2 - iM\Gamma}{q^2}$$

$$\to \frac{1}{q^2\left(1-\overline{\Pi}_\gamma^{(2)}\left(q^2\right)\right)} \approx \frac{1}{q^2 - M^2 + iM\Gamma}$$

Propagator of a massive, unstable particle

$$q^2 = s = E_{CM}^2 \quad \to \frac{1}{q^2\left(1-\overline{\Pi}_\gamma^{(2)}\left(q^2\right)\right)} \approx \frac{1}{q^2 - M\left(M - i\Gamma\right)} = \frac{1}{E_{CM}^2 - M^2 + iM\Gamma}$$

$$\to \frac{1}{q^2\left(1-\overline{\Pi}_\gamma^{(2)}\left(q^2\right)\right)} \approx \frac{1}{\left(E_{CM} - M\right)\underbrace{\left(E_{CM} + M\right)}_{\approx 2M} + iM\Gamma}$$

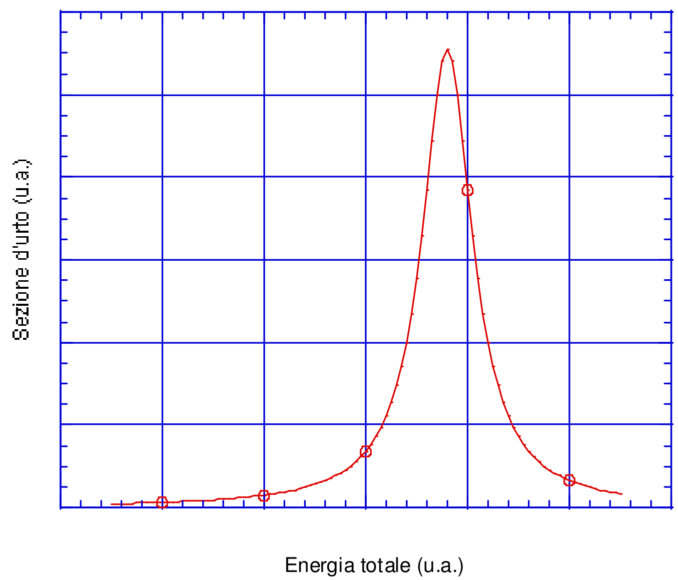
$$\to \frac{1}{q^2\left(1-\overline{\Pi}_\gamma^{(2)}\left(q^2\right)\right)} \approx \frac{1}{2M}\frac{1}{\left(E_{CM} - M\right) + i\Gamma/2}$$

Total cross section: Strongly peaked at $E_{CM} \approx M$

Propagators in the *s*-channel - IV

General rule:

Every time the intermediate state is coupled to an unstable state (excited bound state, genuine elementary particle coupled to decay channels, …), the s-channel propagator and cross section show resonant behavior when the total energy is close to the mass of the unstable state



Propagators in the *t*-channel - I

The same propagator describes the *t*-channel amplitude, $t=q^2<0$:

$$\frac{1}{q^2\left(1-\overline{\Pi}_\gamma^{(2)}\left(q^2\right)\right)} \approx \frac{1}{q^2-M\left(M-i\Gamma\right)} \underset{\Gamma\ll M}{\approx} \frac{1}{q^2-M^2}$$ 'Pole' amplitude

In this case, there is *no* resonant behavior: $q^2-M^2<0$ strictly

Rather, the amplitude can be seen as an extension of the virtual photon idea, corresponding to the exchange of a *virtual particle*, with mass M and width Γ, or lifetime $1/\Gamma$. In the previous example, it would be a *virtual muonic atom.*

As for the virtual photon , the virtual particle exchanged is said to be *off mass-shell*: $q^2 \neq M^2$

Largest contribution from lightest (virtual) particles:

Exchange of virtual pions dominating at low q^2

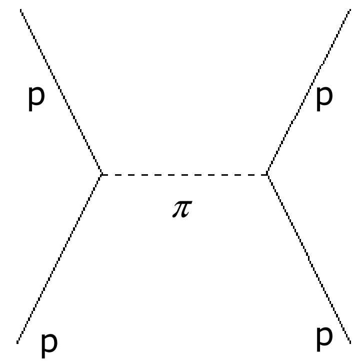
Propagators in the *t*-channel - II

Take *NN* scattering at small q^2 as dominated by *one pion exchange*:
This *can* be maintained, to some extent (or so one believes).
Then

$$A \propto \frac{1}{q^2 - m_\pi^2}$$

In the static potential limit

$$E_C \approx E_A$$
$$q^2 = (E_C - E_A)^2 - (\mathbf{p}_C - \mathbf{p}_A)^2 \approx -(\mathbf{p}_C - \mathbf{p}_A)^2 = -|\mathbf{q}|^2$$
$$\to \frac{1}{q^2 - m_\pi^2} \approx \frac{1}{-|\mathbf{q}|^2 - m_\pi^2} = -\frac{1}{|\mathbf{q}|^2 + m_\pi^2}$$



Assuming Born approximation

$$V(r) \propto \int e^{i\mathbf{q}\cdot\mathbf{r}} \left(-\frac{1}{|\mathbf{q}|^2 + m_\pi^2} \right) d^3\mathbf{q} \propto -\frac{e^{-m_\pi r}}{r} \quad \text{Yukawa potential}$$

$\to$ Potential scattering formalism useful

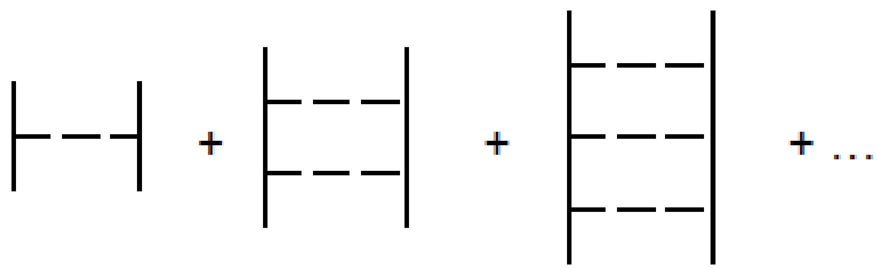
Propagators in the *t*-channel - III

Very appealing as a qualitative visualization of processes
Also superficially consistent with perturbative expansion:
Just include diagrams with 2,3,… virtual particles

But:

…Unfortunately not very useful as a tool for quantitative work in strong interactions physics: perturbative expansion cannot be maintained for large coupling constant ...

Most simply: Diagrams with more than one particle exchanged yielding amplitudes *larger* than diagrams with just one



$$|A_1| \quad < \quad |A_2| \quad < \quad |A_3| < \ldots$$

Resonances - I

Generalize concept of stationary state:

$$\psi(\mathbf{r},t)=\varphi(\mathbf{r})e^{-iE_0t} \rightarrow \int_{-\infty}^{+\infty} e^{-iE_0t}e^{iEt}dt=\delta(E-E_0)$$

(Amplitude to find energy E when system is prepared in the state ψ)

to a kind of non-stationary, decaying state

Complex E: Just meaning "System is unstable"

$$e^{-iEt}=e^{-i(E_0-i\Gamma)t}=e^{-iE_0t}e^{-\Gamma t}, \quad t > 0$$

$$\int_0^{+\infty} e^{-i(E_0-i\Gamma)t}e^{iEt}dt=\int_0^{+\infty} e^{-i(E_0-E-i\Gamma)t}dt=-\frac{1}{i(E_0-E-i\Gamma)}e^{-i(E_0-E-i\Gamma)t}\Big|_0^{+\infty}=\frac{i}{(E-E_0+i\Gamma)}$$

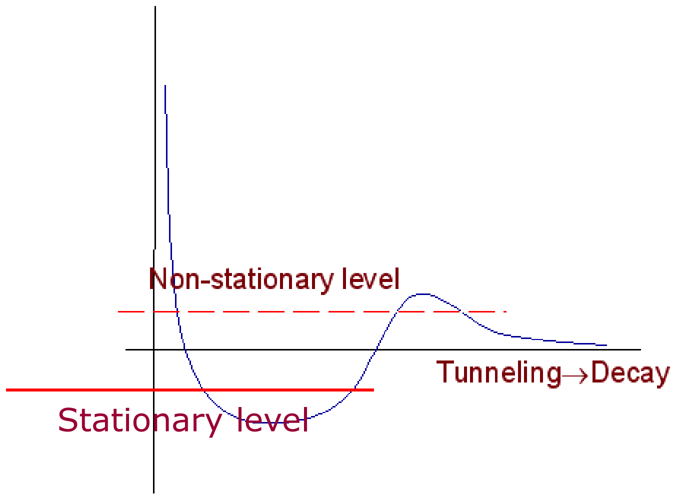
(Breit-Wigner:
Amplitude to find energy E when system prepared in the state ψ)

$$|\psi|^2 \propto \left|\frac{i}{E-E_0+i\Gamma}\right|^2=\left|\frac{E-E_0-i\Gamma}{(E-E_0)^2+\Gamma^2}\right|^2=\frac{(E-E_0-i\Gamma)(E-E_0+i\Gamma)}{\left[(E-E_0)^2+\Gamma^2\right]^2}=\frac{\left((E-E_0)^2+\Gamma^2\right)}{\left[(E-E_0)^2+\Gamma^2\right]^2}=\frac{1}{(E-E_0)^2+\Gamma^2}$$

Resonances - II

Non-stationary levels may result from a particular shape of the effective potential
Non stationary, scattering state
But: *Almost* stationary…

Long lifetime, sharp quantum numbers: Like a *stable* state (Bohr, '30s)



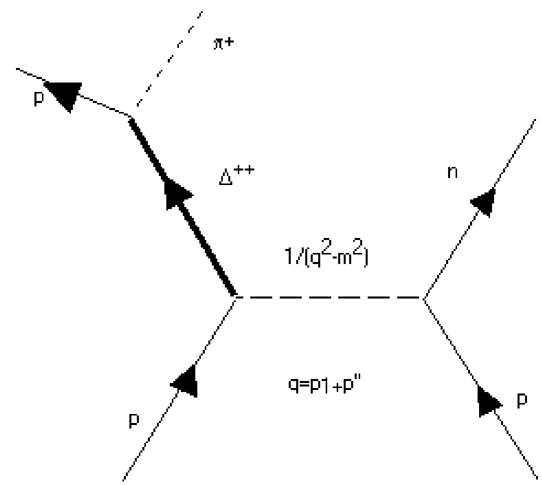
$$\Gamma = \left.\begin{array}{c} \text{1/time constant of decaying state} \approx \text{time uncertainty} \\ \text{Half width at half maximum} \approx \text{energy uncertainty} \end{array}\right\} \rightarrow \Delta E \Delta t \sim \Gamma \frac{1}{\Gamma} = 1$$

Resonances - III

With higher energy beams available, new processes become possible.
Use *virtual pions* to excite nucleon levels

$$p + p \to n + \Delta^{++} \to n + p + \pi^{+}$$

Resonance is *produced* in the *t*-channel, rather than *formed* in the *s*-channel.



Strong interaction between exchanged *virtual* pion and real proton similar to interaction between *real* pion and proton

Not directly observed in the cross-section vs. energy plot.

But: Resonance mass and quantum numbers are *invariant* properties, like the corresponding quantities of a stable particle

Resonances - IV

Expect resonant behavior also for mesonic systems, e.g. $\pi\pi$:
Virtual and real pion coupled at the strong vertex

Observation of meson resonances possible in production experiments

Remark:

Taking baryon resonances only, possible isospin:

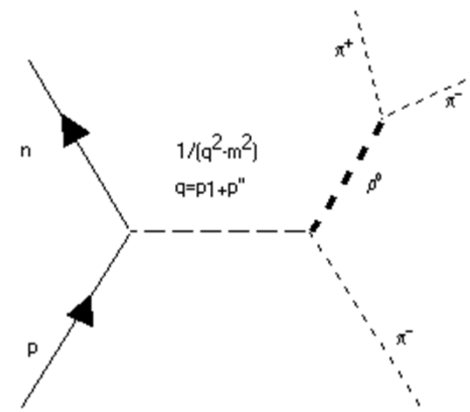
Minimum coupling is between nucleon and pion

$\rightarrow$ Expect $1 \oplus 1/2 = 1/2, 3/2$ as observed

Take meson resonances:

Minimum coupling is between pion and pion

$\rightarrow$ Expect $1 \oplus 1 = 0, 1, 2$ I=2 mesons not observed



Resonances - V

Take reaction

$$\pi^- + p \rightarrow n + \pi^+ + \pi^-$$

Observe strong enhancements for

$$m_{\pi\pi} \sim 760,\ 1260,\ 1550 \quad MeV$$
$$m_{\pi n} \sim 1230 - 1550 \quad MeV$$

Interpretation:

Meson Baryon Resonances

$$\left.\begin{array}{l} \rho(760) \\ f_0(1250) \\ g(1550) \end{array}\right\} \rightarrow \pi^{\pm}\pi^{\mp}, \quad \Delta^{+,-}(1232) \rightarrow n\pi^{\pm}$$

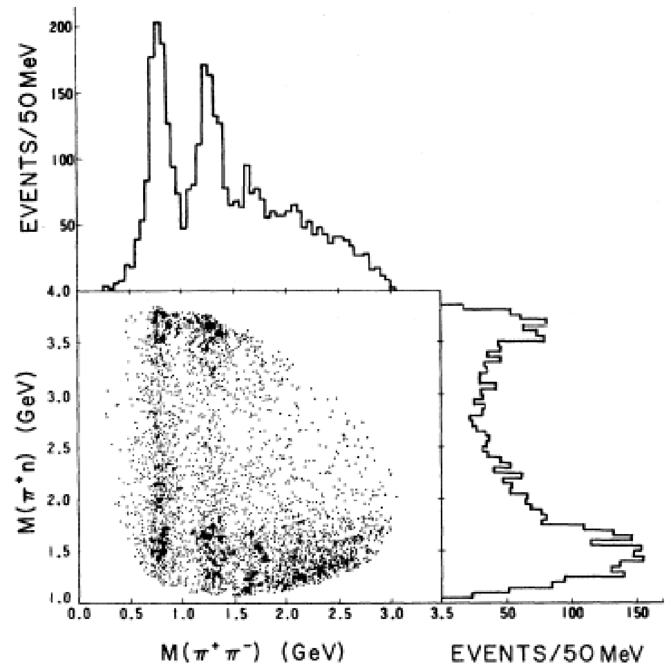
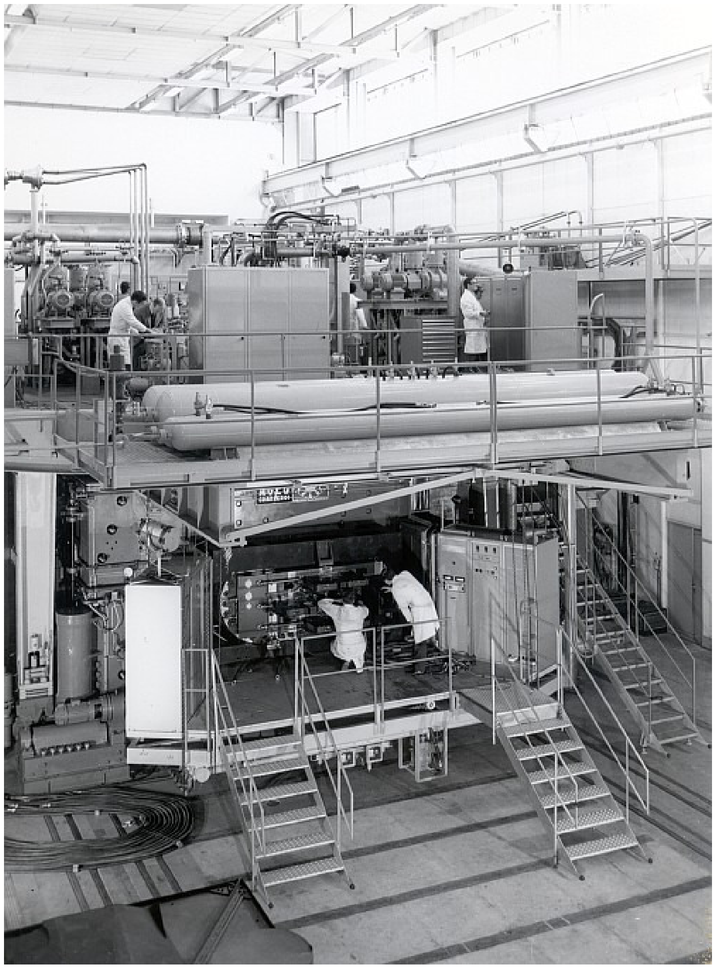


FIG. 2. Scatter plot of $M(\pi^+\pi^-)$ versus $M(\pi^+ n)$ with the projections on both axes.

Bubble Chambers - I



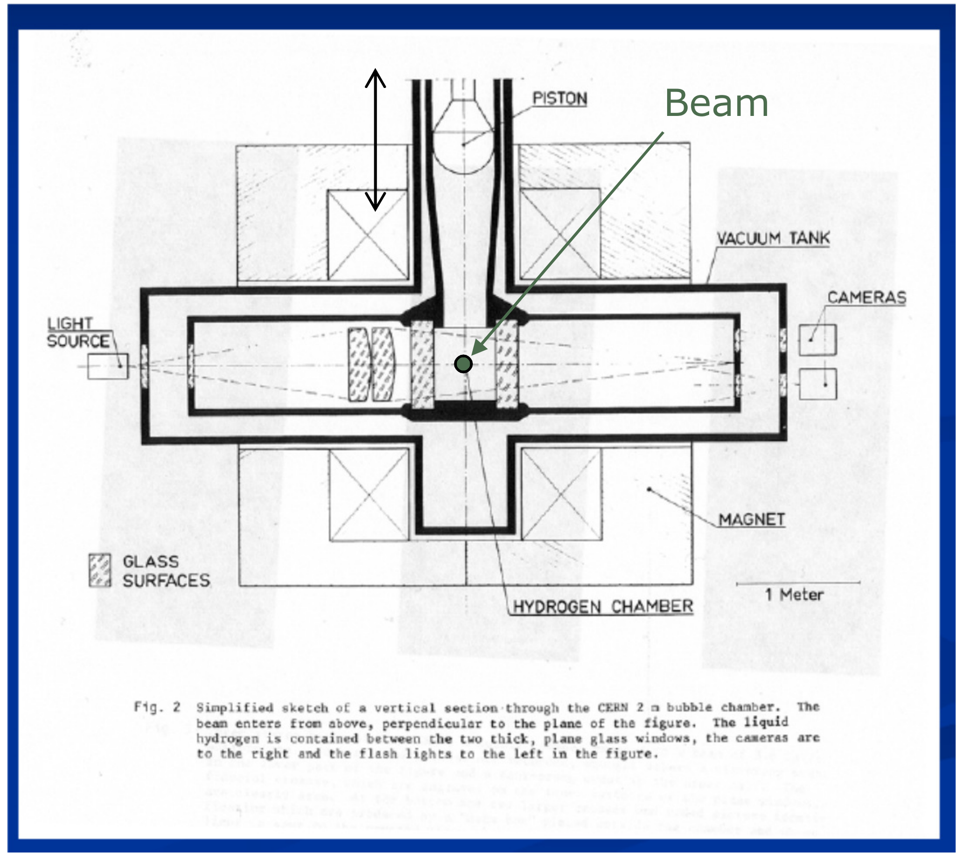
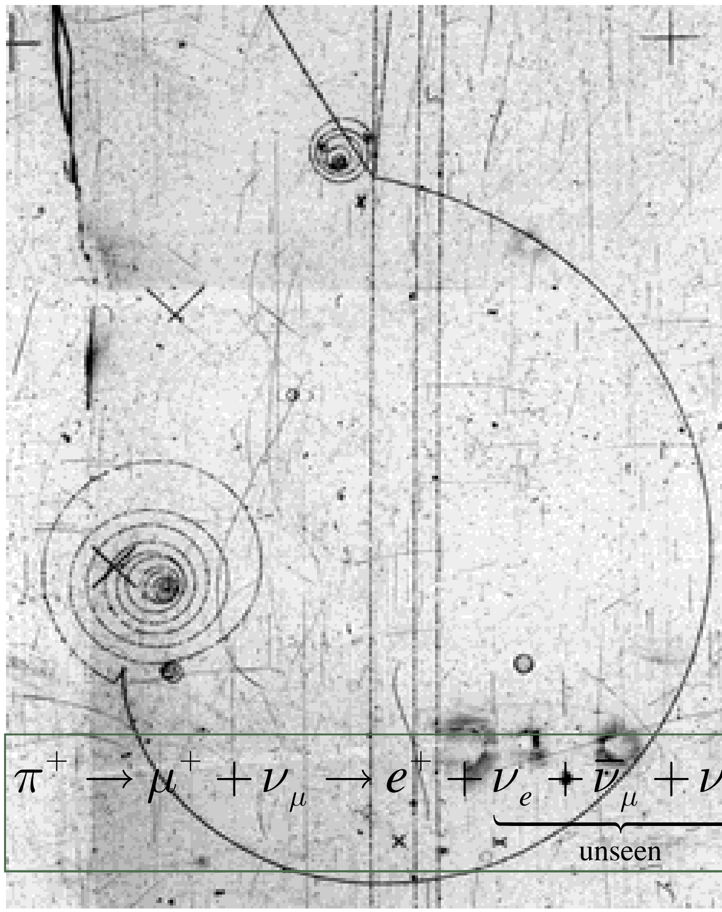


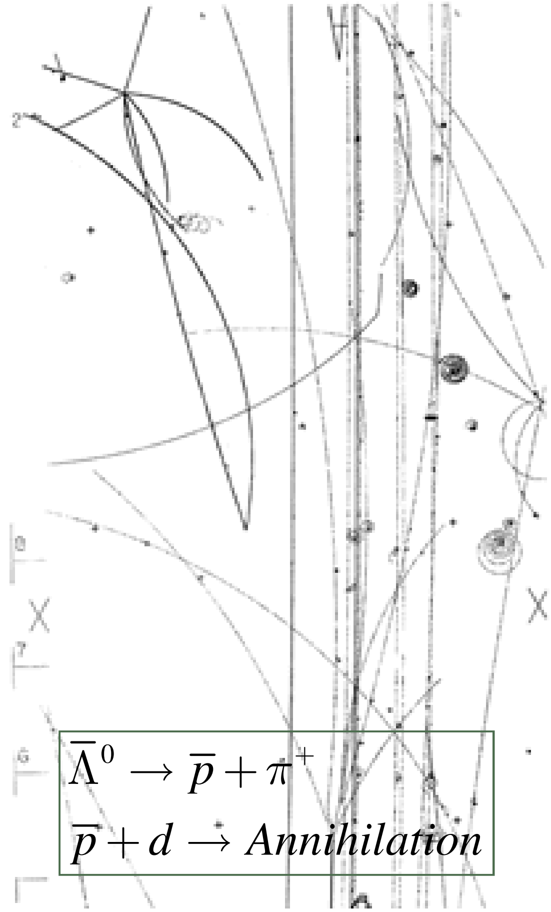
Fig. 2 Simplified sketch of a vertical section through the CERN 2 m bubble chamber. The beam enters from above, perpendicular to the plane of the figure. The liquid hydrogen is contained between the two thick, plane glass windows, the cameras are to the right and the flash lights to the left in the figure.

CERN 2m Bubble Chamber

Bubble Chambers - II



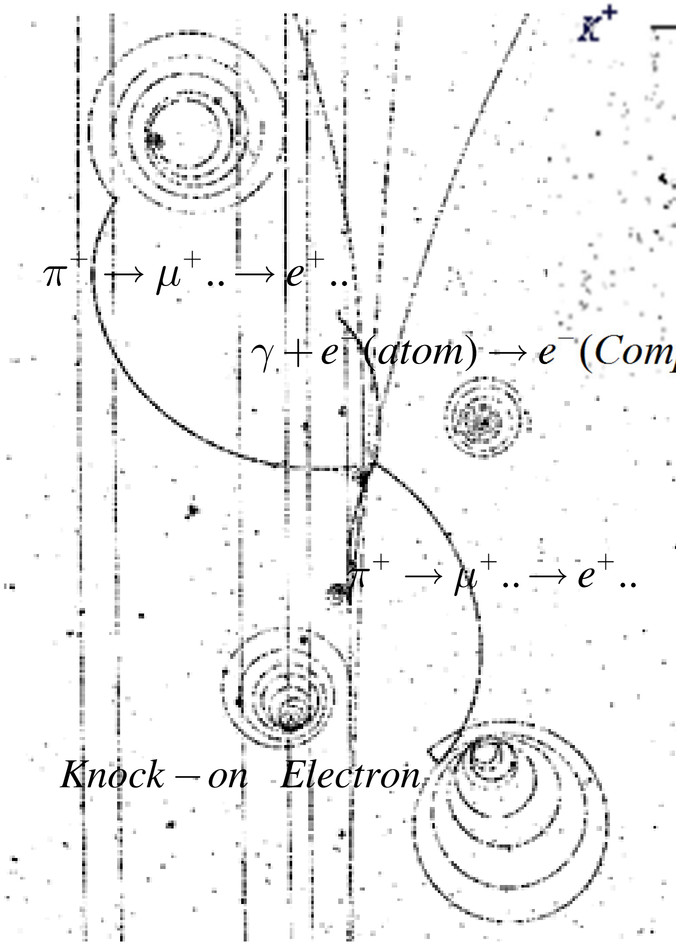
$$\pi^+ \to \mu^+ + \nu_\mu \to e^+ + \underbrace{\nu_e + \bar{\nu}_\mu + \nu_\mu}_{\text{unseen}}$$

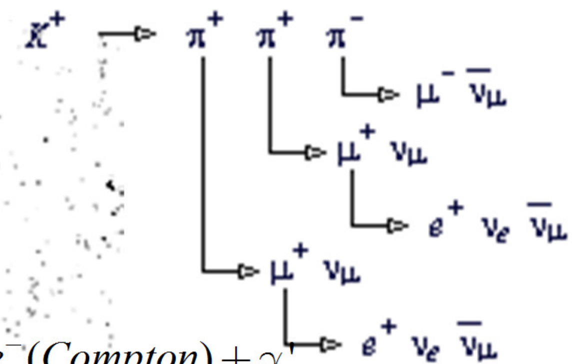


$$\bar{\Lambda}^0 \to \bar{p} + \pi^+$$

$$\bar{p} + d \to Annihilation$$

Bubble Chambers - III





<u>Knock-On electron</u>

$Beam + Atomic\ Electron$

$\to Beam + \ Free\ Electron$

Electromagnetic process

Usually modest energy

Bubble Chambers - IV

$\pi\mu\varepsilon$ kinematics

π^+ only: π^- is usually captured to a π - *mesic* atom

π decays after stopping: 'long' lifetime..

μ Energy, momentum:

$$E_\mu = \frac{1}{2m_\pi}\left(m_\pi^2 + m_\mu^2 - 0\right) \sim 109.9\ MeV \rightarrow p_\mu = \sqrt{109.9^2 - 106^2} \sim 29.1\ MeV$$

$$\rightarrow \beta_\mu = \frac{p_\mu}{E_\mu} \sim \frac{29.1}{109.9} \sim 0.265, \gamma_\mu \sim 1.04 \text{ when created}$$

Would expect typical path length $\sim \beta_\mu \gamma_\mu c\tau_\mu \sim 182\ m$

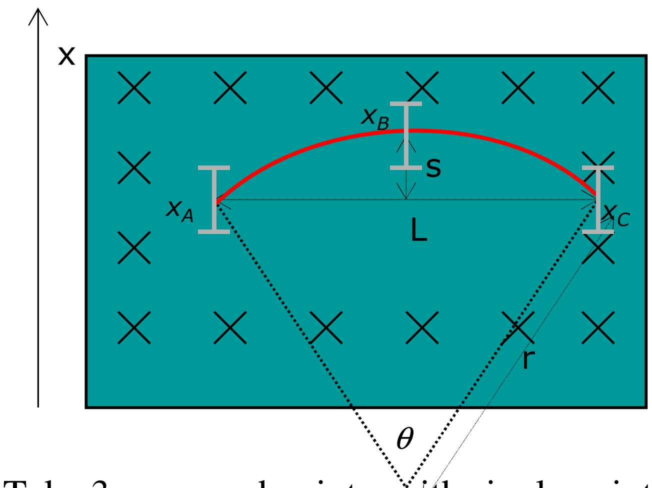
But: μ quickly slows down by $\frac{dE}{dx} \rightarrow$ Total path length $\sim$ few cm

Positron spiralling down: Energy loss by $\begin{cases} ionization \\ radiation \end{cases}$

Bubble Chambers - V

Motion of a charged particle in a uniform magnetic field: Cylindrical helix coaxial to ***B***

$$r = \frac{p_\perp}{0.3B} \qquad r:\ m,\ p_\perp:\ GeV,\quad B:\ T$$



Get p from s

$$\sin\frac{\theta}{2} = \frac{L}{2r} \xrightarrow[L \ll 2r]{} \frac{\theta}{2} \approx \frac{L}{2r} \to \theta \approx \frac{0.3BL}{p_\perp}$$

$$s = r - r\cos\frac{\theta}{2} \approx r\left[1 - \left(1 - \frac{\theta^2}{4}\right)\right] = r\frac{\theta^2}{8} \approx \frac{0.3BL^2}{8p_\perp}$$

$$\to p_\perp \approx \frac{0.3BL^2}{8s}$$

Take 3 measured points, with single point accuracy σ

Then:

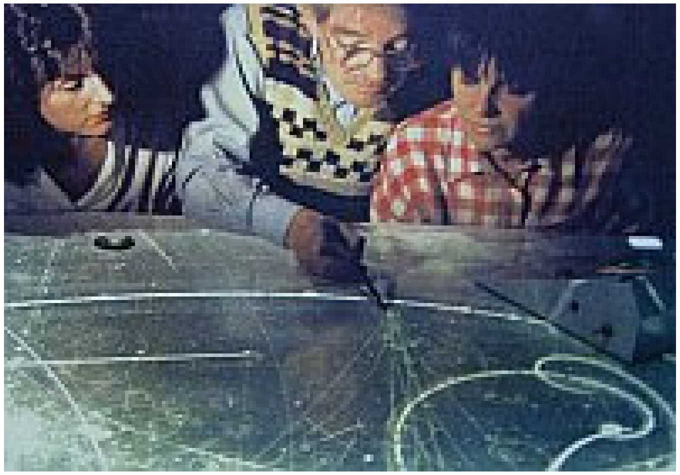
$$s = x_B - \frac{x_A + x_B}{2} \to \sigma_s^2 = \sigma^2 + \frac{1}{2}\sigma^2 = \frac{3}{2}\sigma^2$$

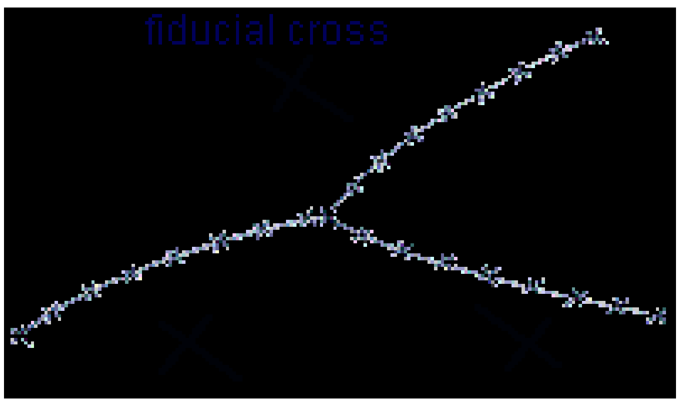
$$\frac{\sigma_{p_\perp}}{p_\perp} = \frac{\sigma_s}{s} = \sqrt{\frac{3}{2}}\frac{\sigma}{s} = \sqrt{\frac{3}{2}}\frac{\sigma 8 p_\perp}{0.3BL^2} = \sqrt{\frac{300 \cdot 64}{18}}\frac{\sigma p_\perp}{BL^2} \approx 32.7\frac{\sigma p_\perp}{BL^2}$$

$N \geq 10$, uniformly spaced points:

$$\frac{\sigma_{p_\perp}}{p_\perp} \approx 28.3\frac{\sigma p_\perp}{BL^2\sqrt{N+4}}$$

Bubble Chambers - VI





Particle	p_x	p_y	p_z	E
K-	8213.4	-248.3	15.2	8232
p	0	0	0	938.3
Sum	8213.4	-248.3	15.2	9170.3
K-	1481.8	27.8	224	1578.1
p-	149.7	-11.3	38.8	208.6
p+	37.9	-122.2	-22.7	190.7
p	1508.6	128.5	-70.5	1782.6
K0	3545.6	-162.9	-245	3592.4
Sum	6723.6	-140.1	-75.4	7352.4
Difference	-1489.8	108.2	-90.6	-1817.9

mass	1032.153

This mass doesn't correspond to a known particle - so there must be at least two neutral particles from the collision leaving the bubble chamber undetected.

Isospin - I

Charge independence leads to a new classification scheme:
All hadrons cast into *isospin multiplets*
Strong interaction identical for all members of each multiplet

$$\left.\begin{array}{l}\text{proton } p\\ \text{neutron } n\end{array}\right\}\ 2\ \text{ states of the } \textit{nucleon} \quad N=\begin{pmatrix}\alpha\\ \beta\end{pmatrix} \quad 2 \text{ states system - isospinor}$$

$$\text{Base } \begin{pmatrix}1\\0\end{pmatrix}\equiv p,\ \begin{pmatrix}0\\1\end{pmatrix}\equiv n \qquad \text{Base states: } \textit{doublet}$$

$$\left.\begin{array}{l}\pi^+\\ \pi^0\\ \pi^-\end{array}\right\}\ 3\ \text{ states of the } \textit{pion} \quad \pi=\begin{pmatrix}\alpha\\ \beta\\ \gamma\end{pmatrix} \quad 3 \text{ state system - isovector}$$

$$\text{Base } \begin{pmatrix}1\\0\\0\end{pmatrix}\equiv \pi^+,\begin{pmatrix}0\\1\\0\end{pmatrix}\equiv \pi^0,\begin{pmatrix}0\\0\\1\end{pmatrix}\equiv \pi^- \qquad \text{Base states: } \textit{triplet}$$

Isospin - II

Isospins add up as angular momenta (Astonished? More on this later…)
For πN system obtain:

$$\left.\begin{array}{l}\pi : I = 1 \\ N : I = 1/2\end{array}\right\} \to \pi N : I = 1 \oplus 1/2 = \left\{\begin{array}{ll}1/2 & \text{doublet} \\ 3/2 & \text{quadruplet}\end{array}\right.$$

By using Clebsch-Gordan coefficients, expand physical (particle) states into total isospin eigenstates

Single particle: Base states

$$I_N = 1/2 \;\; ; \;\; \left|\mathrm{p}\right\rangle = \left|1/2, +1/2\right\rangle \;\; , \;\; \left|\mathrm{n}\right\rangle = \left|1/2, -1/2\right\rangle$$

$$I_\pi = 1 \;\; ; \;\; \left|\pi^+\right\rangle = \left|1, +1\right\rangle \;\; , \;\; \left|\pi^0\right\rangle = \left|1, 0\right\rangle \;\; , \;\; \left|\pi^-\right\rangle = \left|1, -1\right\rangle$$

Isospin - III

Expand physical, 2 particle states into total isospin eigenstates:

$$\left|\pi^- p\right\rangle = \left|1,-1,1/2,+1/2\right\rangle = \sqrt{\frac{1}{3}}\left|3/2,-1/2\right\rangle - \sqrt{\frac{2}{3}}\left|1/2,-1/2\right\rangle$$

$$\left|\pi^+ n\right\rangle = \left|1,+1,1/2,-1/2\right\rangle = \sqrt{\frac{1}{3}}\left|3/2,+1/2\right\rangle + \sqrt{\frac{2}{3}}\left|1/2,+1/2\right\rangle$$

$$\left|\pi^+ p\right\rangle = \left|1,+1,1/2,+1/2\right\rangle = \left|3/2,+3/2\right\rangle$$

$$\left|\pi^- n\right\rangle = \left|1,-1,1/2,-1/2\right\rangle = \left|3/2,-3/2\right\rangle$$

$$\left|\pi^0 p\right\rangle = \left|1,0,1/2,+1/2\right\rangle = \sqrt{\frac{2}{3}}\left|3/2,+1/2\right\rangle - \sqrt{\frac{1}{3}}\left|1/2,+1/2\right\rangle$$

$$\left|\pi^0 n\right\rangle = \left|1,0,1/2,-1/2\right\rangle = \sqrt{\frac{2}{3}}\left|3/2,-1/2\right\rangle + \sqrt{\frac{1}{3}}\left|1/2,-1/2\right\rangle$$

Isospin - IV

Guess isospin is a new *symmetry* for hadrons: connect to some *invariance* property (like angular momentum).
Non-trivial conservation rule follows:

Total isospin conserved by all strong processes

Interesting predictions for πN scattering and reactions:

$$\begin{cases} (A)\pi^+ p \to \pi^+ p \\ (B)\pi^- n \to \pi^- n \end{cases} \to A_A = A_B = A_{3/2} \quad \text{pure I = 3/2}$$

$$\begin{cases} (A)\pi^+ n \to \pi^+ n \\ (B)\pi^- n \to \pi^- n \end{cases} \to A_A = \frac{1}{3}A_{3/2} + \frac{2}{3}A_{1/2}, A_B = A_{3/2}$$

$$\begin{cases} (A)\pi^+ p \to \pi^+ p \\ (B)\pi^- p \to \pi^- p \end{cases} \to A_A = A_{3/2}, A_B = \frac{1}{3}A_{3/2} - \frac{2}{3}A_{1/2}$$

$$\begin{cases} (A)\pi^+ p \to \pi^+ p \\ (B)\pi^- p \to \pi^0 n \end{cases} \to A_A = A_{3/2}, A_B = \sqrt{\frac{2}{9}}A_{3/2} - \sqrt{\frac{2}{9}}A_{1/2}$$

Isospin - V

If $A_{3/2} >> A_{1/2}$

$$\begin{cases} (A)\,\pi^+ p \to \pi^+ p \\ (B)\,\pi^- n \to \pi^- n \end{cases} \to \sigma_A = \sigma_B$$

$$\begin{cases} (A)\,\pi^+ n \to \pi^+ n \\ (B)\,\pi^- n \to \pi^- n \end{cases} \to \sigma_A \simeq \frac{1}{9}\sigma_B$$

$$\begin{cases} (A)\,\pi^+ p \to \pi^+ p \\ (B)\,\pi^- p \to \pi^- p \end{cases} \to \sigma_A \simeq 9\sigma_B$$

$$\begin{cases} (A)\,\pi^+ p \to \pi^+ p \\ (B)\,\pi^- p \to \pi^0 n \end{cases} \to \sigma_A \simeq \frac{9}{2}\sigma_B$$

Still lacking: *What exactly is isospin?*

Continuous Groups - I

Transformations of coordinates :

$S : x \rightarrow x' = S(x) \rightarrow S^{-1} : x' \rightarrow x = S^{-1}(x')$

Example: Space Translations along x

$T : x \rightarrow x' = T(x) = x + a$

$T^{-1} : x' \rightarrow x = T^{-1}(x') = x' - a$

Continuous transformation $\leftrightarrow -\infty < a < +\infty$ real parameter

$T(b)T(a) = T(a+b)$ Group property

$T(0) = 1$ Identity element

$T(-a) = T^{-1}(a)$

$\rightarrow T(-a)T(a) = T^{-1}(a)T(a) = 1$

$T(-a)$ Inverse element

$\rightarrow$ Translations along x form a continuous group

$T(b)T(a) = T(a)T(b)$ Group is *commutative*

$\rightarrow$ Translations along x are a *commutative* ($=$ *Abelian*), *order* 1, *continuous* group

Continuous Groups - II

Key point for continuous groups:

Any finite element can be obtained by iteration of infinitesimal transformation

$$T : x \rightarrow x' = T(x) = x + a$$

$$T(0) \rightarrow T(\delta x) \rightarrow T(\delta x + \delta x) \rightarrow .... \rightarrow T(x)$$

$$\rightarrow T(x + \delta x) = T(x) T(\delta x)$$

$$T(x + \delta x) = T(x) + \frac{dT}{dx}\delta x + ...$$

$$\rightarrow T(x) + \frac{dT}{dx}\delta x \simeq T(x) T(\delta x)$$

$$\rightarrow \frac{dT}{dx}\delta x \simeq T(x) T(\delta x) - T(x) \rightarrow \frac{dT}{dx} \simeq T(x) \frac{\left[T(\delta x) - T(0)\right]}{\delta x} \underset{\delta x \rightarrow 0}{=} T(x) \underbrace{\left.\frac{dT}{dx}\right|_{x=0}}_{\equiv G}, \ G \text{ generator}$$

$$\rightarrow \frac{dT}{T} = G \rightarrow T(x) = T(0) e^{Gx}$$

Continuous Groups - III

Translations along *any* axis form an Abelian group:

$$T(\mathbf{a})T(\mathbf{b}) = T(\mathbf{a}+\mathbf{b}) = T(\mathbf{b})T(\mathbf{a})$$

Translations in 3D:

N. of generators = order of the 3D translations group:

$N = 3$

$\rightarrow$ Translations along *different* axes form an *order* 3, *Abelian*, *continuous* group

Continuous Groups - IV

Rotations around a *fixed* axis (e.g. z): $SO(2) \equiv$ *S*pecial *O*rthogonal in 2 D

Parameter: φ

$$R(\varphi_2)R(\varphi_1) = R(\varphi_1 + \varphi_2) = R(\varphi_2)R(\varphi_1)$$

N. of generators = order of the 1D rotations group:

$N = 1$

$\rightarrow$ Rotations around a *fixed* axis form an *Abelian*, *order* 1, *continuous* group

Continuous Groups - V

Rotations in $3D$: $SO(3) \equiv \mathit{S}$pecial Orthogonal in 3 D

Parameters: $e.g.$ axis (θ, φ)+ angle (α)

$$R(\theta_2,\varphi_2;\alpha_2)R(\theta_1,\varphi_1;\alpha_1) = R(\theta,\varphi;\alpha) \neq R(\theta_1,\varphi_1;\alpha_1)R(\theta_2,\varphi_2;\alpha_2)$$

$$\left.\begin{aligned}\theta &= \theta(\theta_1,\varphi_1;\alpha_1,\theta_2,\varphi_2;\alpha_2)\\ \varphi &= \varphi(\theta_1,\varphi_1;\alpha_1,\theta_2,\varphi_2;\alpha_2)\\ \alpha &= \alpha(\theta_1,\varphi_1;\alpha_1,\theta_2,\varphi_2;\alpha_2)\end{aligned}\right\} \text{ analytic functions}$$

N. of generators = order of the 3D rotations group:

$N = 3$

Non-Abelian : Rotations are *not* vectors

No. of commuting generators = 1 $\rightarrow$ Rank = 1

Rotations in 3D form a *Non - Abelian*, *order* 3, *rank* 1, *continuous* group

Representations - I

Use a set of matrices to represent a given group:

$$\left.\begin{array}{l} \textit{Product} \\ \textit{Inverse} \\ \textit{Unit element} \end{array}\right\} \text{Correspondance}$$

Matrices $\equiv$ Linear transformations in a (complex) vector space

Different vector spaces $\rightarrow$ Different representations
e.g. different dimensions

Reducible / Irreducible representation:
Can/Can't be reduced to a *block diagonal* form
by some coordinate transformation
Note: Abelian group $\rightarrow$ Irr.Reps. *unidimensional*

$$\begin{bmatrix} a_{11} & a_{12} & 0 & 0 & 0 & 0 & 0 \\ a_{21} & a_{22} & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & b_{11} & b_{12} & b_{13} & 0 & 0 \\ 0 & 0 & b_{21} & b_{22} & b_{23} & 0 & 0 \\ 0 & 0 & b_{31} & b_{32} & b_{33} & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & c_{11} & c_{12} \\ 0 & 0 & 0 & 0 & 0 & c_{21} & c_{22} \end{bmatrix} = \begin{bmatrix} A & 0 & 0 \\ 0 & B & 0 \\ 0 & 0 & C \end{bmatrix}$$

Representations - II

Representations of continuous groups:

$$T : x \to x' = T(x; a)$$

$$R \leftrightarrow T$$

$$\frac{dR}{da} \underset{a \to 0}{=} R(a) \underbrace{\left.\frac{dR}{da}\right|_{a=0}}_{\equiv X}, \; X \text{ generator}$$

$$G \leftrightarrow X$$

$$\to \frac{dR}{R} = X da$$

$$\to R(a) = R(0) e^{Xa}$$

Representations - III

Example: $SO(2)$

'Natural' representation in $\mathbb{R}^2$:

$$\begin{pmatrix} x' \\ y' \end{pmatrix} = \begin{pmatrix} \cos\varphi & -\sin\varphi \\ \sin\varphi & \cos\varphi \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix}$$

(Abelian) 1-parameter group of 2×2, real, orthogonal matrices: $SO(2)$

Reducible representation

Less natural representation in $\mathbb{C}^1$:

$$z' = e^{\alpha\varphi} z, \quad \alpha \in \mathbb{C}^1$$

$$\varphi = 2\pi \to z' = z$$

$$\to e^{\alpha 2\pi} = 1 \to \alpha = im, \quad m = 0, \pm 1, \pm 2, \ldots$$

$$z' = e^{im\varphi} z \leftrightarrow x' + iy' = \left(\cos m\varphi + i \sin m\varphi\right)\left(x + iy\right)$$

(Abelian) 1-parameter group of phase transformations: $U\left(1\right)$

Irreducible representations

Representations - IV

Generators

$$R(\varphi)=\begin{pmatrix}\cos\varphi & -\sin\varphi\\ \sin\varphi & \cos\varphi\end{pmatrix}$$

Generator:

$$X=\left.\frac{dR}{d\varphi}\right|_{\varphi=0}=\left.\begin{pmatrix}-\sin\varphi & -\cos\varphi\\ \cos\varphi & -\sin\varphi\end{pmatrix}\right|_{\varphi=0}=\begin{pmatrix}0 & -1\\ +1 & 0\end{pmatrix}$$

$$R(\varphi)=e^{im\varphi},\quad m=0,\pm1,\pm2,\ldots$$

Generator:

$$X=im\left.e^{im\varphi}\right|_{\varphi=0}=im=me^{i\frac{\pi}{2}}$$

Representations - V

Example: $SO(3)$

'Natural' representation in $\mathbb{R}^3$:

Either Axis $\boldsymbol{u}$ + Angle θ

$$R=\begin{pmatrix} \cos\theta+u_x^2\left(1-\cos\theta\right) & u_xu_y\left(1-\cos\theta\right)-u_z\sin\theta & u_xu_z\left(1-\cos\theta\right)+u_y\sin\theta \\ u_xu_y\left(1-\cos\theta\right)+u_z\sin\theta & \cos\theta+u_y^2\left(1-\cos\theta\right) & u_yu_z\left(1-\cos\theta\right)-u_x\sin\theta \\ u_xu_z\left(1-\cos\theta\right)-u_y\sin\theta & u_yu_z\left(1-\cos\theta\right)+u_x\sin\theta & \cos\theta+u_z^2\left(1-\cos\theta\right) \end{pmatrix}$$

or 3 'Cartesian' angles

$$R=R_x\left(\alpha\right)R_y\left(\beta\right)R_z\left(\gamma\right)$$

$$R_x\left(\alpha\right)=\begin{pmatrix} 1 & 0 & 0 \\ 0 & \cos\alpha & -\sin\alpha \\ 0 & \sin\alpha & \cos\alpha \end{pmatrix}, R_y\left(\beta\right)=\begin{pmatrix} \cos\beta & 0 & \sin\beta \\ 0 & 1 & 0 \\ -\sin\beta & 0 & \cos\beta \end{pmatrix}, R_z\left(\gamma\right)=\begin{pmatrix} \cos\gamma & -\sin\gamma & 0 \\ \sin\gamma & \cos\gamma & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

Representations - VI

Generators: Use 'Cartesian angles'

$$X_x = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & -1 \\ 0 & 1 & 0 \end{pmatrix}, X_y = \begin{pmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ -1 & 0 & 0 \end{pmatrix}, X_z = \begin{pmatrix} 0 & -1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}$$

Non-commuting matrices :

$\left[X_x, X_y\right] = X_z$ & cyclic

$\rightarrow$ (Non-Abelian) 3-parameters group of 3×3, real, orthogonal matrices: $SO(3)$

Note: Rank 1 group $\rightarrow$ Should have one diagonal generator

OK by a unitary transformation

Representations - VII

Transformations in state space : State vectors

$\left|\psi'\right\rangle = U\left|\psi\right\rangle \rightarrow U$ unitary: $U^{\dagger} = U^{-1}$

For any H Hermitian $\rightarrow U = e^{iH}$ unitary [H generic, *not* the Hamiltonian..]

Indeed:

$$H^{\dagger} = H \rightarrow U^{\dagger} = \left(e^{iH}\right)^{\dagger} = e^{-iH^{\dagger}} = e^{-iH} = U^{-1}$$

Continuous symmetry:

$U = U\left(a\right)$, a continuous parameter(s)

$$H \underset{a\rightarrow 0}{=} aY$$

Y : Generator(s) of the transformation

$U\left(a\right) = e^{iaY} \rightarrow U \simeq 1 + iaY$, Y Hermitian

$$\lim_{a\rightarrow 0} U\left(a\right) = 1$$

Representations - VIII

Generalize $3D$ space to ∞D space:

$$\underset{n<\infty}{Vectors} \rightarrow \underset{n\rightarrow\infty(continuous)}{(Wave)\,Functions}$$

Transformation in wave functions space :

Unitary operators

sometimes realized as *differential operators*

Ex:

$$U : \psi(x) \rightarrow \psi(x+a)$$

$$\psi(x+a) = U[\psi(x)] = \sum_{n=0}^{\infty} \frac{1}{n!} \left.\frac{\partial^n \psi}{\partial x^n}\right|_x a^n = \left(\sum_{n=0}^{\infty} \frac{1}{n!} \left.\frac{\partial^n}{\partial x^n}\right|_x a^n \right)[\psi] = e^{a\frac{\partial}{\partial x}}[\psi]$$

$$\rightarrow U = e^{a\frac{\partial}{\partial x}} = e^{iap_x}$$

Representations - IX

Example: Space Translations

$$S : x \to x' = S(x) = x + a, \quad S^{-1} : x' \to x = S^{-1}\left(x'\right) = x' - a$$

$$\psi\left(x'\right) \simeq \psi\left(x\right) + \frac{\partial \psi}{\partial x} da$$

$$\to \psi\left(x + da\right) \simeq \left(1 + da \frac{\partial}{\partial x}\right) \psi\left(x\right)$$

$$\to U\left(da\right) \simeq 1 + da \frac{\partial}{\partial x}$$

$$= 1 - i^2 da \frac{\partial}{\partial x} = 1 - ida\, \underbrace{i \frac{\partial}{\partial x}}_{-p_x} = 1 + idap_x$$

$$\to U\left(a\right) = e^{iap_x}$$

$$p_x = -i \frac{\partial}{\partial x}$$ Generator of translations along x

Representations - X

Example: $SO(2)$

$$S:\begin{pmatrix}x'\\y'\end{pmatrix}=\begin{pmatrix}\cos\varphi & -\sin\varphi\\ \sin\varphi & \cos\varphi\end{pmatrix}\begin{pmatrix}x\\y\end{pmatrix}$$

$$\begin{pmatrix}\cos\varphi & -\sin\varphi\\ \sin\varphi & \cos\varphi\end{pmatrix}\underset{\varphi\to d\varphi\sim 0}{\longrightarrow}\begin{pmatrix}1 & -d\varphi\\ d\varphi & 1\end{pmatrix}\to\begin{pmatrix}x'\\y'\end{pmatrix}\simeq\begin{pmatrix}1 & -d\varphi\\ d\varphi & 1\end{pmatrix}\begin{pmatrix}x\\y\end{pmatrix}=\begin{pmatrix}x-yd\varphi\\ xd\varphi+y\end{pmatrix}$$

$$\psi(x',y')\simeq\psi(x-yd\varphi,xd\varphi+y)$$

$$\psi(x',y')\simeq\psi(x,y)+\frac{\partial\psi}{\partial x}(-yd\varphi)+\frac{\partial\psi}{\partial y}(xd\varphi)$$

$$\to\psi(x',y')\simeq\psi(x,y)+\underbrace{\left(-y\frac{\partial}{\partial x}+x\frac{\partial}{\partial y}\right)}_{X}\psi d\varphi$$

$$\to U(d\varphi)\simeq 1+\left(x\frac{\partial}{\partial y}-y\frac{\partial}{\partial x}\right)d\varphi$$

$$J_z=xp_y-yp_x=x\left(-i\frac{\partial}{\partial y}\right)-y\left(-i\frac{\partial}{\partial x}\right)=-i\left(x\frac{\partial}{\partial y}-y\frac{\partial}{\partial x}\right)\to X=iJ_z \text{ Generator of rotations around } z$$

$$\to U(d\varphi)\simeq 1+iJ_z d\varphi\to U(\varphi)=e^{i\varphi J_z}$$

Representations - XI

Example: $SO(3)$

Use 'Cartesian angles':

Then everything similar to $SO(2)$

$$\begin{cases} X_x = iJ_x \\ X_y = iJ_y \\ X_z = iJ_z \end{cases}$$

But:

$\left[J_x, J_y\right] = iJ_z$ & cyclic: Angular momentum components

Non-Abelian group, can diagonalize just *one* generator $\rightarrow J_z \left(\leftarrow Rank\ 1\right)$:

$\rightarrow$ Just *one* (quadratic), omni-commuting operator built out of generators:

$\boldsymbol{J}^2 = J_x^2 + J_y^2 + J_z^2$ Casimir operator

Representations - XII

Identify $SO(3)$ irreps by eigenvalues of commuting operators:

$$\boldsymbol{J}^2 \to j(j+1) \qquad\qquad j \quad integer\ 0,1,2...$$

$$J_z \to m = -j,...,+j \qquad 2j+1 \text{ values, 1-stepping between } -j \text{ and } +j$$

Bottom line:

Infinite sequence of irreducible representations,

each one identified by j, integer

$\to$ dim $= 2j+1$, odd $=$

Use dim to identify representations:

1, 3, 5, ...

Representations - XIII

Extend to finite rotations as done for $SO(2)$:

$$U(\alpha,\beta,\gamma) = e^{i\alpha J_x} e^{i\alpha J_y} e^{i\alpha J_z}$$

Use $2j+1$ dimensional matrices to rotate $2j+1$ components of 'vectors'

($\leftarrow$ *spherical tensors*):

1 - Scalars

3 - Vectors (Spherical components)

5 - Rank 2 Tensors (Spherical components)

etc

Use differential operators to rotate wave functions

$SU(2)$ - I

Consider the set of 2×2 *unitary* matrices $U(2)$:

$$U \equiv \begin{pmatrix} a & b \\ c & d \end{pmatrix} \text{ unitary} \rightarrow UU^{\dagger} = 1 \rightarrow \begin{pmatrix} a & b \\ c & d \end{pmatrix} \begin{pmatrix} \bar{a} & \bar{c} \\ \bar{b} & \bar{d} \end{pmatrix} = \begin{pmatrix} a\bar{a} + b\bar{b} & a\bar{c} + b\bar{d} \\ c\bar{a} + d\bar{b} & c\bar{c} + d\bar{d} \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$$

$$\rightarrow \begin{cases} a\bar{a} + b\bar{b} = 1 \\ c\bar{c} + d\bar{d} = 1 \\ a\bar{c} + b\bar{d} = 0 \\ c\bar{a} + d\bar{b} = 0 \end{cases} \text{ \& } a, b, c, d \text{ complex} \rightarrow 4 \text{ free parameters}$$

Require extra condition:

$\det M = 1 \rightarrow ad - bc = 1 \rightarrow 3$ free parameters

Can be shown to be a group:

Special Unitary group of 2×2 matrices = $SU(2)$

$SU(2)$ - II

Any matrix of $SU(2)$:

$$U = \begin{pmatrix} a & b \\ c & d \end{pmatrix} = \begin{pmatrix} a & \bar{b} \\ b & -a \end{pmatrix},\ a \text{ real} \rightarrow 3 \text{ real parameters}$$

Generators:

$$\left.\begin{aligned} \sigma_1 &= \left.\frac{\partial U}{\partial(\mathrm{Re}\,b)}\right|_{\mathrm{Re}\,b=0} = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \\ \sigma_2 &= \left.\frac{\partial U}{\partial(\mathrm{Im}\,b)}\right|_{\mathrm{Im}\,b=0} = \begin{pmatrix} 0 & +i \\ -i & 0 \end{pmatrix} \\ \sigma_3 &= \left.\frac{\partial U}{\partial a}\right|_{a=0} = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \end{aligned}\right\} \rightarrow \text{Pauli matrices}$$

$$\left[\frac{\sigma_1}{2}, \frac{\sigma_2}{2}\right] = i\frac{\sigma_3}{2} \Rightarrow \textit{Same algebra} \text{ as } SO(3)$$

$\rightarrow$ Non-Abelian, order 3, rank 1 continuous group

$SU(2)$ - III

$$\left[\frac{\sigma_1}{2}, \frac{\sigma_2}{2}\right] = i\frac{\sigma_3}{2}$$

Casimir: $\boldsymbol{\sigma}^2 = \sigma_x^2 + \sigma_y^2 + \sigma_z^2 = j\left(j+1\right),\ j\ \ \textit{integer \& half - integer}$

$\rightarrow \dim = 2j+1 = 1,2,3,4,5,\ldots$

1, 2, 3, 4, 5, ...

Bottom line:

Odd dimension irreps $\equiv$ Same as $SO(3) \rightarrow \mathbf{3}$: *adjoint* representation

Even dimension irreps $\equiv$ $SU(2)$ own $\rightarrow$ **2**: *fundamental* representation

$SU(2)$ - IV

Extending to finite transformations as done for $SO(3)$:

$$\psi' = U(\mathbf{v})\psi$$

$$\mathbf{v} = (a, \mathrm{Re}\, b, \mathrm{Im}\, b)$$

$$\boldsymbol{\sigma} = (\sigma_1, \sigma_2, \sigma_3)$$

$$U(\mathbf{v}) = e^{i\mathbf{v}\cdot\boldsymbol{\sigma}}$$

But:

Can't extend to differential operators in Hilbert space as done for $SO(3)$

$SU(2)$ - V

$SU(2)$ generators: Pauli matrices

$\rightarrow$ Any $SU(2)$ matrix = Linear combination of generators

But:

$SU(2)$ matrix *generic* unitary operator for any 2-state system

$\rightarrow SU(2)$ symmetry may have very different origins

e.g. spin 1/2 particle, 2-level atom, NMR, neutral Kaon, oscillating neutrino, ...
...and, of course, isospin

SU(2) - VI

Question: What is the observable we have called *isospin*?

Answer: *There is no classical analogy!*

Observe the neutron and proton to be almost degenerate in mass

→ Assume they are just *two states* of the same physical system, the *nucleon.*

Nuclear constituents and their relatives (the whole family of *hadron*s) have internal degrees of freedom with no classical analogue

Guess the two states of the nucleon are two 'vectors' spanning the fundamental representation of a symmetry group, which we identify with *SU(2).*

SU(2) - VII

When looking at strongly interacting particles, observe particle states grouping themselves into multiplets of size *1,2,3,4*

States of a multiplet $\cong$ *Same mass*

$\rightarrow$States belonging to different multiplets must be distinguished by some internal quantum number:

By analogy, call the corresponding observable the particle *isospin*

$\rightarrow$States of any given multiplet must be identified by some *internal* quantum number:

Call the corresponding observable the *3rd component of the particle isospin*

SU(2) - VIII

Notice: Isospin symmetry is not exact (broken), still is quite good
Indeed, looking at symmetry breaking mass splittings:

$$\frac{m_n - m_p}{m_n} \simeq \frac{939.57 - 938.27}{939.57} \simeq 0.0014 \ \text{ Nucleon doublet}$$

$$\frac{m_{\pi^\pm} - m_{\pi^0}}{m_{\pi^\pm}} \simeq \frac{139.6 - 135.0}{139.6} \simeq 0.011 \ \text{ Pion triplet}$$

For a long time:
Breaking entirely blamed on electromagnetic effects, which is only partially true (e.g. neutral and charged members indeed have quite different e.m. interactions contributing to their mass).

Today:
Isospin taken as an ‘accidental’ symmetry, not due to some fundamental property of hadron constituents or strong interaction

SU(2) - IX

Guess: *SU(2)* is a symmetry of all the strongly interacting particles.

Therefore:

All strongly interacting particles should fill some SU(2) representation

This is actually true, after neglecting small symmetry breaking effects within each multiplet (see later)

As for any other symmetry, expect the invariance property to yield a conservation law

SU(2) - X

What is conserved in this case?

Since there is no classical analogy, stick to our algebraic skills to get insight

SU(2) algebra is just the same as *SO(3),* so we can expect the same conserved observables for a closed system of strongly interacting particles:

$$\boldsymbol{J}^2, J_3 \leftrightarrow \boldsymbol{I}^2, I_3$$

This is the origin of the common wisdom:

'Isospin is like Angular Momentum'

SU(2) - XI

Within any given *SU(2)* multiplet, states can be represented as points on a straight line

Reason is the group structure of *SU(2):*

3 parameters → 3 generators

Just 1 invariant function of generators:
$\boldsymbol{I}^2$ *→ Multiplets identified just by I*

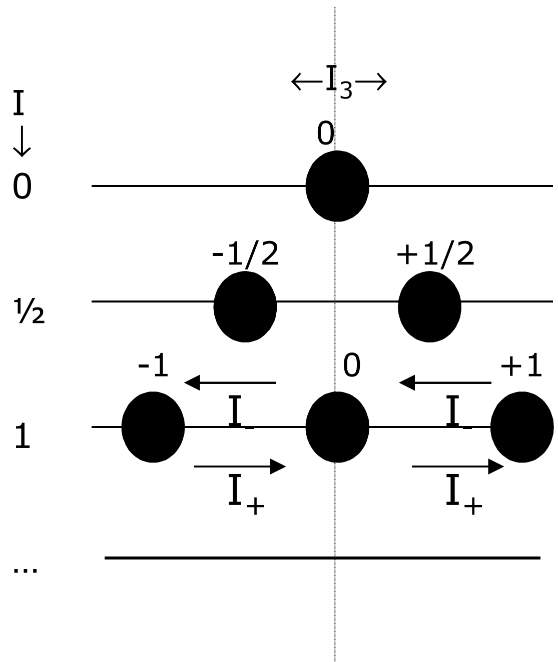
Generators do not commute with each other
→ States in any multiplet identified just by I_3

$$I_{\pm} = I_1 \pm iI_2$$

Define 2 *ladder operators*:

Action: Shift states right or left on the multiplet line, i.e. increment/ decrement I_3 by *1*

Observe:
I_3 eigenvalues *symmetric* wrt *0*



$SU(2)$ - XII

D : Any representation

$\psi^{'} = D(\alpha)\psi$

$\rightarrow D(\alpha) = e^{i\alpha F}$

α set of 3 parameters

F hermitian $\leftarrow$ True because D is unitary

Take complex conjugate of equations

$\psi^{'*} = D^{*}\psi^{*}$

Get another representation

$D^{*} = e^{-i\alpha(F)^{*}} = e^{i\alpha\left[-(F)^{*}\right]} \equiv e^{i\alpha\tilde{F}}$

Relation bewteen new and old generators

$\rightarrow \tilde{F} = -\left(F^{*}\right)$

$SU(2)$ - XIII

Take D of $SU(2)$ fundamental representation:

F Hermitian $\rightarrow \tilde{F}$ Hermitian

$\rightarrow$ Real eigenvalues for both $F, \tilde{F}$, and $f_i = -f_i^*$

$\rightarrow$ Since f_i are symmetric wrt 0, so are f_i^*

$\rightarrow \{f_i\} \equiv \{f_i^*\}$

$\rightarrow \tilde{F}$ eigenvalues are just a re-labeling of F's

Direct and conjugate representations are said to be *equivalent*

True for SU(2), generally false

SU(2) - XIV

Take a system made of 2 nucleons: *What is the total isospin?*

SU(2) is equivalent to *O(3)* $\rightarrow$ *Can use Clebsch-Gordan coefficients*

But: Can also re-formulate the problem in a different way
Each nucleon spans the fundamental representation of *SU(2),* ***2***

Thus a 2 nucleon system spans the *direct product rep.*

$$\mathbf{2} \otimes \mathbf{2}$$

Question:

What are the irreducible representations of SU(2) contained in any state of 2 nucleons?

Need to decompose $\mathbf{2} \otimes \mathbf{2}$ into a *direct sum* of irr.rep.

SU(2) - XV

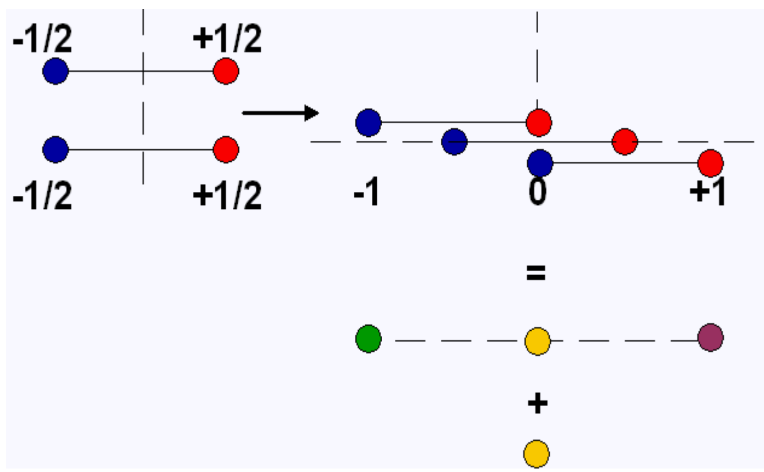
Answer (After a little group theory):

$$\mathbf{2} \otimes \mathbf{2} = \mathbf{1} \oplus \mathbf{3}$$

Answer (Graphical):

Center the segment carrying the 2 states of representation ***2*** *(1st nucleon) over the 2 states of representation* ***2*** *(2nd nucleon)*

$\rightarrow$ *Get a set of 4 states, decomposing into 2 sets of 1 and 3 states*



I-Spin Multiplets

Amazingly *large* number of resonant states

$L_{2J+1,2I+1}$ $\quad L = S, P, D, ...$

p, n	P_{11}	****	$\Delta(1232)$	P_{33}	****
$N(1440)$	P_{11}	****	$\Delta(1600)$	P_{33}	***
$N(1520)$	D_{13}	****	$\Delta(1620)$	S_{31}	****
$N(1535)$	S_{11}	****	$\Delta(1700)$	D_{33}	****
$N(1650)$	S_{11}	****	$\Delta(1750)$	P_{31}	*
$N(1675)$	D_{15}	****	$\Delta(1900)$	S_{31}	**
$N(1680)$	F_{15}	****	$\Delta(1905)$	F_{35}	****
$N(1700)$	D_{13}	***	$\Delta(1910)$	P_{31}	****
$N(1710)$	P_{11}	***	$\Delta(1920)$	P_{33}	***
$N(1720)$	P_{13}	****	$\Delta(1930)$	D_{35}	***
$N(1900)$	P_{13}	**	$\Delta(1940)$	D_{33}	*
$N(1990)$	F_{17}	**	$\Delta(1950)$	F_{37}	****
$N(2000)$	F_{15}	**	$\Delta(2000)$	F_{35}	**
$N(2080)$	D_{13}	**	$\Delta(2150)$	S_{31}	*
$N(2090)$	S_{11}	*	$\Delta(2200)$	G_{37}	*
$N(2100)$	P_{11}	*	$\Delta(2300)$	H_{39}	**
$N(2190)$	G_{17}	****	$\Delta(2350)$	D_{35}	*
$N(2200)$	D_{15}	**	$\Delta(2390)$	F_{37}	*
$N(2220)$	H_{19}	****	$\Delta(2400)$	G_{39}	**
$N(2250)$	G_{19}	****	$\Delta(2420)$	$H_{3,11}$	****
$N(2600)$	$I_{1,11}$	***	$\Delta(2750)$	$I_{3,13}$	**
$N(2700)$	$K_{1,13}$	**	$\Delta(2950)$	$K_{3,15}$	**

Baryons

I=1/2 *I=3/2*

LIGHT UFLAVORED
($S = C = B = 0$)

	$I^G(J^{PC})$
• $\pi^{\pm}$	$1^-(0^-)$
• π^0	$1^-(0^{-+})$
• η	$0^+(0^{-+})$
• $f_0(400\text{–}1200)$	$0^+(0^{++})$
• $\rho(770)$	$1^+(1^{--})$
• $\omega(782)$	$0^-(1^{--})$
• $\eta'(958)$	$0^+(0^{-+})$
• $f_0(980)$	$0^+(0^{++})$
• $a_0(980)$	$1^-(0^{++})$
• $\phi(1020)$	$0^-(1^{--})$
• $h_1(1170)$	$0^-(1^{+-})$
• $b_1(1235)$	$1^+(1^{+-})$
• $a_1(1260)$	$1^-(1^{++})$
• $f_2(1270)$	$0^+(2^{++})$
• $f_1(1285)$	$0^+(1^{++})$
• $\eta(1295)$	$0^+(0^{-+})$
• $\pi(1300)$	$1^-(0^{-+})$
• $a_2(1320)$	$1^-(2^{++})$
• $f_0(1370)$	$0^+(0^{++})$
$h_1(1380)$	$?^-(1^{+-})$
$\pi_1(1400)$	$1^-(1^{-+})$
• $f_1(1420)$	$0^+(1^{++})$
• $\omega(1420)$	$0^-(1^{--})$
$f_2(1430)$	$0^+(2^{++})$
• $\eta(1440)$	$0^+(0^{-+})$
• $a_0(1450)$	$1^-(0^{++})$
• $\rho(1450)$	$1^+(1^{--})$
• $f_0(1500)$	$0^+(0^{++})$
$f_1(1510)$	$0^+(1^{++})$
• $f'_2(1525)$	$0^+(2^{++})$
$f_2(1565)$	$0^+(2^{++})$
$\pi_1(1600)$	$1^-(1^{-+})$
$X(1600)$	$2^+(2^{++})$
$a_1(1640)$	$1^+(1^{++})$
$f_2(1640)$	$0^+(2^{++})$
$\eta_2(1645)$	$0^+(2^{-+})$
• $\omega(1650)$	$0^-(1^{--})$
$X(1650)$	$0^-(?^{?-})$
$a_2(1660)$	$1^-(2^{++})$
• $\omega_3(1670)$	$0^-(3^{--})$

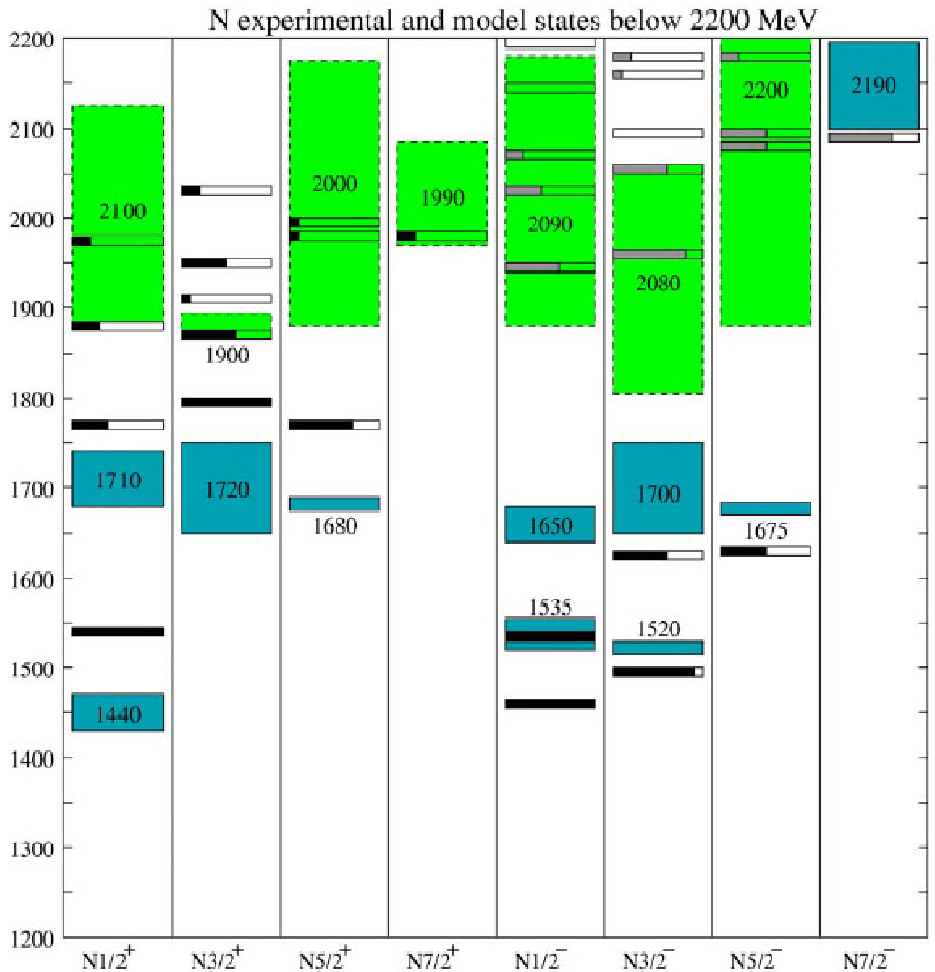
	$I^G(J^{PC})$
• $\pi_2(1670)$	$1^-(2^{-+})$
• $\phi(1680)$	$0^-(1^{--})$
• $\rho_3(1690)$	$1^+(3^{--})$
• $\rho(1700)$	$1^+(1^{--})$
• $f_0(1710)$	$0^+(0^{++})$
$a_2(1750)$	$1^-(2^{++})$
$\eta(1760)$	$0^+(0^{-+})$
$X(1775)$	$1^-(?^{-+})$
• $\pi(1800)$	$1^-(0^{-+})$
$f_2(1810)$	$0^+(2^{++})$
• $\phi_3(1850)$	$0^-(3^{--})$
$\eta_2(1870)$	$0^+(2^{-+})$
$X(1910)$	$0^+(?^{?+})$
$f_2(1950)$	$0^+(2^{++})$
$X(2000)$	$1^-(?^{?+})$
• $f_2(2010)$	$0^+(2^{++})$
$f_0(2020)$	$0^+(0^{++})$
• $a_4(2040)$	$1^-(4^{++})$
• $f_4(2050)$	$0^+(4^{++})$
$f_0(2060)$	$0^+(0^{++})$
$\pi_2(2100)$	$1^-(2^{-+})$
$f_2(2150)$	$0^+(2^{++})$
$\rho(2150)$	$1^+(1^{--})$
$f_0(2200)$	$0^+(0^{++})$
$f_J(2220)$	$0^+(2^{++}$ or $4^{++})$
$\eta(2225)$	$0^+(0^{-+})$
$\rho_3(2250)$	$1^+(3^{--})$
• $f_2(2300)$	$0^+(2^{++})$
$f_4(2300)$	$0^+(4^{++})$
• $f_2(2340)$	$0^+(2^{++})$
$\rho_5(2350)$	$1^+(5^{--})$
$a_6(2450)$	$1^-(6^{++})$
$f_6(2510)$	$0^+(6^{++})$
$X(3250)$	$?^?(?^{??})$

I=2 ???

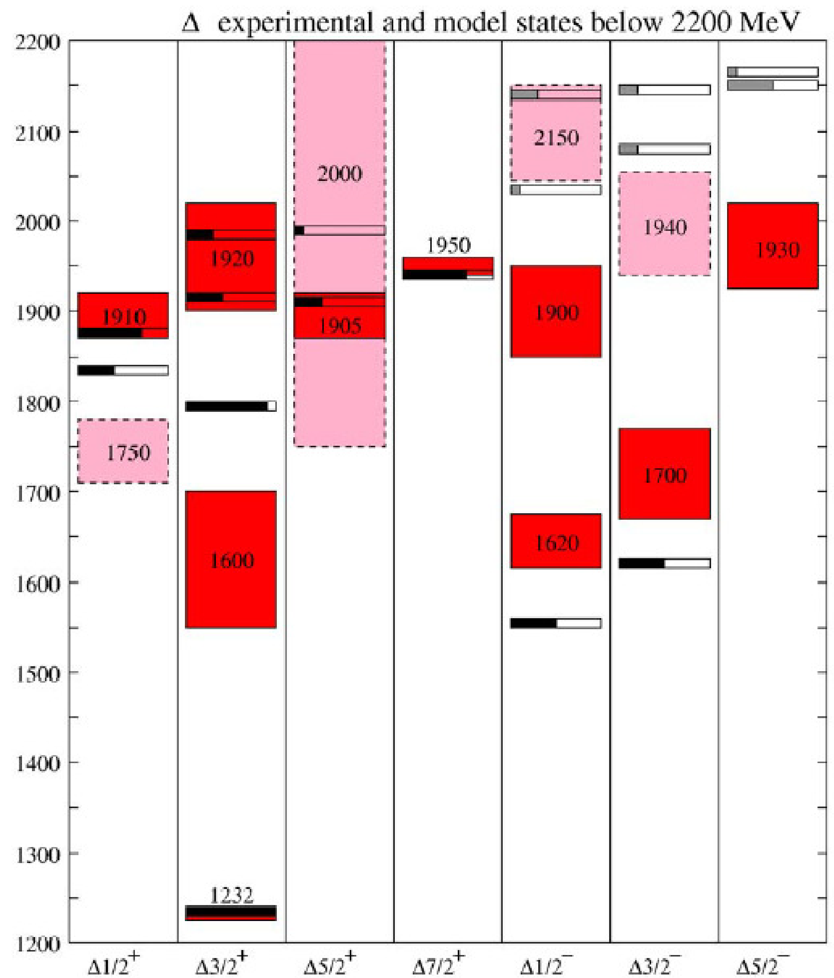
Mesons

I=0,1

Baryons – I

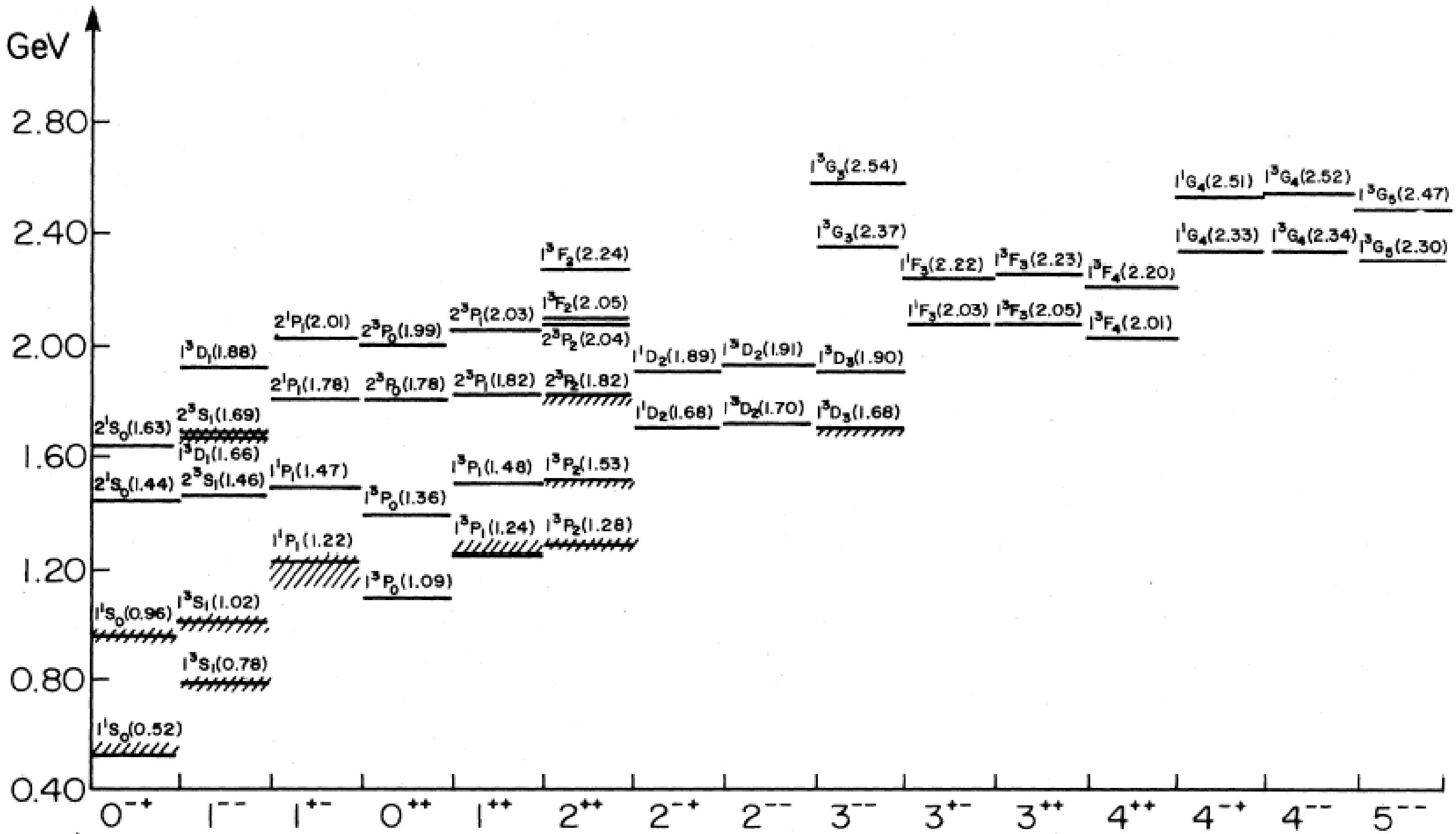

N experimental and model states below 2200 MeV
2200
2100
2000
1900
1800
1700
1600
1500
1400
1300
1200
2100
1900
2000
1990
2090
2080
2200
2190
1710
1720
1680
1650
1700
1675
1535
1520
1440
N1/2+
N3/2+
N5/2+
N7/2+
N1/2−
N3/2−
N5/2−
N7/2−

Baryons – II



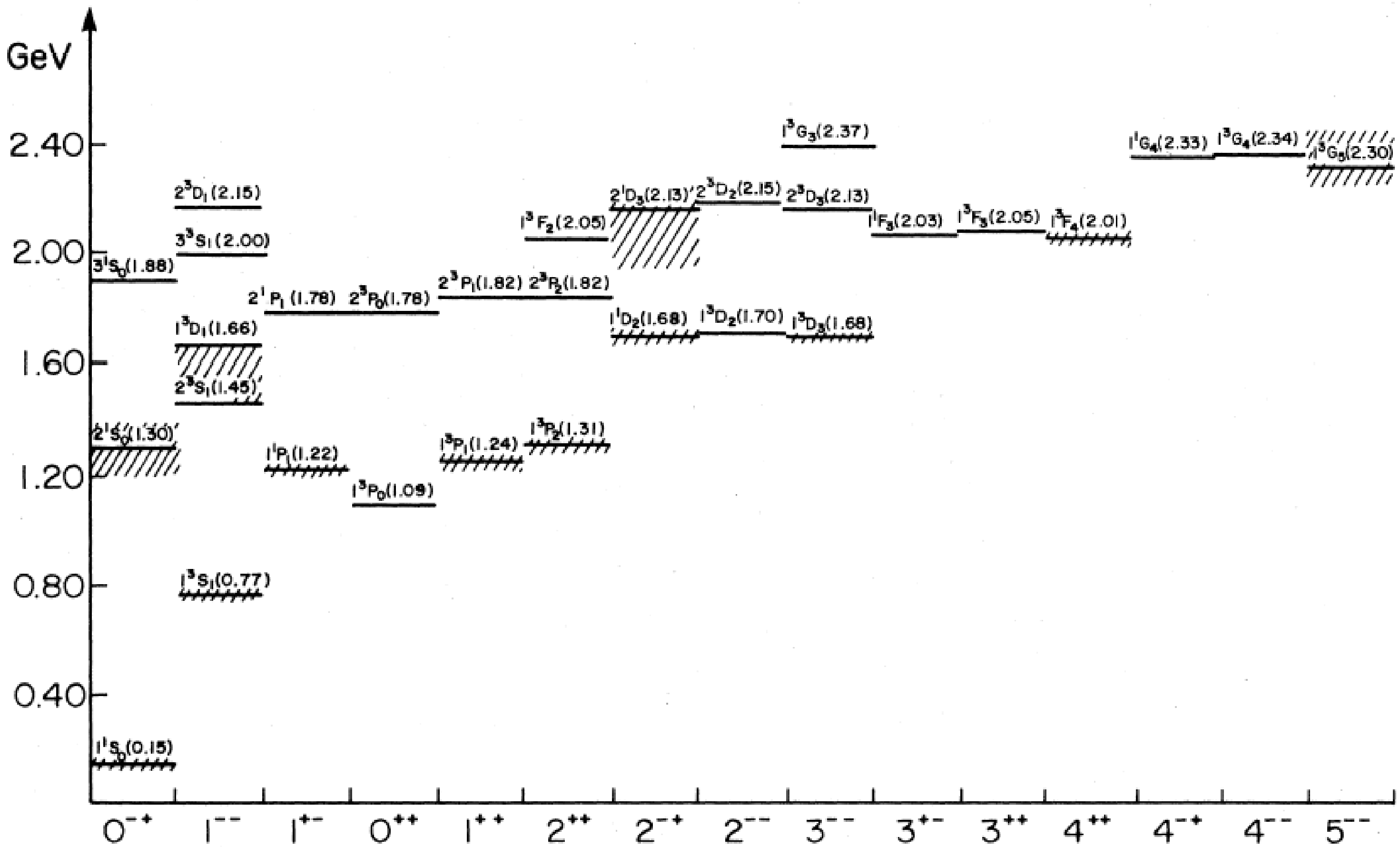
Δ experimental and model states below 2200 MeV
2200
2100
2000
1900
1800
1700
1600
1500
1400
1300
1200
1910
1750
2000
1920
1600
1232
1905
1950
2150
1900
1620
1940
1700
1930
Δ1/2+
Δ3/2+
Δ5/2+
Δ7/2+
Δ1/2−
Δ3/2−
Δ5/2−

Mesons – I



GeV
2.80
2.40
2.00
1.60
1.20
0.80
0.40
0^{-+}
1^{--}
1^{+-}
0^{++}
1^{++}
2^{++}
2^{-+}
2^{--}
3^{--}
3^{+-}
3^{++}
4^{++}
4^{-+}
4^{--}
5^{--}
$1^1S_0(0.52)$
$1^1S_0(0.96)$
$2^1S_0(1.44)$
$2^1S_0(1.63)$
$1^3S_1(0.78)$
$1^3S_1(1.02)$
$2^3S_1(1.46)$
$1^3D_1(1.66)$
$2^3S_1(1.69)$
$1^3D_1(1.88)$
$1^1P_1(1.22)$
$1^1P_1(1.47)$
$2^1P_1(1.78)$
$2^1P_1(2.01)$
$1^3P_0(1.09)$
$1^3P_0(1.36)$
$2^3P_0(1.78)$
$2^3P_0(1.99)$
$1^3P_1(1.24)$
$1^3P_1(1.48)$
$2^3P_1(1.82)$
$2^3P_1(2.03)$
$1^3P_2(1.28)$
$1^3P_2(1.53)$
$2^3P_2(1.82)$
$2^3P_2(2.04)$
$1^3F_2(2.05)$
$1^3F_2(2.24)$
$1^1D_2(1.68)$
$1^1D_2(1.89)$
$1^3D_2(1.70)$
$1^3D_2(1.91)$
$1^3D_3(1.68)$
$1^3D_3(1.90)$
$1^3G_3(2.37)$
$1^3G_3(2.54)$
$1^1F_3(2.03)$
$1^1F_3(2.22)$
$1^3F_3(2.05)$
$1^3F_3(2.23)$
$1^3F_4(2.01)$
$1^3F_4(2.20)$
$1^1G_4(2.33)$
$1^1G_4(2.51)$
$1^3G_4(2.34)$
$1^3G_4(2.52)$
$1^3G_5(2.30)$
$1^3G_5(2.47)$

Mesons – II


GeV
2.40
2.00
1.60
1.20
0.80
0.40
0⁻⁺ 1⁻⁻ 1⁺⁻ 0⁺⁺ 1⁺⁺ 2⁺⁺ 2⁻⁺ 2⁻⁻ 3⁻⁻ 3⁺⁻ 3⁺⁺ 4⁺⁺ 4⁻⁺ 4⁻⁻ 5⁻⁻
$1^1S_0(0.15)$
$2^1S_0(1.30)$
$3^1S_0(1.88)$
$1^3S_1(0.77)$
$2^3S_1(1.45)$
$1^3D_1(1.66)$
$3^3S_1(2.00)$
$2^3D_1(2.15)$
$1^1P_1(1.22)$
$2^1P_1(1.78)$
$1^3P_0(1.09)$
$2^3P_0(1.78)$
$1^3P_1(1.24)$
$2^3P_1(1.82)$
$1^3P_2(1.31)$
$2^3P_2(1.82)$
$1^3F_2(2.05)$
$1^1D_2(1.68)$
$2^1D_2(2.13)$
$1^3D_2(1.70)$
$2^3D_2(2.15)$
$1^3D_3(1.68)$
$2^3D_3(2.13)$
$1^3G_3(2.37)$
$1^1F_3(2.03)$
$1^3F_3(2.05)$
$1^3F_4(2.01)$
$1^1G_4(2.33)$
$1^3G_4(2.34)$
$1^3G_5(2.30)$

Gell-Mann – Nishijima Rule

B = Baryon number
Q = Charge in e units
I_3 = Isospin 3rd component

Empirical relationship for pions:

$$Q = I_3$$

Linking electromagnetic and strong properties of pions:
Electric charge as *3rd component* of isospin vector

Extend to nucleons:
$Q = I_3 + B/2$ Gell-Mann - Nishijima relation
More complicated properties:
Electric charge as both *isoscalar* and *3rd component of isovector*

Strangeness - I

Strange particles discovered in cosmic rays at the end of the '40s, and then quicky observed at the first GeV accelerators

Why strange?

Large production cross section $\rightarrow$ *Like ordinary hadrons*
Long lifetime $\rightarrow$ *Like weak decays*

Understood as carriers of a new quantum number: *Strangeness*

Ordinary hadrons $S = 0$
Strange particles $S \# 0$

Strangeness conserved by strong, e.m. processes, violated by weak
Explain funny behavior, also predicting *associated production* to guarantee *S* conservation in strong & EM processes:

Strange particles always produced in pairs

Strangeness - II

For strong processes, S similar to electric charge and to baryon or lepton numbers
But:

S not absolutely conserved

S not the source of a physical field

Large variety of strange particles, both baryons and mesons, including many strange resonances

Generalize Gell-Mann Nishijima relation to

$$Q = I_3 + \frac{B+S}{2} = I_3 + \frac{Y}{2}$$

$Y = B + S$ Hypercharge

Strangeness - III

Mesons

I_3	S=+1	S=-1
+1/2	K^+	K^0
-1/2	$\bar{K}^0$	K^-

Spin 0

I_3	S=+1	S=-1
+1/2	K^{*+}	$\bar{K}^{*0}$
-1/2	K^{*0}	K^{*-}

Spin 1

I_3	S	name
0	-1	Λ^0
+1,0,-1	-1	$\Sigma^+, \Sigma^-, \Sigma^0$
+1/2,-1/2	-2	Ξ^0, Ξ^-
0	-3	Ω^-

Baryons

I_3	S	name
0	+1	$\bar{\Lambda}^0$
+1,0,-1	+1	$\bar{\Sigma}^+, \bar{\Sigma}^0, \bar{\Sigma}^-$
+1/2,-1/2	+2	$\bar{\Xi}^0, \bar{\Xi}^-$
0	+3	$\bar{\Omega}^-$

Antibaryons

Strangeness - IV

Isospin conservation in

$$\pi^- + p \to \pi^- + p$$

leads in a natural way to extend to virtual states like

$$\pi^- + p \to \left(K^0 + \Lambda^0\right)^* \to \pi^- + p$$

→Strange particles should group into I-spin multiplets.

Λ^0 only observed as a neutral state $\to$ Singlet , $I = 0$

Observe 3 charge states for K: Triplet?

$\pi^- + p : I = 1/2, 3/2 \to K$ must be $I = 1/2, 3/2$

Quartets not observed $\to$ 2 Doublets! Predict *two* neutral K states, with opposite S

Would imply charge +2

$$\pi^- + p \to K^0 + \Lambda^0$$

$$p + \overline{p} \to K^0 + \overline{K}^0$$

Must be different particles!

Strangeness - V

Example: Historical Picture

$$K^- + p \to K^0 + K^+ + \Omega^-$$

$$K^0 \to \pi^+ + \pi^-$$

$$K^+ \to \pi^+ + \pi^0 \,(unseen)$$

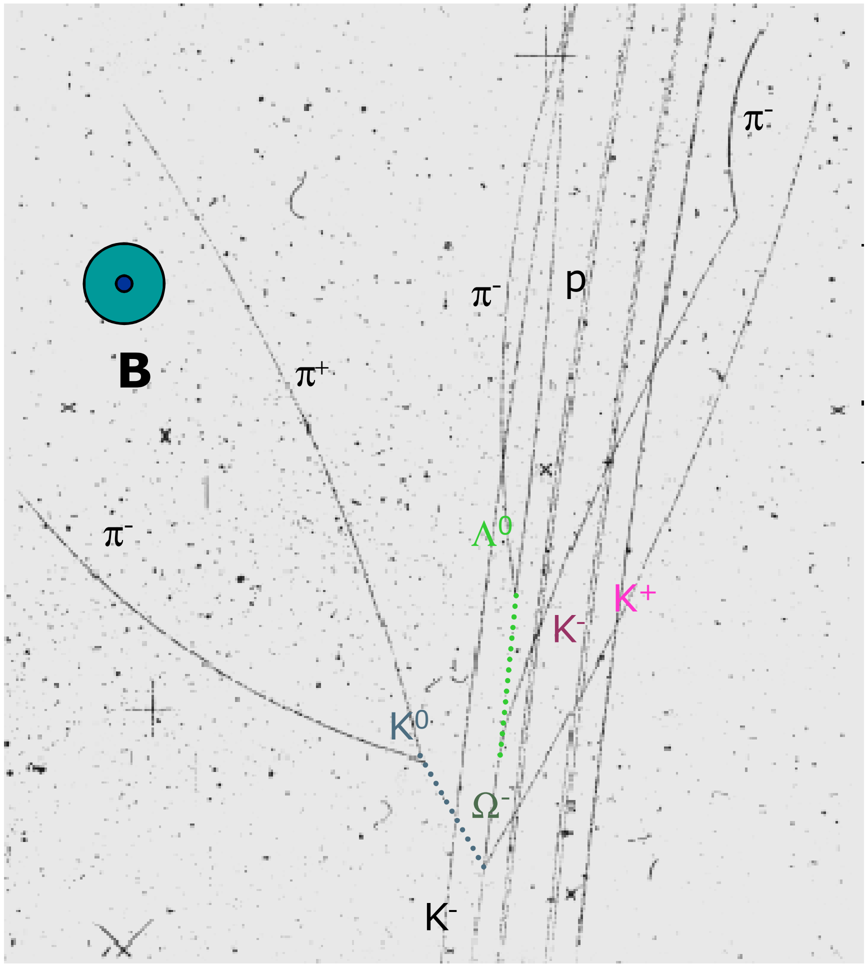
$$\Omega^- \to \Lambda^0 + K^-$$

$$\Lambda^0 \to p + \pi^-$$

$$K^- \to \pi^- + \pi^0 \,(unseen)$$

Beam momentum 4.2 GeV

Magnetic field 2 T



Strangeness Zoology

Baryons, S=*-1,-2,-3* (Antibaryons not shown)

Λ	P_{01}	****
$\Lambda(1405)$	S_{01}	****
$\Lambda(1520)$	D_{03}	****
$\Lambda(1600)$	P_{01}	***
$\Lambda(1670)$	S_{01}	****
$\Lambda(1690)$	D_{03}	****
$\Lambda(1800)$	S_{01}	***
$\Lambda(1810)$	P_{01}	***
$\Lambda(1820)$	F_{05}	****
$\Lambda(1830)$	D_{05}	****
$\Lambda(1890)$	P_{03}	****
$\Lambda(2000)$		*
$\Lambda(2020)$	F_{07}	*
$\Lambda(2100)$	G_{07}	****
$\Lambda(2110)$	F_{05}	***
$\Lambda(2325)$	D_{03}	*
$\Lambda(2350)$	H_{09}	***
$\Lambda(2585)$		**

Ω^-	****
$\Omega(2250)^-$	***
$\Omega(2380)^-$	**
$\Omega(2470)^-$	**

Ξ^0	P_{11}	****
Ξ^-	P_{11}	****
$\Xi(1530)$	P_{13}	****
$\Xi(1620)$		*
$\Xi(1690)$		***
$\Xi(1820)$	D_{13}	***
$\Xi(1950)$		***
$\Xi(2030)$		***
$\Xi(2120)$		*
$\Xi(2250)$		**
$\Xi(2370)$		**
$\Xi(2500)$		*

Σ^+	P_{11}	****
Σ^0	P_{11}	****
Σ^-	P_{11}	****
$\Sigma(1385)$	P_{13}	****
$\Sigma(1480)$		*
$\Sigma(1560)$		**
$\Sigma(1580)$	D_{13}	*
$\Sigma(1620)$	S_{11}	**
$\Sigma(1660)$	P_{11}	***
$\Sigma(1670)$	D_{13}	****
$\Sigma(1690)$		**
$\Sigma(1750)$	S_{11}	***
$\Sigma(1770)$	P_{11}	*
$\Sigma(1775)$	D_{15}	****
$\Sigma(1840)$	P_{13}	*
$\Sigma(1880)$	P_{11}	**
$\Sigma(1915)$	F_{15}	****
$\Sigma(1940)$	D_{13}	***
$\Sigma(2000)$	S_{11}	*
$\Sigma(2030)$	F_{17}	****
$\Sigma(2070)$	F_{15}	*
$\Sigma(2080)$	P_{13}	**
$\Sigma(2100)$	G_{17}	*
$\Sigma(2250)$		***
$\Sigma(2455)$		**
$\Sigma(2620)$		**
$\Sigma(3000)$		*
$\Sigma(3170)$		*

Mesons, S=±*1*

• $K^{\pm}$	$1/2(0^-)$
• K^0	$1/2(0^-)$
• K^0_S	$1/2(0^-)$
• K^0_L	$1/2(0^-)$
$K_0^*(800)$	$1/2(0^+)$
• $K^*(892)$	$1/2(1^-)$
• $K_1(1270)$	$1/2(1^+)$
• $K_1(1400)$	$1/2(1^+)$
• $K^*(1410)$	$1/2(1^-)$
• $K_0^*(1430)$	$1/2(0^+)$
• $K_2^*(1430)$	$1/2(2^+)$
$K(1460)$	$1/2(0^-)$
$K_2(1580)$	$1/2(2^-)$
$K(1630)$	$1/2(?^?)$
$K_1(1650)$	$1/2(1^+)$
• $K^*(1680)$	$1/2(1^-)$
• $K_2(1770)$	$1/2(2^-)$
• $K_3^*(1780)$	$1/2(3^-)$
• $K_2(1820)$	$1/2(2^-)$
$K(1830)$	$1/2(0^-)$
$K_0^*(1950)$	$1/2(0^+)$
$K_2^*(1980)$	$1/2(2^+)$
• $K_4^*(2045)$	$1/2(4^+)$
$K_2(2250)$	$1/2(2^-)$
$K_3(2320)$	$1/2(3^+)$
$K_5^*(2380)$	$1/2(5^-)$
$K_4(2500)$	$1/2(4^-)$
$K(3100)$	$?^?(?^{??})$

SU(3) - I

Experimental evidence for several 'multiplets of multiplets'

$J^P=0^-$

I	S=+1	S=0	S=-1
0		η,η'	
1/2	K		$\bar{K}$
1		π	

$J^P=1^-$

I	S=+1	S=0	S=-1
0		ω,φ	
1/2	K^*		$\bar{K}^*$
1		ρ	

$J^P=2^+$

I	S=+1	S=0	S=-1
0		f_0,f_1	
1/2	K^{**}		$\bar{K}^{**}$
1		a_2	

Mesons

$J^P=1/2^+$

I	S=-2	S=-1	S=0
0		Λ^0	
1/2	Ξ		N
1		Σ	

$J^P=3/2^+$

I	S=-3	S=-2	S=-1	S=0
0	Ω^-			
1/2		Ξ^*		
1			Σ^*	
3/2				Δ

Baryons

Remember: Each square is a *I-spin multiplet,* with size (2I+1)
Total of 45 particle states in this page!

SU(3) - II

Try to find a larger group to encompass both strangeness and isospin into a unified symmetry scheme.

Requirements:

2 commuting generators, since both S and I_3 are defined within any observed supermultiplet

NB SU(2) has just one, I_3

Multiplet structure matching experimental data

SU(3) - III

Take *SU(3)* as candidate to extend *SU(2):*

Group of unitary, unimodular 3x3 matrices

9 complex parameters $\rightarrow$ 18 real parameters

9 unitary conditions: $\left.\begin{array}{r} UU^{\dagger}=1 \\ \left(U^{\dagger}\right)_{ij}=U^{*}_{ji} \end{array}\right\} \rightarrow \sum_{j=1}^{3} a_{ij}a^{*}_{jk}=\delta_{ik}, \quad i,k=1,..,3$

1 unimodularity condition: $\det U=1$

$\rightarrow 18-10=8$ free, real parameters

$SU(3)$ - IV

As usual, for any unitary matrix

$$U = e^{iH}, \quad H \text{ Hermitian}$$

$$\det U = 1 \rightarrow \det e^{iH} = 1 \rightarrow e^{itr(H)} = 1 \rightarrow tr(H) = 0$$

8 parameters $\rightarrow$ 8 generators

Generalize Pauli matrices to *Gell-Mann matrices*

$$\lambda_1 = \begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \lambda_2 = \begin{pmatrix} 0 & -i & 0 \\ i & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \lambda_3 = \begin{pmatrix} 1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 0 \end{pmatrix} \lambda_4 = \begin{pmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ 1 & 0 & 0 \end{pmatrix},$$

$$\lambda_5 = \begin{pmatrix} 0 & 0 & -i \\ 0 & 0 & 0 \\ i & 0 & 0 \end{pmatrix}, \lambda_6 = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{pmatrix}, \lambda_7 = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & -i \\ 0 & i & 0 \end{pmatrix}, \lambda_8 = \frac{1}{\sqrt{3}} \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & -2 \end{pmatrix}$$

SU(3) - V

Commutators:

$$\left[\lambda_i,\lambda_j\right]=f_{ijk}\lambda_k,\quad f_{ijk}\ \text{structure constants}$$

Two diagonal generators, l_3 and l_8

$\rightarrow$ *Rank 2 group*
$\rightarrow$ 2 invariant functions of generators

Quadratic: $$C^{(2)}=\sum_{i,j=1}^{8}\delta_{ij}\lambda_i\lambda_j$$

Cubic: $$C^{(3)}=\sum_{i,j,k=1}^{8}f_{ijk}\lambda_i\lambda_j\lambda_k$$

$F_i\equiv\dfrac{\lambda_i}{2}$ Definition

Identify: $$\begin{cases} I_3=F_3 & \text{Isospin 3rd component} \\ Y=\dfrac{2}{\sqrt{3}}F_8 & \text{Hypercharge} \end{cases}$$

Compare to *SU(2):*

$$\left[\sigma_i,\sigma_j\right]=i\varepsilon_{ijk}\sigma_k$$

One diagonal generator, σ_3

$\rightarrow$ *Rank 1 group*

$\rightarrow$ *1* invariant function of generators

Quadratic: $$C^{(2)}=\sum_{i,j=1}^{3}\delta_{ij}\sigma_i\sigma_j$$

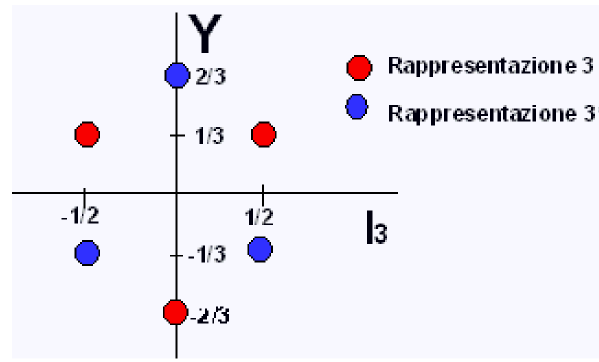
$SU(3)$ - VI

Fundamental representation (3 x 3 matrices): ***3***

Find eigenvalues & eigenvectors for **3**:

$$\begin{pmatrix}1\\0\\0\end{pmatrix}\to\begin{cases}I_3=\dfrac{1}{2}\\[2ex] Y=\dfrac{1}{3}\end{cases};\quad \begin{pmatrix}0\\1\\0\end{pmatrix}\to\begin{cases}I_3=-\dfrac{1}{2}\\[2ex] Y=\dfrac{1}{3}\end{cases};\quad \begin{pmatrix}0\\0\\1\end{pmatrix}\to\begin{cases}I_3=0\\[1ex] Y=-\dfrac{2}{3}\end{cases}$$

$Y= B+S$



$\to$ 3 independent base states

$\to I_3, Y$ eigenvalues not symmetrical wrt origin

$\to$ Conjugate representation: $\mathbf{3}^*$ different from **3**

$\to$ For both $\mathbf{3}, \mathbf{3}^*$ hypercharge eigenvalues fractionary

$\to Q = I_3 + Y/2$ fractionary!!!

$SU(3)$ - VII

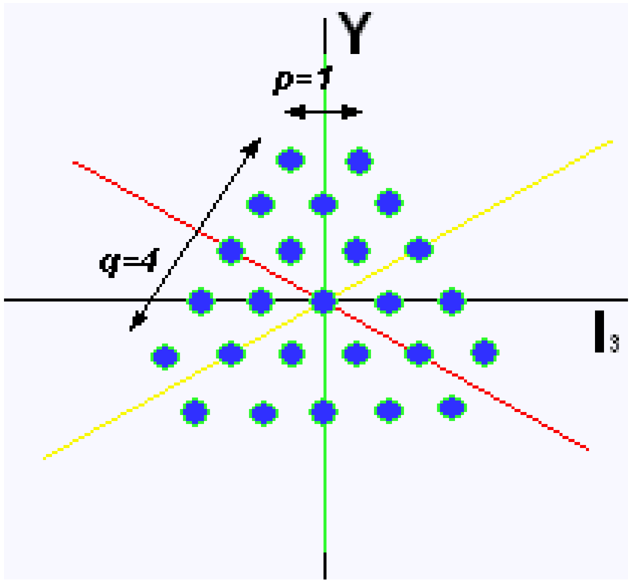
States identified by Y,I_3 eigenvalues
$\rightarrow$Points in a plane

Hexagonal/Triangular symmetry

Specified by *2* integers *(p,q)*

Multiplicity (i.e. size)

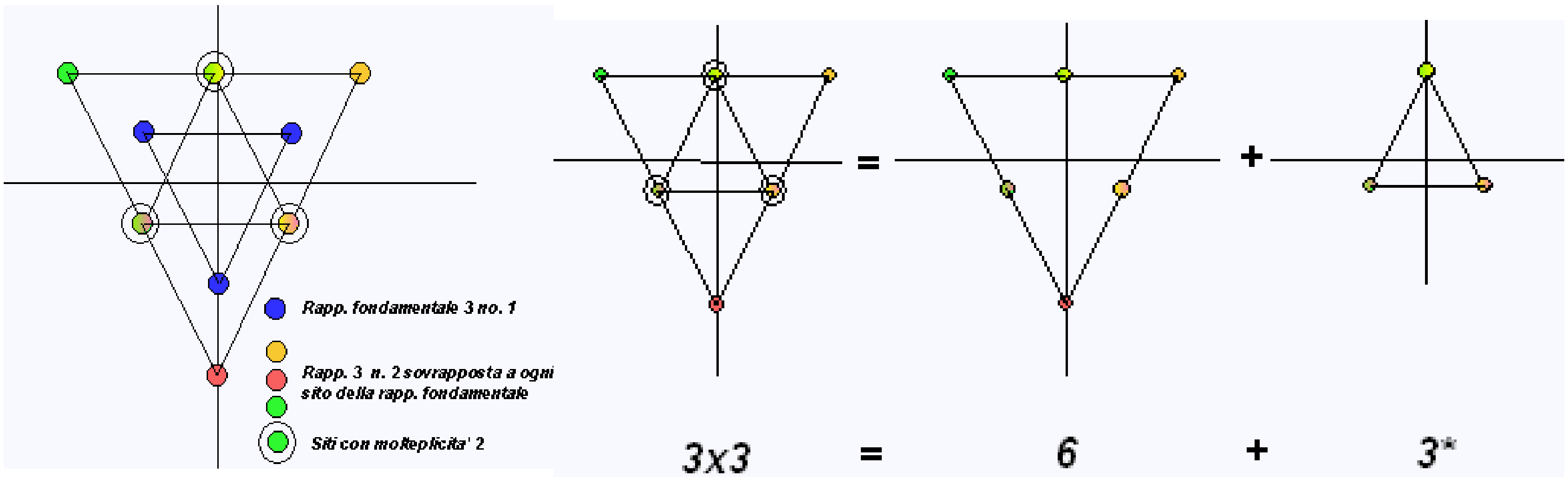
$$n = \frac{1}{2}(p+1)(q+1)(p+q+2)$$



Multiplet *(1,4)*
Frequently indicated by ***n**=35*

SU(3) - VIII

Products and decomposition into irr.rep.:
Proceed graphically as for *SU(2)*



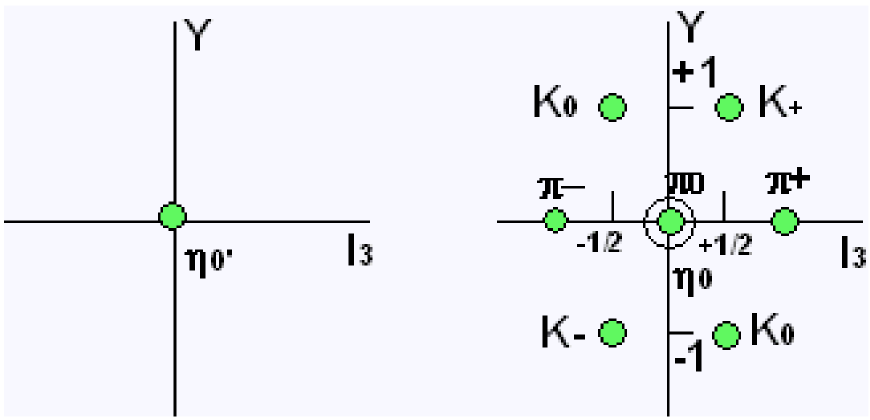
SU(3) - IX

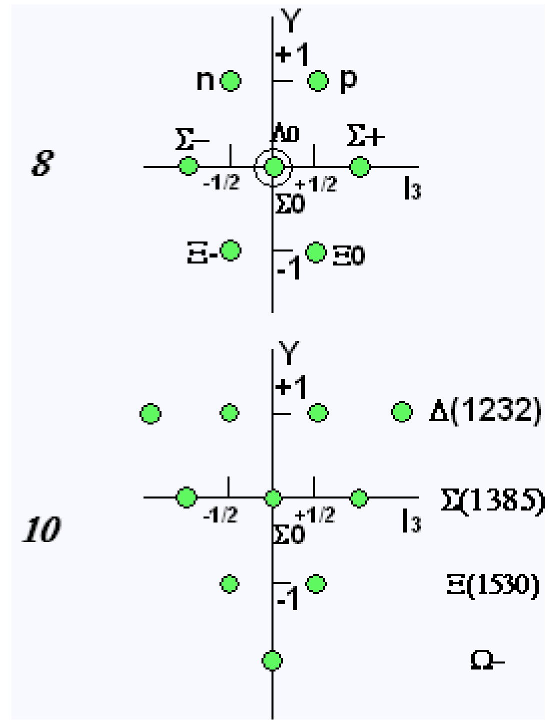
All the hadronic multiplets nicely fit some SU(3) representation
No hadron found which does not fit

Mesons $J^{PC}=0^{-+}$

Baryons $J^{P}=1/2^{+},3/2^{+}$

1 **8**





SU(3) Breaking

$J^P=0^-$

I	S=-1	S=0	S=+1
0		$\eta(547), \eta'(958)$	
1/2	$\bar{K}(496)$		$K(496)$
1		$\pi(137)$	

$J^P=1/2^+$

I	S=-2	S=-1	S=0
0		$\Lambda^0(1116)$	
1/2	$\Xi(1317)$		$N(938)$
1		$\Sigma(1192)$	

$J^P=1^-$

I	S=-1	S=0	S=+1
0		$\omega(782), \varphi(1020)$	
1/2	$\bar{K}^*(892)$		$K^*(892)$
1		$\rho(770)$	

$J^P=3/2^+$

I	S=-3	S=-2	S=-1	S=0
0	$\Omega^-(1672)$			
1/2		$\Xi^*(1530)$		
1			$\Sigma^*(1385)$	
3/2				$\Delta(1232)$

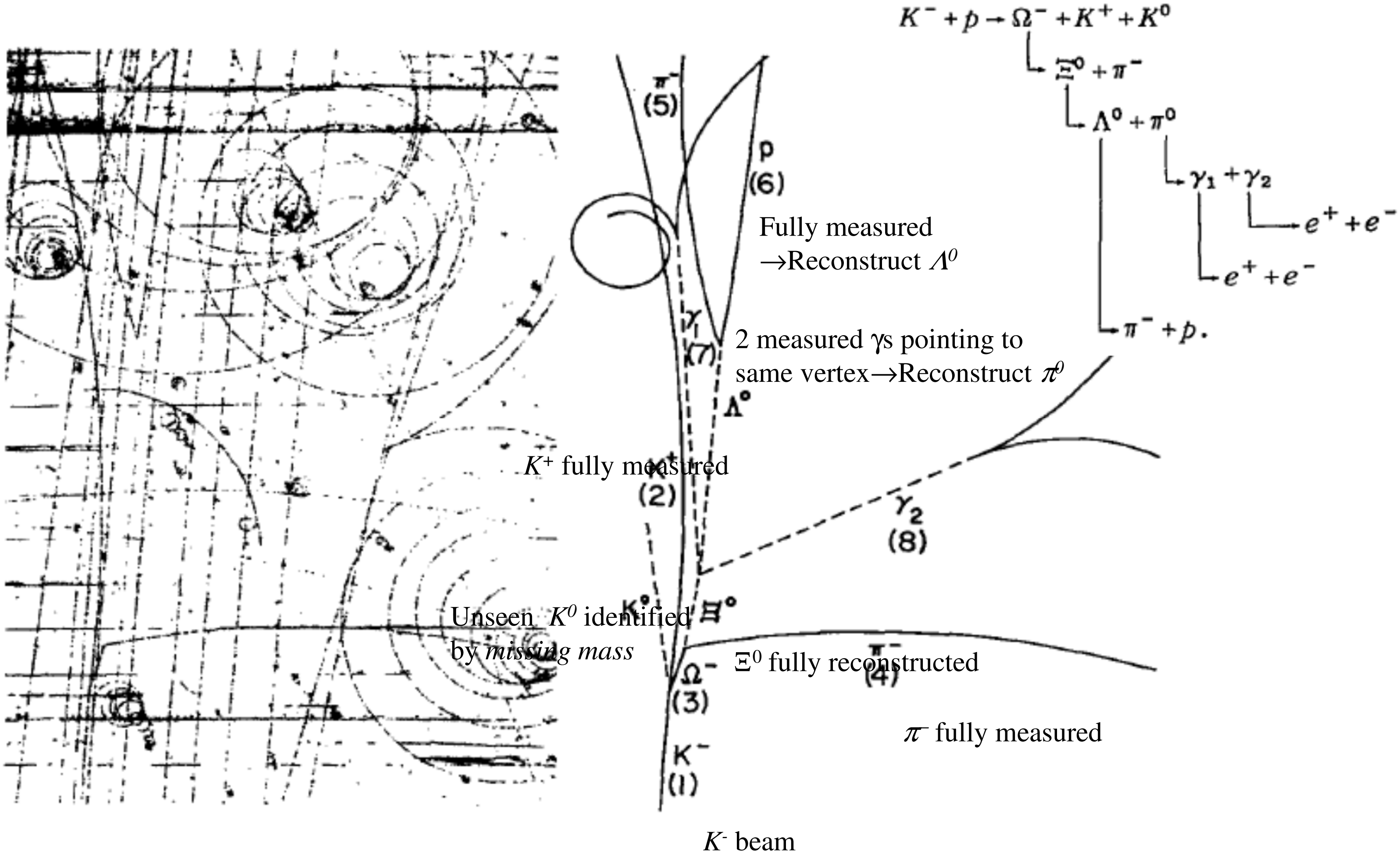
$J^P=2^+$

I	S=-1	S=0	S=+1
0		$f_2(1270), f_2'(1525)$	
1/2	$\bar{K}^{**}(1430)$		$K^{**}(1430)$
1		$a_2(1320)$	

As before, but including masses: *SU(3)* is not an exact symmetry

Mass differences within a multiplet are large, typ. $\Delta m/m \sim 10\text{-}20\%$

Ω^- Discovery at BNL



$\to$ Get $M_{\Omega^-} = 1675\ \ MeV\,!$

Quark Model - I

Fundamental hypothesis:

Mesons = Bound states $q\bar{q}$

Baryons, Antibaryons = Bound states $qqq, \overline{qqq}$

What are states $q, \bar{q}$? They are called *quark, antiquark*
Building blocks of ordinary hadrons:
A new level of structure for the hadronic matter

Quarks fill the fundamental representation of *SU(3)*
Quarks are spin *1/2*, point-like fermions
Guess:

$$\left.\begin{array}{l}\textit{They are never observed as free particles}\\ \textit{The only bound states observed are } q\bar{q}, qqq, \overline{qqq}\end{array}\right\} \text{ Why ?}$$

Quark Model - II

Fundamental and conjugate irr.rep. of *SU(3)*: ***3***, ***3****
Each made of 3 states

Quantum numbers: From Gell-Mann – Nishijima & SU(3) $Q = I_3 + Y/2$

Symbol	Flavor	Spin	Q	B	S	Y	I	I_3
u	*Up*	½	*2/3*	*1/3*	*0*	*1/3*	*1/2*	*+1/2*
d	*Down*	½	*-1/3*	*1/3*	*0*	*1/3*	*1/2*	*-1/2*
s	*Strange*	½	*-1/3*	*1/3*	*-1*	*-2/3*	*0*	*0*

} isospin doublet

isospin singlet

Quarks are predicted to carry fractional charge, baryon number!

Should they show up as free particles, would be easy to detect :

Expect unusual electromagnetic rates $\propto Q^2$

Expect bound states with fractional mass numbers $\propto B$

Quark Model - III

Hadrons: Expected to fill product representations

From our group theory rudiments:

Mesons $$3 \otimes 3^* = 1 \oplus 8$$

Baryons $$3 \otimes 3 \otimes 3 = 1 \oplus 8 \oplus 8 \oplus 10$$

Expect:

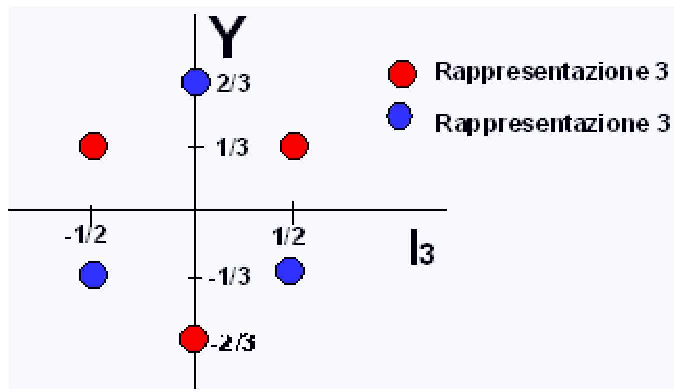
Nonets of mesons with given spin, parity
Singlets, octets, decuplets of baryons, as above

Quark Model - IV

More quantum numbers

Relative space parity = -1 (Fermions)
Absolute space parity = +1 quarks, -1 antiquarks (Conventional)

Flavor	Spin	Q	B	S	Y	I	I_3
Up	½	*2/3*	*1/3*	*0*	*1/3*	*1/2*	*+1/2*
Down	½	*-1/3*	*1/3*	*0*	*1/3*	*1/2*	*-1/2*
Strange	½	*-1/3*	*1/3*	*-1*	*-2/3*	*0*	*0*



Flavor	Spin	Q	B	S	Y	I	I_3
Anti-Up	½	*-2/3*	*-1/3*	*0*	*-1/3*	*1/2*	*-1/2*
Anti-Down	½	*+1/3*	*-1/3*	*0*	*-1/3*	*1/2*	*+1/2*
Anti-Strange	½	*+1/3*	*-1/3*	*+1*	*+2/3*	*0*	*0*

Quark Model - V

Q: *Why are isospin 3rd components swapped for antiquarks?*

A: *Want to stick to Gell-Mann – Nishijima for them too*

Required in order to deal with qqq, $q\bar{q}, \overline{qqq}$

E.g. all present in the same process

$$u = \begin{pmatrix} 1 \\ 0 \end{pmatrix}, d = \begin{pmatrix} 0 \\ 1 \end{pmatrix}$$

$$Q = I_3 + \frac{Y}{2} = I_3 + \frac{B}{2}$$

$$Q(\bar{u}) = -\frac{2}{3} = I_3(\bar{u}) + \frac{B(\bar{u})}{2} = I_3(\bar{u}) - \frac{1}{6} \rightarrow I_3(\bar{u}) = -\frac{2}{3} + \frac{1}{6} = -\frac{1}{2}$$

$$Q(\bar{d}) = +\frac{1}{3} = I_3(\bar{d}) + \frac{B(\bar{d})}{2} = I_3(\bar{d}) - \frac{1}{6} \rightarrow I_3(\bar{d}) = +\frac{1}{3} + \frac{1}{6} = +\frac{1}{2}$$

Quark Model - VI

Q: *Why there is a -1 extra phase for u antiquark?* $\begin{pmatrix} \bar{d} \\ -\bar{u} \end{pmatrix}$

A: *Want to stick to same C-G coefficient for both quarks and antiquarks*

Same C-G ↔ Same I-spin rotation matrices

Indeed, required because mesons *are* made of quark-antiquark pairs

$$\begin{pmatrix} u' \\ d' \end{pmatrix} = e^{-i\tau_2\theta/2}\begin{pmatrix} u \\ d \end{pmatrix} = \begin{pmatrix} \cos(\theta/2)u - \sin(\theta/2)d \\ \sin(\theta/2)u + \cos(\theta/2)d \end{pmatrix}$$ Rotation of generic state

$$\rightarrow \begin{pmatrix} \bar{d}' \\ \bar{u}' \end{pmatrix} = \begin{pmatrix} \sin(\theta/2)\bar{u} + \cos(\theta/2)\bar{d} \\ \cos(\theta/2)\bar{u} - \sin(\theta/2)\bar{d} \end{pmatrix} = \begin{pmatrix} \cos(\theta/2)\bar{d} + \sin(\theta/2)\bar{u} \\ -\sin(\theta/2)\bar{d} + \cos(\theta/2)\bar{u} \end{pmatrix}$$

$$\rightarrow \begin{pmatrix} \bar{d}' \\ \bar{u}' \end{pmatrix} = \begin{pmatrix} \cos(\theta/2)\bar{d} - \sin(\theta/2)(-\bar{u}) \\ -\sin(\theta/2)\bar{d} - \cos(\theta/2)(-\bar{u}) \end{pmatrix}$$

$$\rightarrow \begin{pmatrix} \bar{d}' \\ -\bar{u}' \end{pmatrix} = \begin{pmatrix} \cos(\theta/2)\bar{d} - \sin(\theta/2)(-\bar{u}) \\ \sin(\theta/2)\bar{d} + \cos(\theta/2)(-\bar{u}) \end{pmatrix} = e^{-i\tau_2\theta/2}\begin{pmatrix} \bar{d} \\ -\bar{u} \end{pmatrix}$$

Light Mesons - I

Combine 3 quarks with 3 antiquarks: Get 9 combinations

Quark content $u\bar{d}, u\bar{s}, u\bar{u}, d\bar{u}, d\bar{s}, d\bar{d}, s\bar{u}, s\bar{d}, s\bar{s}$

Identified mesons

'State'	Q	I_3	I	S	Remarks	$J^{PC}=0^{-+}$	$J^{PC}=1^{--}$	$J^{PC}=2^{++}$
$u\bar{d}$	+1	+1	1	0		π^+	ρ^+	a_2^+
$u\bar{s}$	+1	+1/2	½	+1		K^+	K^{+*}	K^{+**}
$u\bar{u}$	0	0	0,1	0	*I-spin undefined*	π^0, η, η'	ρ^0, ω, φ	a_2^0, f_2, f_2'
$d\bar{u}$	-1	-1	1	0		π^-	ρ^-	a_2^-
$d\bar{s}$	0	-1/2	1/2	+1		K^0	K^{0*}	K^{0**}
$d\bar{d}$	0	0	0,1	0	*I-spin undefined*	π^0, η, η'	ρ^0, ω, φ	a_2^0, f_2, f_2'
$s\bar{u}$	-1	-1/2	1/2	-1		K^-	K^{-*}	K^{-**}
$s\bar{d}$	0	+1/2	1/2	-1		$\bar{K}^0$	$\bar{K}^{0*}$	$\bar{K}^{0**}$
$s\bar{s}$	0	0	0	0		π^0, η, η'	ρ^0, ω, φ	a_2^0, f_2, f_2'

L # 0

Light Mesons - II

$$\mathbf{J} = \mathbf{L} + \mathbf{S}$$

$$P = (-1)^{l+1}$$

$$C = (-1)^{l+s}$$

Ground state $L = 0 \rightarrow J = S$

Singlets $\rightarrow J = 0 \rightarrow P = -1, C = +1 \rightarrow J^{PC} = 0^{-+}$

Triplets $\rightarrow J = 1 \rightarrow P = -1, C = -1 \rightarrow J^{PC} = 1^{--}$

Remark 1:
Very simple and clear, but: Not covariant!
J separation into ***L,S*** contributions is frame dependent
$\rightarrow$We are assuming small quark speed: Is this correct?

Remark 2:
Higher spin multiplets more difficult to explain, due to orbital degrees of freedom

Light Mesons - III

Physical particles must have *I* defined: *I*-spin is a good symmetry
Build isospin eigenstates from $S=0, I_3=0$ states:

$$\frac{1}{\sqrt{2}}\left(u\bar{u}-d\bar{d}\right),\frac{1}{\sqrt{6}}\left(u\bar{u}+d\bar{d}-2s\bar{s}\right),\frac{1}{\sqrt{3}}\left(u\bar{u}+d\bar{d}+s\bar{s}\right)$$

6 unambiguous states are octet members
Left with 3 ambiguous states: $I_3=0 \rightarrow$2 octets, 1 singlet ambiguous
SU(3) singlet: Invariant wrt SU(3) rotations

$$\frac{1}{\sqrt{3}}\left(u\bar{u}+d\bar{d}+s\bar{s}\right): \quad \eta_1$$

SU(3) Octets: 1 *SU(2)* triplet, 1 *SU(2)* singlet

$$\frac{1}{\sqrt{2}}\left(u\bar{u}-d\bar{d}\right): \quad \pi^0; \qquad \frac{1}{\sqrt{6}}\left(u\bar{u}+d\bar{d}-2s\bar{s}\right): \quad \eta_8$$

η_1, η_8 cannot be identified with physical particles

Light Mesons - IV

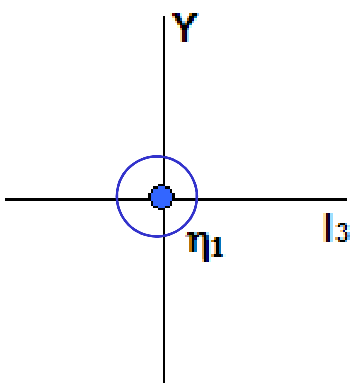
Particle identification with SU(3) eigenstates not always straightforward
Example: Take pseudoscalars

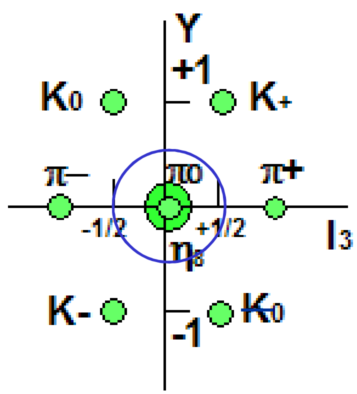
$$\left|\mathbf{8};1,0\right\rangle = \frac{1}{\sqrt{2}}\left(u\bar{u} - d\bar{d}\right) \rightarrow \pi^0$$

Must be true because I-spin is a good symmetry

$$\left.\begin{array}{l} \left|\mathbf{8};0,0\right\rangle = \frac{1}{\sqrt{6}}\left(u\bar{u} + d\bar{d} - 2s\bar{s}\right) \\ \left|\mathbf{1};0,0\right\rangle = \frac{1}{\sqrt{2}}\left(u\bar{u} + d\bar{d} + s\bar{s}\right) \end{array}\right\}$$

Not identified
Get some insight
from decay modes





π^0, η_1, η_8 Central states, $I_3 = Y = Q = 0$

Light Mesons - V

Use *SU(2)* shift operators: First, π^+

$I^-|\pi^+\rangle = \sqrt{2}|\pi^0\rangle$ From definition (and multiplet diagram)

From π^+ wave function:

$$I^-|\pi^+\rangle = I^-|u\bar{d}\rangle = |d\bar{d} - u\bar{u}\rangle \Rightarrow \pi^0 = -\frac{1}{\sqrt{2}}|d\bar{d} - u\bar{u}\rangle$$

Then re-define π^+ $as\ -u\bar{d} \rightarrow \pi^0 = \frac{1}{\sqrt{2}} I^- \pi^+$

Repeat for π^0:

$$I^-\pi^0 = \sqrt{2}\pi^- = I^- \frac{1}{\sqrt{2}}|u\bar{u} - d\bar{d}\rangle = \frac{1}{\sqrt{2}}|d\bar{u} + d\bar{u}\rangle \Rightarrow \pi^- = d\bar{u}$$

Isosinglet (with *u* and *d* only), is *h*:

$$I^-\eta = I^-\left(\frac{d\bar{d} + u\bar{u}}{\sqrt{2}}\right) = \frac{-d\bar{u} + d\bar{u}}{\sqrt{2}} = 0$$

Conclude the π^0 is an octet, don't know about η_1, η_8

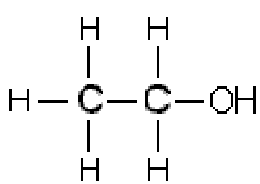
Light Baryons - I

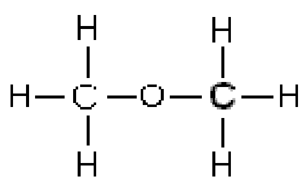
Combine 3 quarks: Get 3x3x3 = 27 combinations
But: Only 10 different quark contents

$$3+3\cdot 2+1=10: \quad uuu, ddd, sss, uud, uus, ddu, dds, ssu, ssd, uds$$

Remember:
Same composition does not imply same quantum state

Somewhat similar to difference between *raw* and *structural* formulae
Examples:





Same atomic content, 2 different chemicals

$$\frac{1}{\sqrt{2}}\left(|pn\rangle+|np\rangle\right) \quad \text{symmetric}$$

No bound states

Same nuclear content, 2 different states

$$\frac{1}{\sqrt{2}}\left(|pn\rangle-|np\rangle\right) \quad \text{antisymmetric}$$

One bound state

Light Baryons - II

SU(3) Multiplets: ***1, 8, 8, 10***

What about different quark masses?
Well, that's all out of SU(3) *breaking*..

Reminder:

Quarks of different flavor to be taken as *different states of identical particles* (like electrons with spin up, down)
$\rightarrow$ Multi-quark states expected to have definite *exchange symmetry*

Can derive flavor exchange symmetry of each multiplet

1 – Singlet
Fully antisymmetric

8 – Two Octets
Undefined symmetry

10 – Decuplet
Fully symmetric

Light Baryons - III

Now look at the remaining part of the wave function:

$\left|a\right\rangle = \left|space\right\rangle \left|spin\right\rangle \left|flavor\right\rangle$ NB: This expression is incomplete! See later

Space: Expect S-Wave $\rightarrow$ *Symmetric*
Difficult to guess an effective potential originating a ground state with L#0

Spin: Quarks are Fermions
Combine 3 spin ½:

$$1/2 \oplus 1/2 = \begin{cases} 0 \rightarrow 0 \oplus 1/2 = 1/2 & \text{2 sub-states} \\ 1 \rightarrow 1 \oplus 1/2 = 1/2, 3/2 & \text{2+4 sub-states} \end{cases}$$

$\rightarrow$ Expect 1 quartet, 2 doublets

$$\left.\begin{array}{l} \left|3/2,+3/2\right\rangle = \left(\uparrow\uparrow\uparrow\right), \quad \left|3/2,-3/2\right\rangle = \left(\downarrow\downarrow\downarrow\right) \\ \left|3/2,+1/2\right\rangle = 1/\sqrt{3}\left(\uparrow\uparrow\downarrow + \uparrow\downarrow\uparrow + \downarrow\uparrow\uparrow\right), \quad \left|3/2,-1/2\right\rangle = 1/\sqrt{3}\left(\downarrow\downarrow\uparrow + \downarrow\uparrow\downarrow + \uparrow\downarrow\downarrow\right) \end{array}\right\} \text{Quartet - Symmetric}$$

$$\left.\left|1/2,+1/2\right\rangle_A = 1/\sqrt{2}\left(\uparrow\downarrow - \downarrow\uparrow\right)\uparrow, \quad \left|1/2,-1/2\right\rangle_A = 1/\sqrt{2}\left(\uparrow\downarrow - \downarrow\uparrow\right)\downarrow\right\} \text{Doublet - Antisymmetric 1-2}$$

$$\left.\left|1/2,+1/2\right\rangle_S = 1/\sqrt{2}\uparrow\left(\uparrow\downarrow - \downarrow\uparrow\right), \quad \left|1/2,-1/2\right\rangle_S = 1/\sqrt{2}\downarrow\left(\uparrow\downarrow - \downarrow\uparrow\right)\right\} \text{Doublet - Antisymmetric 2-3}$$

Light Baryons - IV

Can use another bit of group theory to write:

$$2 \otimes 2 \otimes 2 = 4 \oplus 2_S \oplus 2_A \quad \text{spin}$$

$$3 \otimes 3 \otimes 3 = 1 \oplus 8_S \oplus 8_A \oplus 10 \quad \text{flavor}$$

Summary of flavor, spin symmetry of different representations:

SU(3)

Flavor	Symmetry	Spin	Symmetry
10_S	*S*	4_S	*S*
$8_{M,S}$	*n.a.; symmetric 1-2*	$2_{M,S}$	*n.a.; symmetric 1-2*
$8_{M,A}$	*n.a.; antisymmetric 1-2*	$2_{M,A}$	*n.a.; antisymmetric 1-2*
1_A	*A*		

SU(2)

Now combine flavor *and* spin:

S,A,M referring to *flavor*spin*

	10_S	$8_{M,S}$	$8_{M,A}$	1_A
4_S	**(10,4) S**	**(8,4) M**	**(8,4) M**	**(1,4) A**
$2_{M,S}$	**(10,2) M**	**(8,2) M**	**(8,2) M**	**(1,2) M**
$2_{M,A}$	**(10,2) M**	**(8,2) M**	**(8,2) M**	**(1,2) M**

Singlet, Decuplet - I

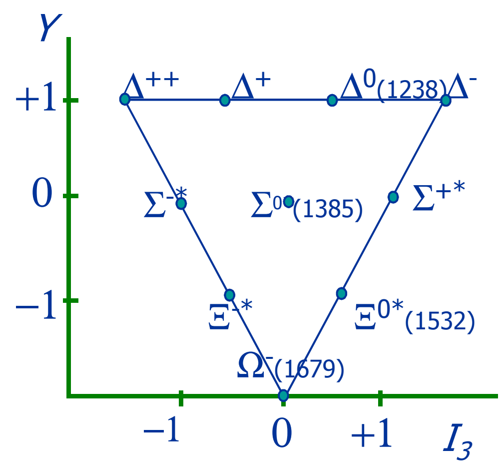
Observed multiplets

(SU(3), SU(2)) flavor spin	Flavor Wave-Function
Singlet: (***1***, ***?***) Tricky..	$\frac{1}{\sqrt{6}}\left(uds - usd + dsu - dus + sud - sdu\right)$
Decuplet: (***10, 4***) Astonishing..	$\begin{cases} uuu, ddd, sss, \frac{1}{\sqrt{6}}\left(uds + usd + dsu + dus + sud + sdu\right) \\ \frac{1}{\sqrt{3}}\left(ddu + dud + udd\right), \frac{1}{\sqrt{3}}\left(uud + udu + duu\right), \\ \frac{1}{\sqrt{3}}\left(dds + dsd + sdd\right), \frac{1}{\sqrt{3}}\left(uus + usu + suu\right), \\ \frac{1}{\sqrt{3}}\left(ssd + sds + dss\right), \frac{1}{\sqrt{3}}\left(ssu + sus + uss\right) \end{cases}$

Singlet, Decuplet - II

Most unexpected:

Total wave function appears to be exchange symmetric for decuplet!
Would expect it *anti-symmetric* for a bundle of identical fermions
Are we forgetting something in this game?



Baryon resonances, except Ω^-

Octet - I

Assume a globally *symmetric* wave-function for octet too:
Very difficult to account for a multiplet-dependent symmetry!
Guess the symmetric spin-flavor part:

Flavor: Two sets, 8 states each

$$\frac{1}{\sqrt{2}}(ud-du)d,\frac{1}{\sqrt{2}}(ud-du)u,$$
$$\frac{1}{\sqrt{2}}(ds-sd)d,\frac{1}{\sqrt{2}}(ds-sd)s,$$
$$\frac{1}{\sqrt{2}}(us-su)u,\frac{1}{\sqrt{2}}(us-su)s,$$
$$\frac{1}{2}\left[(us-su)d+(ds-sd)u\right],$$
$$\frac{1}{\sqrt{12}}\left[2(ud-du)s+(us-su)d-(ds-sd)u\right]$$
$$\varphi_{A12}^{(i)},i=1,8$$

Antisymmetric 1↔2

$$\frac{1}{\sqrt{2}}d(ud-du),\frac{1}{\sqrt{2}}u(ud-du),$$
$$\frac{1}{\sqrt{2}}d(ds-sd),\frac{1}{\sqrt{2}}s(ds-sd),$$
$$\frac{1}{\sqrt{2}}u(us-su),\frac{1}{\sqrt{2}}s(us-su),$$
$$\frac{1}{2}\left[d(us-su)+u(ds-sd)\right],$$
$$\frac{1}{\sqrt{12}}\left[2s(ud-du)+d(us-su)-u(ds-sd)\right]$$
$$\varphi_{A23}^{(i)},i=1,8$$

Antisymmetric 2↔3

Octet - II

Spin: Two sets, 2 states each

$$\left.\begin{array}{l} |1/2,+1/2\rangle_A = 1/\sqrt{2}\,(\uparrow\downarrow - \downarrow\uparrow)\uparrow \\ |1/2,-1/2\rangle_A = 1/\sqrt{2}\,(\uparrow\downarrow - \downarrow\uparrow)\downarrow \end{array}\right\} \chi_{A12}^{(j)},\ j=1,2$$

$$\left.\begin{array}{l} |1/2,+1/2\rangle_S = 1/\sqrt{2}\uparrow(\uparrow\downarrow - \downarrow\uparrow) \\ |1/2,-1/2\rangle_S = 1/\sqrt{2}\downarrow(\uparrow\downarrow - \downarrow\uparrow) \end{array}\right\} \chi_{A23}^{(j)},\ j=1,2$$

Can also write down a 3rd set of 8 (flavor) or 2 (spin) wave functions, antisymmetric wrt 1↔3:

$\varphi_{A13}^{(i)}, i=1,8,\ \chi_{A13}^{(j)}, j=1,2$

Not independent from the former

Octet - III

Question:

What is the spin-flavor wave function of, say, a proton with spin up?

Answer:

Must consider all symmetric spin-flavor products with the proper quark content and s_z

The appropriate functions are n.*2* (flavor) and n.*1* (spin)

$$\varphi = \begin{cases} \varphi_{A12}^{(2)} = \dfrac{1}{\sqrt{2}}(ud - du)u \\ \varphi_{A23}^{(2)} = \dfrac{1}{\sqrt{2}}u(ud - du) \\ \varphi_{A13}^{(2)} = \dfrac{1}{\sqrt{2}}(uud - duu) \end{cases} \qquad \chi = \begin{cases} \chi_{A12}^{(1)} = \dfrac{1}{\sqrt{2}}(\uparrow\downarrow - \downarrow\uparrow)\uparrow \\ \chi_{A23}^{(1)} = \dfrac{1}{\sqrt{2}}\uparrow(\uparrow\downarrow - \downarrow\uparrow) \\ \chi_{A13}^{(1)} = \dfrac{1}{\sqrt{2}}(\uparrow\uparrow\downarrow - \downarrow\uparrow\uparrow) \end{cases}$$

$$\text{Products:} \quad \begin{cases} \varphi_{A12}^{(2)}\chi_{A12}^{(1)} = \dfrac{1}{\sqrt{2}}(ud - du)u\dfrac{1}{\sqrt{2}}(\uparrow\downarrow - \downarrow\uparrow)\uparrow \\ \varphi_{A23}^{(2)}\chi_{A23}^{(1)} = \dfrac{1}{\sqrt{2}}u(ud - du)\dfrac{1}{\sqrt{2}}\uparrow(\uparrow\downarrow - \downarrow\uparrow) \\ \varphi_{A13}^{(2)}\chi_{A13}^{(1)} = \dfrac{1}{\sqrt{2}}(uud - duu)\dfrac{1}{\sqrt{2}}(\uparrow\uparrow\downarrow - \downarrow\uparrow\uparrow) \end{cases}$$

Octet - IV

In order to get a fully exchange-symmetric state, must take a linear combination of *all* contributions

$$\left|p,+1/2\right\rangle = \sum_{k=A12}^{A13} \varphi_k^{(2)} \chi_k^{(1)}$$

$$\left|p,+1/2\right\rangle = \frac{1}{\sqrt{3}} \begin{bmatrix} \frac{1}{\sqrt{2}}\left(ud-du\right)u\frac{1}{\sqrt{2}}\left(\uparrow\downarrow-\downarrow\uparrow\right)\uparrow \\ +\frac{1}{\sqrt{2}}u\left(ud-du\right)\frac{1}{\sqrt{2}}\uparrow\left(\uparrow\downarrow-\downarrow\uparrow\right) \\ +\frac{1}{\sqrt{2}}\left(uud-duu\right)\frac{1}{\sqrt{2}}\left(\uparrow\uparrow\downarrow-\downarrow\uparrow\uparrow\right) \end{bmatrix}$$

Unconvinced? Still it's true, because any sum of 2 products is equivalent to the 3rd…

Octet - V

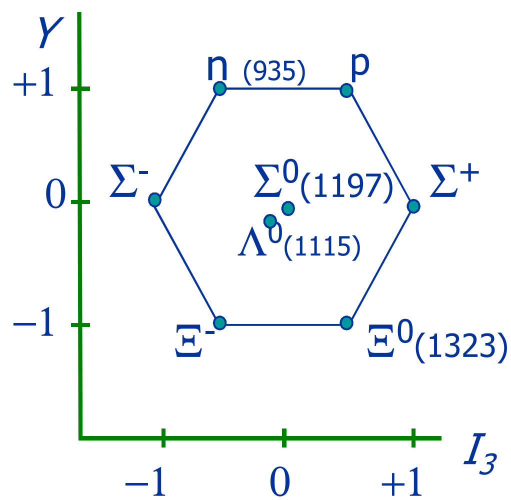
Finally: The *proton, spin up* wave function!

$$\left|p,+1/2\right\rangle = N\begin{pmatrix} 2u\uparrow d\downarrow u\uparrow +2u\uparrow u\uparrow d\downarrow +2d\downarrow u\uparrow u\uparrow \\ -u\downarrow d\uparrow u\uparrow -d\uparrow u\downarrow u\uparrow -u\uparrow u\downarrow d\uparrow -u\uparrow d\uparrow u\downarrow -u\downarrow u\uparrow d\uparrow -d\uparrow u\uparrow u\downarrow \end{pmatrix}$$

N = Normalization constant

$$N=\frac{1}{\sqrt{6\cdot 1^2+3\cdot 2^2}}=\frac{1}{\sqrt{6+12}}=\frac{1}{\sqrt{18}}$$

Baryon Octet



Summary: Decuplet

State	*Q*	I_3	*I*	*S*	$J^{PC}=3/2^+$
uuu	+2	+3/2	3/2	0	Δ^{++}
$1/\sqrt{3}\left(uud+udu+duu\right)$	+1	+1/2	3/2	0	Δ^{+}
$1/\sqrt{3}\left(udd+dud+duu\right)$	0	-1/2	3/2	0	Δ^{0}
ddd	-1	-1/2	3/2	0	Δ^{-}
$1/\sqrt{3}\left(uus+usu+suu\right)$	+1	+1	1	-1	Σ^{*+}
$1/\sqrt{6}\left(uds+sud+dsu+sdu+dus+usd\right)$	0	0	1	-1	Σ^{*0}
$1/\sqrt{3}\left(dds++dsd+sdd\right)$	-1	-1	1	-1	Σ^{*-}
$1/\sqrt{3}\left(uss+sus+ssu\right)$	0	+1/2	1/2	-2	Ξ^{*0}
$1/\sqrt{3}\left(dss+sds+ssd\right)$	-1	-1/2	1/2	-2	Ξ^{*-}
sss	-1	0	0	-3	Ω^{-}

Wave functions

Summary: Octet

Quarks	Q	I_3	I	S	J^{PC}=1/2$^+$
uud	+1	+1/2	1/2	0	p
udd	0	-1/2	1/2	0	n
dds	-1	-1	1	-1	Σ^-
uds	0	0	1,0	-1	Σ^0, Λ^0
uus	+1	+1	1	-1	Σ^-
dss	-1	-1/2	1/2	-2	Ξ^-
uss	0	+1/2	1/2	-2	Ξ^0

Quark content only
(no wave function)

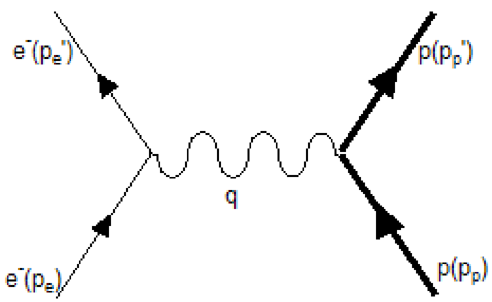
e-p Effective Interaction - I

Go for some dynamics…

Examine first electron-positron bound states: *Positronium*
Somewhat similar to mesons: *Particle-antiparticle bound state*

Can be dealt with by use of non-relativistic potential models
Useful insight by connecting to perturbation theory..

Start first from electron-proton interaction:



$$T_{fi} = e^2 \frac{\left(\overline{u}\left(p_e^{'}\right)\gamma^{\mu}u\left(p_e\right)\right)\left(\overline{u}\left(p_p{}'\right)\gamma_{\mu}u\left(p\right)\right)}{q^2}$$

e-p Effective Interaction - II

Expand matrix element to low speed approximation

Get a non-relativistic matrix element

The Bottom Line:

At low speed/energy neglect radiation, pair production (real & virtual)

$\rightarrow$Left with corrections:

Relativistic Energy/Momentum

Magnetic Moments

More

e-p Effective Interaction - III

Effective *e-p* potential: Valid for *S* states

$$V_C = -\frac{e^2}{r} \qquad \text{Coulomb term}$$

$$\left.\begin{aligned} V_{SO} &= \frac{e^2}{4m_e^2 r^3}\boldsymbol{\sigma}\cdot\mathbf{L} \qquad \text{Spin-orbit} \\ V_D &= \frac{e^2}{8m_e^2}4\pi\delta^{(3)}(\mathbf{r}) \qquad \text{'Darwin term'} \end{aligned}\right\} \text{Fine structure terms}$$

$$V_{dip-dip} = \underbrace{\frac{8\pi e^2}{3m_e m_p} g_p \mathbf{s}_e\cdot\mathbf{s}_p \delta^{(3)}(\mathbf{r})}_{\text{Hyperfine structure}} + \underbrace{\frac{e^2}{m_e m_p r^3} g_p \left[3(\mathbf{s}_e\cdot\mathbf{r})(\mathbf{s}_p\cdot\mathbf{r}) - \mathbf{s}_e\cdot\mathbf{s}_p\right]}_{\text{Tensor interaction}} \qquad \text{Dipole-dipole interaction}$$

Astonishing: Everything included in our modest 1-photon diagram…

e-p Effective Interaction - IV

Effect of hyperfine interaction on ground state energy:

$$\left\langle \frac{8\pi e^2}{3m_e m_p} g_p \mathbf{s}_e \cdot \mathbf{s}_p \delta^{(3)}(\mathbf{r}) \right\rangle_{\text{ground state}}$$

$$\to \Delta E_{hyp} = \frac{8\pi e^2}{3m_e m_p} g_p \cdot \frac{1}{2}\left[j(j+1) - s_e(s_e+1) - s_p(s_p+1) \right] \cdot \left|\psi(0)\right|^2$$

$$\left|\psi(0)\right|^2 = \frac{(m_e\alpha)^3}{\pi} \to \Delta E_{hyp} = \frac{8\pi e^2}{3m_e m_p} g_p \frac{1}{2} \frac{(m_e\alpha)^3}{\pi}\left[j(j+1) - \frac{3}{4} - \frac{3}{4} \right]$$

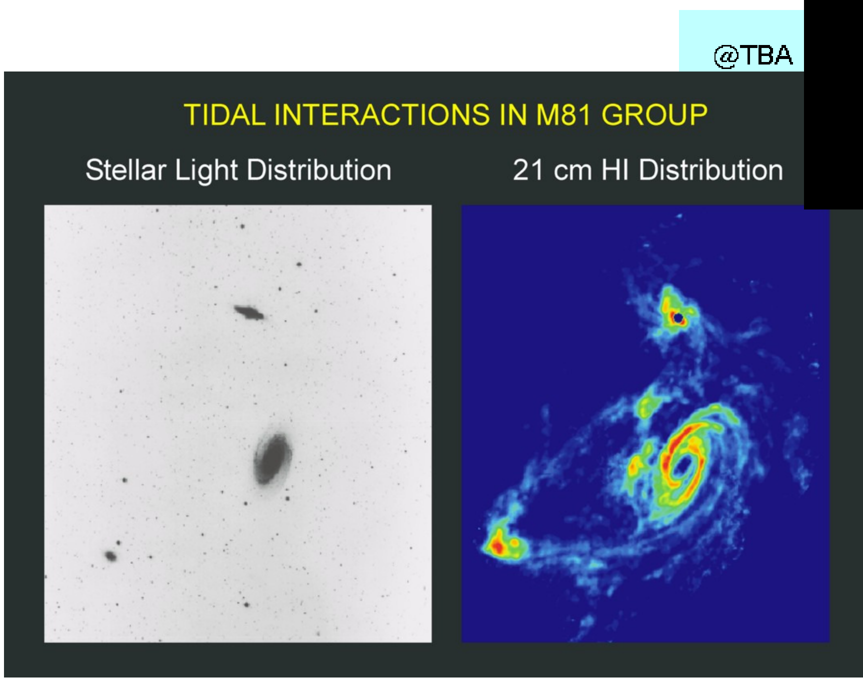
$$\to \Delta E_{hyp} = \frac{4}{3} g_p \frac{e^2}{m_e m_p}\left[j(j+1) - \frac{3}{2} \right](m_e\alpha)^3$$

$$\Delta E_{hyp} = \begin{cases} \dfrac{4}{3} g_p \dfrac{e^2 m_e}{m_p}(2-3/2)(m_e\alpha^3) = \dfrac{4}{3} g_p \dfrac{m_e}{m_p}\dfrac{1}{2}(m_e\alpha^4) & \text{hyperfine shift -triplet} \\ \dfrac{4}{3} g_p \dfrac{2e^2 m_e}{m_p}(0-3/2)(m_e\alpha^3) = -\dfrac{12}{3} g_p \dfrac{1}{2}\dfrac{m_e}{m_p}(m_e\alpha^4) & \text{hyperfine shift - singlet} \end{cases}$$

$$\to \Delta\left(\Delta E_{hyp}\right)_{\text{triplet-singlet}} = \frac{8}{3} g_p \frac{m_e}{m_p}(m_e\alpha^4)$$

21 cm *H* Line

Important radio-astronomical tool: *Mapping the Hydrogen content of the Universe*

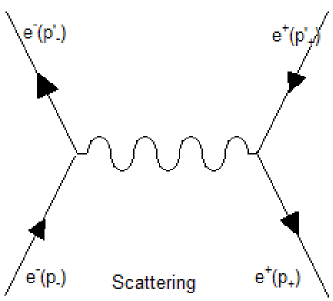


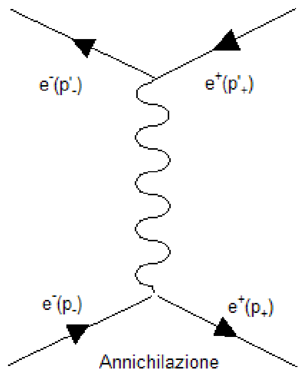
Lots of physics and cosmology..
Example:
How is the transition excited?

A measurement of the galactic/ intergalactic temperature

Positronium - I

There are now 2 diagrams:



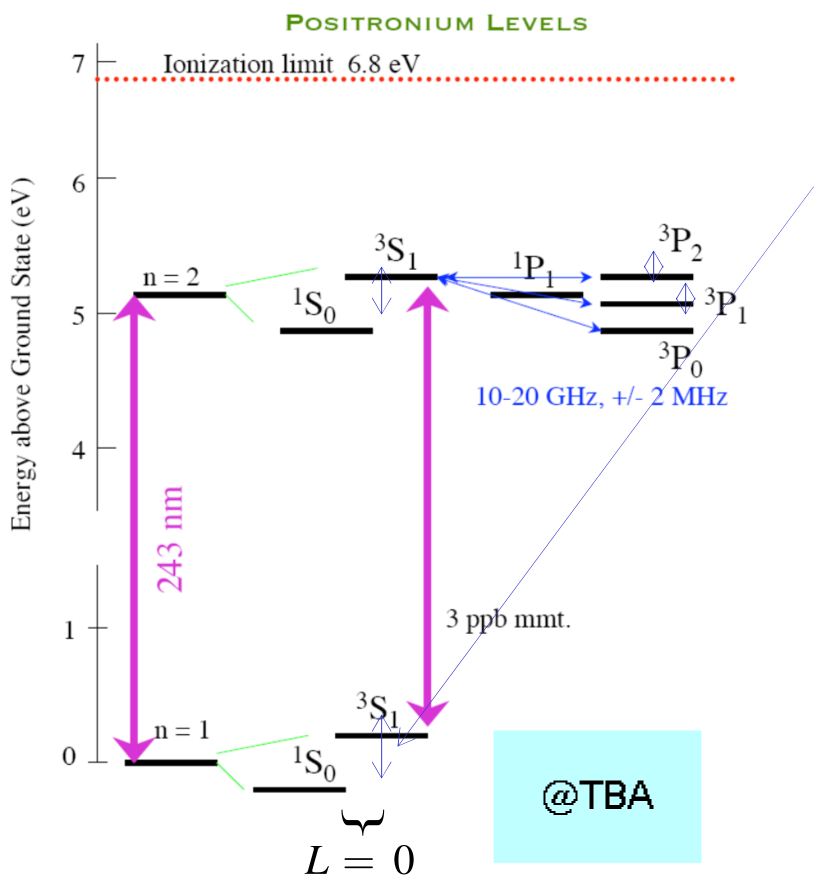


$$\mathcal{T}_{fi} = e^2 \left[-\frac{\left(\bar{u}\left(p_-^{'}\right)\gamma^\mu u\left(p_-\right)\right)\left(\bar{v}\left(p_+\right)\gamma_\mu v\left(p_+^{'}\right)\right)}{\left(p_- - p_-{'}\right)^2} + \frac{\left(\bar{v}\left(p_+\right)\gamma^\mu u\left(p_-\right)\right)\left(\bar{u}\left(p_-^{'}\right)\gamma_\mu v\left(p_+^{'}\right)\right)}{\left(p_+ + p_-\right)^2} \right]$$

Net effect: Add another term to the effective interaction

$V_A = \frac{e^2\pi}{2m^2}\left(3 + \boldsymbol{\sigma}_1 \cdot \boldsymbol{\sigma}_2\right)\delta^{(3)}\left(\mathbf{r}\right)$ Same structure as hyperfine term

Positronium - II



Form of hyperfine term:

$$\Delta E_{hyp} = \left\langle \frac{8\pi e^2}{3m_e^2} \mathbf{s}_{e^-} \cdot \mathbf{s}_{e^+} \delta^{(3)}(\mathbf{r}) \right\rangle$$

Ground state

More complicated for $n>1,\ l>0$

Observe:
Levels labeled by $^S L_J$
S: *Total* spin

Previous pictures:
Levels labeled by $^S L_J$
S: *Electron* spin
Proton spin only in hyperfine term

Hadron Masses - I

Observe large mass splitting between singlet and triplet mesons:

Guess effective strong interaction has some term similar to hyperfine electromagnetic

$$\Delta E = \frac{A}{m_1 m_2}\left(\mathbf{S}_1 \cdot \mathbf{S}_2\right)$$

Then expect for the hadron mass:

$$M = m_1 + m_2 + A\frac{\left(\mathbf{S}_1 \cdot \mathbf{S}_2\right)}{m_1 m_2}$$

$$\mathbf{J} = \mathbf{S}_1 + \mathbf{S}_2 \rightarrow J^2 = S_1^2 + S_2^2 + 2\mathbf{S}_1 \cdot \mathbf{S}_2$$

$$\rightarrow \mathbf{S}_1 \cdot \mathbf{S}_2 = 1/2\left(J^2 - S_1^2 - S_2^2\right) = 1/2\left(J\left(J+1\right) - 2S\left(S+1\right)\right)$$

$$\mathbf{S}_1 \cdot \mathbf{S}_2 = \begin{cases} +1/4 & \text{triplets} \\ -3/4 & \text{singlets} \end{cases}$$

Hadron Masses - II

1) About the expected, large hyperfine splitting:

Can be shown to be true, to some extent..

When perturbative expansion can be granted, color quark-(anti)quark interaction in the static limit yields a *chromomagnetic term* with the proper hyperfine structure

2) About the quark masses:

m_1, m_2 *constituent* quark mass

Somewhat difficult idea, basically similar to *effective mass* for electrons bound in a crystal
Different from the *current*, i.e. the free quark mass
Will be (somewhat) clarified when discussing QCD.

Hadron Masses - III

Free parameter counting:

3 quark masses (m_u,m_d,m_s)+1 constant A
Hope to fit 7 meson masses:
Pseudoscalars + Vectors

$\rightarrow$ *Go for a 3 constraints fit*

Results:

$$m_u = m_d \simeq 310 \quad MeV$$

$$m_s \simeq 483 \quad MeV$$

$$A \simeq 160 \ m_{u,d}^2 \quad MeV^3$$

Meson	ΔE_{HF}	Fitted mass (MeV)
p	$-\frac{3a}{m_u^2}$	140
K	$-\frac{3a}{m_u m_s}$	485
η	$-\frac{a}{m_u^2}-\frac{2a}{m_s^2}$	559
ρ,ω	$\frac{a}{m_u^2}$	780
K^*	$\frac{a}{m_u m_s}$	896
ϕ	$\frac{a}{m_s^2}$	1032

Hadron Masses - IV

Extend the idea to baryons: Sum over 3 quark pairs

$$m(q_1,q_2,q_3)=\sum_{i=1}^{3} m_i + A'\frac{1}{2}\sum_{i,j=1}^{3}\frac{\mathbf{S}_i\cdot\mathbf{S}_j}{m_i m_j}$$

As an exercise, first neglect differences between quark masses:

$$\mathbf{J}=\mathbf{S}_1+\mathbf{S}_2+\mathbf{S}_3 \rightarrow J^2=\left(\mathbf{S}_1+\mathbf{S}_2+\mathbf{S}_3\right)^2$$

$$=S_1^2+S_2^2+S_3^2+2\left(\mathbf{S}_1\cdot\mathbf{S}_2+\mathbf{S}_1\cdot\mathbf{S}_3+\mathbf{S}_2\cdot\mathbf{S}_3\right)$$

$$S^2=S\left(S+1\right)=3/4 \rightarrow S_1^2+S_2^2+S_3^2=9/4$$

$$\rightarrow \mathbf{S}_1\cdot\mathbf{S}_2+\mathbf{S}_1\cdot\mathbf{S}_3+\mathbf{S}_2\cdot\mathbf{S}_3=1/2\left[J^2-9/4\right]=1/2\,J\left(J+1\right)-9/4$$

$$\rightarrow \frac{1}{2}\sum_{i,j=1}^{3}\mathbf{S}_i\cdot\mathbf{S}_j=\begin{cases}+3/4 & j=3/2 \text{ decuplet}\\ -3/4 & j=1/2 \text{ octet}\end{cases}$$

Hadron Masses - V

Now take into account different quark masses:

$$m(q_1, q_2, q_3) = \sum_{i=1}^{3} m_i + A' \frac{1}{2} \sum_{i,j=1}^{3} \frac{\mathbf{S}_i \cdot \mathbf{S}_j}{m_i m_j}$$

Must consider quark pairs in order to evaluate the hyperfine contribute

$$J_{ik}^2 = \left(\mathbf{S}_i + \mathbf{S}_k\right)^2 = S_i^2 + S_k^2 + 2\mathbf{S}_i \cdot \mathbf{S}_k$$

$$\rightarrow \mathbf{S}_i \cdot \mathbf{S}_k = 1/2\left[J_{ik}\left(J_{ik}+1\right) - S_i\left(S_i+1\right) - S_k\left(S_k+1\right)\right]$$

Quarks i, k in a spin triplet state:

$$\mathbf{S}_i \cdot \mathbf{S}_k = 1/2\left[1(1+1) - 1/2(1/2+1) - 1/2(1/2+1)\right]$$

$$\rightarrow \mathbf{S}_i \cdot \mathbf{S}_k = 1/4$$

Quarks i, k in a spin singlet state:

$$\mathbf{S}_i \cdot \mathbf{S}_k = 1/2\left[0(0+1) - 1/2(1/2+1) - 1/2(1/2+1)\right]$$

$$\rightarrow \mathbf{S}_i \cdot \mathbf{S}_k = -3/4$$

Hadron Masses - VI

N: Only u,d quarks $\to$ Same mass

$$\to m_N = 3m_u - \frac{3A'}{4m_u^2},\quad m_u = m_d$$

Λ: u,d spin & isospin singlet

$$m_\Lambda = 2m_u + m_s + A'\left(\frac{\mathbf{S}_u\cdot\mathbf{S}_d}{m_u m_d} + \frac{\mathbf{S}_u\cdot\mathbf{S}_s}{m_u m_s} + \frac{\mathbf{S}_s\cdot\mathbf{S}_d}{m_s m_d}\right)$$

$$m_\Lambda = 2m_u + m_s + A'\left(\frac{\mathbf{S}_u\cdot\mathbf{S}_d}{m_u^2} + \frac{\mathbf{S}_u\cdot\mathbf{S}_s + \mathbf{S}_s\cdot\mathbf{S}_d}{m_u m_s}\right)$$

$$\mathbf{S}_u\cdot\mathbf{S}_s + \mathbf{S}_s\cdot\mathbf{S}_d = \frac{1}{2}\sum_{i,j=1}^{3}\mathbf{S}_i\cdot\mathbf{S}_j - \mathbf{S}_u\cdot\mathbf{S}_d = -3/4 - (-3/4) = 0$$

$$\to m_\Lambda = 2m_u + m_s - \frac{3A'}{4m_u^2}$$

Hadron Masses - VII

Σ: u, d spin & isospin triplet

$$m_\Sigma = 2m_u + m_s + A'\left(\frac{\mathbf{S}_u\cdot\mathbf{S}_d}{m_u m_d} + \frac{\mathbf{S}_u\cdot\mathbf{S}_s}{m_u m_s} + \frac{\mathbf{S}_s\cdot\mathbf{S}_d}{m_s m_d}\right)$$

$$m_\Sigma = 2m_u + m_s + A'\left(\frac{\mathbf{S}_u\cdot\mathbf{S}_d}{m_u^2} + \frac{\mathbf{S}_u\cdot\mathbf{S}_s + \mathbf{S}_s\cdot\mathbf{S}_d}{m_u m_s}\right)$$

$$\mathbf{S}_u\cdot\mathbf{S}_s + \mathbf{S}_s\cdot\mathbf{S}_d = \frac{1}{2}\sum_{i,j=1}^{3}\mathbf{S}_i\cdot\mathbf{S}_j - \mathbf{S}_u\cdot\mathbf{S}_d = -3/4 - (+1/4) = -1 \rightarrow m_\Sigma = 2m_u + m_s + A'\left(\frac{1}{4m_u^2} - \frac{1}{m_u m_s}\right)$$

Ξ: $s1, s2$ spin triplet

Why? Flavor = ss Symmetric $\rightarrow$ Spin must be symmetric too

$$m_\Xi = 2m_s + m_u + A'\left(\frac{\mathbf{S}_u\cdot\mathbf{S}_{s1}}{m_u m_s} + \frac{\mathbf{S}_u\cdot\mathbf{S}_{s2}}{m_u m_s} + \frac{\mathbf{S}_{s1}\cdot\mathbf{S}_{s2}}{m_s^2}\right)$$

$$m_\Xi = 2m_s + m_u + A'\left(\frac{\mathbf{S}_{s1}\cdot\mathbf{S}_{s2}}{m_s^2} + \frac{\mathbf{S}_u\cdot\mathbf{S}_{s1} + \mathbf{S}_u\cdot\mathbf{S}_{s2}}{m_u m_s}\right)$$

$$\mathbf{S}_u\cdot\mathbf{S}_{s1} + \mathbf{S}_u\cdot\mathbf{S}_{s2} = \frac{1}{2}\sum_{i,j=1}^{3}\mathbf{S}_i\cdot\mathbf{S}_j - \mathbf{S}_{s1}\cdot\mathbf{S}_{s2} = -3/4 - 1/4 = -1 \rightarrow m_\Xi = 2m_s + m_u + A'\left(\frac{1}{4m_s^2} - \frac{1}{m_u m_s}\right)$$

Hadron Masses - VIII

Fit all octet + decuplet:
8 masses $\rightarrow$ 4 constraints

Interesting questions:
Is A = A' ?
Are the quark masses the same in mesons as in baryons?

$$m_u = m_d \simeq 363 \quad MeV$$

$$m_s \simeq 538 \quad MeV$$

$$A' \simeq 50 \; m_{u,d}^2 \quad MeV^3$$

Baryons vs. Mesons:

Masses ~+*50 MeV* ~ *10%* higher
Constant ~*1/3* Hyperfine splitting reduced

Baryon	ΔE^{HF}	Fitted mass (MeV)
$N(938)$	$-\frac{3a'}{m_{u,d}^2}$	939
$\Lambda(1116)$	$-\frac{3a'}{m_{u,d}^2}$	1114
$\Sigma(1193)$	$\frac{a'}{m_{u,d}^2} - \frac{4a'}{m_u m_s}$	1179
$\Xi(1318)$	$\frac{a'}{m_{u,d}^2} - \frac{4a'}{m_u m_s}$	1327
$\Delta(1232)$	$+\frac{3a'}{m_{u,d}^2}$	1239
$\Sigma^*(1384)$	$\frac{a'}{m_{u,d}^2} + \frac{4a'}{m_u m_s}$	1381
$\Xi^*(1533)$	$\frac{a'}{m_{u,d}^2} - \frac{4a'}{m_u m_s}$	1529
$\Omega(1672)$	$+\frac{3a'}{m_s^2}$	1682

Baryon Magnetic Moments - I

Take total dipole moment operator as the sum of the single quark operators:

NB: Can this be really granted??

$$\boldsymbol{\mu} = \sum_{i=1}^{3} \boldsymbol{\mu}_i \rightarrow \mu_p = \langle p, +1/2 | \mu | p, +1/2 \rangle = \langle p, +1/2 | (\mu_1 + \mu_2 + \mu_3) | p, +1/2 \rangle$$

Each operator acting on the corresponding factor of the wave function

$$| p, +1/2 \rangle = \frac{1}{\sqrt{18}} \begin{pmatrix} 2u\uparrow u\uparrow d\downarrow + 2d\downarrow u\uparrow u\uparrow + 2u\uparrow d\downarrow u\uparrow - \\ -u\downarrow d\uparrow u\uparrow - u\uparrow u\downarrow d\uparrow - u\downarrow u\uparrow d\uparrow - \\ -d\uparrow u\downarrow u\uparrow - u\uparrow d\uparrow u\downarrow - d\uparrow u\uparrow u\downarrow \end{pmatrix}$$

Baryon Magnetic Moments - II

Some really dull algebra:

$$4\langle u\uparrow u\uparrow d\downarrow|(\mu_1+\mu_2+\mu_3)|u\uparrow u\uparrow d\downarrow\rangle = 4\left[\langle u|\mu_1|u\rangle + \langle u|\mu_2|u\rangle - \langle d|\mu_3|d\rangle\right]$$

$$= 4[\mu_u + \mu_u - \mu_d] = 8\mu_u - 4\mu_d$$

$$4\langle d\downarrow u\uparrow u\uparrow|(\mu_1+\mu_2+\mu_3)|d\downarrow u\uparrow u\uparrow\rangle$$

$$= 4\langle u\uparrow d\downarrow u\uparrow|(\mu_1+\mu_2+\mu_3)|u\uparrow d\downarrow u\uparrow\rangle = 8\mu_u - 4\mu_d$$

$$\langle u\downarrow d\uparrow u\uparrow|(\mu_1+\mu_2+\mu_3)|u\downarrow d\uparrow u\uparrow\rangle = \langle u\uparrow u\downarrow d\uparrow|(\mu_1+\mu_2+\mu_3)|u\uparrow u\downarrow d\uparrow\rangle$$

$$\langle d\uparrow u\downarrow u\uparrow|(\mu_1+\mu_2+\mu_3)|d\uparrow u\downarrow u\uparrow\rangle = \ldots = \mu_d$$

$$\rightarrow \langle p,+1/2|(\mu_1+\mu_2+\mu_3)|p,+1/2\rangle = \frac{1}{18}\left[3(8\mu_u - 4\mu_d) + 6\mu_d\right] = \frac{1}{18}\left[24\mu_u - 6\mu_d\right]$$

$$\rightarrow \mu_p \equiv \frac{1}{3}(4\mu_u - \mu_d)$$

Then take neutron: Just swap $u \leftrightarrow d$

$$|n,+1/2\rangle = \frac{1}{\sqrt{18}}\begin{pmatrix} 2d\uparrow d\uparrow u\downarrow + 2u\downarrow d\uparrow d\uparrow + 2d\uparrow u\downarrow d\uparrow - \\ -d\downarrow u\uparrow d\uparrow - d\uparrow d\downarrow u\uparrow - d\downarrow d\uparrow u\uparrow - \\ -u\uparrow d\downarrow d\uparrow - d\uparrow u\uparrow d\downarrow - u\uparrow d\uparrow d\downarrow \end{pmatrix} \rightarrow \mu_n = \frac{1}{3}(4\mu_d - \mu_u)$$

Baryon Magnetic Moments - III

Take quarks as Dirac particles:Can this be really granted??

$$\mu = \frac{e}{2m}$$

$$\rightarrow \mu_u = \frac{2}{3}\frac{e}{2m_u}, \mu_d = -\frac{1}{3}\frac{e}{2m_d}, \mu_s = -\frac{1}{3}\frac{e}{2m_s}$$

Predict moment ratio:

$$\frac{\mu_n}{\mu_p} = \frac{4\mu_d - \mu_u}{4\mu_u - \mu_d} = \frac{4\left(-\frac{e}{3\cdot 2m_d}\right) - \frac{2e}{3\cdot 2m_u}}{4\frac{2e}{3\cdot 2m_u} - \left(-\frac{e}{3\cdot 2m_d}\right)} \simeq \frac{-4e-2e}{8e+e} = -\frac{2}{3} \approx -0.667$$

Experimental ratio:

$$\frac{\mu_n}{\mu_p} \approx -0.685 \quad \text{Amazingly close!}$$

Absolute moments difficult to estimate, as involving unknown quark mass.
Nevertheless..

Baryon Magnetic Moments - IV

If one insists in believing the constituent quark masses have something to do with reality, then one can compute the expected magnetic moments for octet:

Baryon	*Moment*	*Predicted*	*Observed*
p	$\frac{4}{3}\mu_u - \frac{1}{3}\mu_d$	2.79	2.793
n	$\frac{4}{3}\mu_d - \frac{1}{3}\mu_u$	-1.86	-1.913
Λ^0	μ_s	-0.58	-0.614
Σ^+	$\frac{4}{3}\mu_u - \frac{1}{3}\mu_s$	2.68	2.33
Σ^0	$\frac{2}{3}(\mu_u + \mu_d) - \frac{1}{3}\mu_s$	0.82	Unstable
Σ^-	$\frac{4}{3}\mu_d - \frac{1}{3}\mu_s$	-1.05	-1.41
Ξ^0	$\frac{4}{3}\mu_s - \frac{1}{3}\mu_u$	-1.40	-1.253
Ξ^-	$\frac{4}{3}\mu_s - \frac{1}{3}\mu_d$	-0.47	-0.69

Not too bad for such a simple attempt…

We are taking baryons as composed only by valence quarks, which is wildly *incomplete*

Vector Mesons - I

Take radiative decays of vector mesons to pseudoscalars:

$V \rightarrow P + \gamma$

$1^{--} \rightarrow 0^{-+} + \gamma$

$\rightarrow \gamma : 1^{+} \rightarrow magnetic \quad dipole$

For any magnetic dipole transition:

$Rate \propto \omega^{3},\ \omega:$ Photon energy

From quark model perspective: *Triplet* $\rightarrow$ *Singlet, S-wave*
As before: Spin flip of one quark
(I = Fraction of space overlap between initial/final meson wave functions)

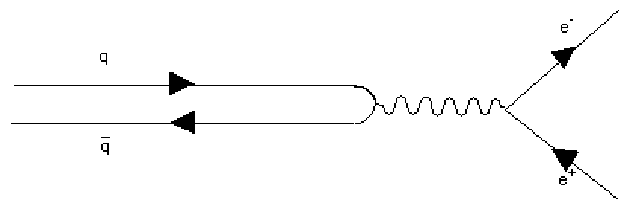
@TBA

Process	Rate in units of $\omega^3 \lvert I \rvert^2/3\pi$	Prediction (keV)	Experiment (keV)	$\lvert I \rvert^2$
$\omega \rightarrow \pi^0\gamma$	$(\mu_u - \mu_d)^2$	$1390\lvert I \rvert^2$	890 ± 50	0.64 ± 0.04
$\rho \rightarrow \pi\gamma$	$((\mu_u + \mu_d)^2$	$148\lvert I \rvert^2$	67 ± 7	0.45 ± 0.05
$\omega \rightarrow \eta\gamma$	$(\mu_u + \mu_d)^2/2$	$11\lvert I \rvert^2$	$3^{+2.5}_{-1.8}$	$0.27^{+0.23}_{-0.16}$
$\rho \rightarrow \eta\gamma$	$(\mu_u - \mu_d)^2/2$	$92\lvert I \rvert^2$	50 ± 13	0.54 ± 0.14
$\eta' \rightarrow \omega\gamma$	$3(\mu_u + \mu_d)^2/2$	$17\lvert I \rvert^2$	7.6 ± 3	0.45 ± 0.18
$\eta' \rightarrow \rho\gamma$	$3(\mu_u - \mu_d)^2/2$	$171\lvert I \rvert^2$	83 ± 30	0.48 ± 0.18
$\phi \rightarrow \eta\gamma$	$2\mu_s^2$	$110\lvert I \rvert^2$	62 ± 9	0.56 ± 0.08
$\phi \rightarrow \pi^0\gamma$	0	0	5.7 ± 2	
$K^{*+} \rightarrow K^+\gamma$	$(\mu_u + \mu_s)^2$	$153\lvert I \rvert^2$	60 ± 15	0.39 ± 0.10
$K^{*0} \rightarrow K^0\gamma$	$(\mu_d - \mu_s)^2$	$224\lvert I \rvert^2$	75 ± 35	0.34 ± 0.16

Quite consistent with simple *SU(3)* symmetry: Same space wave function

Vector Mesons - II

$$\rho^0 \to e^+ + e^-$$
$$\omega \to e^+ + e^-$$
$$\varphi \to e^+ + e^-$$



$$\Gamma_{e^+e^-} = \frac{16\pi\alpha^2}{q^2\left(=M_V^2\right)}\left|\psi(0)\right|^2\left|\sum_i a_i Q_i\right|^2$$ Van Royen-Weisskopf formula

$$\rho^0 : \frac{1}{\sqrt{2}}\left(u\bar{u} - d\bar{d}\right) \to \left|\sum_i a_i Q_i\right|^2 = \left|\frac{1}{\sqrt{2}}\left(\frac{2}{3} - \left(-\frac{1}{3}\right)\right)\right|^2 = \frac{1}{2}$$

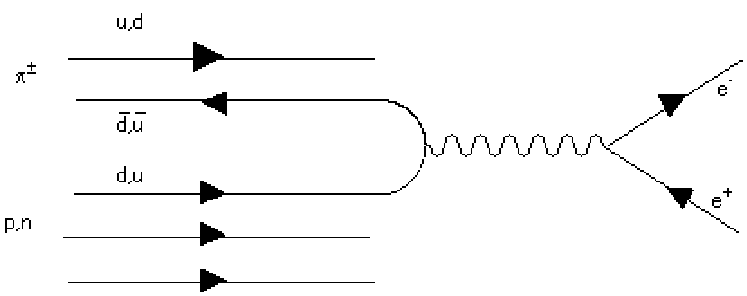
$$\omega : \frac{1}{\sqrt{2}}\left(u\bar{u} + d\bar{d}\right) \to \left|\sum_i a_i Q_i\right|^2 = \left|\frac{1}{\sqrt{2}}\left(\frac{2}{3} + \left(-\frac{1}{3}\right)\right)\right|^2 = \frac{1}{18}$$

$$\varphi : \quad s\bar{s} \to \left|\sum_i a_i Q_i\right|^2 = \left|\left(-\frac{1}{3}\right)\right|^2 = \frac{1}{9}$$

$$\to \Gamma_{e^+e^-}\left(\rho^0\right) : \Gamma_{e^+e^-}(\omega) : \Gamma_{e^+e^-}(\varphi) = 9:1:2$$

Drell-Yan

Take production of electron pairs from pion beams: *Drell-Yan*



Rate $\propto Q^2_{quark}$

Cross section: Electromagnetic, counting antiquark content in π
For isoscalar targets: N_p=N_n $\rightarrow N_u$=N_d

$$\left.\begin{aligned}\sigma\left(\pi^+\right)\propto Q_{\bar{d}}^2=\frac{1}{9}\\ \sigma\left(\pi^-\right)\propto Q_{\bar{u}}^2=\frac{4}{9}\end{aligned}\right\}\rightarrow\frac{\sigma\left(\pi^-\right)}{\sigma\left(\pi^+\right)}=4$$

More Quarks

Flavor	*Mass*	*Q*	*I*	I_3	*S*	*C*	*B*	*T*
Up	5.6 MeV	2/3	½	+1/2	0	0	0	0
Down	9.9 MeV	-1/3	½	-1/2	0	0	0	0
strange	199 MeV	-1/3	0	0	-1	0	0	0
charm	1350 MeV	2/3	0	0	0	1	0	0
bottom	4400 MeV	-1/3	0	0	0	0	-1	0
Top	174 GeV	2/3	0	0	0	0	0	1

Charm

Predicted to exist in order to solve a major puzzle in weak interactions via the *GIM Mechanism* (see later)

Expected mass much higher than *u,d,s*

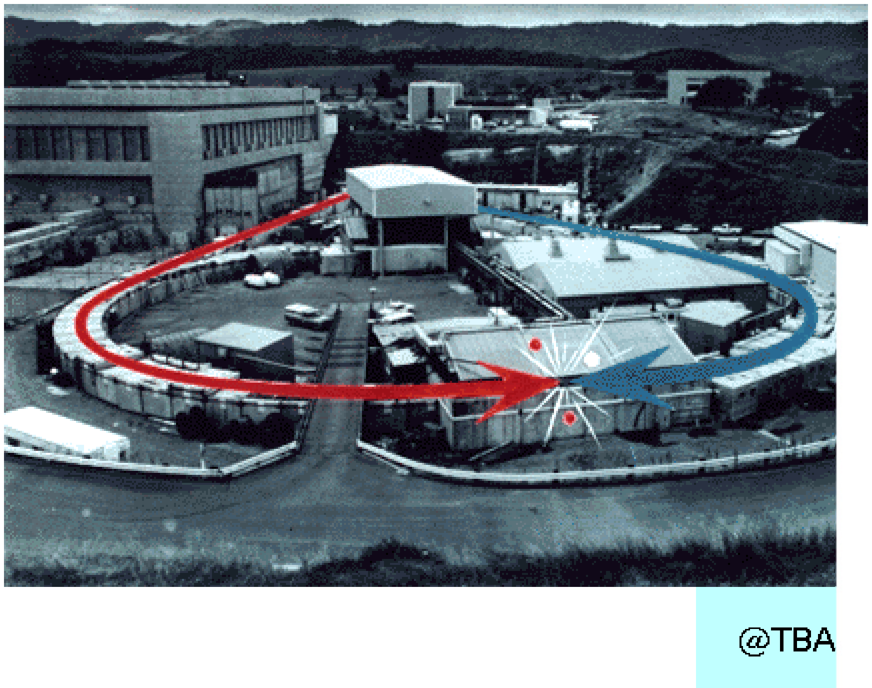
Phenomenology similar to strange quark *s*:
New breed of *charmed particles*, both mesons and baryons
Difference: Much larger mass

→Many channels open to weak decays → *Shorter lifetime* $\sim 10^{-13}$ *s*
→Extended symmetry severely broken → *SU(4) not useful*

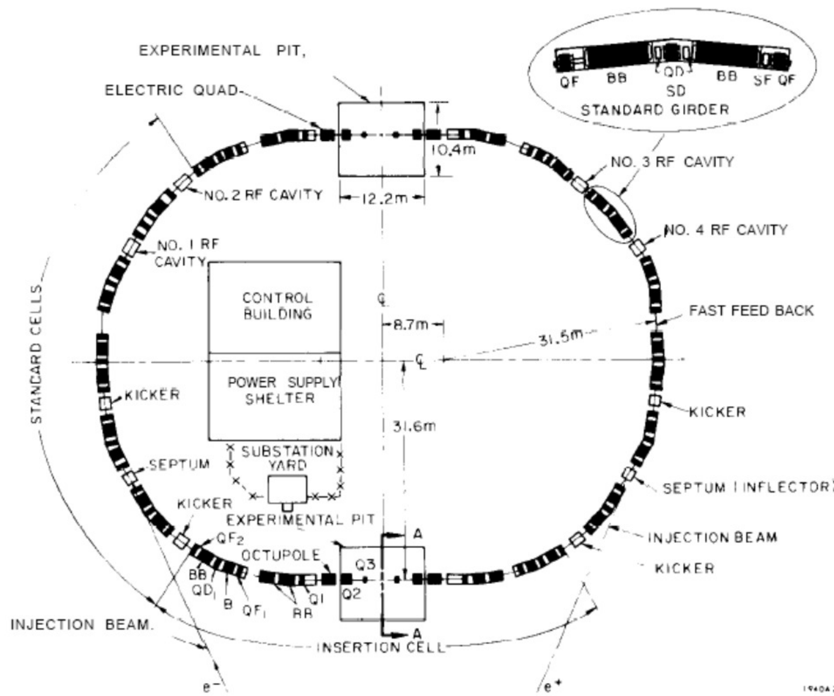
Based on *asymptotic freedom* of QCD (see later), guess an exciting physics case for a *heavy, hidden charm bound state*

Discovered simultaneosly at SLAC (Mark I) and BNL (E598)

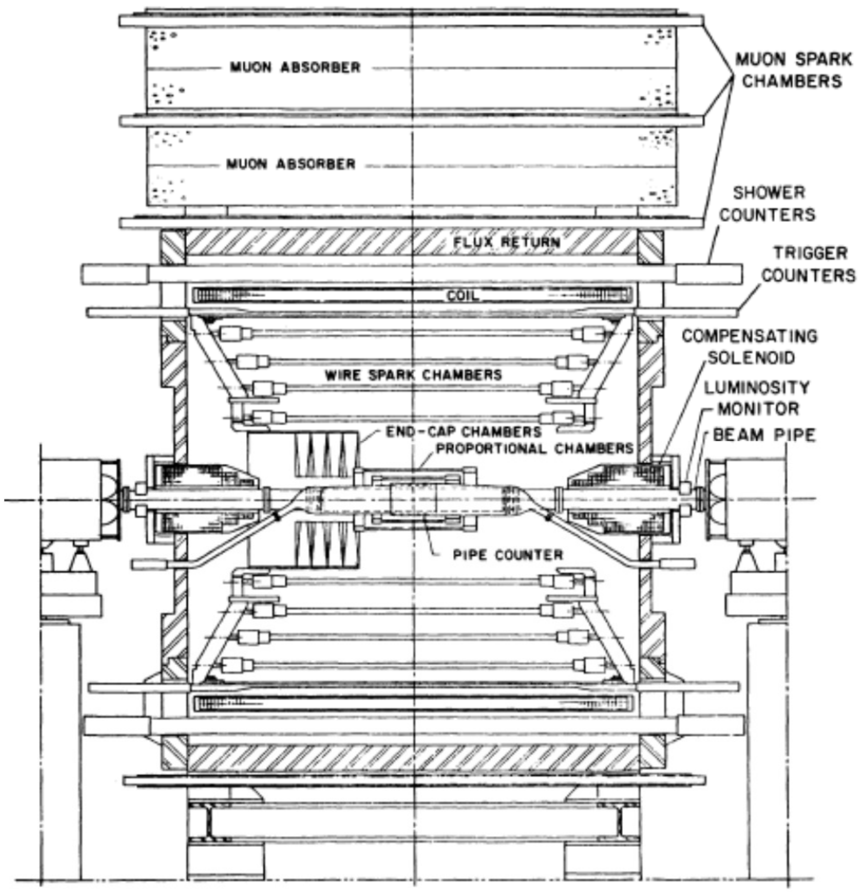
J/ψ at SLAC - I



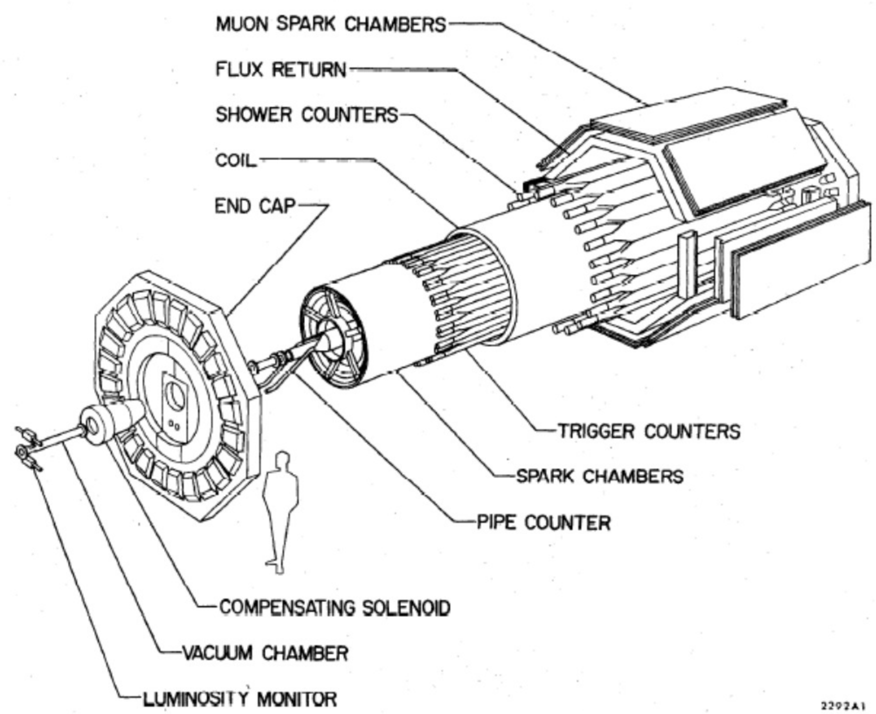
SPEAR



J/ψ at SLAC - II



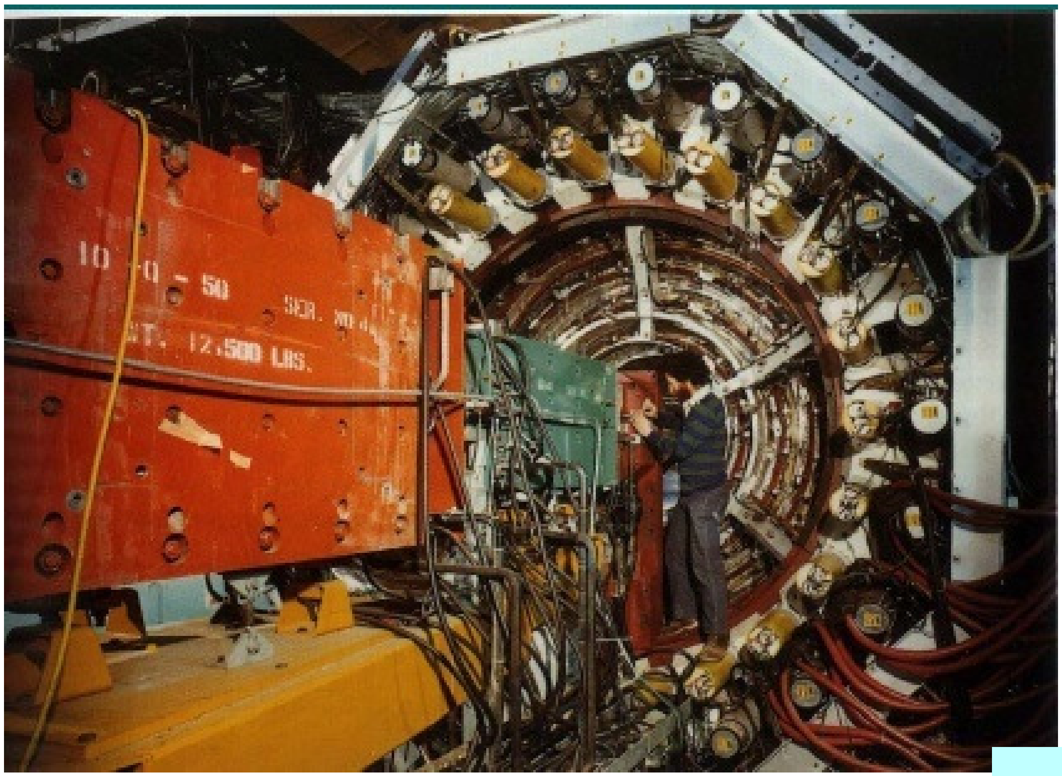
TBA

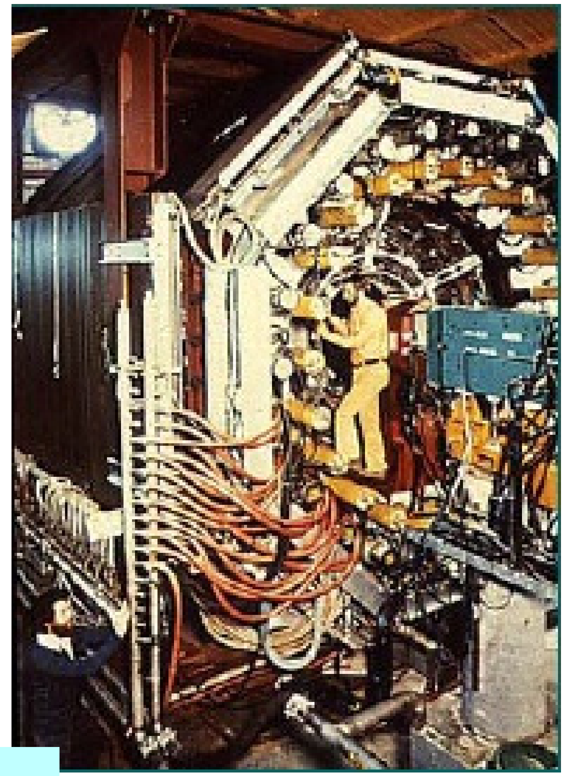


The Mark I Detector

J/ψ at SLAC - III

Mark I: First example of multi-purpose, collider detector





@TBA

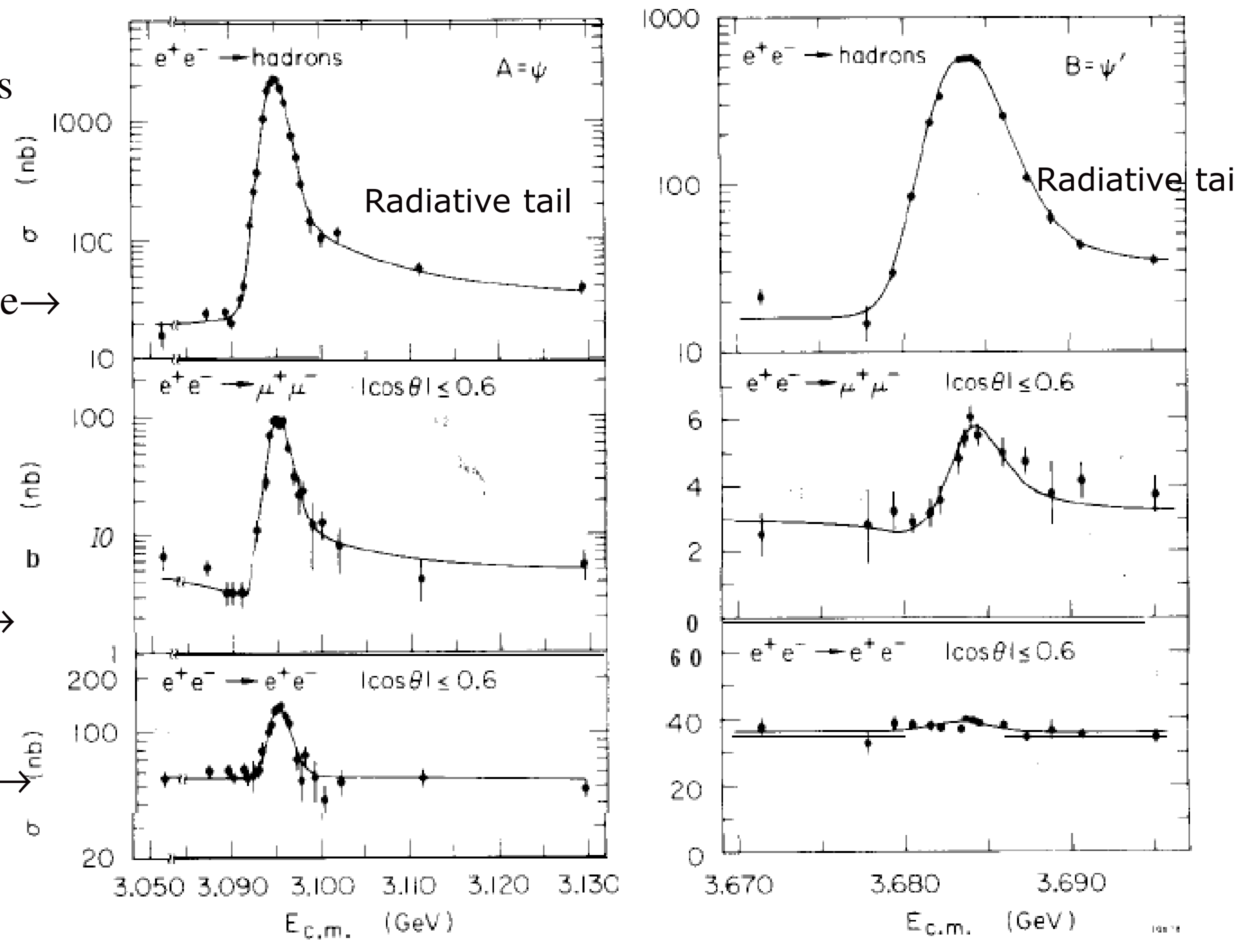
J/ψ at SLAC - IV

J/ψ and *ψ'* as seen
in different decay channels

No interference→

Interference →

No interference→



@TBA

J/ψ Quantum Numbers - I

Quickly understood as the first, indirect evidence for charm

Bound state of quark-antiquark pair $c, \overline{c}$

Another member of the vector mesons family

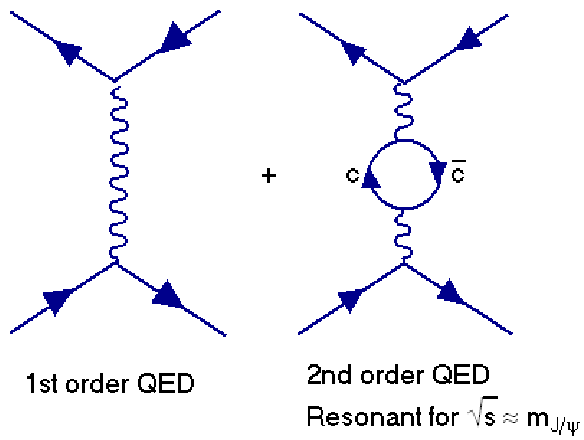
Main differences:

Charm quark has a large mass 1.5 GeV

Lightest charmed particles are so heavy the J/ψ cannot decay into a pair of them$\rightarrow$Most decays channels are closed

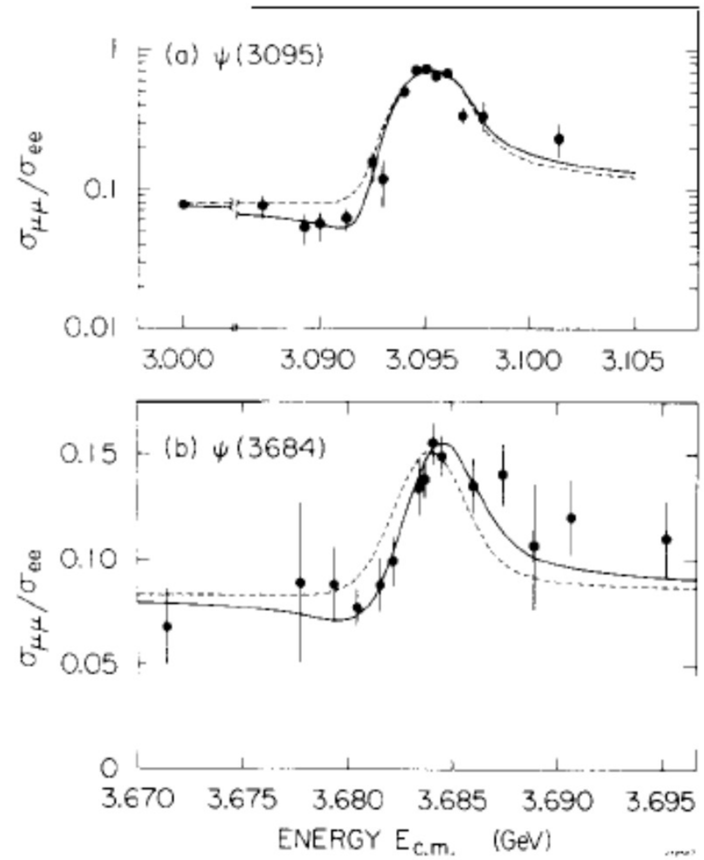
J/ψ Quantum Numbers - II

An interesting example of quantum interference
Take the 2 annihilation diagrams:



@TBA

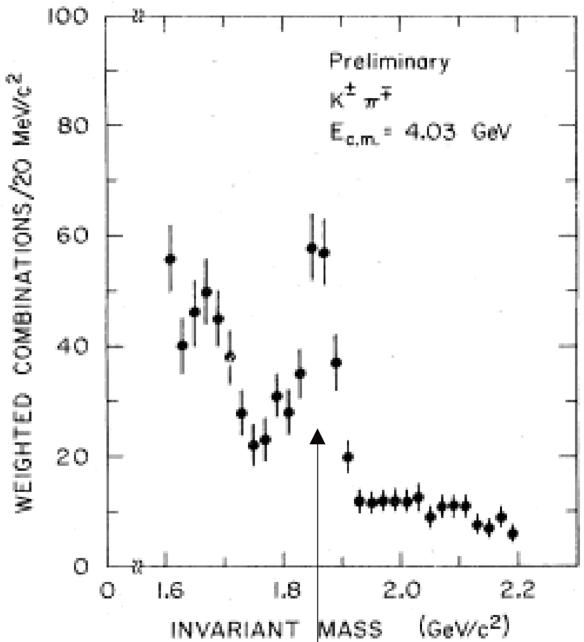
Take the ratio to minimize
point-to-point luminosity systematics



Interference between the 2 shows up in the total cross-section as a result of the resonant state being $J^{PC}=1^{--}$ like the photon

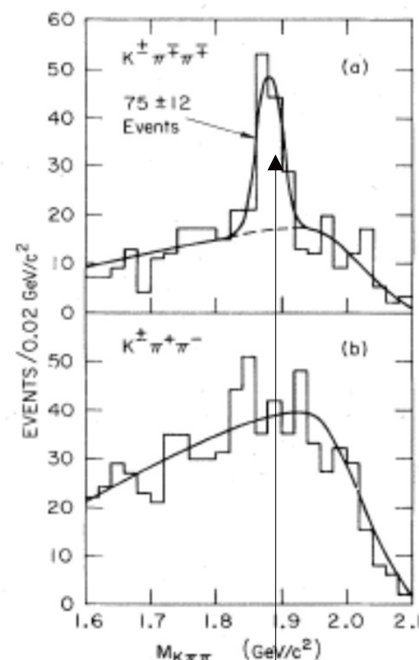
Charmed Particles - I

SLAC-LBL Collaboration – Mark I



D^0

@TBA



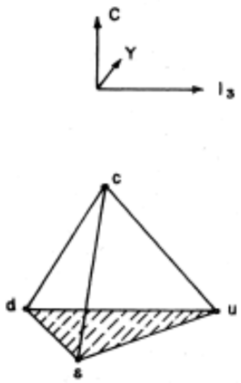
D^{0*}

Charmed Particles - II

Fundamental rep. ***4, 4*, 6***

$4 \cdot 4 - 1 = 15$ generators

3 fundamental, non equivalent irr. reps.



@TBA

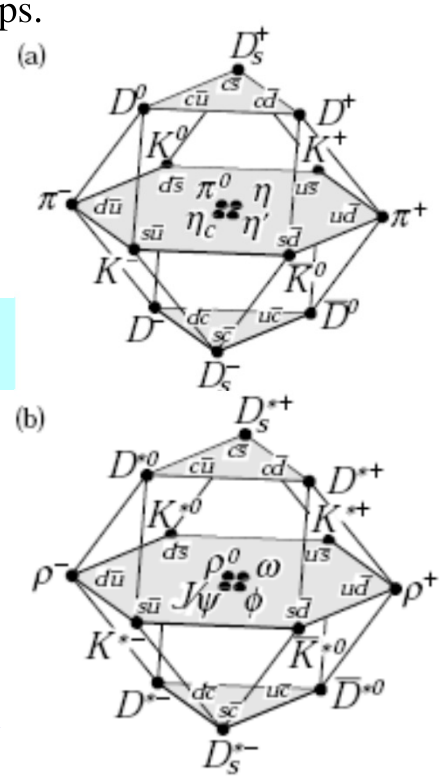
Spin 3/2 Spin 1/2

$$\mathbf{4} \otimes \mathbf{4}^* = \mathbf{1} \oplus \mathbf{15}$$

$$\mathbf{4} \otimes \mathbf{4} \otimes \mathbf{4} = \mathbf{20}_S \oplus \mathbf{20}_M \oplus \mathbf{20}_M \oplus \cancel{\mathbf{4}^*_A}$$

Mesons

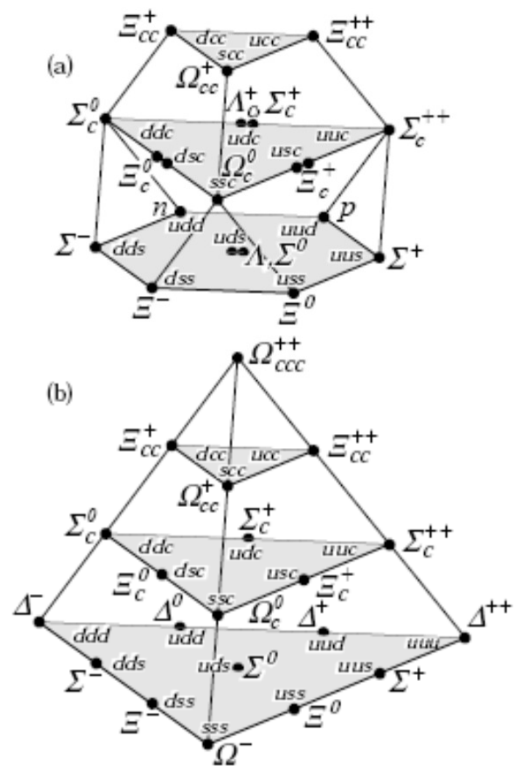
Pseudoscalars



Vectors

Baryons

Spin 1/2



Spin 3/2

Charmed Particle - III

Λ_c^+	****
$\Lambda_c(2593)^+$	***
$\Lambda_c(2625)^+$	***
$\Lambda_c(2765)^+$	*
$\Lambda_c(2880)^+$	**
$\Sigma_c(2455)$	****
$\Sigma_c(2520)$	***
$\Sigma_c(2800)$	***
Ξ_c^+	***
Ξ_c^0	***
$\Xi_c'^+$	***
$\Xi_c'^0$	***
$\Xi_c(2645)$	***
$\Xi_c(2790)$	***
$\Xi_c(2815)$	***
Ω_c^0	***
Ξ_{cc}^+	*

Baryons

@TBA

CHARMED ($C = \pm1$)	
• $D^\pm$	$1/2(0^-)$
• D^0	$1/2(0^-)$
• $D^*(2007)^0$	$1/2(1^-)$
• $D^*(2010)^\pm$	$1/2(1^-)$
$D_0^*(2400)^0$	$1/2(0^+)$
$D_0^*(2400)^\pm$	$1/2(0^+)$
• $D_1(2420)^0$	$1/2(1^+)$
$D_1(2420)^\pm$	$1/2(?^?)$
$D_1(2430)^0$	$1/2(1^+)$
• $D_2^*(2460)^0$	$1/2(2^+)$
• $D_2^*(2460)^\pm$	$1/2(2^+)$
$D^*(2640)^\pm$	$1/2(?^?)$
CHARMED, STRANGE ($C = S = \pm1$)	
• $D_s^\pm$	$0(0^-)$
• $D_s^{*\pm}$	$0(?^?)$
• $D_{s0}^*(2317)^\pm$	$0(0^+)$
• $D_{s1}(2460)^\pm$	$0(1^+)$
• $D_{s1}(2536)^\pm$	$0(1^+)$
• $D_{s2}(2573)^\pm$	$0(?^?)$

Mesons

Bottom

3rd family (*Bottom, Top*) predicted in order to ‘explain’ (Better: Account for) CP violation (see later) – Kobayashi & Maskawa, 1973

Bound $b\bar{b}$ states first observed at Fermilab in 1977

Discovery subsequently confirmed at e^+e^- machines (DESY, Cornell)

Several b-hadrons observed

Very large b-quark mass ~ 4-5 GeV

Situation somewhat similar to charm

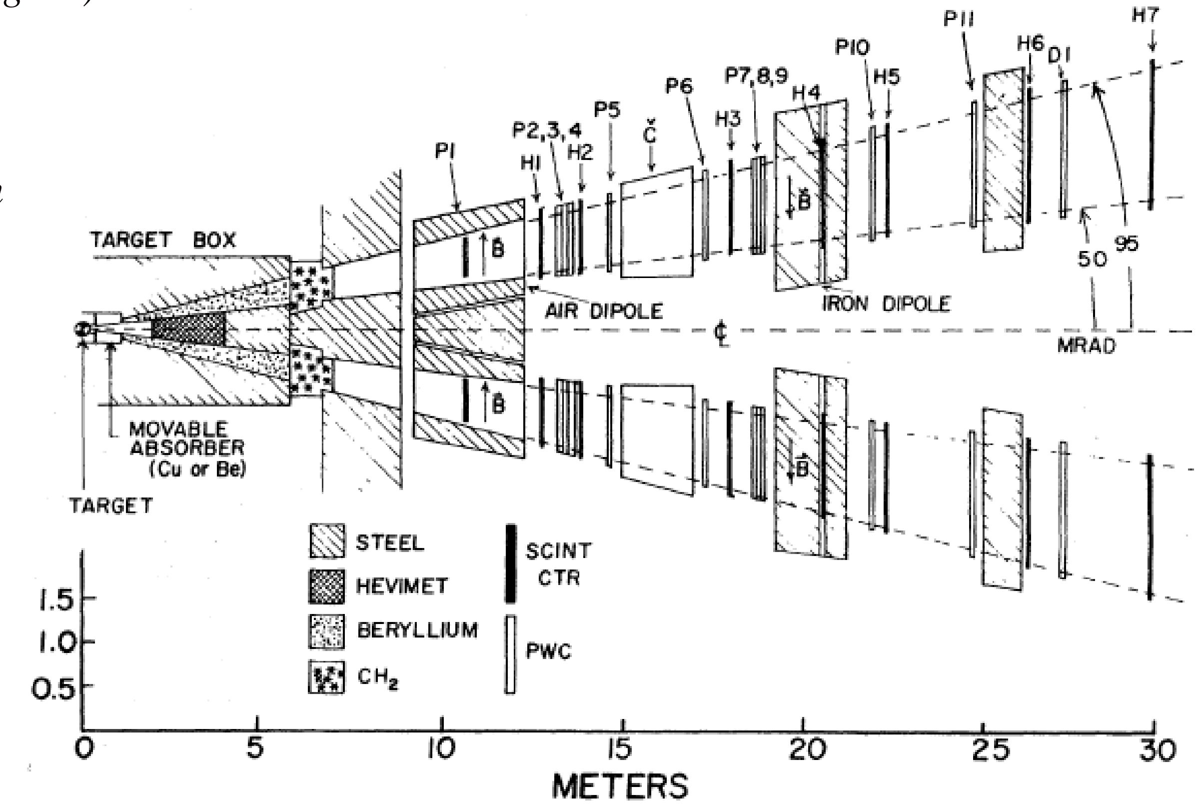
Υ at FNAL - I

Design similar to J/ψ experiment:

Switch from electrons to muons
(Easier to handle at high E)

High intensity

~Good mass resolution



@TBA

Υ at FNAL - II

Mass distribution for exclusive process:
$p + Be \rightarrow \mu^+ + \mu^- + X$

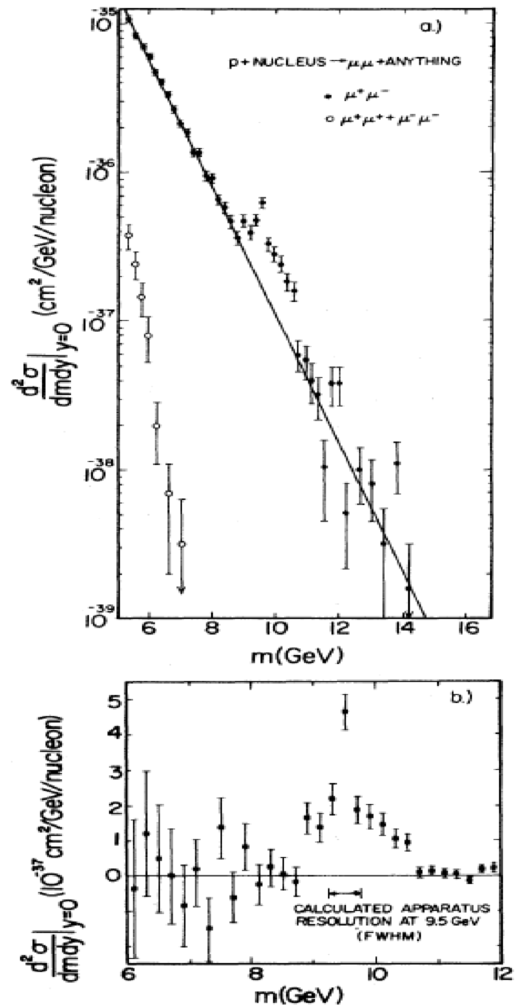
y: Pseudorapidity of the muon pair
(Related to CM angle)

y=0 Central region

High mass region shown
Exponential trend + peak

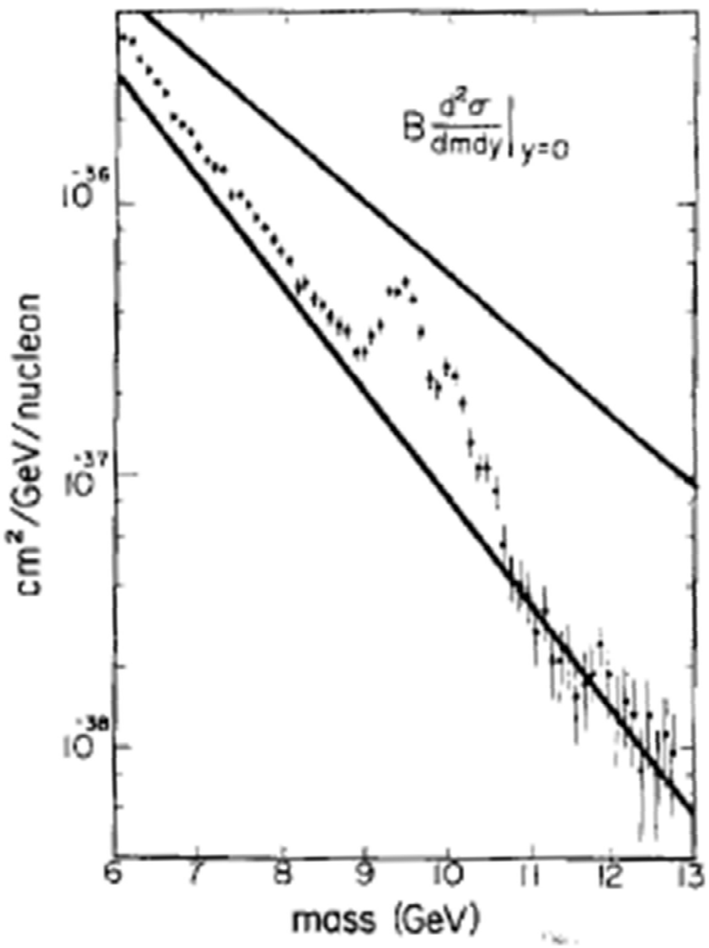
Mass resolution ~ 180 MeV

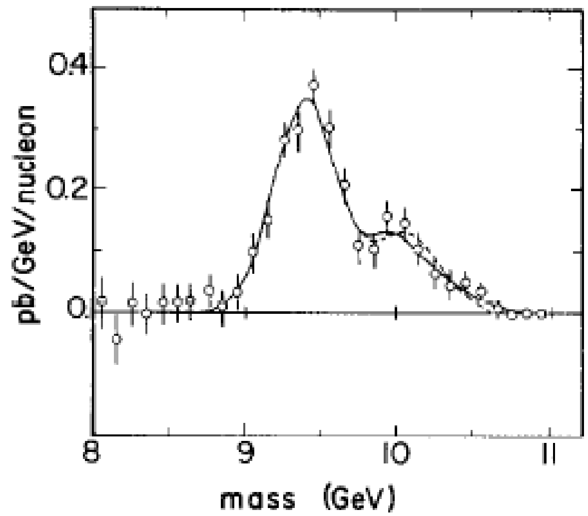
@TBA



Υ at FNAL - III

With some more statistics

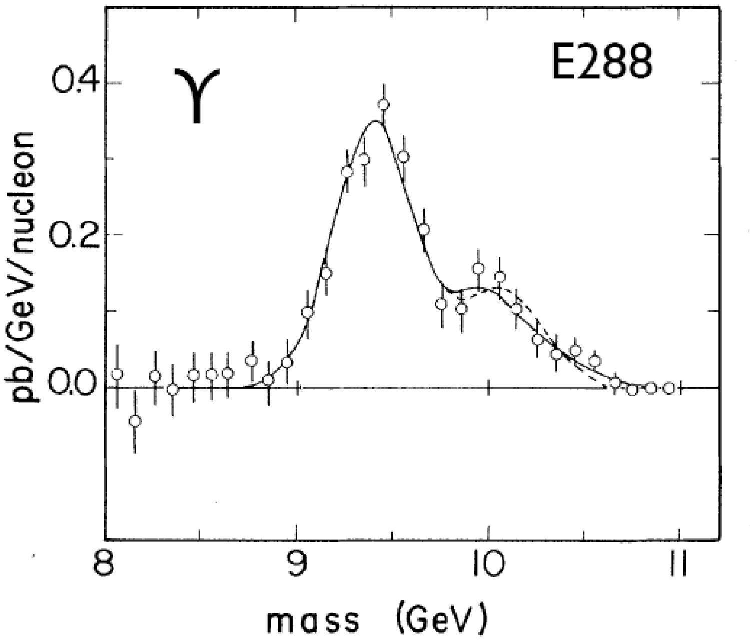




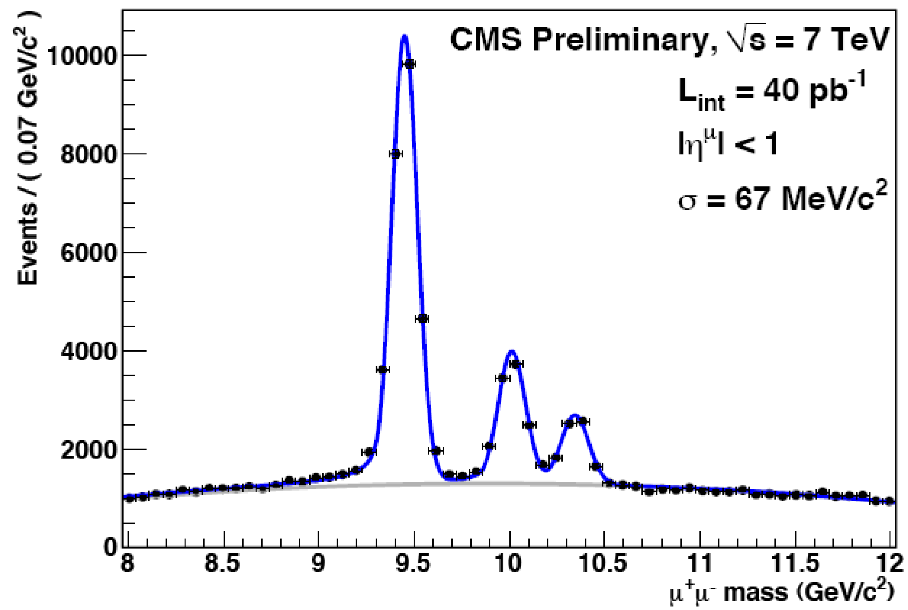
Background subtracted

Υ at FNAL - IV

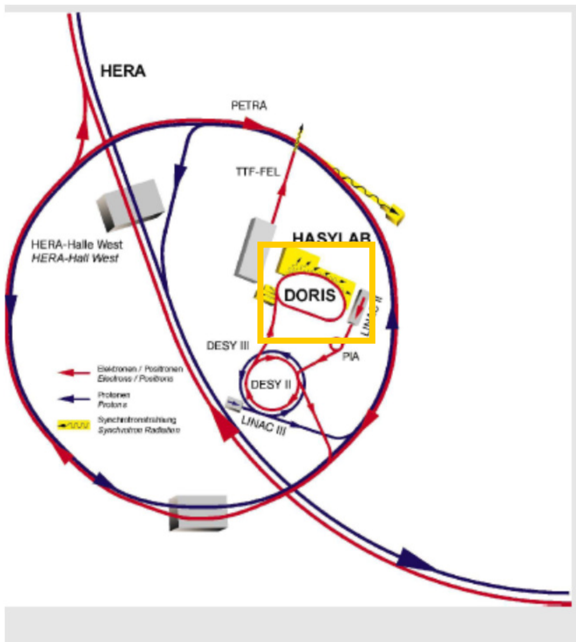
Yesterday



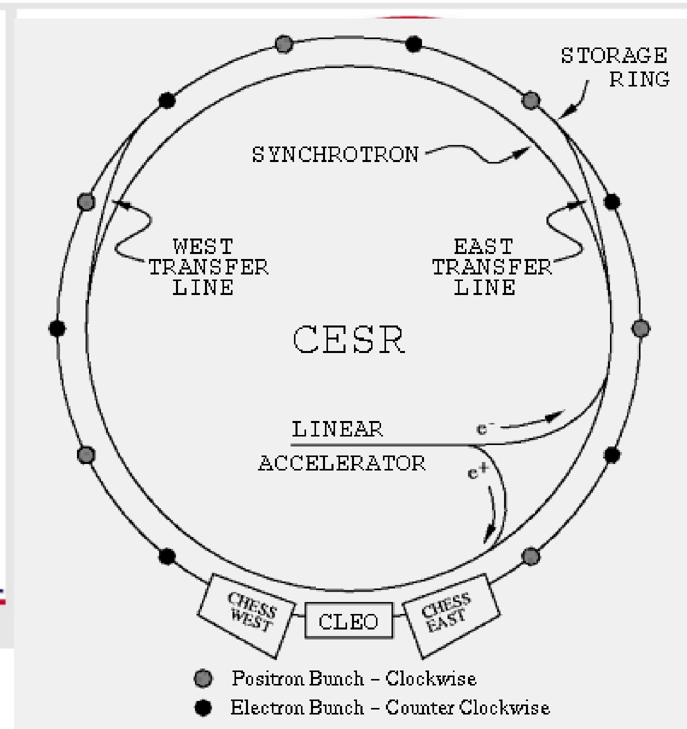
Today



Heavy Quarkonium - I



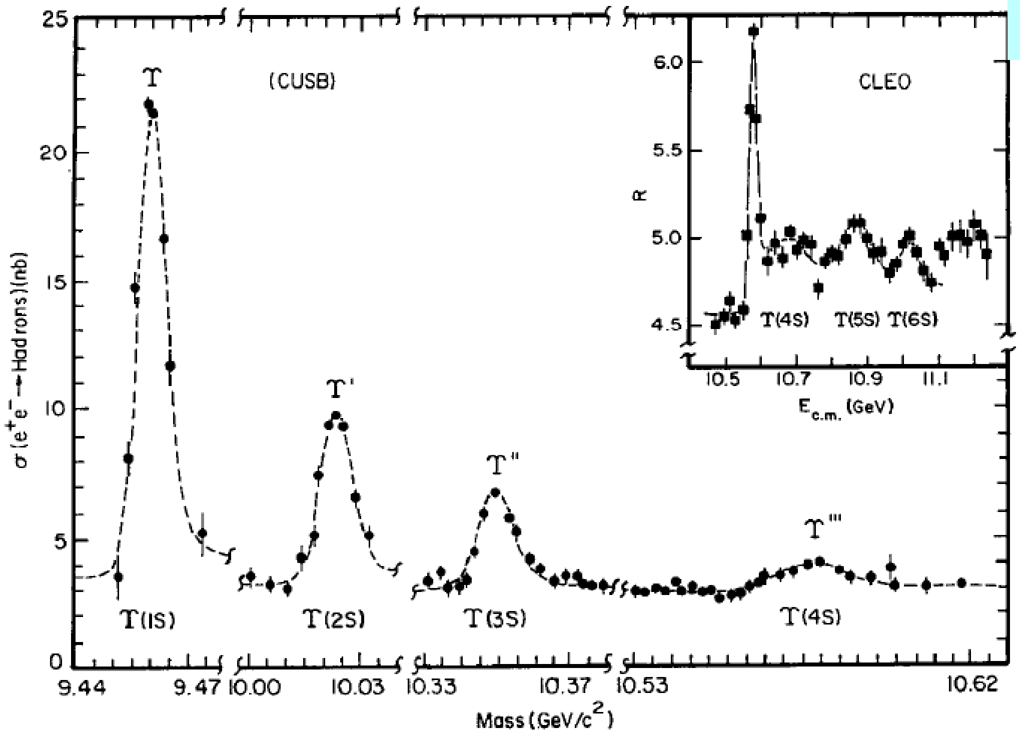
DESY, Hamburg



Cornell, Ithaca, NY

Heavy Quarkonium - II

@TBA



3 *radial excitations* of the *Υ* observed as narrow peaks

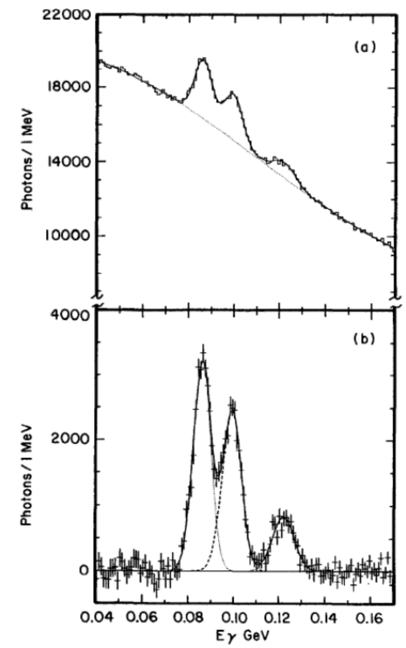


Figure 5 (a) Inclusive Υ(3S) photon energy spectrum from the CLEO-II collaboration. .
(b) Background subtracted spectrum.

Inclusive γ-ray spectrum from *Υ(3S)*

Beautiful Particles

Λ_b^0	***
Ξ_b^0, Ξ_b^-	*

Baryons

BOTTOM ($B = \pm 1$)	$I^G(J^{PC})$
• $B^\pm$	$1/2(0^-)$
• B^0	$1/2(0^-)$
• $B^\pm/B^0$ ADMIXTURE	
• $B^\pm/B^0/B_s^0/b$-baryon ADMIXTURE	
V_{cb} and V_{ub} CKM Matrix Elements	
• B^*	$1/2(1^-)$
$B_J^*(5732)$	$?(?^?)$
BOTTOM, STRANGE ($B = \pm 1, S = \mp 1$)	
• B_s^0	$0(0^-)$
B_s^*	$0(1^-)$
$B_{sJ}^*(5850)$	$?(?^?)$
BOTTOM, CHARMED ($B = C = \pm 1$)	
• $B_c^\pm$	$0(0^-)$

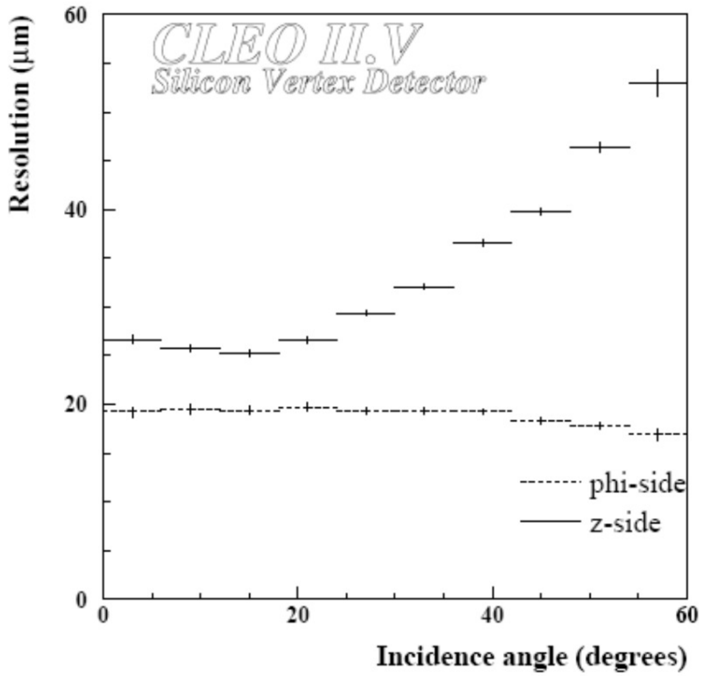
@TBA

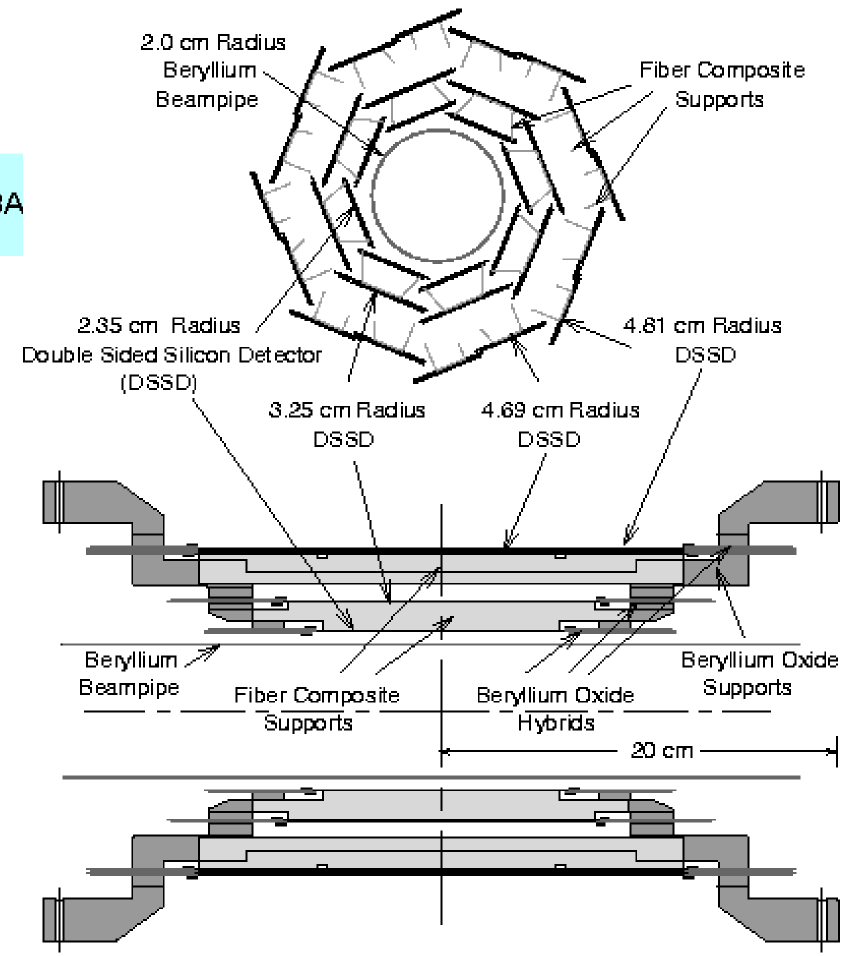
Mesons

Vertex Detection - I

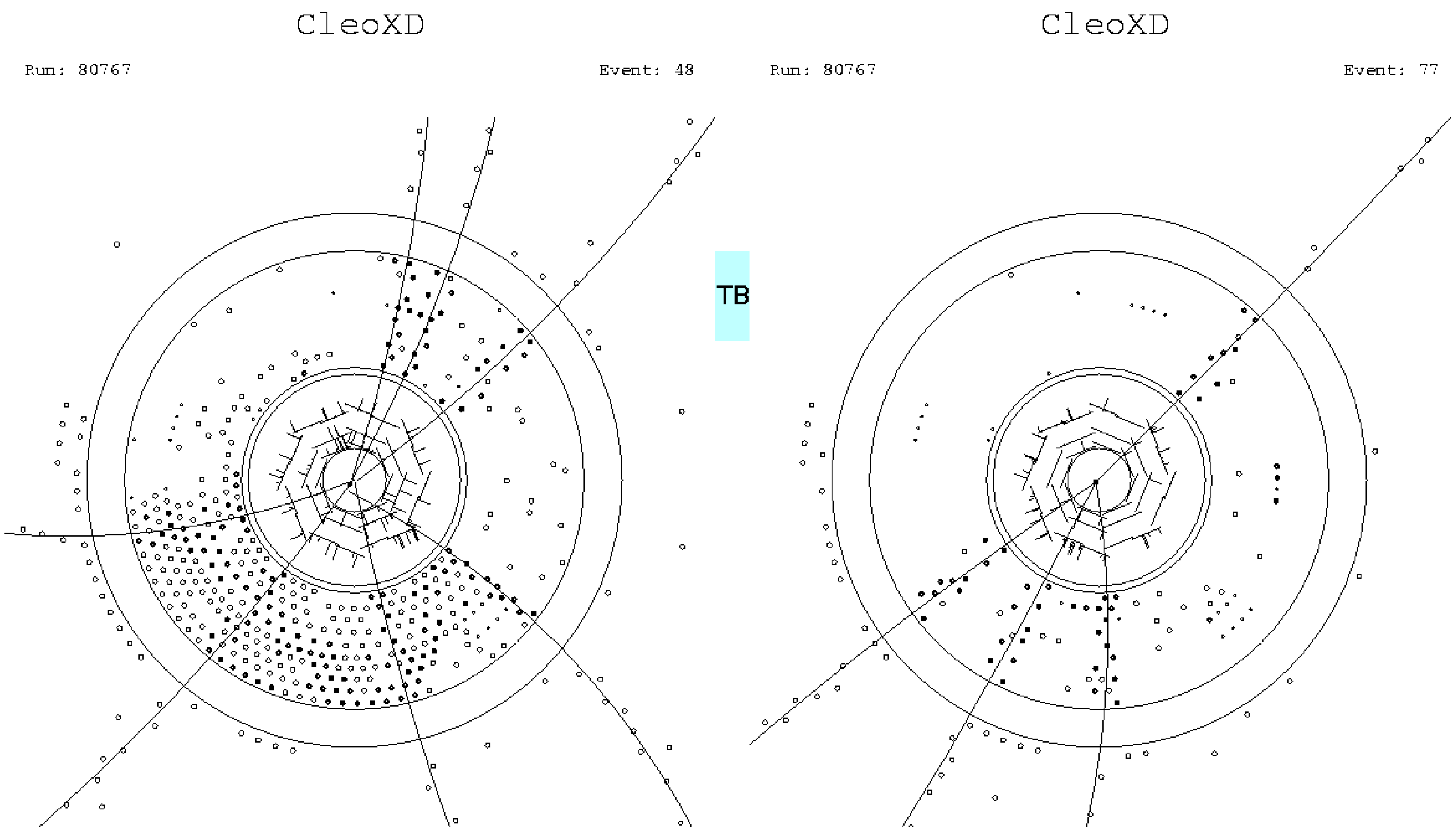
20208 strips

@TBA

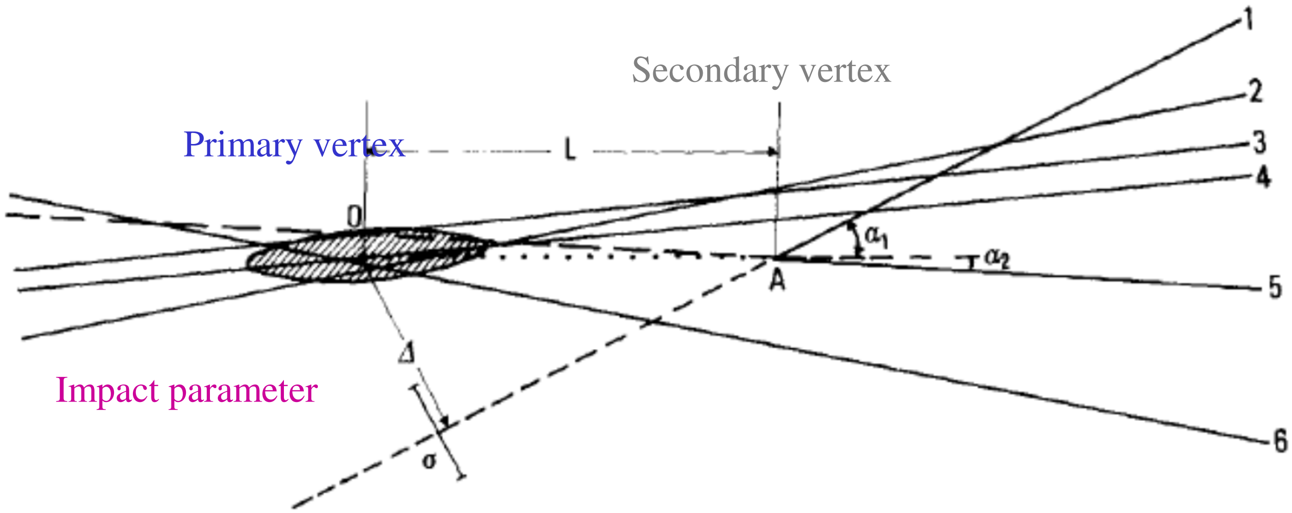




Vertex Detection - II



Vertex Detection - III



$$\theta \equiv \alpha_1$$

Plane defined by primary vertex,track direction

Consider a particle produced at primary vertex with speed β

When it decays to another particle, call speed β^*, decay angle in CM θ^*

$$\tan\theta = \frac{\sin\theta^*}{\gamma\cos\theta^* + \beta/\beta^*}$$ Lorentz transformation to LAB

Vertex Detection - IV

$L = \beta\gamma\tau$ Decay length

Define impact parameter Δ in terms of decay length, L, and angle θ:

$$\Delta = L\sin\theta = L\frac{\tan\theta}{\sqrt{1+\tan^2\theta}} = L\frac{\dfrac{\sin\theta^*}{\gamma\left(\cos\theta^* + \beta/\beta^*\right)}}{\sqrt{1+\left(\dfrac{\sin\theta^*}{\gamma\left(\cos\theta^* + \beta/\beta^*\right)}\right)^2}} = L\frac{\sin\theta^*}{\sqrt{\left(\gamma\left(\cos\theta^* + \beta/\beta^*\right)\right)^2 + \sin^2\theta^*}}$$

$$\to \Delta = L\frac{1}{\gamma}\frac{\sin\theta^*}{\sqrt{\left(\cos\theta^* + \beta/\beta^*\right)^2 + \dfrac{1}{\gamma^2}\sin^2\theta^*}} = \beta\tau\frac{\sin\theta^*}{\sqrt{\left(\cos\theta^* + \beta/\beta^*\right)^2 + \dfrac{1}{\gamma^2}\sin^2\theta^*}}$$

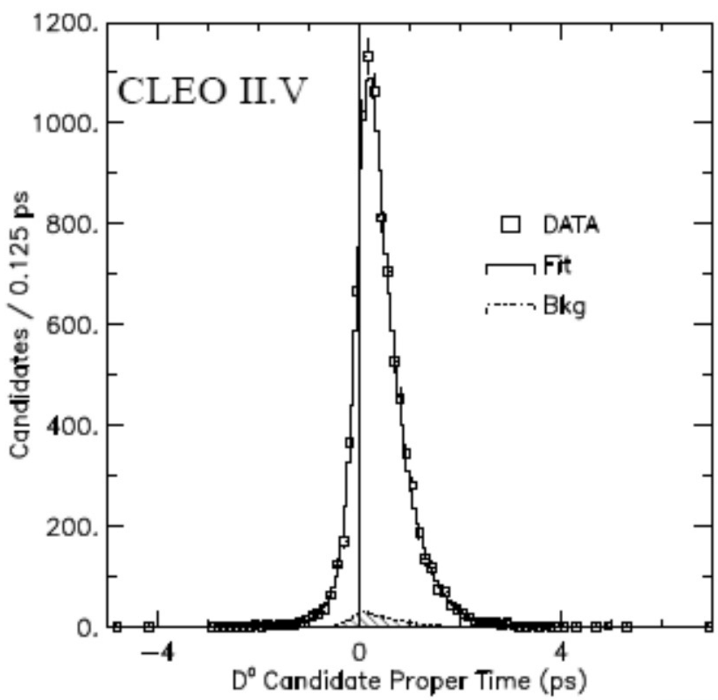
$$\Delta \underset{\beta,\beta^*\to 1}{\to} \beta\tau\frac{\sin\theta^*}{1+\cos\theta^*} = \beta\tau\tan\frac{\theta^*}{2}$$

$y \equiv \dfrac{\Delta}{\tau} \to$ Find statistical distribution of y for isotropic θ^*, exponential τ

$\to \langle y\rangle = \dfrac{\pi}{2} \to \langle\Delta\rangle = \dfrac{\langle\tau\rangle\pi}{2}$ Get a measurement of the decay lifetime

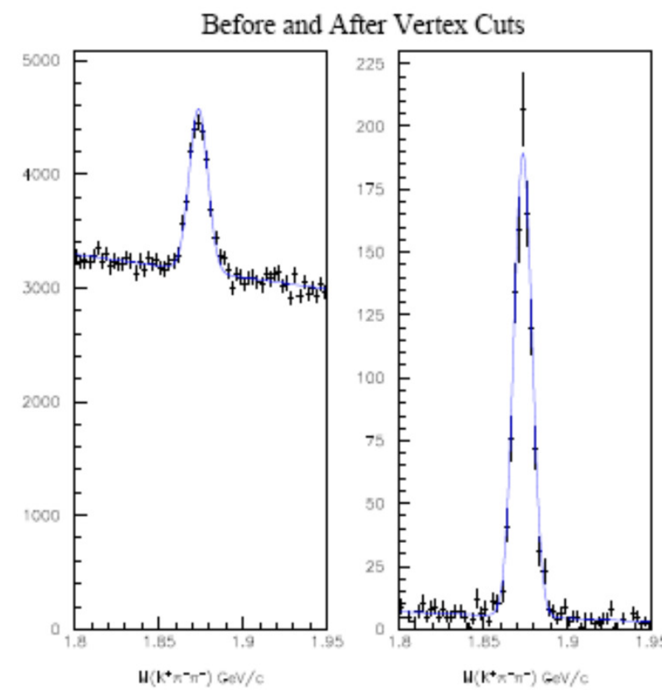
In the limit of relativistic speeds, only from impact parameter !
Full decay reconstruction not required

Vertex Detection - V



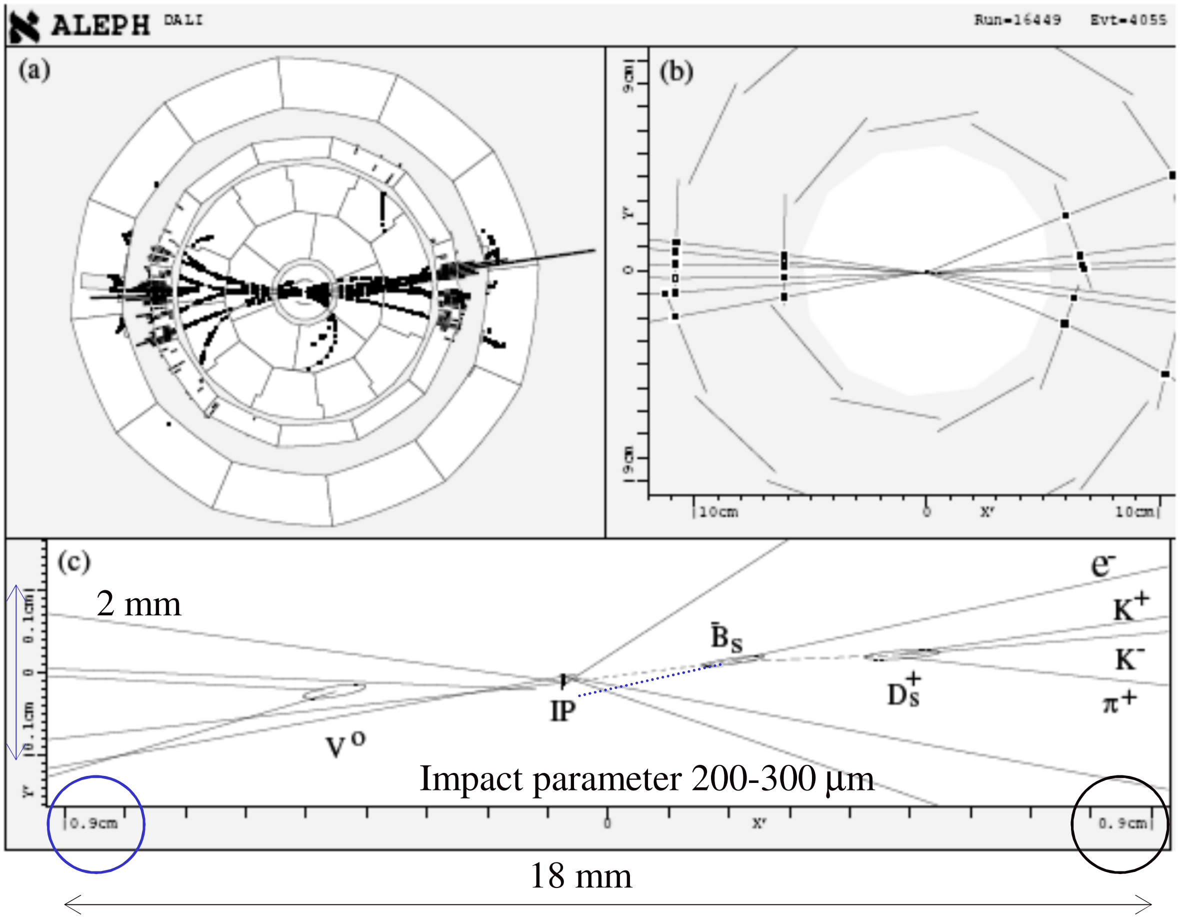
D^0 Lifetime

@TBA



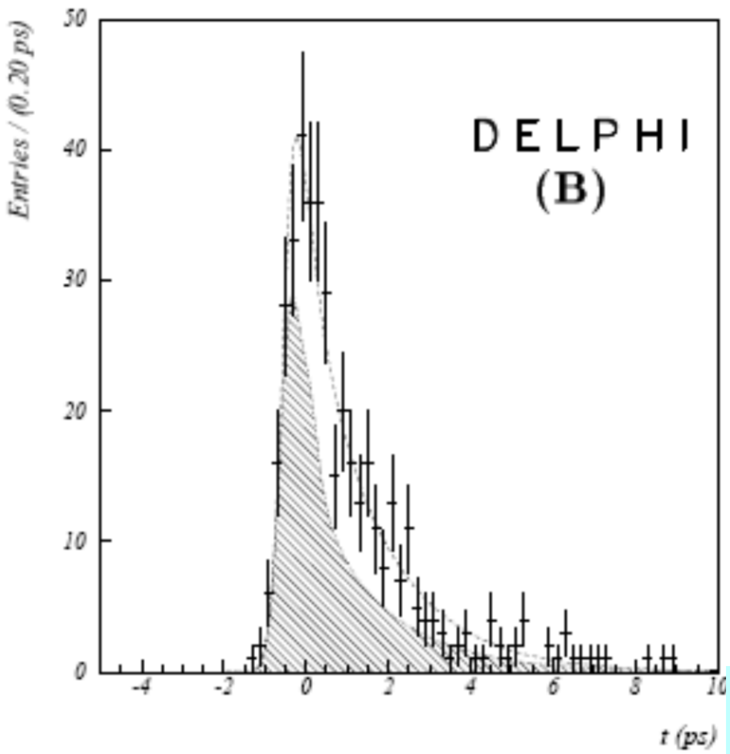
*D** selection,
with and without
secondary vertex

Vertex Detection - VI

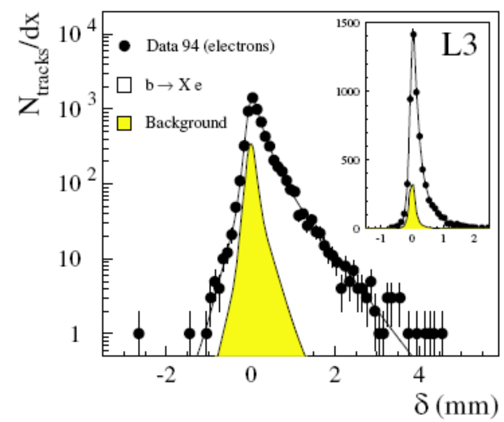


@TBA

Vertex Detection - VII



@TBA



Top

Heaviest quark, predicted together with *b* as a member of the 3rd family

Finally found at Fermilab in 1995, after more than 20 years of theoretical and experimental hunting

Very peculiar quark, whose mass of 171 GeV is so large as to allow for weak decays into b + *real* W,Z^0

$\rightarrow$ *Very large weak decay rate, short lifetime similar to strong interaction resonances*

$\rightarrow$ *Does not bind into mesons, baryons*

Best understood while discussing weak interactions (see later)

Quark Parton Model - I

Write down F_2 in terms of PDFs

$$F_2 = \left(\sum_i z_i^2 n_i\right) x\delta\left(x - \frac{m}{M}\right)$$

$$F_2(x) = x\left(\sum_i z_i^2 q_i(x)\right)$$

$$p \;=\; uud$$

$$F_2^p(x) = x\left[\left(\frac{2}{3}\right)^2 u_p(x) + \left(-\frac{1}{3}\right)^2 d_p(x)\right]$$

$$\rightarrow F_2^p(x) = x\left[\frac{4}{9}u_p(x) + \frac{1}{9}d_p(x)\right]$$

$$n \;=\; ddu$$

$$F_2^n(x) = x\left[\left(-\frac{1}{3}\right)^2 d_n(x) + \left(\frac{2}{3}\right)^2 u_n(x)\right]$$

From isospin symmetry:

$$F_2^n(x) = x\left[\left(-\frac{1}{3}\right)^2 u_p(x) + \left(\frac{2}{3}\right)^2 d_p(x)\right]$$

$$\rightarrow F_2^n(x) = x\left[\frac{1}{9}u_p(x) + \frac{4}{9}d_p(x)\right]$$

Quark Parton Model - II

Consider the deuteron structure function:

$$F_2^d(x) = \frac{1}{2}\left(F_2^p + F_2^n\right) = \frac{5}{9}\frac{x}{2}\left[u_p(x) + d_p(x)\right]$$

$$\rightarrow F_2^n(x) = F_2^d(x) - F_2^p(x)$$

$$= \frac{5}{18}x\left[u_p(x) + d_p(x)\right] - \frac{1}{9}x\left[u_p(x) - 4d_p(x)\right]$$

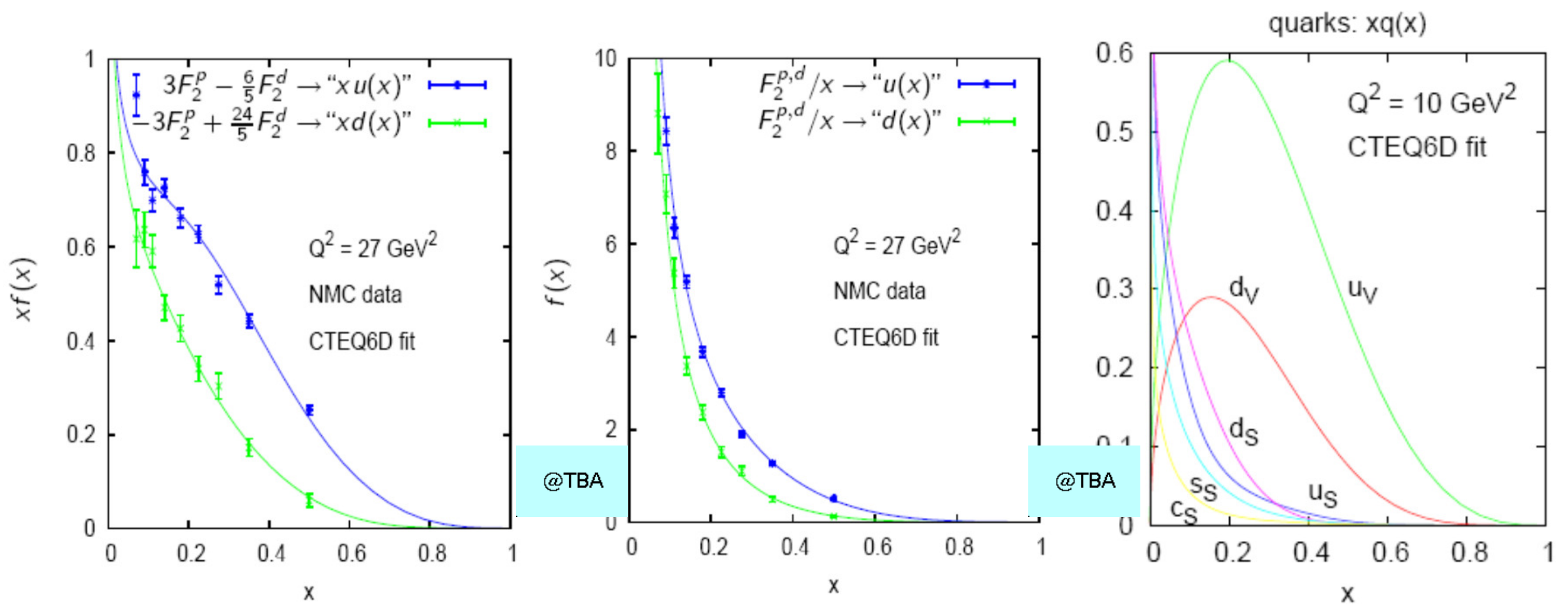
$$= \frac{3}{18}x\left[u_p(x) - d_p(x)\right]$$

Finally extract PDFs from measured F_2

$$xu_p(x) = xd_n(x) = 3F_2^p(x) - \frac{6}{5}F_2^d(x)$$

$$xu_n(x) = xd_p(x) = 3F_2^p(x) + \frac{24}{5}F_2^d(x)$$

Quark Parton Model - III



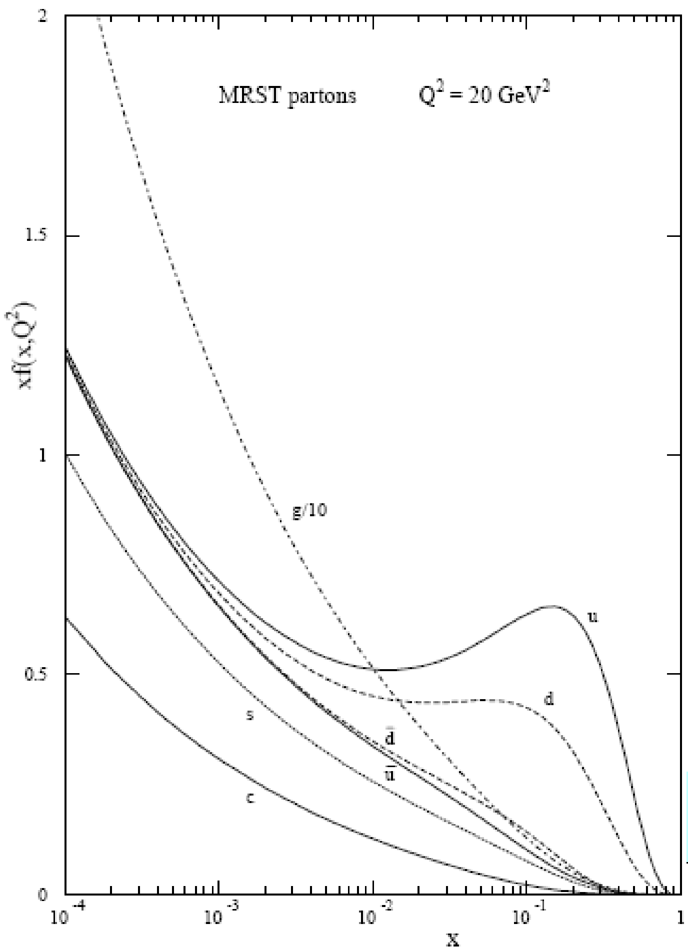
Among parton model predictions: *Sum Rules* (= Integral relations) for PDFs
Examples: Proton quark content is *uud*

$$\int \left[u_p(x) - \bar{u}_p(x)\right] dx = 2$$

$$\int \left[d_p(x) - \bar{d}_p(x)\right] dx = 1$$

$$\int \left[s_p(x) - \bar{s}_p(x)\right] dx = 0$$

Quark Parton Model - IV

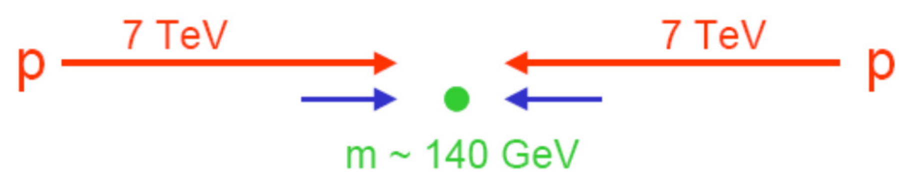


Data-based calculation

Low-x region very important at LHC

Example:

Production of a Higgs with m_H = *140 GeV*



$$\Rightarrow \quad x \sim \frac{70\ GeV}{7\ TeV} \sim 0.01$$

@TBA