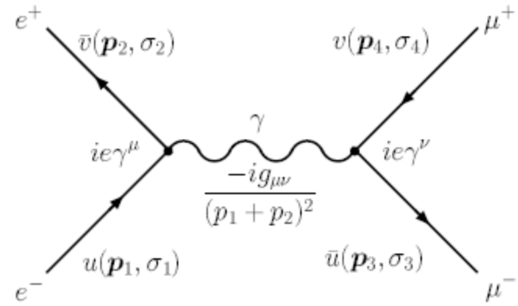
Electron-Positron Annihilation into a Muon-Antimuon Pair

With Everything (Hopefully) Spelled Out



Initial state:

$e^+ \quad \overline{v}(p_2, \sigma_2)$

$e^- \quad u(p_1, \sigma_1)$

Final state:

$\mu^+ \quad v(p_4, \sigma_4)$

$\mu^- \quad \overline{u}(p_3, \sigma_3)$

$\rightarrow$ Matrix element:

$$T_{fi} = \frac{e^2}{q^2}\overline{u}(p_3, \sigma_3)\gamma_\mu v(p_4, \sigma_4)\overline{v}(p_2, \sigma_2)\gamma^\mu u(p_1, \sigma_1)$$

$\rightarrow$ Squared matrix element:

$$\left|T_{fi}\right|^2 = \frac{e^4}{q^4}\left[\overline{u}(p_3, \sigma_3)\gamma_\mu v(p_4, \sigma_4)\overline{v}(p_2, \sigma_2)\gamma^\mu u(p_1, \sigma_1)\right]$$
$$\cdot \left[\overline{u}(p_3, \sigma_3)\gamma_\nu v(p_4, \sigma_4)\overline{v}(p_2, \sigma_2)\gamma^\nu u(p_1, \sigma_1)\right]^*$$

Observe:

$$\left[\overline{u}(p_3,\sigma_3)\gamma_\nu v(p_4,\sigma_4)\overline{v}(p_2,\sigma_2)\gamma^\nu u(p_1,\sigma_1)\right]^* =$$

$$\left[\overline{u}(p_3,\sigma_3)\gamma_\nu v(p_4,\sigma_4)\right]^*\left[\overline{v}(p_2,\sigma_2)\gamma^\nu u(p_1,\sigma_1)\right]^*$$

Complex number = 1×1 'matrix' $\rightarrow [\ ]^* \equiv [\ ]^\dagger$

$$\rightarrow \left[\overline{u}(p_3,\sigma_3)\gamma_\nu v(p_4,\sigma_4)\right]^* = \left[u^\dagger(p_3,\sigma_3)\gamma_0\gamma_\nu v(p_4,\sigma_4)\right]^\dagger = \left[v^\dagger(p_4,\sigma_4)\gamma_\nu^\dagger\gamma_0^\dagger u(p_3,\sigma_3)\right]$$

$$\gamma_0^2 = 1, \gamma_0^\dagger = \gamma_0, \overline{\gamma}_\nu \equiv \gamma_0\gamma_\nu^\dagger\gamma_0$$

$$\rightarrow \left[v^\dagger(p_4,\sigma_4)\gamma_\nu^\dagger\gamma_0^\dagger u(p_3,\sigma_3)\right] = \left[v^\dagger(p_4,\sigma_4)\gamma_0^2\gamma_\nu^\dagger\gamma_0 u(p_3,\sigma_3)\right]$$

$$= \left[\overline{v}(p_4,\sigma_4)\gamma_0\gamma_\nu^\dagger\gamma_0 u(p_3,\sigma_3)\right] = \left[\overline{v}(p_4,\sigma_4)\overline{\gamma}_\nu u(p_3,\sigma_3)\right]$$

$$\rightarrow \left[\overline{u}(p_3,\sigma_3)\gamma_\nu v(p_4,\sigma_4)\right]^* = \left[\overline{v}(p_4,\sigma_4)\overline{\gamma}_\nu u(p_3,\sigma_3)\right]$$

Similarly:

$$\left[\overline{v}(p_2,\sigma_2)\gamma^\nu u(p_1,\sigma_1)\right]^* = \left[\overline{u}(p_1,\sigma_1)\overline{\gamma}^\nu v(p_2,\sigma_2)\right]$$

$$\rightarrow \left|T_{fi}\right|^2 = \frac{e^4}{q^4}\left[\overline{u}(p_3,\sigma_3)\gamma_\mu v(p_4,\sigma_4)\overline{v}(p_2,\sigma_2)\gamma^\mu u(p_1,\sigma_1)\right]$$
$$\cdot\left[\overline{v}(p_4,\sigma_4)\overline{\gamma}_\nu u(p_3,\sigma_3)\overline{u}(p_1,\sigma_1)\overline{\gamma}^\nu v(p_2,\sigma_2)\right]$$

Averaging and summing over initial and final spins:

$$\left\langle\left|T_{fi}\right|^2\right\rangle = \frac{1}{4}\sum_{\sigma_1,\sigma_2,\sigma_3,\sigma_4}\left|T_{fi}\right|^2$$

$$\left\langle\left|T_{fi}\right|^2\right\rangle = \frac{1}{4}\frac{e^4}{q^4}\sum_{\sigma_1,\sigma_2,\sigma_3,\sigma_4}\left[\overline{u}(p_3,\sigma_3)\gamma_\mu v(p_4,\sigma_4)\overline{v}(p_2,\sigma_2)\gamma^\mu u(p_1,\sigma_1)\right]$$
$$\cdot\left[\overline{v}(p_4,\sigma_4)\overline{\gamma}_\nu u(p_3,\sigma_3)\overline{u}(p_1,\sigma_1)\overline{\gamma}^\nu v(p_2,\sigma_2)\right]$$

$$\left\langle\left|T_{fi}\right|^2\right\rangle = \frac{1}{4}\frac{e^4}{q^4}\sum_{\sigma_1,\sigma_2,\sigma_3,\sigma_4}\left[\overline{u}(p_3,\sigma_3)\gamma_\mu v(p_4,\sigma_4)\right]\left[\overline{v}(p_2,\sigma_2)\gamma^\mu u(p_1,\sigma_1)\right]$$
$$\cdot\left[\overline{v}(p_4,\sigma_4)\overline{\gamma}_\nu u(p_3,\sigma_3)\right]\left[\overline{u}(p_1,\sigma_1)\overline{\gamma}^\nu v(p_2,\sigma_2)\right]$$

$$\left\langle\left|T_{fi}\right|^2\right\rangle = \frac{1}{4}\frac{e^4}{q^4}\sum_{\sigma_1,\sigma_2}\left[\overline{u}(p_1,\sigma_1)\overline{\gamma}^\nu v(p_2,\sigma_2)\right]\left[\overline{v}(p_2,\sigma_2)\gamma^\mu u(p_1,\sigma_1)\right]$$
$$\cdot\sum_{\sigma_3,\sigma_4}\left[\overline{u}(p_3,\sigma_3)\gamma_\mu v(p_4,\sigma_4)\right]\left[\overline{v}(p_4,\sigma_4)\overline{\gamma}_\nu u(p_3,\sigma_3)\right]$$

This sum of products can be reduced in several ways. Simplest trick:

a) Write explicitly spinor indexes:

$$\sum_{\sigma_1,\sigma_2}\sum_{\alpha,\beta,\gamma,\delta}\bar{u}_\alpha(p_1,\sigma_1)\bar{\gamma}^\nu_{(\alpha\beta)}v_\beta(p_2,\sigma_2)\bar{v}_\gamma(p_2,\sigma_2)\gamma^\mu_{(\gamma\delta)}u_\delta(p_1,\sigma_1)\quad \text{(\& similar for } \sum_{\sigma_3,\sigma_4}\text{)}$$

b) Move u_δ to the front in each term (OK because it is just a number)

$$\sum_{\sigma_1,\sigma_2}\sum_{\alpha,\beta,\gamma,\delta}u_\delta(p_1,\sigma_1)\bar{u}_\alpha(p_1,\sigma_1)\bar{\gamma}^\nu_{(\alpha\beta)}v_\beta(p_2,\sigma_2)\bar{v}_\gamma(p_2,\sigma_2)\gamma^\mu_{(\gamma\delta)}$$

c) Exchange the sum order and split the spin sums:

$$\sum_{\alpha,\beta,\gamma,\delta}\sum_{\sigma_1}u\ (p_1,\sigma_1)\bar{u}_\alpha(p_1,\sigma_1)\bar{\gamma}^\nu_{(\alpha\beta)}\sum_{\sigma_2}v_\beta(p_2,\sigma_2)\bar{v}_\gamma(p_2,\sigma_2)\gamma^\mu_{(\gamma\delta)}$$

d) Observe the appearance of

$$\sum_{\sigma_1}u_\delta(p_1,\sigma_1)\bar{u}_\alpha(p_1,\sigma_1)=\left(\not{p}_1+m_1\right)_{\delta\alpha}$$

$$\sum_{\sigma_2}v_\beta(p_2,\sigma_2)\bar{v}_\gamma(p_2,\sigma_2)=\left(\not{p}_2-m_2\right)_{\beta\gamma}$$

e) Then

$$\sum_{\alpha,\beta,\gamma,\delta}\left(\not{p}_1+m_1\right)_{\delta\alpha}\bar{\gamma}^\nu_{(\alpha\beta)}\left(\not{p}_2-m_2\right)_{\beta\gamma}\gamma^\mu_{(\gamma\delta)}=Tr\left[\left(\not{p}_1+m_1\right)\bar{\gamma}^\nu\left(\not{p}_2-m_2\right)\gamma^\mu\right]$$

Similarly for the sum over σ_3,σ_4:

$$\sum_{\alpha,\beta,\gamma,\delta}\left(\not{p}_3+m_3\right)_{\delta\alpha}\bar{\gamma}_{\nu(\alpha\beta)}\left(\not{p}_4-m_4\right)_{\beta\gamma}\gamma_{\mu(\gamma\delta)}=Tr\left[\left(\not{p}_3+m_3\right)\bar{\gamma}_\nu\left(\not{p}_4-m_4\right)\gamma_\mu\right]$$

Now

$\overline{\gamma}_\nu \equiv \gamma_0 \gamma_\nu^\dagger \gamma_0 = \gamma_\nu$ by Dirac matrices algebra

$$\rightarrow \left\langle \left| T_{fi} \right|^2 \right\rangle = \frac{1}{4}\frac{e^4}{q^4} Tr\left[\left(\not{p}_1 + m_1\right)\gamma^\nu \left(\not{p}_2 - m_2\right)\gamma^\mu\right] Tr\left[\left(\not{p}_3 + m_3\right)\gamma_\nu \left(\not{p}_4 - m_4\right)\gamma_\mu\right]$$

$$= \frac{1}{4}\frac{e^4}{q^4} Tr\left[\not{p}_1\gamma^\nu \not{p}_2\gamma^\mu - \not{p}_1\gamma^\nu m_2\gamma^\mu + m_1\gamma^\nu \not{p}_2\gamma^\mu - m_1\gamma^\mu m_2\gamma^\nu\right]$$

$$\cdot Tr\left[\left(\not{p}_3\gamma_\nu \not{p}_4\gamma_\mu - \not{p}_3\gamma_\nu m_4\gamma_\mu + m_3\gamma_\nu \not{p}_4\gamma_\mu - m_3\gamma_\nu m_4\gamma_\mu\right)\right]$$

$m_1 = m_2 = m, m_3 = m_4 = M$

$$\rightarrow \left\langle \left| T_{fi} \right|^2 \right\rangle = \frac{1}{4}\frac{e^4}{q^4} Tr\left[\not{p}_1\gamma^\nu \not{p}_2\gamma^\mu - m\not{p}_1\gamma^\nu\gamma^\mu + m\gamma^\nu \not{p}_2\gamma^\mu - m^2\gamma^\mu\gamma^\nu\right]$$

$$\cdot Tr\left[\left(\not{p}_3\gamma_\nu \not{p}_4\gamma_\mu - M\not{p}_3\gamma_\nu\gamma_\mu + M\gamma_\nu \not{p}_4\gamma_\mu - M^2\gamma_\nu\gamma_\mu\right)\right]$$

Trace of product of odd number of γ matrices = 0

$$\rightarrow Tr\left(\not{p}_1\gamma^\nu\gamma^\mu\right) = Tr\left(\gamma^\nu \not{p}_2\gamma^\mu\right) = Tr\left(\not{p}_3\gamma_\nu\gamma_\mu\right) = Tr\left(\gamma_\nu \not{p}_4\gamma_\mu\right) = 0$$

$$\rightarrow \left\langle \left| T_{fi} \right|^2 \right\rangle = \frac{1}{4}\frac{e^4}{q^4} Tr\left[\not{p}_1\gamma^\nu \not{p}_2\gamma^\mu - m^2\gamma^\mu\gamma^\nu\right]\cdot Tr\left[\left(\not{p}_3\gamma_\nu \not{p}_4\gamma_\mu - M^2\gamma_\nu\gamma_\mu\right)\right]$$

$Tr\left(\gamma_\nu\gamma_\mu\right) = 4g_{\nu\mu}$, similar for contravariant

$$\rightarrow \left\langle \left| T_{fi} \right|^2 \right\rangle = \frac{1}{4}\frac{e^4}{q^4} \left[Tr\left(\not{p}_1\gamma^\nu \not{p}_2\gamma^\mu\right) - 4m^2 g^{\mu\nu}\right]\cdot\left[Tr\left(\not{p}_3\gamma_\nu \not{p}_4\gamma_\mu\right) - 4M^2 g_{\nu\mu}\right]$$

$$Tr\left(\not{p}_3\gamma_\nu \not{p}_4\gamma_\mu\right) = Tr\left(p_{3\alpha}\gamma_\alpha\gamma_\nu p_{4\beta}\gamma_\beta\gamma_\mu\right) = p_{3\alpha}p_{4\beta} Tr\left(\gamma_\alpha\gamma_\nu\gamma_\beta\gamma_\mu\right)$$

$$\rightarrow Tr\left(\not{p}_3\gamma_\nu \not{p}_4\gamma_\mu\right) = p_{3\alpha}p_{4\beta} 4\left(g_{\alpha\nu}g_{\beta\mu} - g_{\alpha\beta}g_{\nu\mu} + g_{\alpha\mu}g_{\nu\beta}\right)$$

$$\rightarrow Tr\left(\not{p}_3\gamma_\nu \not{p}_4\gamma_\mu\right) = 4p_{3\nu}p_{4\mu} + 4p_{3\mu}p_{4\nu} - 4p_3 p_4 g_{\nu\mu}$$

Similarly:

$$Tr\left(\not{p}_1\gamma^\nu \not{p}_2\gamma^\mu\right) = 4p^{1\nu}p^{2\mu} + 4p^{1\nu}p^{2\mu} - 4p_1 p_2 g^{\nu\mu}$$

$$\rightarrow \left\langle \left| T_{fi} \right|^2 \right\rangle = \frac{1}{4}\frac{e^4}{q^4} \left[4p_1^\nu p_2^\mu + 4p_1^\mu p_2^\nu - 4\left(p_1 p_2 + m^2\right)g^{\mu\nu}\right]$$

$$\cdot\left[4p_{3\nu}p_{4\mu} + 4p_{3\mu}p_{4\nu} - 4\left(p_3 p_4 + M^2\right)g_{\nu\mu}\right]$$

$$s = \left(p_1 + p_2\right)^2 = p_1^2 + p_2^2 + 2p_1 p_2 = 2m^2 + 2p_1 p_2$$

$$\rightarrow 4\left(p_1 p_2 + m^2\right) = 4\left(p_3 p_4 + M^2\right) = 2s$$

$$\rightarrow \left\langle \left| T_{fi} \right|^2 \right\rangle = \frac{1}{4}\frac{e^4}{q^4} \left[4p_1^\nu p_2^\mu + 4p_1^\mu p_2^\nu - 2sg^{\mu\nu}\right]\cdot\left[4p_{3\nu}p_{4\mu} + 4p_{3\mu}p_{4\nu} - 2sg_{\nu\mu}\right]$$

$$\rightarrow \left\langle \left|T_{fi}\right|^2 \right\rangle = \frac{1}{4}\frac{e^4}{q^4}\left[32(p_1p_4)(p_2p_3)+32(p_1p_3)(p_2p_4)-16s(p_1p_2)-16s(p_4p_3)+16s^2\right]$$

$$\rightarrow \left\langle \left|T_{fi}\right|^2 \right\rangle = \frac{1}{4}\frac{e^4}{q^4}\Bigg[32(p_1p_4)(p_2p_3)+32(p_1p_3)(p_2p_4)$$

$$+16s\left[\underbrace{\left(p_1p_2+m^2\right)}_{s/2}+\underbrace{\left(p_4p_3+M^2\right)}_{s/2}-(p_1p_2)-(p_4p_3)\right]\Bigg]$$

$q^2 \equiv s$

$$\rightarrow \left\langle \left|T_{fi}\right|^2 \right\rangle = \frac{1}{4}\frac{e^4}{s^2}\left[32(p_1p_4)(p_2p_3)+32(p_1p_3)(p_2p_4)+16s\left(m^2+M^2\right)\right]$$

$$\rightarrow \left\langle \left|T_{fi}\right|^2 \right\rangle = \frac{4e^4}{s^2}\left[2(p_1p_4)(p_2p_3)+2(p_1p_3)(p_2p_4)+s\left(m^2+M^2\right)\right]$$

When $m \approx 0$:

$$\left\langle \left|T_{fi}\right|^2 \right\rangle = \frac{4e^4}{s^2}\left[2(p_1p_4)(p_2p_3)+2(p_1p_3)(p_2p_4)+sM^2\right]$$

When $m \approx 0,\ M \approx 0$:

$$\left\langle \left|T_{fi}\right|^2 \right\rangle = \frac{8e^4}{s^2}\left[(p_1p_4)(p_2p_3)+(p_1p_3)(p_2p_4)\right]$$

By neglecting m:

$$CM:\ \begin{cases} |\mathbf{p}_1| = |\mathbf{p}_2| = |\mathbf{p}_{in}| = E \\ |\mathbf{p}_3| = |\mathbf{p}_4| = |\mathbf{p}_{out}| = \sqrt{E^2 - M^2} \end{cases}$$

$$\rightarrow \begin{cases} p_1 = \begin{pmatrix} E & 0 & 0 & E \end{pmatrix}, p_2 = \begin{pmatrix} E & 0 & 0 & -E \end{pmatrix} \\ p_3 = \begin{pmatrix} E & |\mathbf{p}_{out}|\sin\theta & 0 & |\mathbf{p}_{out}|\cos\theta \end{pmatrix}, p_4 = \begin{pmatrix} E & -|\mathbf{p}_{out}|\sin\theta & 0 & -|\mathbf{p}_{out}|\cos\theta \end{pmatrix} \end{cases}$$

$$\left.\begin{aligned} p_1 p_4 = E^2 + E\sqrt{E^2 - M^2}\cos\theta \\ p_2 p_3 = E^2 + E\sqrt{E^2 - M^2}\cos\theta \end{aligned}\right\}$$

$$\rightarrow p_1 p_4 p_2 p_3 = \left(E^2 + E\sqrt{E^2 - M^2}\cos\theta\right)^2 = E^4 + E^2\left(E^2 - M^2\right)\cos^2\theta + 2E^2 E\sqrt{E^2 - M^2}\cos\theta$$

$$\rightarrow p_1 p_4 p_2 p_3 = E^4\left[1 + \left(1 - \frac{M^2}{E^2}\right)\cos^2\theta + 2\sqrt{1 - \frac{M^2}{E^2}}\cos\theta\right]$$

$$\left.\begin{aligned} p_1 p_3 = E^2 - E\sqrt{E^2 - M^2}\cos\theta \\ p_2 p_4 = E^2 - E\sqrt{E^2 - M^2}\cos\theta \end{aligned}\right\} \rightarrow p_1 p_3 p_2 p_4 = \left(E^2 - E\sqrt{E^2 - M^2}\cos\theta\right)^2$$

$$\rightarrow p_1 p_3 p_2 p_4 = E^4\left[1 + \left(1 - \frac{M^2}{E^2}\right)\cos^2\theta - 2\sqrt{1 - \frac{M^2}{E^2}}\cos\theta\right]$$

$$\rightarrow \left\langle |T_{fi}|^2 \right\rangle = \frac{8e^4}{s^2}\left[(p_1 p_4)(p_2 p_3) + (p_1 p_3)(p_2 p_4) + \frac{1}{2} s M^2\right]$$

$$\rightarrow \left\langle |T_{fi}|^2 \right\rangle = \frac{8e^4}{16E^4}\begin{Bmatrix} E^4\left[1 + \left(1 - \frac{M^2}{E^2}\right)\cos^2\theta + 2\sqrt{1 - \frac{M^2}{E^2}}\cos\theta\right] + \\ +E^4\left[1 + \left(1 - \frac{M^2}{E^2}\right)\cos^2\theta - 2\sqrt{1 - \frac{M^2}{E^2}}\cos\theta\right] \\ +2E^2 M^2 \end{Bmatrix}$$

$$\rightarrow \left\langle |T_{fi}|^2 \right\rangle = \frac{e^4}{2}\left[2 + 2\left(1 - \frac{M^2}{E^2}\right)\cos^2\theta + 4\frac{M^2}{E^2}\right]$$

$$\rightarrow \left\langle |T_{fi}|^2 \right\rangle = \frac{e^4}{2}\left[2 + 2\left(1 - \frac{M^2}{E^2}\right)\cos^2\theta + 2\frac{M^2}{E^2}\right]$$

$$\rightarrow \left\langle |T_{fi}|^2 \right\rangle = e^4\left[1 + \frac{M^2}{E^2} + \left(1 - \frac{M^2}{E^2}\right)\cos^2\theta\right]$$

Therefore, for the cross-section in the CM frame:

$$\frac{d\sigma}{d\Omega} = \underbrace{\frac{(2\pi)^{4-2\cdot 3}}{4\left[(p_1\cdot p_2)^2 - m^4\right]^{1/2}}}_{\substack{flux \\ factor}} \left\langle \left|T_{fi}\right|^2 \right\rangle \underbrace{\frac{\left|\mathbf{p}_f\right|}{4\sqrt{s}}}_{\substack{phase\ space \\ factor}} = \frac{1}{4\pi^2}\frac{1}{4\cdot 2E^2}\left\langle \left|T_{fi}\right|^2 \right\rangle \frac{\sqrt{E^2 - M^2}}{4(2E)}$$

$$\rightarrow \frac{d\sigma}{d\Omega} = \frac{1}{32\pi^2 E^2}\left\langle \left|T_{fi}\right|^2 \right\rangle \frac{1}{8}\sqrt{1-\frac{M^2}{E^2}}$$

$$\rightarrow \frac{d\sigma}{d\Omega} = \frac{e^4}{32\pi^2 E^2}\frac{\sqrt{E^2 - M^2}}{E}\frac{1}{8}\left[1+\frac{M^2}{E^2}+\left(1-\frac{M^2}{E^2}\right)\cos^2\theta\right]$$

$$\alpha = \frac{e^2}{4\pi} \rightarrow \frac{e^4}{16\pi^2} = \alpha^2$$

$$\rightarrow \frac{d\sigma}{d\Omega} = \frac{\alpha^2}{16E^2}\sqrt{1-\frac{M^2}{E^2}}\left[1+\frac{M^2}{E^2}+\left(1-\frac{M^2}{E^2}\right)\cos^2\theta\right]$$

Reminder on 2 - body phase space :

P total 4 - momentum, CM components: $\left(\sqrt{s} \quad 0 \quad 0 \quad 0\right)$

p_1, p_2, m_1, m_2 4-momentum & mass of final state particles

$$d^2\Phi_2 = \int \frac{d^3\boldsymbol{p}_1}{2E_1}\frac{d^3\boldsymbol{p}_2}{2E_2}\delta^4\left(P - p_1 - p_2\right)$$

$$\rightarrow d^2\Phi_2 = \int \delta^4\left(P - p_1 - p_2\right)\delta\left[p_1^2 - m_1^2\right]\delta\left[p_2^2 - m_2^2\right]d^4 p_1 d^4 p_2$$

$$\rightarrow d^2\Phi_2 = \int \delta^4\left(P - p_1 - p_2\right)\delta\left[p_1^2 - m_1^2\right]\delta\left[p_2^2 - m_2^2\right]d^4 p_1 d^4 p_2$$

$$\rightarrow d^2\Phi_2 = \int \delta\left[p_1^2 - m_1^2\right]\delta\left[\left(P - p_1\right)^2 - m_2^2\right]d^4 p_1$$

$$\delta\left[\left(P - p_1\right)^2 - m_2^2\right] = \delta\left[\left(\sqrt{s} - E_1^*\right)^2 - \boldsymbol{p}_1^{*2} - m_2^2\right] = \delta\left[\left(s + E_1^{*2} - 2\sqrt{s}E_1^*\right) - \boldsymbol{p}_1^{*2} - m_2^2\right]$$

$$= \delta\left[s - 2\sqrt{s}E_1^* + m_1^2 - m_2^2\right]$$

$$\rightarrow d^2\Phi_2 = \int \frac{d^3\boldsymbol{p}_1^*}{2E_1^*}\delta\left[s - 2\sqrt{s}E_1^* + m_1^2 - m_2^2\right]$$

$$E^2 = p^2 + m^2 \to EdE = pdp \to dp = \frac{E}{p} dE$$

$$d^3\boldsymbol{p} = p^2 dp d\Omega = p^2 \frac{E}{p} dE d\Omega = pEdEd\Omega$$

$$\to d^2\Phi_2 = \int \frac{d^3\boldsymbol{p}_1^*}{2E_1^*} \delta\left[d - 2\sqrt{s}E_1^* + m_1^2 - m_2^2\right] = \int \frac{p_1^* E_1^*}{2E_1^*} \delta\left[s - 2\sqrt{s}E_1^* + m_1^2 - m_2^2\right] dE_1^* d\Omega_1^*$$

$$\to d^2\Phi_2 = \int \frac{1}{2} p_1^* \delta\left[s - 2\sqrt{s}E_1^* + m_1^2 - m_2^2\right] dE_1^* d\Omega_1^*$$

$$E_1^* = \frac{s + m_1^2 - m_2^2}{2\sqrt{s}} \to \delta\left[s - 2\sqrt{s}E_1^* + m_1^2 - m_2^2\right] = \frac{1}{2\sqrt{s}} \delta\left[E_1^* - \frac{s + m_1^2 - m_2^2}{2\sqrt{s}}\right]$$

$$p_1^* = \sqrt{E_1^{*2} - m_1^2} \equiv p^* = \sqrt{\frac{\left(s + m_1^2 - m_2^2\right)^2 - 4m_1^2 s}{4s}} = \sqrt{\frac{\left(s - m_1^2 - m_2^2\right)^2}{4s}} = \frac{s - m_1^2 - m_2^2}{2\sqrt{s}}$$

$$\to d^2\Phi_2 = \underbrace{\frac{p^*}{4\sqrt{s}}}_{\substack{\text{phase space} \\ \text{factor}}} d\Omega_1^*$$

Reminder on Dirac representation:

$$\begin{cases} \gamma_0^\dagger = \gamma_0 \\ \gamma_k^\dagger = -\gamma_k \end{cases}$$

$$\to \gamma_\nu^\dagger = \gamma_0 \gamma_\nu \gamma_0$$

$$\to \gamma_0 \gamma_\nu^\dagger \gamma_0 = \gamma_0 \gamma_0 \gamma_\nu \gamma_0 \gamma_0$$

$$\gamma_0^2 = 1$$

$$\to \gamma_0 \gamma_\nu^\dagger \gamma_0 = \gamma_\nu$$