Trasformata di Laplace: Generalizzazione della TdF

$$X(\omega) = F[x(t)] = \int_{-\infty}^{+\infty} x(t) e^{-j\omega t} dt$$

$X(\omega)$ esiste se

$$\int_{-\infty}^{+\infty} |x(t)| dt < \infty$$

Molti casi non soddisfano la condizione: Es funzione a gradino

Estensione:

$$x(t) \to x(t) e^{-\sigma t}$$

$$\int_{-\infty}^{+\infty} x(t) e^{-j\omega t} dt \to \int_{-\infty}^{+\infty} x(t) e^{-\sigma t} e^{-j\omega t} dt = \int_{-\infty}^{+\infty} x(t) e^{-(\sigma + j\omega)t} dt \equiv \int_{-\infty}^{+\infty} x(t) e^{-st} dt$$

$s = \sigma + j\omega$ frequenza complessa

Trasformata di Laplace:

$$X(s) = L[x(t)] = \int_{-\infty}^{+\infty} x(t) e^{-st} dt$$

Funzione di variabile complessa definita nel piano complesso:

asse *x*: $\mathrm{Re}(s) = \sigma$

asse *y*: $\mathrm{Im}(s) = \omega$

Trasformazione inversa:

$$X(s) = \int_{-\infty}^{+\infty} x(t) e^{-(\sigma + j\omega)t} dt = \int_{-\infty}^{+\infty} \left[x(t) e^{-\sigma t} \right] e^{-j\omega t} dt$$

$$\to x(t) e^{-\sigma t} = \frac{1}{2\pi} \int_{-\infty}^{+\infty} X(s) e^{+j\omega t} d\omega \to x(t) = \frac{1}{2\pi} \int_{-\infty}^{+\infty} X(s) e^{+j\omega t} e^{+\sigma t} d\omega$$

$$d\omega = \frac{ds}{j} \to \frac{1}{2\pi} \int_{-\infty}^{+\infty} X(s) e^{(\sigma + j\omega)t} d\omega = \frac{1}{2\pi j} \int_{\sigma - j\infty}^{\sigma + j\infty} X(s) e^{(\sigma + j\omega)t} ds$$

Forma piu' usata:

$$X(s) = \int_{0}^{+\infty} x(t) e^{-st} dt$$

Integrale solo su $t > 0$: TdL unilatera

Regione di convergenza:

Parte del piano complesso in cui $X(s)$ esiste

Es:

$$x(t) = e^{-at} \rightarrow \int_0^{+\infty} e^{-at} e^{-st} dt = \frac{1}{a+\sigma+j\omega} e^{-(a+\sigma+j\omega)t} \Big|_0^{\infty}$$

Convergenza: $a+\sigma > 0 \rightarrow \sigma > -a$

$$\rightarrow X(s) = \frac{1}{a+s}$$

Es:

Caso limite $a = 0 \rightarrow x(t) = H(t) \rightarrow X(s) = \frac{1}{s}$

Es:

$$x(t) = e^{-2t} + e^{-t} \cos 3t$$

$$\rightarrow x(t) = e^{-2t} + e^{-t} \frac{e^{j3t} + e^{-j3t}}{2} = e^{-2t} + \frac{1}{2} e^{-(1+j3)t} + \frac{1}{2} e^{-(1-j3)t}$$

$$\rightarrow X(s) = \int_0^{\infty} \left[e^{-2t} + \frac{1}{2} e^{-(1+j3)t} + \frac{1}{2} e^{-(1-j3)t} \right] e^{-st} dt$$

Convergenza: $\sigma > -1$

$$\rightarrow X(s) = \frac{1}{s+2} + \frac{1}{2} \frac{1}{s+(1+j3)} + \frac{1}{2} \frac{1}{s+(1-j3)}$$

$$\rightarrow X(s) = \frac{2s^2 + 5s + 12}{(s^2 + 2s + 10)(s+2)}$$

Esempi di coppie *Funzione-Trasformata di Laplace* di uso frequente

Common Laplace Transform Pairs

Time Domain Function		Laplace Domain Function
Name	**Definition***	
Unit Impulse	$\delta(t)$	1
Unit Step	$\gamma(t)$ †	$\frac{1}{s}$
Unit Ramp	t	$\frac{1}{s^2}$
Parbola	t^2	$\frac{2}{s^3}$
Exponential	e^{-at}	$\frac{1}{s+a}$
Asymptotic Exponential	$\frac{1}{a}(1-e^{-at})$	$\frac{1}{s(s+a)}$
Dual Exponential	$\frac{1}{b-a}\left(e^{-at}-e^{-bt}\right)$	$\frac{1}{(s+a)(s+b)}$
Asymptotic Dual Exponential	$\frac{1}{ab}\left[1+\frac{1}{a-b}\left(\mathrm{b}e^{-at}-\mathrm{a}e^{-bt}\right)\right]$	$\frac{1}{s(s+a)(s+b)}$
Time multiplied Exponential	te^{-at}	$\frac{1}{(s+a)^2}$
Sine	$\sin(\omega_0 t)$	$\frac{\omega_0}{s^2+\omega_0^2}$
Cosine	$\cos(\omega_0 t)$	$\frac{s}{s^2+\omega_0^2}$
Decaying Sine	$e^{-at}\sin(\omega_0 t)$	$\frac{\omega_0}{(s+a)^2+\omega_0^2}$
Decaying Cosine	$e^{-at}\cos(\omega_0 t)$	$\frac{s+a}{(s+a)^2+\omega_0^2}$

Proprieta' fondamentali della TdL derivabili dalla definizione

Common Laplace Transform Properties

Name	Illustration
Definition of Transform	$f(t) \xleftrightarrow{L} F(s)$ $F(s) = \int_{0^-}^{\infty} f(t) e^{-st} dt$
Linearity	$Af_1(t) + Bf_2(t) \xleftrightarrow{L} AF_1(s) + BF_2(s)$
First Derivative	$\frac{df(t)}{dt} \xleftrightarrow{L} sF(s) - f(0^-)$
Second Derivative	$\frac{d^2 f(t)}{dt^2} \xleftrightarrow{L} s^2 F(s) - sf(0^-) - \dot{f}(0^-)$
n^{th} Derivative	$\frac{d^n f(t)}{dt^n} \xleftrightarrow{L} s^n F(s) - \sum_{i=1}^{n} s^{n-i} f^{(i-1)}(0^-)$
Integral	$\int_0^t f(\lambda) d\lambda \xleftrightarrow{L} \frac{1}{s} F(s)$
Time Multiplication	$tf(t) \xleftrightarrow{L} -\frac{dF(s)}{ds}$
Time Delay	$f(t-a)\gamma(t-a) \xleftrightarrow{L} e^{-as} F(s)$ $\gamma(t)$ is unit step
Complex Shift	$f(t) e^{-at} \xleftrightarrow{L} F(s+a)$
Scaling	$f\left(\frac{t}{a}\right) \xleftrightarrow{L} aF(as)$
Convolution Property	$f_1(t) * f_2(t) \xleftrightarrow{L} F_1(s) F_2(s)$
Initial Value	$\lim_{t \to 0^+} f(t) = \lim_{s \to \infty} sF(s)$
Final Value (if final value exists)	$\lim_{t \to \infty} f(t) = \lim_{s \to 0} sF(s)$

TdL: Ulteriore formalismo per la soluzione di eq. differenziali
Piu' generale di molti altri

Applicazione ai circuiti:

a) Capacita'

$$i(t) = C\frac{dv(t)}{dt}$$

$$\rightarrow I(s) = C\mathcal{L}\left[\frac{dv(t)}{dt}\right] = CsV(s)$$

$$\rightarrow Z_C(s) = \frac{V(s)}{I(s)} = \frac{1}{sC}$$ Impedenza generalizzata della capacita' C

b) Induttanza

$$v(t) = L\frac{di(t)}{dt}$$

Segno +: $v(t)$ *tensione* ai capi di L $\rightarrow$ opposta a *fem* autoindotta

$$\rightarrow V(s) = L\mathcal{L}\left[\frac{di(t)}{dt}\right] = LsI(s)$$

$$\rightarrow Z_L(s) = sL$$ Impedenza generalizzata della induttanza L

c) Resistenza:

$$\rightarrow Z_R(s) = R$$

Impedenze generalizzate: utilizzate come le impedenze complesse
Ma: funzioni della frequenza complessa s

$\rightarrow$ Generalizzazione delle leggi di Kirchoff

Da proprieta' elementari TdL:

$$x(t) \overset{L}{\leftrightarrow} X(s)$$

$$\frac{dx}{dt} \overset{L}{\leftrightarrow} sX(s)$$

$$\int_0^t x(t')dt' \overset{L}{\leftrightarrow} \frac{1}{s} X(s)$$

$\rightarrow$ Impedenze generalizzate di L e C:

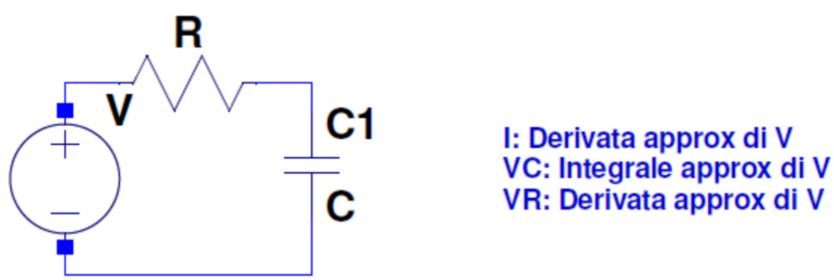
$$V_C(s) = Z_C(s) I_C(s) = \frac{1}{sC} I(s)$$

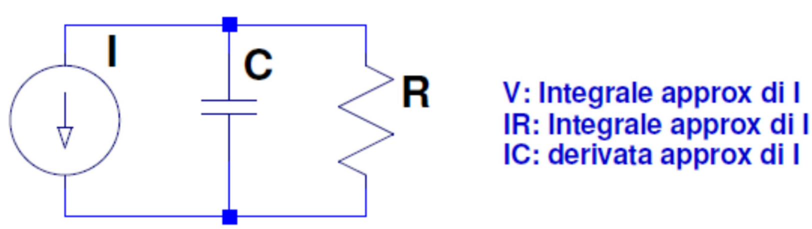
$$V_L(s) = Z_L(s) I_L(s) = sLI(s)$$

$\rightarrow$ Derivatore e Integratore ideali

Circuiti pratici: Approssimazioni

Circuiti con 1 elemento reattivo: I ordine





Per brevita', solo circuito RC serie: altri (RC parallelo, RL serie e parallelo) del tutto simili. Grafici rilevanti: v. lezione precedente su trattamento 'euristico'

Risposta al gradino:

$$v_{in}(t) = V_0 H(t) \to V_{in}(s) = \frac{V_0}{s}$$

Variabile di uscita: tensione su C

TdL:

$$V_{out}(s) = V_{in}(s) \frac{\frac{1}{sC}}{\frac{1}{sC} + R} = \frac{V_0}{s} \frac{1}{1 + sCR} = V_0 \left(\frac{A}{s} + \frac{B}{1 + sCR} \right)$$

$$\frac{A(1 + sCR) + Bs}{s(1 + sCR)} = \frac{s(ACR + B) + A}{s(1 + sCR)} = \frac{1}{s(1 + sCR)}$$

$$\to A = 1, A = -\frac{B}{CR} \to B = -CR$$

$$\to V_{out}(s) = V_0 \left(\frac{1}{s} - \frac{CR}{1 + sCR} \right) = V_0 \left(\frac{1}{s} - \frac{1}{\frac{1}{RC} + s} \right)$$

Anti-TdL:

$$\to v_{out}(t) = V_0 \left(H(t) - \mathrm{e}^{-\frac{t}{CR}} \right)$$

Come atteso:

$t \ll RC \to v_{out}(t) \approx V_0 \frac{t}{RC}$ Integrale approx di $v_{in}(t)$

Risposta alla δ:

$$v_{in}(t) = V_0 \delta(t) \to V_{in}(s) = V_0$$

TdL:

$$V_{out}(s) = V_{in}(s) \frac{\frac{1}{sC}}{\frac{1}{sC} + R} = V_0 \frac{1}{1 + sCR}$$

Anti-TdL:

$$\to v_{out}(t) = V_0 \,\mathrm{e}^{-\frac{t}{CR}}$$

Come atteso:

$t \ll RC \to v_{out}(t) \approx V_0$ Integrale approx di $v_{in}(t)$

Risposta al gradino:

$$v_{in}(t) = V_0 H(t) \rightarrow V_{in}(s) = \frac{V_0}{s}$$

Variabile di uscita: tensione su R

TdL:

$$V_{out}(s) = V_{in}(s)\frac{R}{\frac{1}{sC}+R} = \frac{V_0}{s}\frac{sRC}{1+sCR} = V_0\left(\frac{RC}{1+sCR}\right) = V_0 RC\frac{1}{1+sCR}$$

Anti-TdL:

$$\rightarrow v_{out}(t) = V_0\,\mathrm{e}^{-\frac{t}{CR}}$$

Come atteso:

$t \gg RC \rightarrow v_{out}(t) \approx 0$ Derivata approx di $v_{in}(t)$

Risposta alla δ:

$$v_{in}(t) = V_0\delta(t) \rightarrow V_{in}(s) = V_0$$

TdL:

$$V_{out}(s) = V_{in}(s)\frac{R}{\frac{1}{sC}+R} = V_0\frac{sCR}{1+sCR} = V_0\left(1-\frac{1}{1+sCR}\right)$$

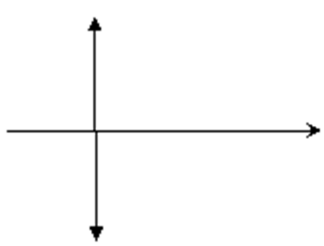
Anti-TdL:

$$\rightarrow v_{out}(t) = V_0\left(\delta(t)-\mathrm{e}^{-\frac{t}{CR}}\right)$$

Come atteso:

$t \gg RC \rightarrow v_{out}(t) \approx 0$ Derivata approx di $v_{in}(t)$

NB Derivata di $\delta(t)$?? $\rightarrow$ 'Doppietto di Dirac'



Moltiplicata per una funzione $x(t)$ e integrata produce $\left.\frac{dx}{dt}\right|_{t=0}$:

$$\int_{-\infty}^{+\infty}\delta'(t)\,x(t)dt = x(t)\,\delta(t)\Big|_{-\infty}^{+\infty} - \int_{-\infty}^{+\infty}\delta(t)\frac{dx}{dt}dt = \left.\frac{dx}{dt}\right|_{t=0}$$

Risposta a impulso rettangolare: Ampiezza V_0, larghezza T

$$v_{in}(t)=V_0\left[H(t)-H(t-T)\right]\rightarrow V_{in}(s)=\frac{V_0}{s}-\frac{V_0}{s}e^{-sT}$$

Variabile di uscita: tensione su R

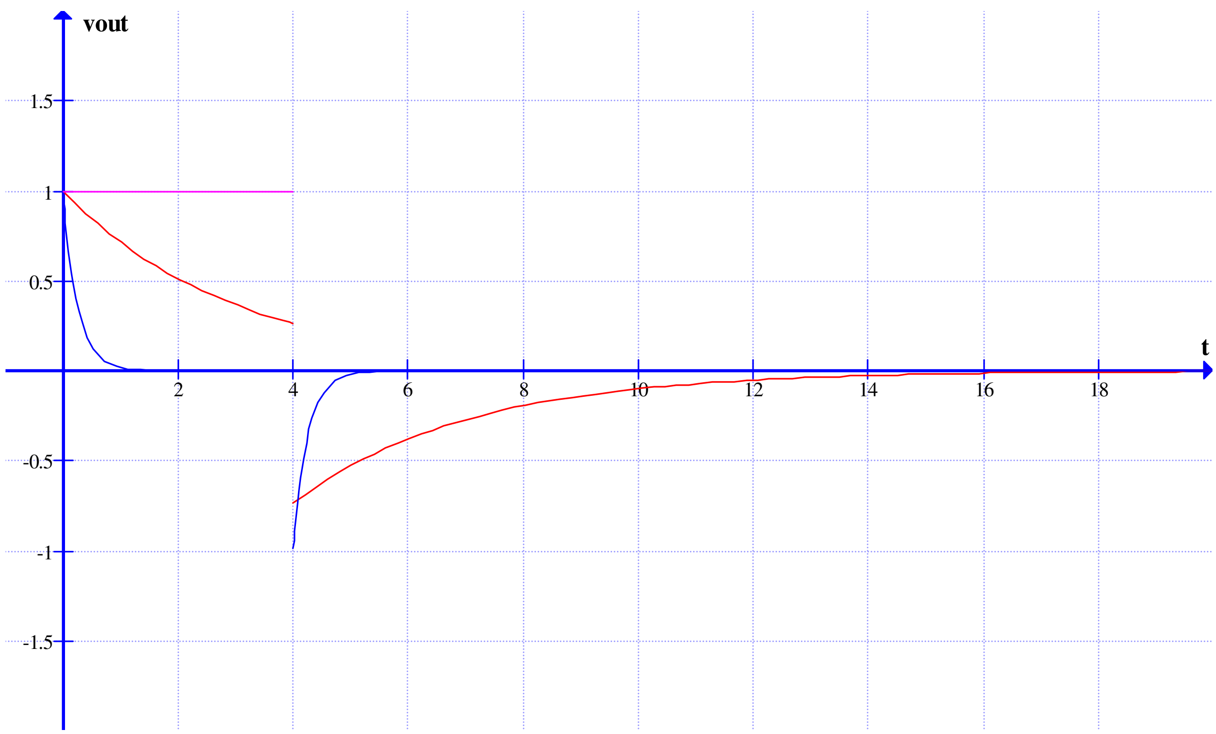
TdL:

$$V_{out}(s)=V_{in}(s)\frac{R}{\frac{1}{sC}+R}=\frac{V_0}{s}\left(1-e^{-sT}\right)\frac{sRC}{1+sCR}=V_0\left(1-e^{-sT}\right)\left(\frac{RC}{1+sCR}\right)$$

$$=V_0\frac{RC}{1+sCR}-V_0\frac{RC}{1+sCR}e^{-sT}$$

Anti-TdL:

$$\rightarrow v_{out}(t)=V_0\left(\mathrm{e}^{-\frac{t}{CR}}-\mathrm{e}^{-\frac{t-T}{CR}}H(t-T)\right)$$



Risposta a impulso rettangolare: Ampiezza V_0, larghezza T

$$v_{in}(t) = V_0\left[H(t) - H(t-T)\right] \rightarrow V_{in}(s) = \frac{V_0}{s} - \frac{V_0}{s}e^{-sT}$$

Variabile di uscita: tensione su C

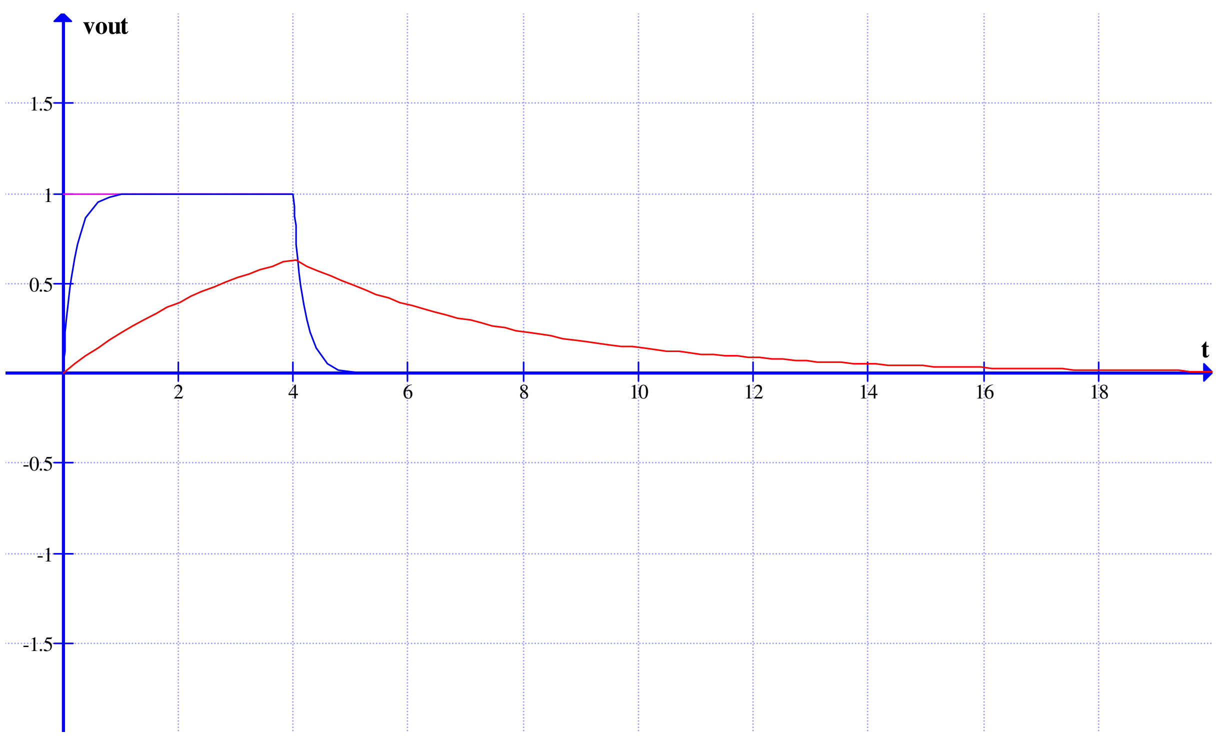
TdL:

$$V_{out}(s) = V_{in}(s)\frac{\frac{1}{sC}}{\frac{1}{sC}+R} = \frac{V_0}{s}\left(1-e^{-sT}\right)\frac{1}{1+sCR}$$

$$= \frac{V_0}{s}\frac{1}{1+sCR} - \frac{V_0}{s}\frac{1}{1+sCR}e^{-sT}$$

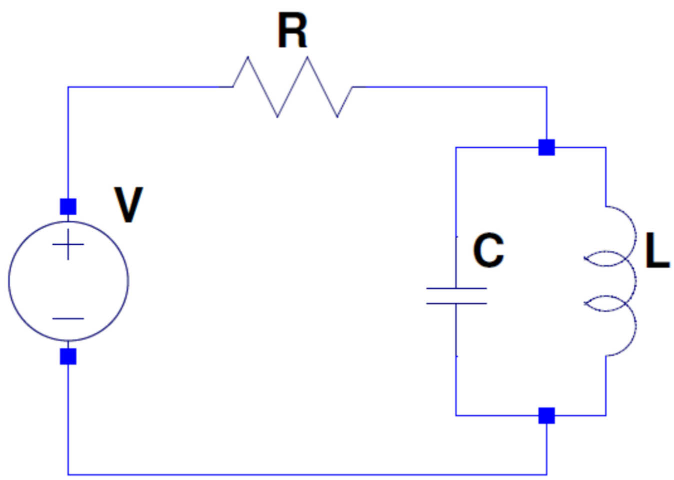
Anti-TdL:

$$\rightarrow v_{out}(t) = V_0\left[1-\mathrm{e}^{-\frac{t}{CR}} - H(t-T)\left(1-\mathrm{e}^{-\frac{t-T}{CR}}\right)\right]$$



Circuito oscillante: risposta al gradino

2 elementi reattivi: circuito del II ordine



$$Z(s) = R + Z_{\parallel}(s)$$

$$Z_{\parallel}(s) = \frac{sL\frac{1}{sC}}{sL + \frac{1}{sC}} = \frac{\frac{L}{C}}{sL + \frac{1}{sC}} = \frac{sL}{s^2 LC + 1}$$

$$\rightarrow Z(s) = R + \frac{sL}{s^2 LC + 1}$$

$$\rightarrow V_{out}(s) = V_{in}(s)\frac{\frac{sL}{s^2 LC + 1}}{R + \frac{sL}{s^2 LC + 1}} = V_{in}(s)\frac{sL}{R(s^2 LC + 1) + sL}$$

$$V_{in}(s) = \frac{V_0}{s} \rightarrow V_{out}(s) = V_0 \frac{L}{RLC\left(s^2 + \frac{1}{LC} + \frac{s}{RC}\right)} = V_0 \frac{1}{RC\left(s^2 + \frac{s}{RC} + \frac{1}{LC}\right)}$$

$$\omega_0 \equiv \frac{1}{\sqrt{LC}}, K \equiv \frac{1}{2R}\sqrt{\frac{L}{C}} \rightarrow 2K\omega_0 = \frac{1}{RC}$$

$$\rightarrow V_{out}(s) = V_0 \frac{1}{RC\left(s^2 + \frac{s}{RC} + \frac{1}{LC}\right)} = 2V_0 K\omega_0 \frac{1}{s^2 + 2K\omega_0 s + \omega_0^2}$$

Radici del denominatore:

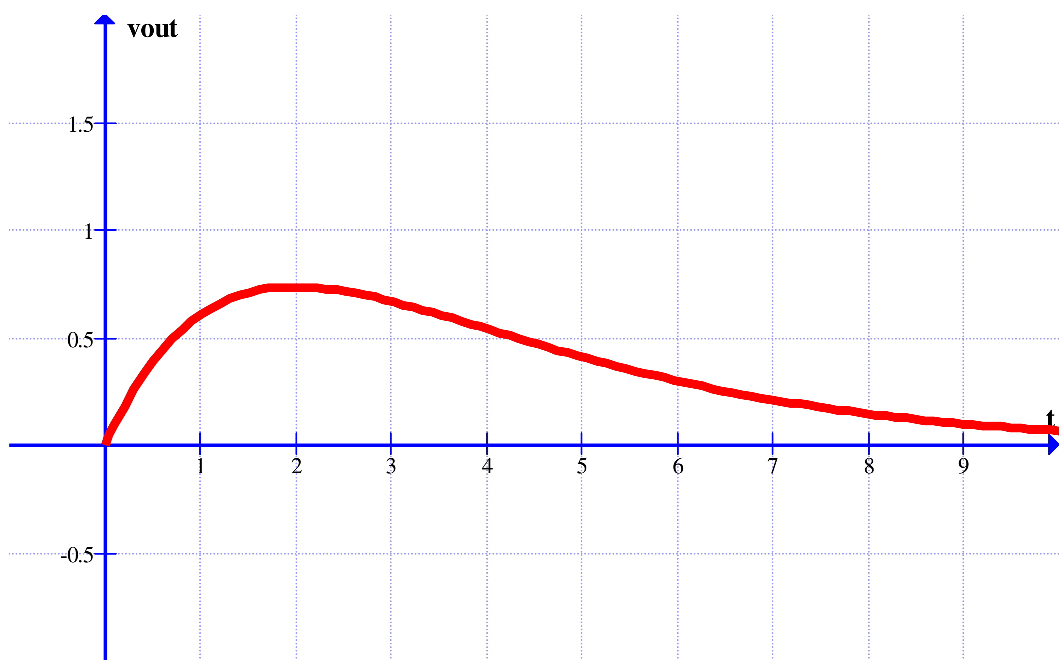
$$s_{\pm} = -K\omega_0 \pm \sqrt{K^2\omega_0^2 - \omega_0^2} = -\omega_0 K \pm \omega_0\sqrt{K^2 - 1}$$

$a) K = 1$ Smorzamento critico

$$s_+ = s_- = -\omega_0$$

$$\rightarrow V_{out}(s) = 2V_0 K\omega_0 \frac{1}{(s+\omega_0)^2} = 2V_0\omega_0 \frac{1}{(s+\omega_0)^2}$$

$$\rightarrow v_{out}(t) = 2V_0 K\omega_0 t e^{-\omega_0 t} = 2V_0\omega_0 t e^{-\omega_0 t}$$



$b) K > 1$ Smorzamento

$$s_{\pm} = -K\omega_0 \pm \sqrt{K^2\omega_0^2 - \omega_0^2} = -\omega_0\left(K \pm \sqrt{K^2-1}\right)$$

$$\rightarrow V_{out}(s) = 2V_0K\omega_0 \frac{1}{\left[s+\omega_0\left(K+\sqrt{K^2-1}\right)\right]\left[s+\omega_0\left(K-\sqrt{K^2-1}\right)\right]}$$

$$\frac{1}{\left[s+\omega_0\left(K+\sqrt{K^2-1}\right)\right]\left[s+\omega_0\left(K-\sqrt{K^2-1}\right)\right]} = \frac{A}{s+\omega_0\left(K+\sqrt{K^2-1}\right)} + \frac{B}{s+\omega_0\left(K-\sqrt{K^2-1}\right)}$$

$$A + B = 0 \rightarrow B = -A$$

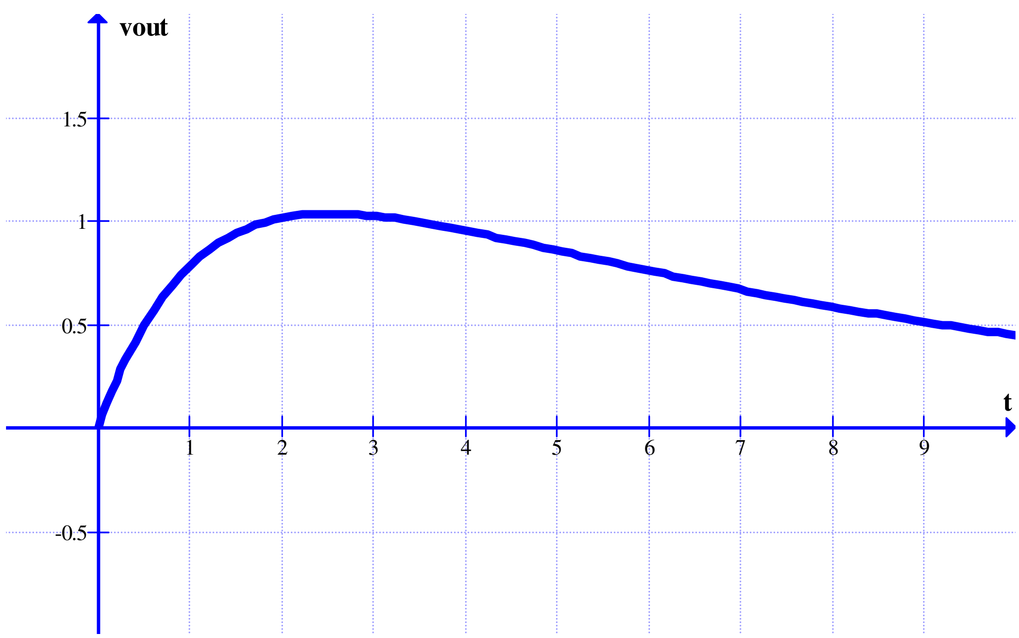
$$A\omega_0\left(K-\sqrt{K^2-1}\right) + B\omega_0\left(K+\sqrt{K^2-1}\right) = 1$$

$$\rightarrow A\omega_0\left[K-\sqrt{K^2-1}-K-\sqrt{K^2-1}\right] = 1 \rightarrow A = -\frac{2\sqrt{K^2-1}}{\omega_0}$$

$$\rightarrow V_{out}(s) = 2V_0K\omega_0 \frac{-\frac{2\sqrt{K^2-1}}{\omega_0}}{s+\omega_0\left(K+\sqrt{K^2-1}\right)} + 2V_0K\omega_0 \frac{\frac{2\sqrt{K^2-1}}{\omega_0}}{s+\omega_0\left(K-\sqrt{K^2-1}\right)}$$

$$\rightarrow V_{out}(s) = 4V_0K\sqrt{K^2-1}\left[\frac{1}{s+\omega_0\left(K-\sqrt{K^2-1}\right)} - \frac{1}{s+\omega_0\left(K+\sqrt{K^2-1}\right)}\right]$$

$$\rightarrow v_{out}(t) = 4V_0K\sqrt{K^2-1}\left[e^{-\omega_0\left(K-\sqrt{K^2-1}\right)t} - e^{-\omega_0\left(K+\sqrt{K^2-1}\right)t}\right]$$



$c)K<1$ Oscillazioni smorzate

$$s_{\pm} = -K\omega_0 \pm \sqrt{K^2\omega_0^2 - \omega_0^2} = -\omega_0\left(K \pm i\sqrt{1-K^2}\right)$$

$$\rightarrow V_{out}(s) = 2V_0 K\omega_0 \frac{1}{\left[s+\omega_0\left(K+i\sqrt{1-K^2}\right)\right]\left[s+\omega_0\left(K-i\sqrt{1-K^2}\right)\right]}$$

$$\frac{1}{\left[s+\omega_0\left(K+i\sqrt{1-K^2}\right)\right]\left[s+\omega_0\left(K-i\sqrt{1-K^2}\right)\right]} = \frac{A}{s+\omega_0\left(K+i\sqrt{1-K^2}\right)} + \frac{B}{s+\omega_0\left(K-i\sqrt{1-K^2}\right)}$$

$$A+B=0 \rightarrow B=-A$$

$$A\omega_0\left(K-i\sqrt{1-K^2}\right)+B\omega_0\left(K+i\sqrt{1-K^2}\right)=1$$

$$\rightarrow A\omega_0\left[K-i\sqrt{1-K^2}-K-i\sqrt{1-K^2}\right]=1 \rightarrow A=-\frac{1}{2i\omega_0\sqrt{1-K^2}}=\frac{i}{2\omega_0\sqrt{1-K^2}}$$

$$\rightarrow V_{out}(s) = 2V_0K\omega_0\frac{\frac{i}{2\omega_0\sqrt{1-K^2}}}{s+\omega_0\left(K+i\sqrt{1-K^2}\right)} - 2V_0K\omega_0\frac{\frac{i}{2\omega_0\sqrt{1-K^2}}}{s+\omega_0\left(K-i\sqrt{1-K^2}\right)}$$

$$\rightarrow V_{out}(s) = \frac{iV_0K}{\sqrt{1-K^2}}\left[\frac{1}{s+\omega_0\left(K+i\sqrt{1-K^2}\right)} - \frac{1}{s+\omega_0\left(K-i\sqrt{1-K^2}\right)}\right]$$

$$\rightarrow v_{out}(t) = \frac{iV_0K}{\sqrt{1-K^2}}\left[e^{-\omega_0\left(K+i\sqrt{1-K^2}\right)t} - e^{-\omega_0\left(K-i\sqrt{1-K^2}\right)t}\right] = V_0\frac{2K}{\sqrt{1-K^2}}e^{-\omega_0Kt}\sin\left(\sqrt{1-K^2}\,\omega_0 t\right)$$