Da proprieta' elementari TdL:

$$x(t) \overset{L}{\leftrightarrow} X(s)$$

$$\frac{dx}{dt} \overset{L}{\leftrightarrow} sX(s)$$

$$\int_0^t x(t')dt' \overset{L}{\leftrightarrow} \frac{1}{s} X(s)$$

$\rightarrow$ Impedenze generalizzate di L e C:

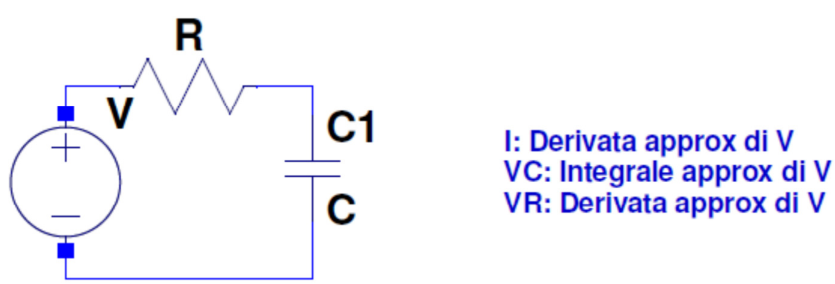
$$V_C(s) = Z_C(s) I_C(s) = \frac{1}{sC} I(s)$$

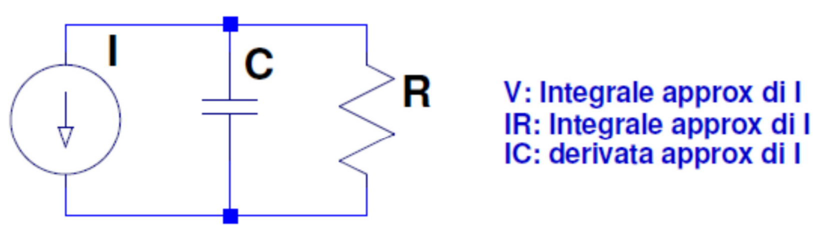
$$V_L(s) = Z_L(s) I_L(s) = sLI(s)$$

$\rightarrow$ Derivatore e Integratore ideali

Circuiti pratici: Approssimazioni

Circuiti con 1 elemento reattivo: I ordine





Per brevita', solo circuito RC serie: altri (RC parallelo, RL serie e parallelo) del tutto simili. Grafici rilevanti: v. lezione precedente su trattamento 'euristico'

Risposta al gradino:

$$v_{in}(t) = V_0 H(t) \rightarrow V_{in}(s) = \frac{V_0}{s}$$

Variabile di uscita: tensione su C

TdL:

$$V_{out}(s) = V_{in}(s) \frac{\frac{1}{sC}}{\frac{1}{sC} + R} = \frac{V_0}{s} \frac{1}{1 + sCR} = V_0 \left(\frac{A}{s} + \frac{B}{1 + sCR} \right)$$

$$\frac{A(1 + sCR) + Bs}{s(1 + sCR)} = \frac{s(ACR + B) + A}{s(1 + sCR)} = \frac{1}{s(1 + sCR)}$$

$$\rightarrow A = 1, A = -\frac{B}{CR} \rightarrow B = -CR$$

$$\rightarrow V_{out}(s) = V_0 \left(\frac{1}{s} - \frac{CR}{1 + sCR} \right) = V_0 \left(\frac{1}{s} - \frac{1}{\frac{1}{RC} + s} \right)$$

Anti-TdL:

$$\rightarrow v_{out}(t) = V_0 \left(H(t) - \mathrm{e}^{-\frac{t}{CR}} \right)$$

Come atteso:

$t \ll RC \rightarrow v_{out}(t) \approx V_0 \frac{t}{RC}$ Integrale approx di $v_{in}(t)$

Risposta alla δ:

$$v_{in}(t) = V_0 \delta(t) \rightarrow V_{in}(s) = V_0$$

TdL:

$$V_{out}(s) = V_{in}(s) \frac{\frac{1}{sC}}{\frac{1}{sC} + R} = V_0 \frac{1}{1 + sCR}$$

Anti-TdL:

$$\rightarrow v_{out}(t) = V_0 \, \mathrm{e}^{-\frac{t}{CR}}$$

Come atteso:

$t \ll RC \rightarrow v_{out}(t) \approx V_0$ Integrale approx di $v_{in}(t)$

Risposta al gradino:

$$v_{in}(t) = V_0 H(t) \rightarrow V_{in}(s) = \frac{V_0}{s}$$

Variabile di uscita: tensione su R

TdL:

$$V_{out}(s) = V_{in}(s) \frac{R}{\frac{1}{sC} + R} = \frac{V_0}{s} \frac{sRC}{1 + sCR} = V_0 \left(\frac{RC}{1 + sCR} \right) = V_0 RC \frac{1}{1 + sCR}$$

Anti-TdL:

$$\rightarrow v_{out}(t) = V_0 \, \mathrm{e}^{-\frac{t}{CR}}$$

Come atteso:

$t \gg RC \rightarrow v_{out}(t) \approx 0$ Derivata approx di $v_{in}(t)$

Risposta alla δ:

$$v_{in}(t) = V_0 \delta(t) \rightarrow V_{in}(s) = V_0$$

TdL:

$$V_{out}(s) = V_{in}(s) \frac{R}{\frac{1}{sC} + R} = V_0 \frac{sCR}{1 + sCR} = V_0 \left(1 - \frac{1}{1 + sCR} \right)$$

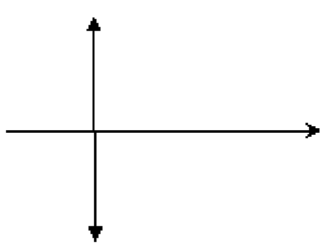
Anti-TdL:

$$\rightarrow v_{out}(t) = V_0 \left(\delta(t) - \mathrm{e}^{-\frac{t}{CR}} \right)$$

Come atteso:

$t \gg RC \rightarrow v_{out}(t) \approx 0$ Derivata approx di $v_{in}(t)$

NB Derivata di $\delta(t)$?? $\rightarrow$ 'Doppietto di Dirac'



Moltiplicata per una funzione $x(t)$ e integrata produce $\left.\frac{dx}{dt}\right|_{t=0}$:

$$\int_{-\infty}^{+\infty} \delta'(t) x(t) dt = x(t) \delta(t) \Big|_{-\infty}^{+\infty} - \int_{-\infty}^{+\infty} \delta(t) \frac{dx}{dt} dt = \left.\frac{dx}{dt}\right|_{t=0}$$

Risposta a impulso rettangolare: Ampiezza V_0, larghezza T

$$v_{in}(t) = V_0\left[H(t) - H(t-T)\right] \rightarrow V_{in}(s) = \frac{V_0}{s} - \frac{V_0}{s}e^{-sT}$$

Variabile di uscita: tensione su R

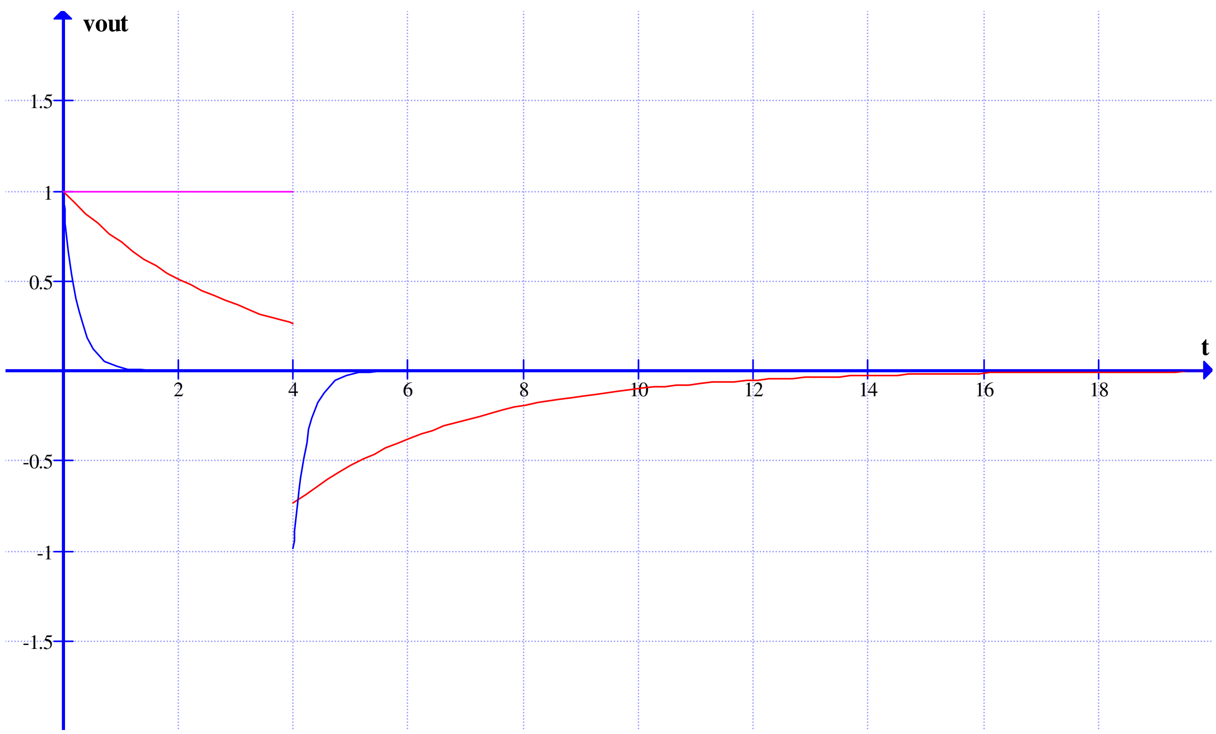
TdL:

$$V_{out}(s) = V_{in}(s)\frac{R}{\frac{1}{sC}+R} = \frac{V_0}{s}\left(1-e^{-sT}\right)\frac{sRC}{1+sCR} = V_0\left(1-e^{-sT}\right)\left(\frac{RC}{1+sCR}\right)$$

$$= V_0\frac{RC}{1+sCR} - V_0\frac{RC}{1+sCR}e^{-sT}$$

Anti-TdL:

$$\rightarrow v_{out}(t) = V_0\left(\mathrm{e}^{-\frac{t}{CR}} - \mathrm{e}^{-\frac{t-T}{CR}}H(t-T)\right)$$



Risposta a impulso rettangolare: Ampiezza V_0, larghezza T

$$v_{in}(t) = V_0\left[H(t) - H(t-T)\right] \rightarrow V_{in}(s) = \frac{V_0}{s} - \frac{V_0}{s}e^{-sT}$$

Variabile di uscita: tensione su C

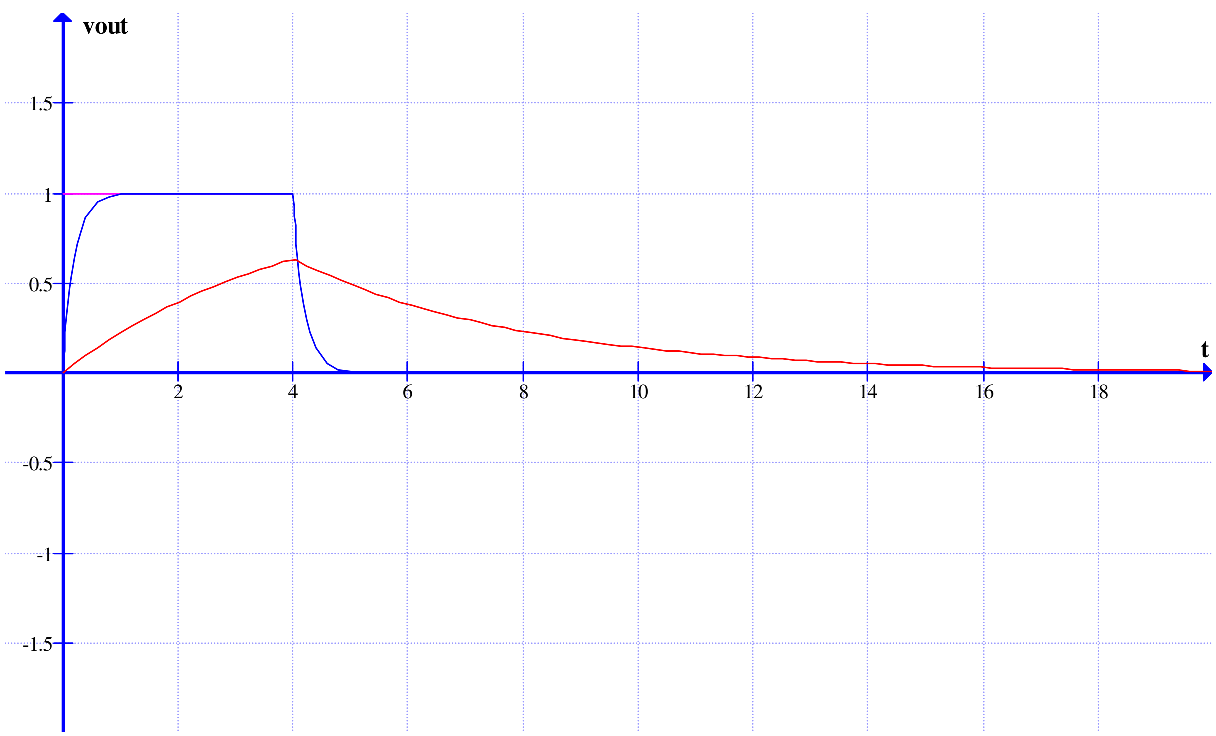
TdL:

$$V_{out}(s) = V_{in}(s)\frac{\frac{1}{sC}}{\frac{1}{sC}+R} = \frac{V_0}{s}\left(1-e^{-sT}\right)\frac{1}{1+sCR}$$

$$= \frac{V_0}{s}\frac{1}{1+sCR} - \frac{V_0}{s}\frac{1}{1+sCR}e^{-sT}$$

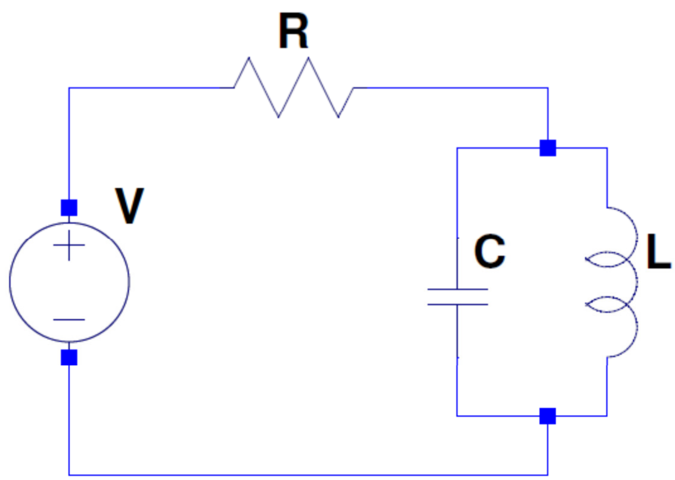
Anti-TdL:

$$\rightarrow v_{out}(t) = V_0\left[1-\mathrm{e}^{-\frac{t}{CR}} - H(t-T)\left(1-\mathrm{e}^{-\frac{t-T}{CR}}\right)\right]$$



Circuito oscillante: risposta al gradino

2 elementi reattivi: circuito del II ordine



$$Z(s) = R + Z_{\|}(s)$$

$$Z_{\|}(s) = \frac{sL\frac{1}{sC}}{sL + \frac{1}{sC}} = \frac{\frac{L}{C}}{sL + \frac{1}{sC}} = \frac{sL}{s^2LC + 1}$$

$$\rightarrow Z(s) = R + \frac{sL}{s^2LC + 1}$$

$$\rightarrow V_{out}(s) = V_{in}(s)\frac{\frac{sL}{s^2LC+1}}{R + \frac{sL}{s^2LC+1}} = V_{in}(s)\frac{sL}{R(s^2LC+1) + sL}$$

$$V_{in}(s) = \frac{V_0}{s} \rightarrow V_{out}(s) = V_0\frac{L}{RLC\left(s^2 + \frac{1}{LC} + \frac{s}{RC}\right)} = V_0\frac{1}{RC\left(s^2 + \frac{s}{RC} + \frac{1}{LC}\right)}$$

$$\omega_0 \equiv \frac{1}{\sqrt{LC}}, K \equiv \frac{1}{2R}\sqrt{\frac{L}{C}} \rightarrow 2K\omega_0 = \frac{1}{RC}$$

$$\rightarrow V_{out}(s) = V_0 \frac{1}{RC\left(s^2 + \frac{s}{RC} + \frac{1}{LC}\right)} = 2V_0 K\omega_0 \frac{1}{s^2 + 2K\omega_0 s + \omega_0^2}$$

Radici del denominatore:

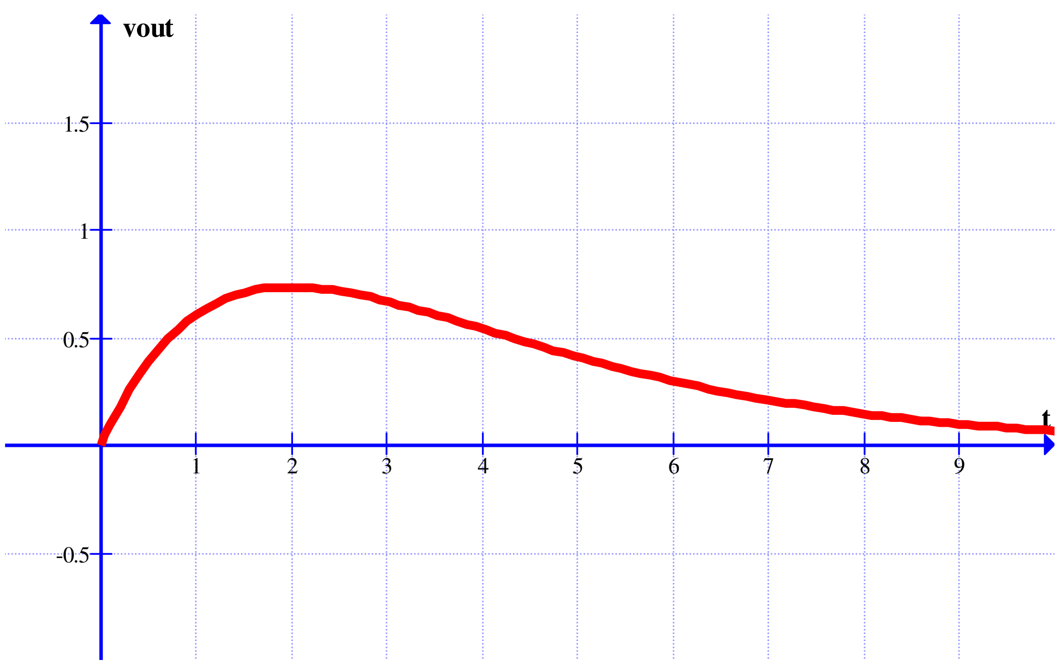
$$s_{\pm} = -K\omega_0 \pm \sqrt{K^2\omega_0^2 - \omega_0^2} = -\omega_0 K \pm \omega_0\sqrt{K^2 - 1}$$

$a)K = 1$ Smorzamento critico

$$s_+ = s_- = -\omega_0$$

$$\rightarrow V_{out}(s) = 2V_0\omega_0 \frac{1}{(s+\omega_0)^2}$$

$$\rightarrow v_{out}(t) = 2V_0\omega_0 t e^{-\omega_0 t}$$



$b) K > 1$ Smorzamento

$$s_{\pm} = -K\omega_0 \pm \sqrt{K^2\omega_0^2 - \omega_0^2} = -\omega_0\left(K \pm \sqrt{K^2 - 1}\right)$$

$$\rightarrow V_{out}(s) = 2V_0\omega_0 \frac{1}{\left[s + \omega_0\left(K + \sqrt{K^2-1}\right)\right]\left[s + \omega_0\left(K - \sqrt{K^2-1}\right)\right]}$$

$$\frac{1}{\left[s + \omega_0\left(K + \sqrt{K^2-1}\right)\right]\left[s + \omega_0\left(K - \sqrt{K^2-1}\right)\right]} = \frac{A}{s + \omega_0\left(K + \sqrt{K^2-1}\right)} + \frac{B}{s + \omega_0\left(K - \sqrt{K^2-1}\right)}$$

$$A + B = 0 \rightarrow B = -A$$

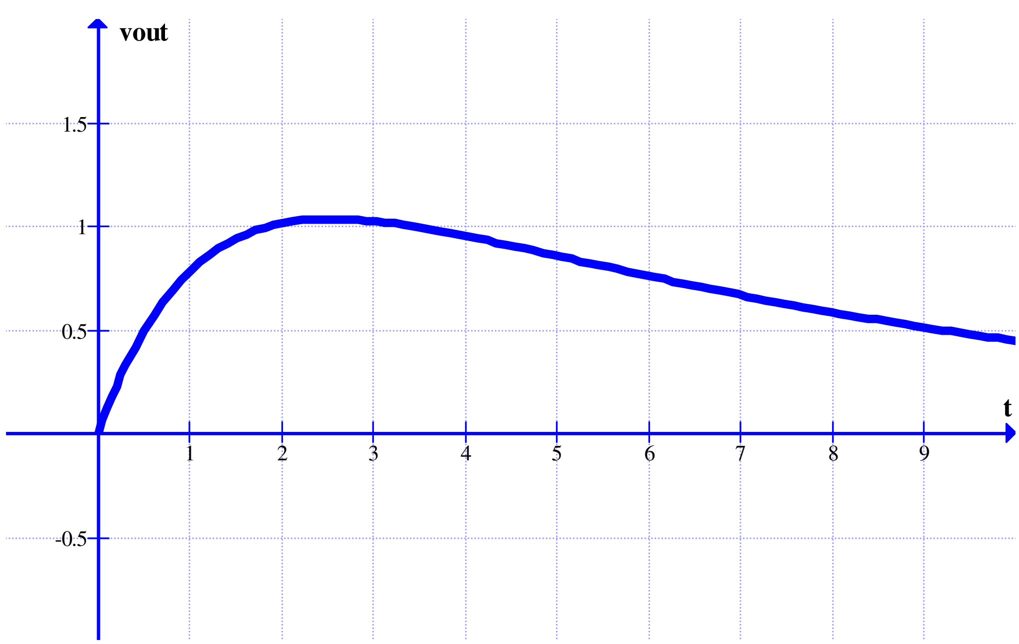
$$A\omega_0\left(K - \sqrt{K^2-1}\right) + B\omega_0\left(K + \sqrt{K^2-1}\right) = 1$$

$$\rightarrow A\omega_0\left[K - \sqrt{K^2-1} - K - \sqrt{K^2-1}\right] = 1 \rightarrow A = -\frac{2\sqrt{K^2-1}}{\omega_0}$$

$$\rightarrow V_{out}(s) = 2V_0\omega_0 \frac{-\frac{2\sqrt{K^2-1}}{\omega_0}}{s + \omega_0\left(K + \sqrt{K^2-1}\right)} + 2V_0\omega_0 \frac{\frac{2\sqrt{K^2-1}}{\omega_0}}{s + \omega_0\left(K - \sqrt{K^2-1}\right)}$$

$$\rightarrow V_{out}(s) = 4V_0\sqrt{K^2-1}\left[\frac{1}{s + \omega_0\left(K - \sqrt{K^2-1}\right)} - \frac{1}{s + \omega_0\left(K + \sqrt{K^2-1}\right)}\right]$$

$$\rightarrow v_{out}(t) = 4V_0\sqrt{K^2-1}\left[e^{-\omega_0\left(K - \sqrt{K^2-1}\right)t} - e^{-\omega_0\left(K + \sqrt{K^2-1}\right)t}\right]$$



$c) K < 1$ Oscillazioni smorzate

$$s_{\pm} = -K\omega_0 \pm \sqrt{K^2\omega_0^2 - \omega_0^2} = -\omega_0\left(K \pm i\sqrt{1-K^2}\right)$$

$$\rightarrow V_{out}(s) = 2V_0\omega_0 \frac{1}{\left[s+\omega_0\left(K+i\sqrt{1-K^2}\right)\right]\left[s+\omega_0\left(K-i\sqrt{1-K^2}\right)\right]}$$

$$\frac{1}{\left[s+\omega_0\left(K+i\sqrt{1-K^2}\right)\right]\left[s+\omega_0\left(K-i\sqrt{1-K^2}\right)\right]} = \frac{A}{s+\omega_0\left(K+i\sqrt{1-K^2}\right)} + \frac{B}{s+\omega_0\left(K-i\sqrt{1-K^2}\right)}$$

$$A + B = 0 \rightarrow B = -A$$

$$A\omega_0\left(K-i\sqrt{1-K^2}\right) + B\omega_0\left(K+i\sqrt{1-K^2}\right) = 1$$

$$\rightarrow A\omega_0\left[K-i\sqrt{1-K^2} - K - i\sqrt{1-K^2}\right] = 1 \rightarrow A = -\frac{2i\sqrt{1-K^2}}{\omega_0}$$

$$\rightarrow V_{out}(s) = 2V_0\omega_0 \frac{-\frac{2i\sqrt{1-K^2}}{\omega_0}}{s+\omega_0\left(K+i\sqrt{1-K^2}\right)} + 2V_0\omega_0 \frac{\frac{2i\sqrt{1-K^2}}{\omega_0}}{s+\omega_0\left(K-i\sqrt{1-K^2}\right)}$$

$$\rightarrow V_{out}(s) = 4V_0 i\sqrt{1-K^2}\left[\frac{1}{s+\omega_0\left(K-i\sqrt{1-K^2}\right)} - \frac{1}{s+\omega_0\left(K+i\sqrt{1-K^2}\right)}\right]$$

$$\rightarrow v_{out}(t) = 4V_0 i\sqrt{1-K^2}\left[e^{-\omega_0\left(K-i\sqrt{1-K^2}\right)t} - e^{-\omega_0\left(K+i\sqrt{1-K^2}\right)t}\right] = 8V_0\sqrt{1-K^2}\,e^{-\omega_0 Kt}\sin\left(\sqrt{1-K^2}\,\omega_0 t\right)$$