Medie temporali di quadrati di funzioni sinusoidali

a) Nella valutazione della media temporale dell'energia trasportata da un segnale, capita di dover calcolare un integrale del tipo:

$$\frac{1}{T}\int_0^T \cos^2(\omega t)\,dt$$

Procedimento:

$$\frac{1}{T}\int_0^T \cos^2(\omega t)\,dt = ?$$

$$u = \omega t \to t = \frac{u}{\omega} \to dt = \frac{du}{\omega}$$

$$\to \int \cos^2(\omega t)\,dt = \frac{1}{\omega}\int \cos^2 u\,du$$

$$\begin{cases} \cos^2 u + \sin^2 u = 1 \\ \cos^2 u - \sin^2 u = \cos 2u \end{cases}$$

$$\to \begin{cases} \cos^2 u = \dfrac{1+\cos 2u}{2} \\ \sin^2 u = \dfrac{1-\cos 2u}{2} \end{cases}$$

$$\to \frac{1}{\omega}\int \cos^2 u\,du = \frac{1}{2\omega}\int (1+\cos 2u)\,du = \frac{1}{2\omega}\left(u + \frac{1}{2}\sin 2u\right)$$

$$\to \frac{1}{T}\int_0^T \cos^2(\omega t)\,dt = \frac{1}{T}\left(\frac{1}{2\omega}\right)\left((\omega t) + \frac{1}{2}\sin 2(\omega t)\right)_0^T$$

$$\omega = \frac{2\pi}{T}$$

$$\to \frac{1}{T}\int_0^T \cos^2(\omega t)\,dt = \frac{1}{4\pi}\left(\left(\underbrace{\omega T}_{=2\pi}\right) + \frac{1}{2}\left(\underbrace{\sin 2\left(\underbrace{\omega T}_{=2\pi}\right)}_{=0}\right)\right)$$

$$= \frac{1}{4\pi}(2\pi) = \frac{1}{2}$$

b) Si trova poi subito che:

$$\sin^2(\omega t) = 1 - \cos^2(\omega t)$$

$$\to \frac{1}{T}\int_0^T \sin^2(\omega t)\,dt = \frac{1}{T}\int_0^T \left[1-\cos^2(\omega t)\right]dt = 1 - \frac{1}{2} = \frac{1}{2}$$

c) La media su periodi molto lunghi e' equivalente alla media su un periodo. Infatti:

$$\tau = NT + \delta, N \gg 1, \delta < T$$

$$\frac{1}{\tau}\int_0^{\tau} \cos^2(\omega t)\,dt = \frac{1}{NT+\delta}\int_0^{NT+\delta} \cos^2(\omega t)\,dt$$

$$= \frac{1}{NT+\delta}\left[\int_0^{NT} \cos^2(\omega t)\,dt + \int_{NT}^{NT+\delta} \cos^2(\omega t)\,dt\right]$$

$$= \underbrace{\frac{1}{NT+\delta}}_{\sim\frac{1}{NT}}\frac{NT}{2} + \underbrace{\frac{1}{NT+\delta}}_{\sim\frac{1}{NT}\ll 1}a, \quad a < \frac{\delta}{2}$$

$$\simeq \frac{1}{2}$$