Serie vs Integrale

Calcolo di un integrale che compare spesso in espressioni contenenti la distribuzione di Fermi-Dirac:

$$\int_0^\infty \frac{x}{1+e^{\alpha x}}dx = I$$

$$I = \int_0^\infty \frac{xe^{-\alpha x}}{1+e^{-\alpha x}}dx$$

$$\frac{1}{1+e^{-\alpha x}} = 1 - e^{-\alpha x} + e^{-2\alpha x} - e^{-3\alpha x} + \ldots = \sum_{n=0}^{\infty}(-1)^n e^{-n\alpha x}$$

$$\rightarrow \int_0^\infty \frac{xe^{-\alpha x}}{1+e^{-\alpha x}}dx = \int_0^\infty xe^{-\alpha x}\sum_{n=0}^{\infty}(-1)^n e^{-n\alpha x}dx$$

$$\rightarrow \int_0^\infty \frac{xe^{-\alpha x}}{1+e^{-\alpha x}}dx = \int_0^\infty x\sum_{n=0}^{\infty}(-1)^n e^{-(n+1)\alpha x}dx = \int_0^\infty x\sum_{n=1}^{\infty}(-1)^{(n-1)} e^{-n\alpha x}dx$$

$$\rightarrow \int_0^\infty \frac{xe^{-\alpha x}}{1+e^{-\alpha x}}dx = \sum_{n=1}^{\infty}(-1)^{(n-1)}\int_0^\infty xe^{-n\alpha x}dx$$

$$\left.\begin{aligned} &u = x \rightarrow du = dx \\ &dv = e^{-n\alpha x}dx \rightarrow v = -\frac{1}{n\alpha}e^{-n\alpha x}\end{aligned}\right\} \rightarrow \int xe^{-n\alpha x}dx = -\frac{x}{n\alpha}e^{-n\alpha x} + \int \frac{1}{n\alpha}e^{-n\alpha x}dx$$

$$\rightarrow \int xe^{-n\alpha x}dx = -\left(\frac{x}{n\alpha}e^{-n\alpha x} + \frac{1}{n^2\alpha^2}e^{-n\alpha x}\right)$$

$$\rightarrow \int_0^\infty xe^{-n\alpha x}dx = -\left(\frac{x}{n\alpha}e^{-n\alpha x} + \frac{1}{n^2\alpha^2}e^{-n\alpha x}\right)_0^\infty = \frac{1}{n^2\alpha^2}$$

$$\rightarrow \int_0^\infty \frac{xe^{-\alpha x}}{1+e^{-\alpha x}}dx = \sum_{n=1}^{\infty}(-1)^{(n-1)}\int_0^\infty xe^{-n\alpha x}dx = \sum_{n=1}^{\infty}(-1)^{(n-1)}\frac{1}{n^2\alpha^2} = \frac{1}{\alpha^2}\sum_{n=1}^{\infty}\frac{(-1)^{(n-1)}}{n^2}$$

$$\sum_{n=1}^{\infty}\frac{(-1)^{(n-1)}}{n^2} = 1 - \frac{1}{2^2} + \frac{1}{3^2} - \frac{1}{4^2} + \ldots = A$$

$$1 + \frac{1}{2^2} + \frac{1}{3^2} + \frac{1}{4^2} + \ldots = \frac{\pi^2}{6} = B \quad \text{Formula di Eulero}$$

$$\rightarrow A = B - 2\frac{1}{2^2} - 2\frac{1}{4^2} - 2\frac{1}{6^2} - \ldots = B - 2\left(\frac{1}{2^2} + \frac{1}{4^2} + \frac{1}{6^2} + \ldots\right) = B - 2\left(\frac{1}{2^2} + \frac{1}{2^2 2^2} + \frac{1}{2^2 3^2} + \ldots\right)$$

$$\rightarrow A = B - 2\frac{1}{2^2}\left(1 + \frac{1}{2^2} + \frac{1}{3^2} + \ldots\right) = B - \frac{1}{2}B = \frac{\pi^2}{12}$$

$$\rightarrow \int_0^\infty \frac{x}{1+e^{\alpha x}}dx = \frac{\pi^2}{12\alpha^2}$$