**Nota sulla proprieta' dell'esponenziale di una matrice hermitiana**

$A = e^{iB}$, B hermitiana $\rightarrow$ A unitaria

B hermitiana : $B^\dagger = B$

A unitaria : $A^{-1} = A^\dagger \rightarrow A^\dagger A = 1$

Si osservi: $B$ hermitiana $\rightarrow$ $B$ sempre diagonalizzabile

$\rightarrow$ Possiamo assumere che $B$ sia diagonale

$$A^\dagger A = \left(e^{iB}\right)^\dagger e^{iB}$$

$$\left(e^{iB}\right)^\dagger = e^{-iB}$$

Infatti:

$$e^{iB} = 1 + iB + \frac{(iB)^2}{2!} + ...$$

$$\left(e^{iB}\right)^\dagger = \left[1 + (iB)^\dagger + \frac{\left[(iB)^2\right]^\dagger}{2!} - ...\right] = 1 - iB + \frac{(iB)^2}{2!} + ... = e^{-iB}$$

Infatti:

$$B \text{ hermitiana } \rightarrow \begin{cases} C = -iB \ = \text{ antihermitiana: } C^\dagger = -C \\ C^2 = (-iB)^2 = \text{ hermitiana} : \left(C^2\right)^\dagger = C^2 \end{cases}$$

Perche'?

$$\begin{cases} B \text{ hermitiana } \rightarrow B_{ij} = B_{ji}^* \\ \text{Complesso coniugato di un prodotto = prodotto dei complessi coniugati:} \\ \qquad a = bc = |b| e^{i\varphi_b} |c| e^{i\varphi_c} = |b||c| e^{i(\varphi_b + \varphi_c)} \\ \qquad \rightarrow a^* = |b||c| e^{-i(\varphi_b + \varphi_c)} = |b| e^{-i\varphi_b} |c| e^{-i\varphi_c} = b^* c^* \\ B \text{ diagonale} \rightarrow \left(B_{ij}\right)^2 = \left(B^2\right)_{ij} \end{cases}$$

$$\rightarrow \begin{cases} C_{ij} = -iB_{ij} \rightarrow C_{ji}^* = (-i)^* B_{ji}^* = +iB_{ij} = -C_{ij} \\ \left(C^2\right)_{ij} = \left[(-iB)^2\right]_{ij} = \left(-B_{ij}\right)^2 = -\left(B_{ji}^*\right)^2 = \left(+iB_{ji}^*\right)^2 = \left(C_{ji}^*\right)^2 = \left(C_{ji}^2\right)^* = \left[\left(C^2\right)_{ji}\right]^* \end{cases}$$

Allora:

$$\begin{cases} (-iB)^\dagger = iB, \text{ etc per potenze dispari} \\ \left[(-iB)^2\right]^\dagger = (-iB)^2 = (iB)^2, \text{ etc per potenze pari} \end{cases}$$

$$\rightarrow A^\dagger A = e^{-iB} e^{iB} = 1$$

$$e^{iB} = 1 + iB + \frac{(iB)^2}{2!} + \ldots$$

$$\rightarrow e^{-i\frac{\theta}{2}\tau_2} = 1 - i\frac{\theta}{2}\tau_2 + \frac{1}{2!}\left(-i\frac{\theta}{2}\tau_2\right)^2 + \ldots$$

$$= 1 - i\frac{\theta}{2}\tau_2 - \frac{1}{2}\left(\frac{\theta}{2}\right)^2 \tau_2^2 + \ldots$$

$$\tau_2^2 = \begin{pmatrix} 0 & -i \\ +i & 0 \end{pmatrix}\begin{pmatrix} 0 & -i \\ +i & 0 \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$$

$$i\tau_2 = i\begin{pmatrix} 0 & -i \\ +i & 0 \end{pmatrix} = \begin{pmatrix} 0 & +1 \\ -1 & 0 \end{pmatrix}$$

$$\rightarrow e^{-i\frac{\theta}{2}\tau_2} = \underbrace{\left[1 - \frac{1}{2!}\left(\frac{\theta}{2}\right)^2 + \frac{1}{4!}\left(\frac{\theta}{2}\right)^4 + \ldots\right]}_{\cos\frac{\theta}{2}}\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} + \underbrace{\left[\frac{\theta}{2} - \frac{1}{3!}\left(\frac{\theta}{2}\right)^3 + \ldots\right]}_{\sin\frac{\theta}{2}}\begin{pmatrix} 0 & -1 \\ +1 & 0 \end{pmatrix}$$

$$\rightarrow e^{-i\frac{\theta}{2}\tau_2} = \cos\frac{\theta}{2}\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} + \sin\frac{\theta}{2}\begin{pmatrix} 0 & -1 \\ +1 & 0 \end{pmatrix}$$

$$\rightarrow e^{-i\frac{\theta}{2}\tau_2}\begin{pmatrix} u \\ d \end{pmatrix} = \begin{pmatrix} u' \\ d' \end{pmatrix} = \cos\frac{\theta}{2}\begin{pmatrix} u \\ d \end{pmatrix} + \sin\frac{\theta}{2}\begin{pmatrix} -d \\ u \end{pmatrix} = \begin{pmatrix} u\cos\frac{\theta}{2} - d\sin\frac{\theta}{2} \\ d\cos\frac{\theta}{2} + u\sin\frac{\theta}{2} \end{pmatrix}$$