On the effects and measurement of the beam-beam (de)correlation in NA48

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1. Introduction

For the purpose of understanding and evaluating some systematic effects on the measurement of the double ratio, it is important to know how well the two beams are correlated in time. For example, a different temporal trend for the two beam currents may give rise to differential effects of accidentals. So one is led to envisage methods to quantify this correlation, using sample readings from beam counters or other estimators to measure how much the two beam intensities diverge instantaneously from one another. It is the purpose of this short note to clarify what is meant by measurement of this correlation, and to indicate quantitatively the extent to which any measurement of this kind will be statistically limited and/or systematically biased. Since nothing new is being presented here, this note should be taken as just a summary of the most relevant formulas and numbers.

1. Beam-beam (de)correlation: an unifying approach

The fundamental idea behind NA48 is to collect *simultaneously* the four decay modes. In this way, many systematic effects on the double ratio vanish to first order. However, this is only true provided the two beam currents are *instantaneously* proportional. There are at least two systematic effects potentially distorting the double ratio, and whose effect strongly depend on the ratio of instantaneous intensities

(a) differential loss/gain of events in the four modes due to accidental activity
(b) differential charged/neutral tagging dilution

It will be shown in the following that the systematic effect of both these can be traced to the correlation between the two beams

(a) **Accidental loss/gain rate**

We start by recalling the well-known formula for the rate of accidental coincidences originating from a pair of input channels

$$\dot{N} = \dot{n}_A \dot{n}_B \tau$$

where

$\dot{N}$ *is the accidental rate*
$\dot{n}_A$ *is the rate at input A*
$\dot{n}_B$ *is the rate at input B*
τ *is the coincidence resolving time*

This expression holds as long as

$$\dot{n}_A\,\tau, \dot{n}_B\,\tau << 1$$

If one assumes that all the accidental activity orignates from the K_L beam only, and for sake of simplicity takes that the only effect is a loss of events, one can write down simple expressions for the accidental losses of events. These are strong assumptions, however they can be easily relaxed to take into account the effect of the K_S beam, and the gain of events as well. One is therefore led to the following expressions for the rate of event losses:

$$\begin{aligned}
\dot{n}_L^{+-} &= k_L^{+-}\tau^{+-} I_L(t) I_L(t) \\
\dot{n}_S^{+-} &= k_S^{+-}\tau^{+-} I_S(t) I_L(t) \\
\dot{n}_L^{00} &= k_L^{00}\tau^{00} I_L(t) I_L(t) \\
\dot{n}_S^{00} &= k_S^{00}\tau^{00} I_S(t) I_L(t)
\end{aligned}$$

Here we have defined

τ^{+-} *charged resolving time * (killing - creating) effectiveness*

$k_{S,L}^{+-}$ *no. of good charged/bem particle*

τ^{00} *neutral resolving time * (killing - creating) effectiveness*

$k_{S,L}^{00}$ *no. of good neutrals/beam particle*

$I_S(t), I_L(t)$ *instantaneous beam intensities*

In our view, there is no serious question on the validity of the algebra above , other than the need of taking into account small effects like K_S-originated accidental activity and little more. The problem is thus restricted to finding the best way of estimating the four numbers $n^{+-,00}_{L,S}$ from the available data set.

A way of measuring these rates takes advantage from the collected *random* events. These are in any respect 'pedestal' events taken at a rate proportional to the beam intensities. These *randoms* are then linearly overlayed to a sizeable fraction of good *COMPACT* candidates, according to their timestamp, which are then passed again through the standard reconstruction chain. It must be stressed that each random is overlayed to a large number of candidates (more than 10). By counting the event losses and gains in each mode one can estimate the double ratio distortion as indicated above. This way one is indeed doing the experiment two times, first with the natural accidental activity (which cannot be eliminated, of course), and then with the activity artificially doubled. Since random triggers are collected at a rate proportional to beam intensities, and one estimates that the losses are proportional to beam intensities, and one is overlaying essentially the whole statistical sample, the added artificial gain-loss (measurable) can be taken as a good estimator of the natural gain-loss (not measurable). The whole algebra involved is quite simple as one compares the real, accidental induced loss to the artificially, random induced one:

$$\dot{n}_l^{(i)}(t) = \dot{n}^{(i)}(t)\int_{-\infty}^{+\infty} \dot{n}_{acc}(t')\, g_{acc}(t-t')\, dt'$$

$$\dot{n}_l^{(i)}(t) = \dot{n}^{(i)}(t)\int_{-\infty}^{+\infty} \dot{n}_{rnd}(t')\, g_{rnd}(t-t')\, dt'$$

where the correlator *g(t-t')* measures the effectiveness for gain-loss at time *t* of accidental activity at time *t'*. One expects for real accidental effects *g(t-t')* to be zero everywhere except in a narrow window around *t'-t=0.* On the other hand, for artificially induced gain-loss, *g(t-t')* is shifted by a random amount *h* within the window *(-d/2,+d/2), d* being the average interval between successive randoms: the higher the random rate, the smaller the window, and the better the approximation. Therefore one can write

$$g_{acc}(t-t') \approx \delta(t-t')\tau$$

$$g_{rnd}(t-t';h) \approx \delta(t-t'-h)\tau$$

Therefore one obtains:

$$n_l^{(i)} = \int_0^T \dot{n}^{(i)}(t)dt \int_{-\infty}^{+\infty} \dot{n}_{acc}(t')\, g_{acc}(t-t')\, dt' \approx \int_0^T \dot{n}^{(i)}(t)\int_{-\infty}^{+\infty} \dot{n}_{acc}(t')\tau\, \delta(t-t')\, dt'$$

$$n_l^{(i)} = \int_0^T \dot{n}^{(i)}(t)\dot{n}_{acc}(t)\,\tau\, dt$$

$$n_l^{(i)} \approx \frac{1}{d}\int_0^T \dot{n}^{(i)}(t)dt \int_{-d/2}^{+d/2} dh \int_{-\infty}^{+\infty} \dot{n}_{rnd}(t')\tau\, \delta(t-t'-h)\, dt' \approx \frac{1}{d}\int_{-d/2}^{+d/2} dh \int_0^T \dot{n}^{(i)}(t)dt \int_{-\infty}^{+\infty} \dot{n}_{rnd}(t')\tau\, \delta(t-t'-h)\, dt'$$

$$n_l^{(i)} \approx \frac{1}{d}\int_{-d/2}^{+d/2} dh \int_0^T \dot{n}^{(i)}(t)\ \dot{n}_{rnd}(t-h)\tau\, dt$$

To the extent the two 'currents' of real accidentals and randoms are proportional to the beam currents, this is easily interpreted as meaning that the total loss is proportional, for real accidentals, to the beam-beam cross-correlation at zero delay , while for randoms it is proportional to the cross-correlation averaged over the delay interval *(-d/2,+d/2).* This observation allows in principle for an evaluation of the random induced loss as estimator of the real loss: If one is able to measure the cross-correlation one can tell how good is the approximation of the zero delay value by the window average.It must be stressed that any deviation of the cross-correlation from being constant over the interval *(-d/2,+d/2)* will introduce a bias in the random estimation of the accidental correction. One then observes that the total loss can be re-written as a sum over all the random and good events, rather than over time:

$$n_l^{(i)} = \int_0^{N_{good}} dn^{(j)} \int_0^{N_{rnd}} dn_{rnd} \approx \sum_{k=1}^{N_{rnd}} \sum_{j=1}^{N_{good}} p_{jk}$$

where p_{jk} is the probability that random k kills event j. The main problem being how to ensure that random triggers really provide a faithful picture of the accidental acitivity seen by each event, some care should be paid to the overlaying strategy. Since one is interested in measuring the loss per decay mode, the correct way of overlapping is to add to each event of a given type the activity seen by the closest random trigger originated by the same beam.. The weak point of this method, arisng from its practical implementation, is that there is frequently a very long delay between the time of the candidate event and the time the random trigger is taken, the average delay being of the order of 70 ms. So the effect of the accidental is verified at the wrong time, due to the very low random rate. Nevertheless, if random triggers do cover the full range of event timestamps with good uniformity, then over the full run period one may have the random time statistically as close as desidered to the event time. It is our understanding that, in this statistical sense, the system bandwith (i.e., the sensitivity of the method to fast, relative variations of intensities) is quite large, however *this is true only in a statistical sense.* It requires the stochastic processes of event generation and accidental activity to be quasi-stationary along the whole run in order to be a faithful description of the effect of accidentals.

(a) **Charged/neutral dilution**

2. The beam currents

The two beam currents can be modeled as two quasi-stationary stochastic processes $I_S(t)$ and $I_L(t)$. Quasi-stationary means that they may have a trend in time, however on a scale much longer than their intrinsic self-correlation time. Without introducing a detailed description of the time structure of the processes, one may assume that both can be written as a combination of a *deterministic* component $s_{S,L}(t)$, with its own trend in time, and a *stochastic* component, originating from the statistical nature of the production process in the target and from the granular time structure of the proton beam current

$$I_{S,L}(t) = s_{S,L}(t) + n_{S,L}(t)$$

In a broad sense, the stochastic component can be understood as a shot noise current adding up to the signal current. The interesting quantities one may want to measure are

$$R_{SL}(\tau) = \frac{1}{\Delta}\int_{-\Delta/2}^{+\Delta/2} I_S(t) I_L(t-\tau)dt \quad \text{cross-correlation of } \mathrm{I_S} \text{ and } \mathrm{I_L}$$

$$R_{SS}(\tau) = \frac{1}{\Delta}\int_{-\Delta/2}^{+\Delta/2} I_S(t) I_S(t-\tau)dt \quad \text{self-correlation of } \mathrm{I_S}$$

$$R_{LL}(\tau) = \frac{1}{\Delta}\int_{-\Delta/2}^{+\Delta/2} I_L(t) I_L(t-\tau)dt \quad \text{self-correlation of } \mathrm{I_L}$$

these expressions being taken in the linit of very large Δ. By taking the above expansion for both I_S and I_L, one sees that of the four crossed products, only $s_S s_L$ will survive for Δ very large. Indeed, what matters in practice for evaluating differential effects is the beam-beam *total* (deterministic+stochastic) correlation over the limited sample time. Any residual noise-noise and noise-signal correlation, due to the finite sample size, builds up what can be called the statistical error on the measurement. I have sketched in App. A a short derivation of the relationship between cross-correlation and cross-spectrum of the two currents.

3. Errors (statistical and bias) in the measurement of cross/self-correlation

The statistical error on the correlation arises from both the finite sample length, and the limited bandwidth of the measurement. For sake of completeness, I'm reporting here the relevant expressions of the fractional error for a for a pair of stochastic signals [1]:

$$\varepsilon\left[R_{ij}(\tau)\right]=\frac{1}{\sqrt{2BT}}\sqrt{1+\frac{R_{ij}^2(\tau)}{R_{ii}(0)R_{jj}(0)}}$$

where

B is the signal bandwith

T is the total recording time

As an example, by taking B = 1 KHz, T = 2.5 s, one has

$$\varepsilon\left[R_{ij}(\tau)\right]=\frac{1}{\sqrt{510^3}}\sqrt{1+\frac{R_{ij}^2(\tau)}{R_{ii}(0)R_{jj}(0)}}$$

$$\approx 0.014\sqrt{1+\frac{R_{ij}^2(\tau)}{R_{ii}(0)R_{jj}(0)}}$$

As for systematic errors, they mostly come from the limited bandwith of the measurement. It is quite obvious that a limited bandwith, besides increasing the statistical error as shown in the formula above, will also reduce the sensitivity to fast oscillations of the correlation under study, thereby effectively introducing some systematics. This is equivalent to say that the correlation function will be effectively time integrated and smoothed, as better seen in the frequency domain. Indeed, the signal cross-/self-power spectrum, consisting of a deterministic and a stochastic component, is effectively distorted by the transfer function of the measurement. Since the deterministic component is different in each particular case, one should not expect the same effects of a given transfer function in all cases.

4. Sampling errors in the measurement of cross/self-correlation

The process of sampling the currents introduces systematic and statistical errors. This should be expected, since the sampling process is nothing else than a special kind of filter applied to the signals. In the present case, where one is taking repeated measurements of the total counts in a time window of width T every τ seconds, the measured sampled current can be taken as given by the following generic expression

$$I^S(t) = \sum_{n=-\infty}^{+\infty} \delta(t - n\tau) \frac{1}{T} \int_{n\tau - T/2}^{n\tau + T/2} I(t')dt'$$

where T is the duration and $n\tau$ is the midpoint of the *n-th* sampling window. The effect of sampling is better understood in the frequency domain. It can be shown (see Appendix B) that the spectrum of the sampled signal , besides being made periodic by the sampling, is damped at frequencies above the cutoff frequency $2/T$ by the amount $\sin(\omega T/2)/(\omega T/2)$. This spectrum distortion makes up some contribution to the systematic error due to sampling. Another, more important systematic effect is originated by the aliasing of the Fourier components above the Nyquist frequency $1/2\tau$. Since normally $\tau >> T$, the effective bandwith of the measurement in fixed by $1/2\tau$. In principle, experimental data should be filtered by removing components above that limit, in order to avoid introducing spurious spectral contributions. Note however that for the important case where $T=\tau$ (i.e. by taking the integration window as wide as the sampling interval), one is effectively damping the frequency spectrum above $\omega_{Nyquist}$ (see App.B).The quoted bandwith corresponds to a rise time of about 4τ for a step variation in time of the correlation, so that the parameter 4τ may be taken as an indicator of the time sensitivity of the measurement. Statistical errors can be evaluated just by taking the expression for $R_{ij}(0)$, as shown in Appendix C. For the practically important case where one is sampling quasi-stationary signals (like we do in NA48), one can get the approximate expressions

$$\frac{\sigma_R}{R} \cong \frac{2}{\sqrt{NT}\sqrt{\bar{r}_i}} \qquad \text{- self}$$

$$\frac{\sigma_R}{R} \cong \frac{1}{\sqrt{NT}\sqrt{\dfrac{\bar{r}_i \bar{r}_j}{\bar{r}_i + \bar{r}_j}}} \qquad \text{- cross}$$

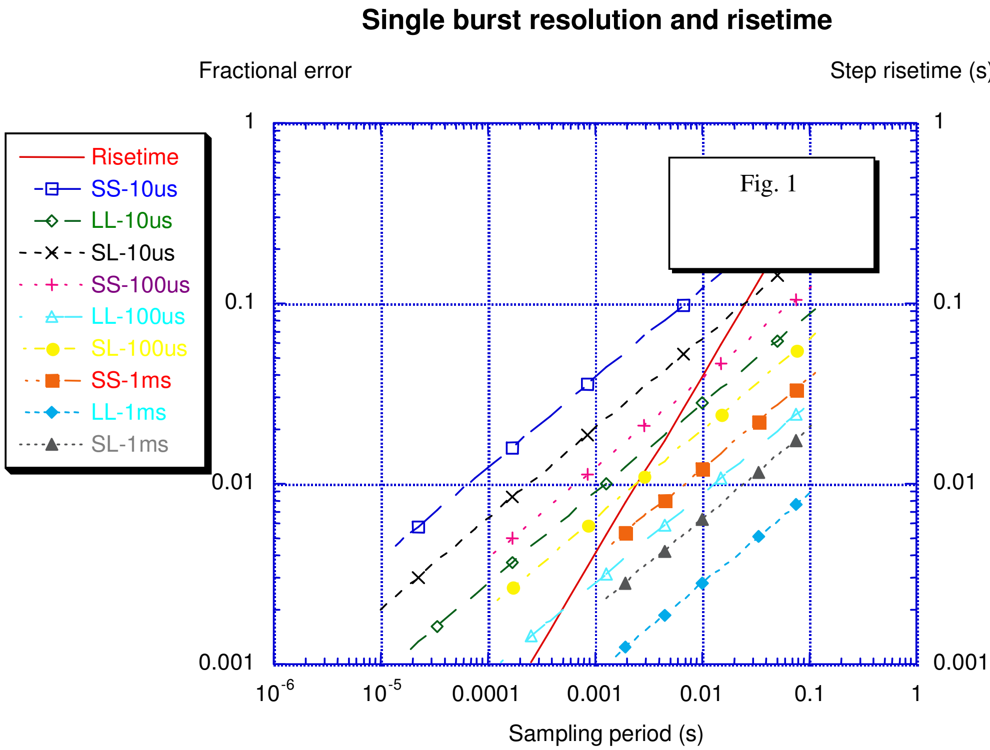
By taking typical values for K_S and K_L beam currents, as measured in NA48 by our beam monitors, of 10^5 Hz and 2 10^6 Hz, one gets

$$\frac{\sigma_R}{R} \cong \frac{0.0061}{\sqrt{NT}} \qquad \text{- S self}$$

$$\frac{\sigma_R}{R} \cong \frac{0.0014}{\sqrt{NT}} \qquad \text{- L self}$$

$$\frac{\sigma_R}{R} \cong \frac{0.0032}{\sqrt{NT}} \qquad \text{- S-L cross}$$

Remembering that $N = T_{burst}/\tau$, by keeping $T<\tau$ I have plotted in fig. 1 these fractional errors, together with the rise time of a step variation of the correlation, as originated by the finite bandwith of the measurement, as functions of τ for different values of sample width T.



This may be useful to appreciate the intrinsic limits of our experimental measurement. The whole point of making such a discussion of the relative weight of statistical and systematic errors is that one does not know *a priori* the deterministic frequency spectrum of the beam currents. So, if one suspects that some periodic components actually modulate the Poissonian flux of the beams, it is quite natural to try to bring them into evidence. However, the analysis sketched above shows that, for each given value of (T, τ), there is a natural lower limit for the detection of any such component, given by the shot noise floor in the beam currents. Conversely, any attempt to measure the beam-beam correlation to a given statistical significance is bound to yield a limited time resolution.

References

[1] P.Bendat, Principles and applications of random noise theory

Appendix A

Cross-correlation and cross-power spectrum of the two currents

The cross-correlation of the two beam currents can be evaluated by taking both as quasi-stationary, δ-correlated Poissonian processes. In this case, the total current can be also taken as a similar process. Then one obtains, by assuming $R_{LS}(\tau)=R_{SL}(\tau)$:

$$
\begin{aligned}
&I_T(t) = I_L(t) + I_S(t) \\
&r_T = r_L + r_S \\
&R_{TT}(\tau) = R_{LL}(\tau) + R_{SS}(\tau) + 2R_{LS}(\tau) \\
&R_{TT}(\tau) = r_T\delta(\tau) + r_T^{\ 2} \\
&R_{LL}(\tau) = r_L\delta(\tau) + r_L^{\ 2} \\
&R_{SS}(\tau) = r_S\delta(\tau) + r_S^{\ 2} \\
&\rightarrow r_T^{\ 2} = r_L^{\ 2} + r_S^{\ 2} + 2R_{LS}(\tau) \\
&\rightarrow R_{LS}(\tau) = r_T r_S
\end{aligned}
$$

In the above expressions for R_{LL}, R_{SS} one can recognize a *signal* term $r_{L,S}^{\ 2}$ and a *shot noise* term $r_{L,S}\delta(\tau)$, as expected for the self-correlation of a Poissonian, constant average current. Conversely, R_{SL} is a *pure signal* cross-correlation, independent noise sources being uncorrelated when integrated over the full time axis. When integrated over a finite time interval, some noise term can be expected to contribute there too. The corresponding cross-power spectrum is evaluated as the Fourier transform of the cross-correlation as

$$R_{LS}(\omega) = \frac{1}{2\pi} r_L r_S \delta(\omega)$$

To the approximation where the original cross-spectrum is δ-like, the sampled cross-spectrum is unchanged, except for the factor $(\tau/T)^2$ .This translates into a still flat cross-correlation with the same factor $(\tau/T)^2$. Any frequency structure above the cutoff frequency $4\pi/T$ in the cross-spectrum is however damped by the factor $(\sin x/x)^2$. Due to the sampling period τ, any structure above the Nyquist frequency π/τ is folded back into the bandwith at a spurious frequency. These expressions for $R_{LS}(\tau)$, $R_{LS}(\omega)$ can be maintained also for time varying currents, provided the time scale of their variation is large. In the latter case, $r_{L,S}$ should be taken as (slow) functions of time.

Appendix B

Derivation of the frequency spectrum for the sampled signal

Having defined the sampled current signal $I_S(t)$, one can compute its frequency spectrum as follows:

By recalling that

$$\int_{n\tau-T/2}^{n\tau+T/2} I(t')dt' = \int_{n\tau-T/2}^{n\tau+T/2} dt' \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{+\infty} e^{i\omega' t'} \widehat{I}(\omega')d\omega'$$

one easily obtains

$$\begin{aligned}\frac{1}{T}\int_{n\tau-T/2}^{n\tau+T/2} I(t')dt' &= \frac{1}{T}\frac{1}{\sqrt{2\pi}} \int_{-\infty}^{+\infty} \widehat{I}(\omega')d\omega' \int_{n\tau-T/2}^{n\tau+T/2} dt' e^{i\omega' t'} \\ &= \frac{1}{T}\frac{1}{\sqrt{2\pi}} \int_{-\infty}^{+\infty} \widehat{I}(\omega')d\omega' \, e^{i\omega' n\tau} \frac{\sin \omega' T/2}{\omega'/2}\end{aligned}$$

Then, by substituting back into the original expression

$$I_S(\omega) = \frac{1}{2\pi} \sum_{n=-\infty}^{+\infty} \int_{-\infty}^{+\infty} I(\omega') \frac{\sin\omega' T/2}{\omega' T/2} d\omega' \, e^{i\omega' n\tau} e^{-i\omega n\tau}$$

$$I_S(\omega) = \int_{-\infty}^{+\infty} I(\omega') \frac{\sin\omega' T/2}{\omega' T/2} \underbrace{\sum_{n=-\infty}^{+\infty} \frac{1}{2\pi} e^{i\omega' n\tau} e^{-i\omega n\tau}}_{\frac{1}{\tau}\sum_{n=-\infty}^{+\infty} \delta(\omega'-\omega-\frac{2\pi n}{\tau})} d\omega'$$

which is

$$\widehat{I}_S(\omega) = \frac{\tau}{T} \sum_{n=-\infty}^{+\infty} \widehat{I}(\omega - \frac{2\pi n}{\tau}) \frac{\sin\left[(\omega - \frac{2\pi n}{\tau}) T/2 \right]}{(\omega - \frac{2\pi n}{\tau}) T/2}$$

This shows that the spectrum of the sampled current is made periodic by the sampling process. It is also damped, within each period, above the cutoff frequency $\omega_{cutoff}{}^{(n)} = \pm(2\pi n/\tau) \pm (4\pi/T)$. One may then look for the effect of sampling on the cross-spectrum of the two currents

$$\widehat{I}_S^{(S)}(\omega)\widehat{I}_S^{(L)}(\omega) = \left(\frac{\tau}{T}\right)^2 \sum_{n'=-\infty}^{+\infty} \sum_{n''=-\infty}^{+\infty} \widehat{I}^{(S)}(\omega - \frac{2\pi n'}{\tau})\widehat{I}^{(L)}(\omega - \frac{2\pi n''}{\tau}) \frac{\sin\left[(\omega - \frac{2\pi n'}{\tau})T/2\right]}{(\omega - \frac{2\pi n'}{\tau})T/2} \frac{\sin\left[(\omega - \frac{2\pi n''}{\tau})T/2\right]}{(\omega - \frac{2\pi n''}{\tau})T/2}$$

$$\cong \left(\frac{\tau}{T}\right)^2 \sum_{n=-\infty}^{+\infty} \widehat{I}^{(S)}(\omega - \frac{2\pi n}{\tau})\widehat{I}^{(L)}(\omega - \frac{2\pi n}{\tau}) \frac{\sin^2\left[(\omega - \frac{2\pi n}{\tau})T/2\right]}{\left[(\omega - \frac{2\pi n}{\tau})T/2\right]^2}$$

Appendix C

Calculation of the statistical error on the sampled cross/self correlation

It has been shown in Appendix A that the original self-correlation contains a shot noise term, giving rise to a statistical error on the signal term, while the cross-correlation is noise-free. One can calculate the statistical error on both self- and cross-correlation after the sampling process as follows: starting from the expression

$$R_{ij}(0) = \frac{1}{NT^2} \sum_{k=1}^{N} n_k^{(i)} n_k^{(j)}$$

by assuming Poissonian fluctuactions in the rates, we derive the following formula for the variance of $R_{ij}(0)$

$$\sigma_R^2 = \frac{1}{N^2 T^4} \sum_{k=1}^{N} \left[\left(n_k^{(i)}\right)^2 n_k^{(j)} + \left(n_k^{(j)}\right)^2 n_k^{(i)} \right] \quad \text{- cross}$$

$$\sigma_R^2 = \frac{4}{N^2 T^4} \sum_{k=1}^{N} \left(n_k^{(i)}\right)^3 \quad \text{- self}$$

These last expressions can be rewritten in terms of the instantaneous rates r_L, r_S:

$$n_k^{(i,j)} = r_k^{i,j} T$$

$$R_{ij}(0) = \frac{1}{N} \sum_{k=1}^{N} r_k^{(i)} r_k^{(j)}$$

$$\sigma_R^2 = \frac{1}{N^2 T} \sum_{k=1}^{N} \left[\left(r_k^{(i)}\right)^2 r_k^{(j)} + \left(r_k^{(j)}\right)^2 r_k^{(i)} \right] \quad \text{- cross}$$

$$\sigma_R^2 = \frac{4}{N^2 T} \sum_{k=1}^{N} \left(r_k^{(i)}\right)^3 \quad \text{- self}$$

yielding the fractional resolution

$$\frac{\sigma_R}{R} = \frac{\sqrt{\sum_{k=1}^{N} \left[\left(r_k^{(i)}\right)^2 r_k^{(j)} + \left(r_k^{(j)}\right)^2 r_k^{(i)} \right]}}{\sqrt{T} \sum_{k=1}^{N} r_k^{(i)} r_k^{(j)}} \quad \text{- cross}$$

$$\frac{\sigma_R}{R} = \frac{2\sqrt{\sum_{k=1}^{N} \left(r_k^{(i)}\right)^3}}{\sqrt{T} \sum_{k=1}^{N} \left(r_k^{(i)}\right)^2} \quad \text{- self}$$

The expression for σ_R – cross does not take into account any correlation between n_S and n_L. Such a correlation is originated, for example, by sampling with a variable gate length (see App. D below for details). For the case of BCTR and KSM rates, it can be shown that the correction is rather small (<15%)

Appendix D

Statistical correlation induced by non-constant gate width

One should point out that one of the available data sets, consisting of incremental readout of the KSM and BCTR scalers between successive triggers, contains a statistical (exponential) distribution of sampling intervals, coming from the Poissonian fluctuations in the trigger rate. Closer analysis shows that these data feature a strong S-L correlation originated by the non-uniform sampling. Indeed, count integration over a variable gate length always originates some level of correlation. By taking an exponential gate length distribution

$$\frac{dN}{dt} = \frac{1}{T} e^{-\frac{t}{T}}$$

the joint statistical distribution for n_S and n_L can be obtained by folding dN/dt to a pair of Poisson distributions with average values

$$\begin{aligned} \overline{n}_S &= r_S t;\, r_S \text{ is the S average rate} \\ \overline{n}_L &= r_L t;\, r_L \text{ is the L average rate} \end{aligned}$$

From

$$P(n_S, n_L) = \int_0^\infty \frac{1}{T} e^{-\frac{t}{T}} \frac{e^{-r_S t} (r_S t)^{n_S}}{n_S!} \frac{e^{-r_L t} (r_L t)^{n_L}}{n_L!} dt$$

one gets

$$P(n_S, n_L) = \frac{r_T}{r_T + r_S + r_L} \frac{(n_S + n_L)!}{n_L! n_S!} \frac{r_S^{n_S} r_L^{n_L}}{(r_T + r_S + r_L)^{n_S + n_L}}$$

Now, this expression for P be cannot factorized as a product

$$P(n_S, n_L) = f_S(n_S) f_L(n_L)$$

so n_S and n_L are statistically correlated. This corresponds to our expectation: when t is large, both n_S and n_L tend to be large, and conversely when t is small. This has no consequences on the estimate of $I_{S,L}$ and $C_{LS,LL}$, which are obtained by taking the ratio between n_S , n_L and Δt, however should not be neglected when computing the statistical errors. On the other hand, for uniform sampling times correlation vanishes.