Meccanica – A.A. 2010/11

Esercizi – 12

12-1) Per una collisione elastica fra due punti materiali di massa m_1 e m_2, trovare la relazione fra l'angolo di deflessione θ di m_1 nel LAB (ossia, nel SRI in cui m_2 e' ferma) e quello ϕ nel CM

Vel. CM nel LAB:

$$\boldsymbol{r}_{CM} = \frac{m_1\boldsymbol{r}_1 + m_2\boldsymbol{r}_2}{m_1+m_2} \rightarrow \boldsymbol{v}_{CM} = \frac{m_1\boldsymbol{v}_1 + m_2\boldsymbol{v}_2}{m_1+m_2}$$

Vel. punti nel CM:

$$\rightarrow \boldsymbol{V}_1 = \boldsymbol{v}_1 - \boldsymbol{v}_{CM}, \boldsymbol{V}_2 = \boldsymbol{v}_2 - \boldsymbol{v}_{CM}$$

$$\boldsymbol{v}_2 = 0 \rightarrow \boldsymbol{v}_{CM} = \frac{m_1}{m_1+m_2}\boldsymbol{v}_1$$

$$\rightarrow \boldsymbol{V}_1 = \boldsymbol{v}_1 - \boldsymbol{v}_{CM} = \boldsymbol{v}_1\left(\frac{m_2}{m_1+m_2}\right)$$

$$\rightarrow \boldsymbol{V}_2 = \boldsymbol{v}_2 - \boldsymbol{v}_{CM} = -\boldsymbol{v}_{CM} = -\frac{m_1}{m_1+m_2}\boldsymbol{v}_1 = -\frac{m_1}{m_2}\boldsymbol{V}_1$$

$$\rightarrow m_2\boldsymbol{V}_2 = -m_1\boldsymbol{V}_1 \rightarrow \boldsymbol{P}_1 + \boldsymbol{P}_2 = 0$$

Conservazione qdm nel CM:

$$\boldsymbol{P}_1 + \boldsymbol{P}_2 = 0 = \boldsymbol{P}'_1 + \boldsymbol{P}'_2 \rightarrow m_2\boldsymbol{V}'_2 = -m_1\boldsymbol{V}'_1$$

Conservazione energia cin. nel CM:

$$\frac{1}{2}m_1V_1^2 + \frac{1}{2}m_2V_2^2 = \frac{1}{2}m_1V'^2_1 + \frac{1}{2}m_2V'^2_2$$

$$\rightarrow m_1\left(V_1^2 - V'^2_1\right) = m_2\left(V'^2_2 - V_2^2\right) = m_2\left(\frac{m_1}{m_2}\right)^2\left(V'^2_1 - V_1^2\right)$$

$$\rightarrow \underbrace{\frac{m_1}{m_2}}_{>0}\left(V_1^2 - V'^2_1\right) = \left(\frac{m_1}{m_2}\right)^2\left(V'^2_1 - V_1^2\right) = \underbrace{-\left(\frac{m_1}{m_2}\right)^2}_{<0}\left(V_1^2 - V'^2_1\right)$$

$$\rightarrow V'_1 = V_1, V'_2 = V_2$$ Nel CM le velocita' non cambiano in modulo

$$\boldsymbol{V}'_1 \cdot \boldsymbol{v}_1 = \left(\boldsymbol{v}'_1 \cdot \boldsymbol{v}_1 - \boldsymbol{v}_{CM} \cdot \boldsymbol{v}_1\right)$$

$$\rightarrow V'_1 v_1 \cos\varphi = v'_1 v_1 \cos\theta - v_1^2 \frac{m_1}{m_1 + m_2}$$

$$\rightarrow \begin{cases} V'_1 \cos\varphi = v'_1 \cos\theta - v_1 \dfrac{m_1}{m_1 + m_2} \\ V'_1 \sin\varphi = v'_1 \sin\theta \end{cases}$$

$$\rightarrow \begin{cases} v'_1 \cos\theta = V'_1 \cos\varphi + v_1 \dfrac{m_1}{m_1 + m_2} \\ v'_1 \sin\theta = V'_1 \sin\varphi \end{cases}$$

$$\rightarrow \tan\theta = \frac{\sin\varphi}{\cos\varphi + \dfrac{v_1}{V'_1} \dfrac{m_1}{m_1 + m_2}}$$

Nel CM:

$$V'_1 = V_1 = v_1 - v_{CM} = v_1 \left(1 - \frac{m_1}{m_1 + m_2}\right) = v_1 \frac{m_2}{m_1 + m_2}$$

$$\rightarrow \tan\theta = \frac{\sin\varphi}{\cos\varphi + \dfrac{v_1}{v_1 \dfrac{m_2}{m_1 + m_2}} \dfrac{m_1}{m_1 + m_2}} = \frac{\sin\varphi}{\cos\varphi + \dfrac{m_1}{m_2}}$$

12-2) Riferendosi al problema precedente, mostrare che se $m_1 > m_2$ c'e' un angolo di deflessione massimo nel LAB

$$\tan\theta = \frac{\sin\varphi}{\cos\varphi + \dfrac{m_1}{m_2}}$$

$$\frac{d\tan\theta}{d\varphi} = \frac{\cos\varphi\left(\cos\varphi + \dfrac{m_1}{m_2}\right) + \sin^2\varphi}{\left(\cos\varphi + \dfrac{m_1}{m_2}\right)^2} = \frac{1 + \cos\varphi \dfrac{m_1}{m_2}}{\left(\cos\varphi + \dfrac{m_1}{m_2}\right)^2}$$

$$\rightarrow \frac{d\tan\theta}{d\varphi} = 0 \rightarrow 1 + \cos\varphi \frac{m_1}{m_2} = 0$$

$$m_1 > m_2 \to \cos\varphi_{\max} = -\frac{m_2}{m_1} \to \sin\varphi_{\max} = \sqrt{1-\left(\frac{m_2}{m_1}\right)^2}$$

$$\to \tan\theta_{\max} = \frac{\sin\varphi_{\max}}{\cos\varphi_{\max} + \frac{m_1}{m_2}} = \frac{\sqrt{1-\left(\frac{m_2}{m_1}\right)^2}}{-\frac{m_2}{m_1} + \frac{m_1}{m_2}} = \frac{\sqrt{1-\left(\frac{m_2}{m_1}\right)^2}}{\frac{-m_2^2 + m_1^2}{m_1 m_2}}$$

$$\to \tan\theta_{\max} = \sqrt{\left[1-\left(\frac{m_2}{m_1}\right)^2\right]} m_1^2 m_2^2 \frac{1}{m_1^2 - m_2^2} = \frac{m_2}{m_1^2 - m_2^2}\sqrt{\left(m_1^2 - m_2^2\right)}$$

$$\to \tan\theta_{\max} = \frac{m_2}{\sqrt{m_1^2 - m_2^2}} = \frac{m_2}{m_1}\frac{1}{\sqrt{1-\frac{m_2^2}{m_1^2}}}$$

Applicazione: si osservano collisioni di particelle α su protoni (idrogeno), notando che gli angoli di deflessione sono sempre minori di 16^0 circa.
Stima della massa delle α:

$$\tan\theta_{\max} = \frac{m_2}{m_1}\frac{1}{\sqrt{1-\frac{m_2^2}{m_1^2}}} = \frac{A}{\sqrt{1-A^2}}$$

$$\to \tan^2\theta_{\max} = \frac{A^2}{\left(1-A^2\right)}$$

$$\to \left(1-A^2\right)\tan^2\theta_{\max} = A^2$$

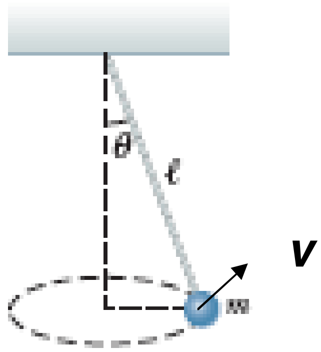
$$\to A^2\left(1+\tan^2\theta_{\max}\right) = \tan^2\theta_{\max}$$

$$\to A = \frac{m_p}{m_\alpha} = \frac{\tan\theta_{\max}}{\sqrt{1+\tan^2\theta_{\max}}} \approx 0.276$$

$$\to m_\alpha \approx \frac{m_p}{0.276} \approx 3.62 m_p$$

Valore corretto: $m_\alpha \approx 3.98 m_p$

12-3) Per il pendolo conico mostrato in figura, calcolare il momento angolare rispetto al punto di sospensione



$$\boldsymbol{L} = \boldsymbol{r} \times \boldsymbol{p} = m\boldsymbol{r} \times \boldsymbol{v}$$

Fissando il polo nel centro dell'orbita:

$$\to \boldsymbol{L} \parallel z,\ L = mrv$$

$$r = l \sin\theta$$

$$m\frac{v^2}{r} = m\frac{v^2}{l\sin\theta} = T\sin\theta$$

$$T\cos\theta = mg \to T = \frac{mg}{\cos\theta}$$

$$\to \frac{v^2}{l\sin\theta} = \frac{g}{\cos\theta}\sin\theta$$

$$\to v^2 = l\frac{g}{\cos\theta}\sin^2\theta$$

$$\to L = mrv = ml\sin^2\theta\sqrt{l\frac{g}{\cos\theta}} = \sqrt{m^2l^3g\frac{\sin^4\theta}{\cos\theta}}$$

$$\boldsymbol{\tau} = \boldsymbol{r} \times \boldsymbol{F}$$

$$\boldsymbol{F} = \boldsymbol{T} + m\boldsymbol{g} \to \boldsymbol{F} \parallel \boldsymbol{r} \to \boldsymbol{\tau} = 0 \to \boldsymbol{L} = \text{cost}$$

Fissando il polo nel punto di sospensione:

$$\to \boldsymbol{L} \perp (\boldsymbol{r}, \boldsymbol{v}) \to L = mlv = ml\sqrt{l\frac{g}{\cos\theta}\sin^2\theta} = \sqrt{m^2l^3g\frac{\sin^2\theta}{\cos\theta}}$$

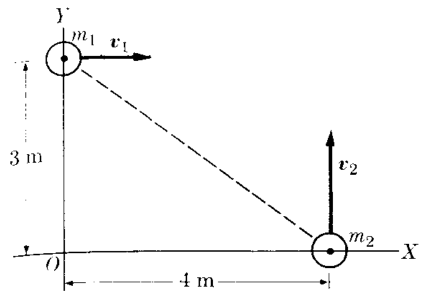
$\to L$ maggiore rispetto all'altra scelta

$$\boldsymbol{\tau} = \boldsymbol{r} \times \boldsymbol{F} = \boldsymbol{r} \times (\boldsymbol{T} + m\boldsymbol{g}), \boldsymbol{T} \parallel \boldsymbol{r} \to \boldsymbol{\tau} = \boldsymbol{r} \times m\boldsymbol{g} \neq 0$$

$\to \boldsymbol{L} \neq$ cost $\to \boldsymbol{L}$ precede attorno alla vel. angolare del punto

$$\boldsymbol{L} \cdot \hat{\boldsymbol{k}} = \sqrt{m^2l^3g\frac{\sin^2\theta}{\cos\theta}}\cos\left(\frac{\pi}{2} - \theta\right) = \sqrt{m^2l^3g\frac{\sin^4\theta}{\cos\theta}} = \text{cost}$$

12-4) Per i due punti materiali mostrati nella figura, determinare il momento angolare totale riferito all'origine e riferito al CM, e verificare esplicitamente la relazione fra i due



$$O:\boldsymbol{L}^{(O)}=\boldsymbol{r}_1\times\boldsymbol{p}_1+\boldsymbol{r}_2\times\boldsymbol{p}_2$$

$$\boldsymbol{r}_{CM}=\frac{m_1\boldsymbol{r}_1+m_2\boldsymbol{r}_2}{m_1+m_2}\rightarrow\boldsymbol{v}_{CM}=\frac{m_1\boldsymbol{v}_1+m_2\boldsymbol{v}_2}{m_1+m_2}$$

$$CM:\boldsymbol{L}^{(CM)}=\boldsymbol{R}_1\times\boldsymbol{P}_1+\boldsymbol{R}_2\times\boldsymbol{P}_2$$

$$\boldsymbol{R}_1=\boldsymbol{r}_1-\boldsymbol{r}_{CM},\boldsymbol{R}_2=\boldsymbol{r}_2-\boldsymbol{r}_{CM}$$

$$\boldsymbol{V}_1=\boldsymbol{v}_1-\boldsymbol{v}_{CM},\boldsymbol{V}_2=\boldsymbol{v}_2-\boldsymbol{v}_{CM}\rightarrow\boldsymbol{P}_1=m_1\boldsymbol{V}_1,\boldsymbol{P}_2=m_2\boldsymbol{V}_2$$

$$\rightarrow\boldsymbol{L}^{(CM)}=\left(\boldsymbol{r}_1-\boldsymbol{r}_{CM}\right)\times m_1\left(\boldsymbol{v}_1-\boldsymbol{v}_{CM}\right)+\left(\boldsymbol{r}_2-\boldsymbol{r}_{CM}\right)\times m_2\left(\boldsymbol{v}_2-\boldsymbol{v}_{CM}\right)$$

$$\rightarrow\boldsymbol{L}^{(CM)}=\boldsymbol{L}^{(O)}-\boldsymbol{r}_{CM}\times\left(m_1\boldsymbol{v}_1+m_1\boldsymbol{v}_2\right)-\left(m_1\boldsymbol{r}_1+m_2\boldsymbol{r}_2\right)\times\boldsymbol{v}_{CM}+\left(m_1+m_1\right)\boldsymbol{r}_{CM}\times\boldsymbol{v}_{CM}$$

$$\rightarrow\boldsymbol{L}^{(CM)}=\boldsymbol{L}^{(O)}-\cancel{\boldsymbol{r}_{CM}\times\left(m_1+m_1\right)\boldsymbol{v}_{CM}}-\left(m_1\boldsymbol{r}_1+m_2\boldsymbol{r}_2\right)\times\boldsymbol{v}_{CM}+\cancel{\left(m_1+m_1\right)\boldsymbol{r}_{CM}\times\boldsymbol{v}_{CM}}$$

$$\rightarrow\boldsymbol{L}^{(CM)}=\boldsymbol{L}^{(O)}-\left(m_1+m_1\right)\boldsymbol{r}_{CM}\times\boldsymbol{v}_{CM}=\boldsymbol{L}^{(O)}-\boldsymbol{r}_{CM}\times\left(m_1+m_1\right)\boldsymbol{v}_{CM}=\boldsymbol{L}^{(O)}-\underbrace{\boldsymbol{r}_{CM}\times\boldsymbol{p}_{CM}}_{L_{CM}}$$

$$\rightarrow\boldsymbol{L}^{(CM)}=\boldsymbol{L}^{(O)}-\boldsymbol{L}_{CM}^{(O)}$$

$$\boldsymbol{L}^{(O)}=y_1\hat{\boldsymbol{j}}\times m_1v_1\hat{\boldsymbol{i}}+x_2\hat{\boldsymbol{i}}\times m_2v_2\hat{\boldsymbol{j}}$$

$$\rightarrow\boldsymbol{L}^{(O)}=m_1v_1y_1\hat{\boldsymbol{j}}\times\hat{\boldsymbol{i}}+m_2v_2x_2\hat{\boldsymbol{i}}\times\hat{\boldsymbol{j}}$$

$$\rightarrow\boldsymbol{L}^{(O)}=-m_1v_1y_1\hat{\boldsymbol{k}}+m_2v_2x_2\hat{\boldsymbol{k}}$$

$$\rightarrow\boldsymbol{L}^{(O)}=\left(m_2v_2x_2-m_1v_1y_1\right)\hat{\boldsymbol{k}}$$

$$\boldsymbol{L}_{CM}^{(O)}=\left(\frac{m_1\boldsymbol{r}_1+m_2\boldsymbol{r}_2}{m_1+m_2}\right)\times\left(m_1\boldsymbol{v}_1+m_2\boldsymbol{v}_2\right)=\frac{m_1y_1\hat{\boldsymbol{j}}+m_2x_2\hat{\boldsymbol{i}}}{m_1+m_2}\times\left(m_1v_1\hat{\boldsymbol{i}}+m_2v_2\hat{\boldsymbol{j}}\right)$$

$$\rightarrow\boldsymbol{L}_{CM}^{(O)}=-\frac{m_1^2y_1v_1\hat{\boldsymbol{k}}-m_2^2x_2v_2\hat{\boldsymbol{k}}}{m_1+m_2}=-\frac{m_1^2y_1v_1-m_2^2x_2v_2}{m_1+m_2}\hat{\boldsymbol{k}}$$

$$\rightarrow\boldsymbol{L}^{(O)}-\boldsymbol{L}_{CM}^{(O)}=\left[\left(m_2v_2x_2-m_1v_1y_1\right)+\frac{m_1^2y_1v_1-m_2^2x_2v_2}{m_1+m_2}\right]\hat{\boldsymbol{k}}$$

$$\boldsymbol{R}_1=\boldsymbol{r}_1-\boldsymbol{r}_{CM}=y_1\hat{\boldsymbol{j}}-\frac{m_1y_1\hat{\boldsymbol{j}}+m_2x_2\hat{\boldsymbol{i}}}{m_1+m_2},\boldsymbol{R}_2=\boldsymbol{r}_2-\boldsymbol{r}_{CM}=x_2\hat{\boldsymbol{i}}-\frac{m_1y_1\hat{\boldsymbol{j}}+m_2x_2\hat{\boldsymbol{i}}}{m_1+m_2}$$

$$\boldsymbol{V}_1=\boldsymbol{v}_1-\boldsymbol{v}_{CM}=v_1\hat{\boldsymbol{i}}-\frac{m_1v_1\hat{\boldsymbol{i}}+m_2v_2\hat{\boldsymbol{j}}}{m_1+m_2},\boldsymbol{V}_2=\boldsymbol{v}_2-\boldsymbol{v}_{CM}=v_2\hat{\boldsymbol{j}}-\frac{m_1v_1\hat{\boldsymbol{i}}+m_2v_2\hat{\boldsymbol{j}}}{m_1+m_2}$$

$$\rightarrow \boldsymbol{L}^{(CM)}=\boldsymbol{R}_1\times\boldsymbol{P}_1+\boldsymbol{R}_2\times\boldsymbol{P}_2$$

$$=\underbrace{\left(y_1\hat{\boldsymbol{j}}-\frac{m_1y_1\hat{\boldsymbol{j}}+m_2x_2\hat{\boldsymbol{i}}}{m_1+m_2}\right)\times m_1\left(v_1\hat{\boldsymbol{i}}-\frac{m_1v_1\hat{\boldsymbol{i}}+m_2v_2\hat{\boldsymbol{j}}}{m_1+m_2}\right)}_{A}$$

$$+\underbrace{\left(x_2\hat{\boldsymbol{i}}-\frac{m_1y_1\hat{\boldsymbol{j}}+m_2x_2\hat{\boldsymbol{i}}}{m_1+m_2}\right)\times m_2\left(v_2\hat{\boldsymbol{j}}-\frac{m_1v_1\hat{\boldsymbol{i}}+m_2v_2\hat{\boldsymbol{j}}}{m_1+m_2}\right)}_{B}$$

$$\boldsymbol{A}=m_1v_1y_1\hat{\boldsymbol{j}}\times\hat{\boldsymbol{i}}-\frac{m_1y_1}{m_1+m_2}\hat{\boldsymbol{j}}\times\left(m_1v_1\hat{\boldsymbol{i}}+m_2v_2\hat{\boldsymbol{j}}\right)$$

$$-\frac{m_1v_1}{m_1+m_2}\left(m_1y_1\hat{\boldsymbol{j}}+m_2x_2\hat{\boldsymbol{i}}\right)\times\hat{\boldsymbol{i}}+m_1\frac{m_1y_1\hat{\boldsymbol{j}}+m_2x_2\hat{\boldsymbol{i}}}{m_1+m_2}\times\frac{m_1v_1\hat{\boldsymbol{i}}+m_2v_2\hat{\boldsymbol{j}}}{m_1+m_2}$$

$$\boldsymbol{B}=m_2v_2x_2\hat{\boldsymbol{i}}\times\hat{\boldsymbol{j}}-\frac{m_2x_2}{m_1+m_2}\hat{\boldsymbol{i}}\times\left(m_1v_1\hat{\boldsymbol{i}}+m_2v_2\hat{\boldsymbol{j}}\right)$$

$$-\frac{m_2v_2}{m_1+m_2}\left(m_1y_1\hat{\boldsymbol{j}}+m_2x_2\hat{\boldsymbol{i}}\right)\times\hat{\boldsymbol{j}}+m_2\frac{m_1y_1\hat{\boldsymbol{j}}+m_2x_2\hat{\boldsymbol{i}}}{m_1+m_2}\times\frac{m_1v_1\hat{\boldsymbol{i}}+m_2v_2\hat{\boldsymbol{j}}}{m_1+m_2}$$

$$\rightarrow \boldsymbol{A}+\boldsymbol{B}=\left(-m_1v_1y_1+m_2v_2x_2\right)\hat{\boldsymbol{k}}+\left(m_1v_1\hat{\boldsymbol{i}}+m_2v_2\hat{\boldsymbol{j}}\right)\times\left(\frac{m_2x_2}{m_1+m_2}\hat{\boldsymbol{i}}+\frac{m_1y_1}{m_1+m_2}\hat{\boldsymbol{j}}\right)$$

$$-\left(m_1y_1\hat{\boldsymbol{j}}+m_2x_2\hat{\boldsymbol{i}}\right)\times\left(\frac{m_1v_1}{m_1+m_2}\hat{\boldsymbol{i}}+\frac{m_2v_2}{m_1+m_2}\hat{\boldsymbol{j}}\right)+\frac{\left(m_1y_1\hat{\boldsymbol{j}}+m_2x_2\hat{\boldsymbol{i}}\right)}{m_1+m_2}\times\left(m_1v_1\hat{\boldsymbol{i}}+m_2v_2\hat{\boldsymbol{j}}\right)$$

$$\rightarrow \boldsymbol{A}+\boldsymbol{B}=\left(-m_1v_1y_1+m_2v_2x_2\right)\hat{\boldsymbol{k}}+\left(m_1v_1\hat{\boldsymbol{i}}+m_2v_2\hat{\boldsymbol{j}}\right)\times\frac{m_2x_2\hat{\boldsymbol{i}}+m_1y_1\hat{\boldsymbol{j}}}{m_1+m_2}$$

$$-\left(m_1y_1\hat{\boldsymbol{j}}+m_2x_2\hat{\boldsymbol{i}}\right)\times\frac{m_1v_1\hat{\boldsymbol{i}}+m_2v_2\hat{\boldsymbol{j}}}{m_1+m_2}+\frac{\left(m_1y_1\hat{\boldsymbol{j}}+m_2x_2\hat{\boldsymbol{i}}\right)}{m_1+m_2}\times\left(m_1v_1\hat{\boldsymbol{i}}+m_2v_2\hat{\boldsymbol{j}}\right)$$

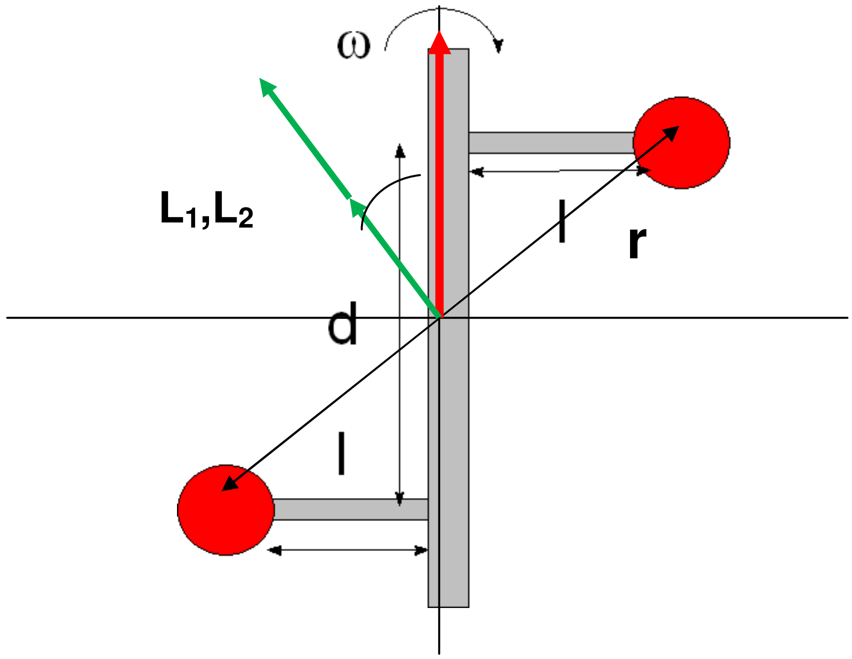
$$\rightarrow \boldsymbol{A}+\boldsymbol{B}=\left(-m_1v_1y_1+m_2v_2x_2\right)\hat{\boldsymbol{k}}+2\frac{m_1v_1\hat{\boldsymbol{i}}+m_2v_2\hat{\boldsymbol{j}}}{m_1+m_2}\times\left(m_1y_1\hat{\boldsymbol{j}}+m_2x_2\hat{\boldsymbol{i}}\right)$$

$$+\frac{m_2x_2\hat{\boldsymbol{i}}+m_1y_1\hat{\boldsymbol{j}}}{m_1+m_2}\times\left(m_1v_1\hat{\boldsymbol{i}}+m_2v_2\hat{\boldsymbol{j}}\right)$$

$$\rightarrow \boldsymbol{A}+\boldsymbol{B}=\boldsymbol{L}^{(CM)}=\left(-m_1v_1y_1+m_2v_2x_2\right)\hat{\boldsymbol{k}}+\frac{m_1v_1\hat{\boldsymbol{i}}+m_2v_2\hat{\boldsymbol{j}}}{m_1+m_2}\times\left(m_1y_1\hat{\boldsymbol{j}}+m_2x_2\hat{\boldsymbol{i}}\right)$$

$$\rightarrow \boldsymbol{L}^{(CM)}=\left[\left(-m_1v_1y_1+m_2v_2x_2\right)+\left(m_1^2y_1v_1-m_2^2x_2v_2\right)\right]\hat{\boldsymbol{k}}$$

12-5) Due masse uguali m sono montate ortogonalmente a sbalzo su un albero motore, tramite sbarre di massa trascurabile e lunghezza l fissate a distanza d. L'albero ruota con velocita' angolare ω. Qual e' l'angolo fra il momento angolare e la velocita' angolare del sistema?



$$\boldsymbol{L}_1 = \boldsymbol{r}_1 \times \boldsymbol{p}_1 = m_1 \boldsymbol{r}_1 \times \boldsymbol{v}_1 = m_1 \boldsymbol{r}_1 \times (\boldsymbol{\omega} \times \boldsymbol{r}_1)$$

$$\boldsymbol{L}_2 = \boldsymbol{r}_2 \times \boldsymbol{p}_2 = m_2 \boldsymbol{r}_2 \times \boldsymbol{v}_2 = m_2 \boldsymbol{r}_2 \times (\boldsymbol{\omega} \times \boldsymbol{r}_2)$$

$$\boldsymbol{r}_2 = -\boldsymbol{r}_1 = -\boldsymbol{r}$$

$$\rightarrow \boldsymbol{L}_1 = m\boldsymbol{r} \times (\boldsymbol{\omega} \times \boldsymbol{r})$$

$$\rightarrow \boldsymbol{L}_2 = m\boldsymbol{r}_2 \times (\boldsymbol{\omega} \times \boldsymbol{r}_2) = m(-\boldsymbol{r}) \times (\boldsymbol{\omega} \times (-\boldsymbol{r})) = m\boldsymbol{r} \times (\boldsymbol{\omega} \times \boldsymbol{r}) = \boldsymbol{L}_1$$

$$\rightarrow \boldsymbol{L} = 2m\boldsymbol{r} \times (\boldsymbol{\omega} \times \boldsymbol{r})$$

$\boldsymbol{\omega} \times \boldsymbol{r} \perp (\boldsymbol{\omega}, \boldsymbol{r})$ Vettore $\perp$ al piano della pagina

$\boldsymbol{r} \times (\boldsymbol{\omega} \times \boldsymbol{r}) \perp \boldsymbol{\omega} \times \boldsymbol{r}$ Vettore nel piano della pagina

Identita' vettoriale:

$$\boldsymbol{a}\times(\boldsymbol{b}\times\boldsymbol{c})=\boldsymbol{b}(\boldsymbol{a}\cdot\boldsymbol{c})-\boldsymbol{c}(\boldsymbol{a}\cdot\boldsymbol{b})$$

$$\rightarrow[\boldsymbol{r}\times(\boldsymbol{\omega}\times\boldsymbol{r})]=\boldsymbol{\omega}r^2-\boldsymbol{r}(\boldsymbol{\omega}\cdot\boldsymbol{r})$$

$$\rightarrow[\boldsymbol{r}\times(\boldsymbol{\omega}\times\boldsymbol{r})]\cdot\boldsymbol{\omega}=\omega^2r^2-(\boldsymbol{\omega}\cdot\boldsymbol{r})^2$$

$$\rightarrow\frac{[\boldsymbol{r}\times(\boldsymbol{\omega}\times\boldsymbol{r})]\cdot\boldsymbol{\omega}}{\left\|[\boldsymbol{r}\times(\boldsymbol{\omega}\times\boldsymbol{r})]\right\|\omega}=\cos\alpha=\frac{\omega^2r^2-(\boldsymbol{\omega}\cdot\boldsymbol{r})^2}{\left|\boldsymbol{\omega}r^2-\boldsymbol{r}(\boldsymbol{\omega}\cdot\boldsymbol{r})\right|\omega}$$

$$\left.\begin{array}{l}\boldsymbol{\omega}\cdot\boldsymbol{r}=\dfrac{\omega d}{2}\\ r^2=l^2+\dfrac{d^2}{4}\end{array}\right\}\rightarrow\left|\boldsymbol{\omega}r^2-\boldsymbol{r}(\boldsymbol{\omega}\cdot\boldsymbol{r})\right|^2=\left[\omega r^2-\frac{d}{2}\omega\frac{d}{2}\right]^2+\left(l\omega\frac{d}{2}\right)^2$$

$$\rightarrow\left|\boldsymbol{\omega}r^2-\boldsymbol{r}(\boldsymbol{\omega}\cdot\boldsymbol{r})\right|^2=\left[\omega\left(l^2+\frac{d^2}{4}\right)-\omega\frac{d^2}{4}\right]^2+\left(\omega\frac{d}{2}l\right)^2=\omega^2l^4+\omega^2l^2\frac{d^2}{4}$$

$$\rightarrow\left|\boldsymbol{\omega}r^2-\boldsymbol{r}(\boldsymbol{\omega}\cdot\boldsymbol{r})\right|=\omega l\sqrt{l^2+\frac{d^2}{4}}$$

$$\rightarrow\cos\alpha=\frac{\omega^2r^2-(\boldsymbol{\omega}\cdot\boldsymbol{r})^2}{\left|\boldsymbol{\omega}r^2-\boldsymbol{r}(\boldsymbol{\omega}\cdot\boldsymbol{r})\right|\omega}=\frac{\omega^2\left(l^2+\frac{d^2}{4}\right)-\omega^2\frac{d^2}{4}}{\omega^2l\sqrt{l^2+\frac{d^2}{4}}}=\frac{\omega^2l^2}{\omega^2l\sqrt{l^2+\frac{d^2}{4}}}$$

$$\rightarrow\cos\alpha=\frac{.24}{\sqrt{l^2+\frac{d^2}{4}}}\approx\frac{.24}{\sqrt{.24^2+\frac{.42^2}{4}}}\approx\frac{.24}{.319}\approx.753$$

$$\rightarrow\alpha\approx41.2^0$$