Periodo del pendolo: calcolo della prima correzione all'approssimazione di piccoli angoli

Come si e' visto, la conservazione dell'energia meccanica consente di ricondurre il problema di trovare l'eq. oraria del pendolo al calcolo di un integrale:

$$E = mgh + \frac{1}{2}mv^2$$

$$mgl\left(1-\cos\theta\right) + \frac{1}{2}ml^2\left(\frac{d\theta}{dt}\right)^2 = mgl\left(1-\cos\theta_0\right)$$

$$\rightarrow g\left(\cos\theta_0 - \cos\theta\right) = -\frac{1}{2}l\left(\frac{d\theta}{dt}\right)^2$$

$$\rightarrow \frac{d\theta}{dt} = \sqrt{\frac{2g}{l}\left(\cos\theta - \cos\theta_0\right)}$$

$$\rightarrow \frac{d\theta}{\sqrt{\frac{2g}{l}\left(\cos\theta - \cos\theta_0\right)}} = dt$$

$$\rightarrow \frac{d\theta}{\sqrt{\left(\cos\theta - \cos\theta_0\right)}} = \sqrt{\frac{2g}{l}}dt$$

$$\rightarrow \int_{\theta_0}^{\theta} \frac{d\theta}{\sqrt{\left(\cos\theta - \cos\theta_0\right)}} = \sqrt{\frac{2g}{l}}t$$

$$\rightarrow \int_{\theta_0}^{\theta} \frac{d\theta}{\sqrt{-\cos\theta_0 + \cos\theta}} = \sqrt{\frac{2g}{l}}t$$

$$\rightarrow \int_{\theta_0}^{\theta} \frac{d\theta}{\sqrt{\left(1-\cos\theta_0\right) - \left(1-\cos\theta\right)}} = \sqrt{\frac{2g}{l}}t$$

$$1-\cos\theta_0 = 2\sin^2\frac{\theta_0}{2}, 1-\cos\theta = 2\sin^2\frac{\theta}{2}$$

$$\to \int_{\theta_0}^{\theta} \frac{d\theta}{\sqrt{\sin^2\frac{\theta_0}{2}-\sin^2\frac{\theta}{2}}} = 2\sqrt{\frac{g}{l}}t$$

$a=\sin\frac{\theta_0}{2}$ definizione

$$\to \int_{\theta_0}^{\theta} \frac{d\theta}{\sqrt{a^2-\sin^2\frac{\theta}{2}}} = \int_{\theta_0}^{\theta} \frac{d\theta}{\sqrt{1-\frac{1}{a^2}\sin^2\frac{\theta}{2}}} = 2a\sqrt{\frac{g}{l}}t$$

Sostituzione:

$$a\sin\varphi = \sin\frac{\theta}{2} \to \theta = 2\arcsin\left(a\sin\varphi\right) \to d\theta = \frac{2a\cos\varphi d\varphi}{\sqrt{1-a^2\sin^2\varphi}}$$

$$\to \int_{\varphi(\theta_0)}^{\varphi} \frac{2a\cancel{\cos\varphi}d\varphi}{\cancel{\cos\varphi}\sqrt{1-a^2\sin^2\varphi}} = 2a\sqrt{\frac{g}{l}}t$$

$$\to \int_{\varphi(\theta_0)}^{\varphi} \frac{d\varphi}{\sqrt{1-a^2\sin^2\varphi}} = \sqrt{\frac{g}{l}}t$$

Questa e’ la forma standard dell’integrale ellittico (incompleto) di I tipo.

Sviluppando la radice e la frazione in serie di Taylor:

$$\sqrt{1-a^2\sin^2\varphi} = 1-\frac{1}{2}a^2\sin^2\varphi+... \to \frac{1}{\sqrt{1-a^2\sin^2\varphi}} \approx 1+\frac{1}{2}a^2\sin^2\varphi+...$$

$$\to \int_{\varphi(\theta_0)}^{\varphi} \frac{d\varphi}{\sqrt{1-a^2\sin^2\varphi}} = \int_{\varphi(\theta_0)}^{\varphi} \left(1+\frac{1}{2}a^2\sin^2\varphi+...\right)d\varphi$$

Integrando la serie termine a termine fra 0 e $\pi/2$ si trova $1/4$ di periodo:

$$\int_0^{\pi/2} \left(1+\frac{1}{2}a^2\sin^2\varphi+...\right)d\varphi \simeq \varphi+\frac{1}{8}a^2\left(2\varphi-\sin 2\varphi\right)\Big|_0^{\pi/2} = \frac{\pi}{2}\left(1+\frac{a^2}{4}\right)$$

$$\to 2\pi\left(1+\frac{a^2}{4}\right) \simeq \sqrt{\frac{g}{l}}T$$

$$\to T \simeq 2\pi\sqrt{\frac{l}{g}}\left(1+\frac{a^2}{4}\right) = 2\pi\sqrt{\frac{l}{g}}\left(1+\frac{1}{4}\sin^2\frac{\theta_0}{2}\right) \underset{\theta_0\ll 1}{\approx} 2\pi\sqrt{\frac{l}{g}}\left(1+\frac{1}{16}\theta_0^2\right)$$

$\to$ Perdita dell' isocronismo: Il periodo dipende dall'ampiezza (anche se poco..)