Oscillatore armonico

$$\frac{d^2u}{dt^2}+\omega_0^2 u=0$$

Punto di partenza: Teorema di esistenza e unicita'
(per eq. diff. ordinarie: Se si trova una soluzione generale,quella e' l'unica:

Funzione di prova:

$$u(t)=Ae^{\lambda t}$$
$$\rightarrow \frac{du}{dt}=A\lambda e^{\lambda t}$$
$$\rightarrow \frac{d^2u}{dt^2}=A\lambda^2 e^{\lambda t}$$
$$\rightarrow A\lambda^2 e^{\lambda t}+\omega_0^2 Ae^{\lambda t}=0$$
$$\rightarrow \lambda^2+\omega_0^2=0\rightarrow \lambda=\pm i\omega_0$$
$$\rightarrow u_\pm(t)=A_\pm e^{\pm i\omega_0 t}$$
$$\rightarrow u(t)=A_+e^{+i\omega_0 t}+A_-e^{-i\omega_0 t}$$
$$\rightarrow v(t)=\frac{du}{dt}=i\omega_0 A_+e^{+i\omega_0 t}-i\omega_0 A_-e^{-i\omega_0 t}=i\omega_0\left(A_+e^{+i\omega_0 t}-A_-e^{-i\omega_0 t}\right)$$

$A_\pm$ costanti arbitrarie complesse (= 4 costanti reali)

Restrizione: $u(t),v(t)$ funzioni reali

$$\rightarrow \operatorname{Im}A_+\sin(\omega_0 t)=-\operatorname{Im}A_-\sin(\omega_0 t)\rightarrow \operatorname{Im}A_+=-\operatorname{Im}A_-$$
$$\rightarrow \operatorname{Re}A_+\cos(\omega_0 t)=\operatorname{Re}A_-\cos(\omega_0 t)\rightarrow \operatorname{Re}A_+=\operatorname{Re}A_-$$
$$\rightarrow A_+=A_-^*=A=C_0e^{i\varphi_0}$$
$$\rightarrow u(t)=Ae^{+i\omega_0 t}+A^*e^{-i\omega_0 t}=C_0\left(e^{i\varphi_0}e^{+i\omega_0 t}+e^{-i\varphi_0}e^{-i\omega_0 t}\right)=2C_0\cos(\omega_0 t+\varphi_0)$$
$$\rightarrow v(t)=i\omega_0\left(Ae^{+i\omega_0 t}-A^*e^{-i\omega_0 t}\right)=i\omega_0 C_0\left(e^{i\varphi_0}e^{+i\omega_0 t}-e^{-i\varphi_0}e^{-i\omega_0 t}\right)=-2\omega_0 C_0\sin(\omega_0 t+\varphi_0)$$

Cost. arbitrarie in termini delle cond. iniziali:

$$\left.\begin{array}{l} u(0)=a\rightarrow 2C_0\cos(\varphi_0)=a \\ v(0)=b\rightarrow -2\omega_0 C_0\sin(\varphi_0)=b \end{array}\right\}\rightarrow \tan(\varphi_0)=-\frac{b}{a\omega_0},C_0=\frac{1}{2}\sqrt{\frac{b^2}{\omega_0^2}+a^2}$$

Oscillatore armonico smorzato

Attrito viscoso:

$$F=-\beta v$$

Massa fissata a molla, in moto in un fluido viscoso:

$$F=-kx-\beta v$$

$$\to m\frac{d^2x}{dt^2}=-kx-\beta\frac{dx}{dt}$$

$$\to \frac{d^2x}{dt^2}+\frac{\beta}{m}\frac{dx}{dt}+\frac{k}{m}x=0$$

$$\to \frac{d^2x}{dt^2}+2\underbrace{\frac{\beta}{2m}}_{\gamma}\frac{dx}{dt}+\underbrace{\frac{k}{m}}_{\omega_0^2}x=0$$

Funzione di prova:

$$x(t)=Ae^{\lambda t}$$

$$\to \frac{dx}{dt}=A\lambda e^{\lambda t},\frac{d^2x}{dt^2}=A\lambda^2e^{\lambda t}$$

$$\frac{d^2x}{dt^2}+2\gamma\frac{dx}{dt}+\omega_0^2x=0\to A\lambda^2e^{\lambda t}+2\gamma A\lambda e^{\lambda t}+\omega_0^2Ae^{\lambda t}=0$$

$$\to \lambda^2+2\gamma\lambda+\omega_0^2=0$$

$$\to \lambda=-\gamma\pm\sqrt{\gamma^2-\omega_0^2}$$

$$\to x(t)=A_1e^{\left(-\gamma+\sqrt{\gamma^2-\omega_0^2}\right)t}+A_2e^{\left(-\gamma-\sqrt{\gamma^2-\omega_0^2}\right)t}$$ Integrale generale

$$\omega_0^2 > \gamma^2$$

$$\sqrt{\gamma^2 - \omega_0^2} = \pm i\sqrt{\omega_0^2 - \gamma^2}$$

$$\rightarrow x_{1,2}(t) = A_{1,2}e^{\left(-\gamma \pm i\sqrt{\omega_0^2 - \gamma^2}\right)t} = A_{1,2}e^{-\gamma t}e^{\pm i\sqrt{\omega_0^2 - \gamma^2}t}$$

$$\rightarrow x_{1,2}(t) = A_{1,2}e^{-\frac{\beta}{2m}t}\left[\cos\left(\sqrt{\omega_0^2 - \gamma^2 t}\right) \pm i\sin\left(\sqrt{\omega_0^2 - \gamma^2 t}\right)\right]$$

Comb. lineare:

$$x_1 + x_2 = e^{-\gamma t} \cdot$$

$$\left\{A_1\left[\cos\left(\sqrt{\omega_0^2 - \gamma^2 t}\right) + i\sin\left(\sqrt{\omega_0^2 - \gamma^2 t}\right)\right] + A_2\left[\cos\left(\sqrt{\omega_0^2 - \gamma^2 t}\right) - i\sin\left(\sqrt{\omega_0^2 - \gamma^2 t}\right)\right]\right\}$$

Imponendo che $x_1 + x_2$ sia reale:

$$\rightarrow x_1 + x_2 = e^{-\gamma t}\left(\underbrace{A_1 + A_2}_{=B,reale}\right)\cos\left(\sqrt{\omega_0^2 - \gamma^2 t}\right) + i\left(\underbrace{A_1 - A_2}_{=-iC,immaginario}\right)\sin\left(\sqrt{\omega_0^2 - \gamma^2 t}\right)$$

$$\rightarrow x(t) = e^{-\gamma t}\left[B\cos\left(\sqrt{\omega_0^2 - \gamma^2 t}\right) + C\sin\left(\sqrt{\omega_0^2 - \gamma^2 t}\right)\right]$$

B, C determinati da condizioni iniziali

Oscillazioni smorzate

$$\omega_0^2 < \gamma^2$$

$$\rightarrow x_{1,2}\left(t\right) = A_{1,2} e^{\left(-\gamma \pm \sqrt{\gamma^2 - \omega_0^2}\right)t}$$

Comb. lineare:

$$x_1 + x_2 = e^{-\gamma t} \left\{ A_1 e^{\sqrt{\gamma^2 - \omega_0^2}\, t} + A_2 e^{-\sqrt{\gamma^2 - \omega_0^2}\, t} \right\}$$

A_1, A_2 determinati da condizioni iniziali

No oscillazione; andamento smorzato

$\omega_0^2 = \gamma^2$

Caso intermedio fra i due: smorzamento critico

$$\frac{d^2x}{dt^2} + 2\gamma\frac{dx}{dt} + \omega_0^2 x = 0$$

$$\frac{d^2x}{dt^2} + \gamma\frac{dx}{dt} + \gamma\frac{dx}{dt} + \omega_0^2 x = 0$$

$$\frac{d^2x}{dt^2} + \gamma\frac{dx}{dt} = \frac{d}{dt}\left(\frac{dx}{dt} + \gamma x\right) \equiv \frac{dz}{dt}$$

$$\gamma\frac{dx}{dt} + \omega_0^2 x = \gamma\frac{dx}{dt} + \gamma^2 x = \gamma z$$

$$\rightarrow \frac{dz}{dt} + \gamma z = 0$$

$$\rightarrow z(t) = Ae^{-\gamma t}$$

$$\rightarrow \frac{dx}{dt} + \gamma x = Ae^{-\gamma t}$$

$$\rightarrow e^{\gamma t}\frac{dx}{dt} + e^{\gamma t}\gamma x = A$$

$$\rightarrow \frac{d}{dt}\left(e^{\gamma t}x\right) = A$$

$$\rightarrow e^{\gamma t}x = At + B$$

$$\rightarrow x(t) = Ate^{-\gamma t} + Be^{-\gamma t} = e^{-\gamma t}\left(At + B\right)$$

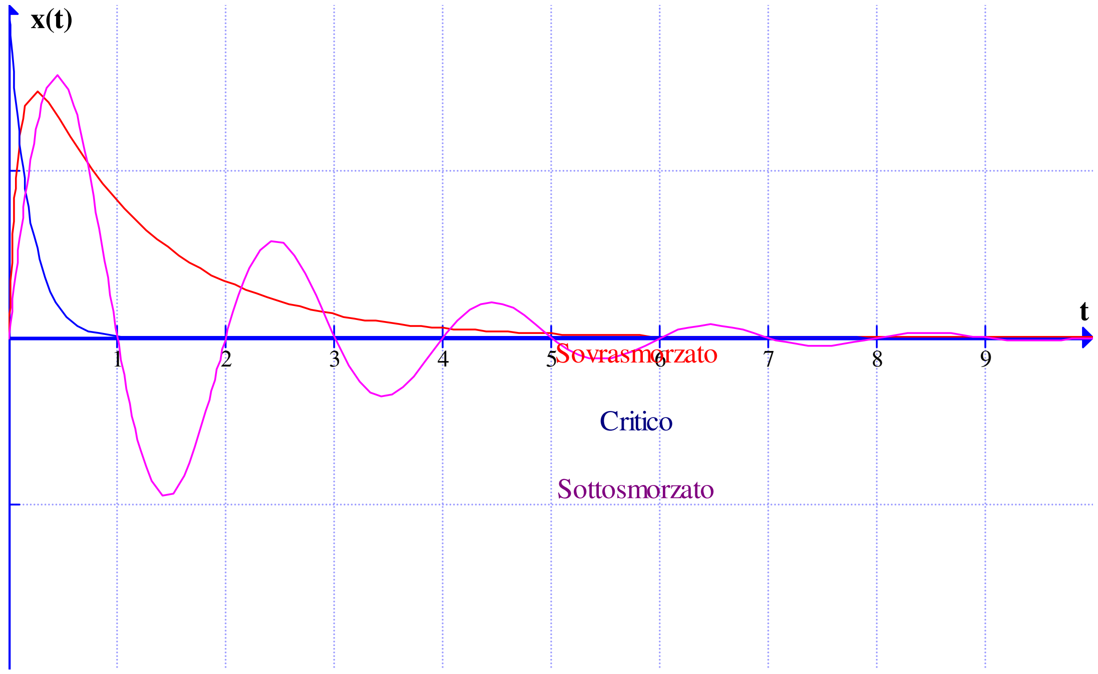
Smorzamento critico:

Minimo tempo di ritorno a zero

Come esempi:

Andamenti temporali nei 3 casi

(Condizioni iniziali arbitrarie e non specificate)



En. meccanica : Oscillazioni non smorzate

$$\frac{d^2x}{dt^2}+\omega_0^2x=0$$

$$\rightarrow\frac{dx}{dt}\frac{d^2x}{dt^2}+\omega_0^2x\frac{dx}{dt}=0$$

$$\rightarrow\frac{1}{2}\frac{d}{dt}\left(\frac{dx}{dt}\right)^2+\frac{1}{2}\frac{d}{dt}\left(\omega_0^2x^2\right)=0$$

$$\rightarrow\frac{d}{dt}\left(\underbrace{\frac{1}{2}mv^2}_{E_k}+\underbrace{\frac{1}{2}kx^2}_{U}\right)=0\rightarrow E=E_k+U=\text{cost}$$

$$x(t)=A\sin\omega_0t+B\cos\omega_0t$$

$$\langle U\rangle_{periodo}=\left\langle\frac{1}{2}kx^2\right\rangle=\frac{1}{2}k\left\langle\left(A\sin\omega_0t+B\cos\omega_0t\right)^2\right\rangle$$

$$\langle E_k\rangle_{periodo}=\left\langle\frac{1}{2}mv^2\right\rangle=\frac{1}{2}\underbrace{m\omega_0^2}_{k}\left\langle\left(A\cos\omega_0t-B\sin\omega_0t\right)^2\right\rangle$$

$$\left\langle\cos^2\omega_0t\right\rangle=\left\langle\sin^2\omega_0t\right\rangle=\frac{1}{2},\left\langle\sin\omega_0t\cos\omega_0t\right\rangle=0$$

$$\rightarrow\langle U\rangle_{periodo}=\frac{1}{2}k\left(\frac{1}{2}A^2+\frac{1}{2}B^2\right)=\frac{1}{4}k\left(A^2+B^2\right)=\langle E_k\rangle_{periodo}$$

$$E=E_k+U=\frac{1}{2}k\left(A^2+B^2\right)$$

En. meccanica : Oscillazioni smorzate

$$\frac{d^2x}{dt^2}+\frac{\beta}{m}\frac{dx}{dt}+\frac{k}{m}x=0$$

$$\frac{dx}{dt}\frac{d^2x}{dt^2}+\frac{\beta}{m}\left(\frac{dx}{dt}\right)^2+\frac{k}{m}\frac{dx}{dt}x=0$$

$$\frac{1}{2}\frac{d}{dt}\left(\frac{dx}{dt}\right)^2+\frac{1}{2}\frac{k}{m}\frac{d}{dt}\left(x^2\right)=-\frac{\beta}{m}\left(\frac{dx}{dt}\right)^2$$

$$\frac{d}{dt}\left[\frac{1}{2}m\left(\frac{dx}{dt}\right)^2\right]+\frac{d}{dt}\left(\frac{1}{2}kx^2\right)=-\beta\left(\frac{dx}{dt}\right)^2$$

$$\rightarrow\frac{dE}{dt}=-\beta\left(\frac{dx}{dt}\right)^2<0$$

$$\rightarrow\frac{dE}{dt}=-\frac{\beta}{m}m\left(\frac{dx}{dt}\right)^2<0$$

v smorzata esponenzialmente

$\rightarrow E$ tende a 0

Eccitazione esterna dell'oscillatore: termine forzante

Es: sinusoidale con frequenza angolare Ω

$$\frac{d^2x}{dt^2}+2\frac{\beta}{2m}\frac{dx}{dt}+\frac{k}{m}x=\frac{F(t)}{m}=\frac{F_0}{m}\cos\Omega t$$

$$\rightarrow\frac{d^2x}{dt^2}+2\gamma\frac{dx}{dt}+\omega_0^2x=\frac{F(t)}{m}=\frac{F_0}{m}\cos\Omega t$$

Formalismo esponenziali complessi:

variabile ausiliaria 'posizione complessa': $\tilde{x}(t)=\alpha e^{i\Omega t}$

$$\rightarrow x(t)=\mathrm{Re}\,\tilde{x}(t)=\mathrm{Re}\left(\alpha e^{i\Omega t}\right)$$

variabile ausiliaria 'forza complessa': $\tilde{F}(t)=F_0e^{i\Omega t}$

$$\rightarrow F(t)=\mathrm{Re}\,\tilde{F}(t)$$

$$\rightarrow\frac{d^2\tilde{x}}{dt^2}+2\gamma\frac{d\tilde{x}}{dt}+\omega_0^2\tilde{x}=\frac{\tilde{F}}{m}$$

$$\rightarrow\alpha\left(-\Omega^2\right)e^{i\Omega t}+2i\alpha\gamma\Omega e^{i\Omega t}+\alpha\omega_0^2e^{i\Omega t}=\frac{F_0}{m}e^{i\Omega t}$$

$$\rightarrow\alpha\left[-\Omega^2+2i\gamma\Omega+\omega_0^2\right]=\frac{F_0}{m}$$

$$\rightarrow\alpha=\frac{F_0}{m}\frac{1}{\left(\omega_0^2-\Omega^2\right)+2i\gamma\Omega}$$ funzione complessa di Ω

$$x(t) = \mathrm{Re}\,\tilde{x}(t) = \mathrm{Re}\left(\alpha e^{i\Omega t}\right)$$

$$\alpha \equiv |\alpha| e^{i\phi}$$

$$\rightarrow x(t) = \mathrm{Re}\left(\alpha e^{i\Omega t}\right) = \mathrm{Re}\left(|\alpha| e^{i(\Omega t+\phi)}\right) = |\alpha|\cos(\Omega t + \phi)$$

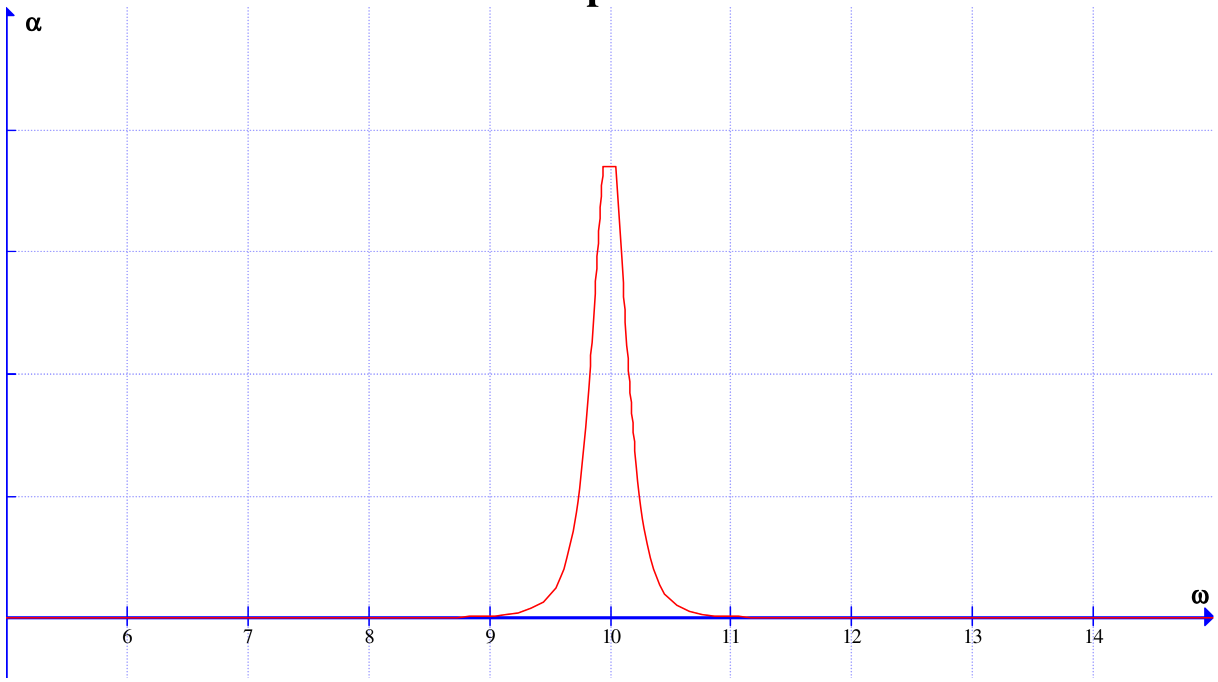
$$\rightarrow \alpha = \frac{F_0}{m}\frac{1}{\left(\omega_0^2 - \Omega^2\right) + 2i\gamma\Omega} = F_0\,\frac{\left(\omega_0^2 - \Omega^2\right) - 2i\gamma\Omega}{\left(\omega_0^2 - \Omega^2\right)^2 - \left(2i\gamma\Omega\right)^2}$$

$$|\alpha|^2 = \frac{F_0^2}{m^2}\frac{\left[\omega_0^2 - \Omega^2\right]^2 + \left(2\lambda\Omega\right)^2}{\left[\left(\omega_0^2 - \Omega^2\right)^2 + 4\lambda^2\Omega^2\right]^2} = \frac{F_0^2}{\left(\omega_0^2 - \Omega^2\right)^2 + 4\lambda^2\Omega^2}$$

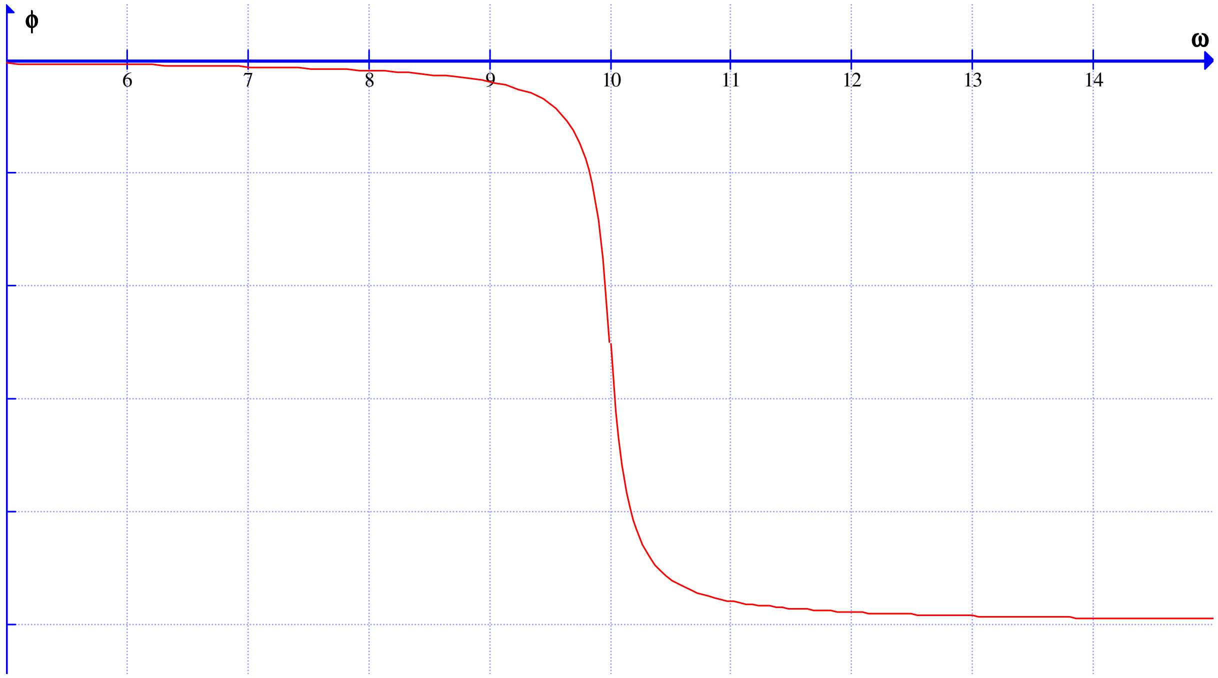
$$\left.\begin{array}{l} |\alpha| = \dfrac{F_0}{m\sqrt{\left(\omega_0^2 - \Omega^2\right)^2 + 4\lambda^2\Omega^2}} \quad \text{modulo} \\[3ex] \phi = \arctan\dfrac{2\lambda\Omega}{\Omega^2 - \omega_0^2} \quad \text{fase} \end{array}\right\} \text{funzioni di } \Omega$$

Max $|\alpha| : \omega_0 = \Omega$

$$|\alpha|_{\max} = \frac{F_0}{\lambda m \omega_0}, \;\; \phi_{\max} = -\frac{\pi}{2}$$



Ampiezza



Fase