Momento angolare:

$$\mathbf{L} = \sum_{i=1}^{N} \mathbf{L}_i = \sum_{i=1}^{N} \mathbf{r}_i \times \mathbf{p}_i , \mathbf{r}_i \text{ riferiti all'origine}$$

$$\frac{d\mathbf{L}_i}{dt} = \mathbf{r}_i \times \mathbf{F}_i^{(ext)} + \mathbf{r}_i \times \mathbf{F}_i^{(int)} = \mathbf{r}_i \times \mathbf{F}_i^{(ext)} + \mathbf{r}_i \times \sum_{\substack{j=1, \\ j \neq i}}^{N} \mathbf{F}_i^{(j)}$$

Sommando su tutti i punti:

$$\frac{d\mathbf{L}}{dt} = \mathbf{M}^{(ext)} + \sum_{i=1}^{N} \sum_{\substack{j=1, \\ j \neq i}}^{N} \mathbf{r}_i \times \mathbf{F}_i^{(j)}$$

$$\mathbf{r}_i \times \mathbf{F}_i^{(j)} + \mathbf{r}_j \times \mathbf{F}_j^{(i)} = \mathbf{r}_i \times \mathbf{F}_i^{(j)} - \mathbf{r}_j \times \mathbf{F}_i^{(j)} = \left(\mathbf{r}_i - \mathbf{r}_j\right) \times \mathbf{F}_i^{(j)}$$

$$\mathbf{F}_i^{(j)} \parallel \left(\mathbf{r}_i - \mathbf{r}_j\right) \leftarrow \text{Stessa linea d'azione}$$

$$\rightarrow \sum_{i=1}^{N} \sum_{\substack{j=1, \\ j \neq i}}^{N} \mathbf{r}_i \times \mathbf{F}_i^{(j)} = 0$$

$$\rightarrow \frac{d\mathbf{L}}{dt} = \mathbf{M}^{(ext)}$$

II equazione cardinale della dinamica dei sistemi

$$\mathbf{M}^{(ext)} = 0 \rightarrow \mathbf{L} = \text{cost}$$

$\rightarrow$ Conservazione del mom. angolare totale

per i sistemi isolati

II eq. cardinale scritta in varie forme equivalenti:

Differenze dipendenti da scelta del polo

Polo nell'origine: scelta particolare, non obbligatoria

Momenti riferiti a polo generico P, anche in movimento:

$$\mathbf{r}_i' = \mathbf{r}_i - \mathbf{r}_P, \mathbf{v}_i' = \mathbf{v}_i - \mathbf{v}_P$$

$$\rightarrow \mathbf{L}_i' = \mathbf{r}_i' \times \mathbf{p}_i' = (\mathbf{r}_i - \mathbf{r}_P) \times m_i (\mathbf{v}_i - \mathbf{v}_P)$$

$$\rightarrow \mathbf{L}_i' = \mathbf{L}_i - \mathbf{r}_P \times m_i \mathbf{v}_i - \mathbf{r}_i \times m_i \mathbf{v}_P + \mathbf{r}_P \times m_i \mathbf{v}_P$$

Sommando su tutti i punti:

$$\sum_{i=1,N} \mathbf{L}_i' = \sum_{i=1,N} \mathbf{L}_i - \mathbf{r}_P \times \sum_{i=1,N} m_i \mathbf{v}_i$$

$$- \sum_{i=1,N} \mathbf{r}_i \times m_i \mathbf{v}_P + \sum_{i=1,N} \mathbf{r}_P \times m_i \mathbf{v}_P$$

$$\rightarrow \mathbf{L}' = \mathbf{L} - \mathbf{r}_P \times \mathbf{P} - \mathbf{r}_{CM} \times M \mathbf{v}_P + \mathbf{r}_P \times M \mathbf{v}_P$$

$$\rightarrow \mathbf{L}' = \mathbf{L} - \mathbf{r}_P \times \mathbf{P} + (\mathbf{r}_P - \mathbf{r}_{CM}) \times M \mathbf{v}_P$$

Se $P = fisso$:

$$\mathbf{v}_P = 0 \rightarrow \mathbf{L}' = \mathbf{L} - \mathbf{r}_P \times \mathbf{P}$$

$$\rightarrow \mathbf{L} = \mathbf{L}' + \mathbf{r}_P \times \mathbf{P} = \mathbf{L}' + \mathbf{L}^{(P)}$$

Se $P = CM$:

$$\mathbf{v}_P = \mathbf{v}_{CM} \rightarrow \mathbf{L}' = \mathbf{L} - \mathbf{r}_{CM} \times \mathbf{P}$$

$$\rightarrow \mathbf{L} = \mathbf{L}' + \mathbf{r}_{CM} \times \mathbf{P} = \mathbf{L}' + \mathbf{L}_{CM}$$

Mom. angolare totale :

somma di quello dovuto al moto del CM e di quello dovuto al moto del sistema rispetto al CM

$$\frac{d\mathbf{L}}{dt} = \frac{d\mathbf{L}_{CM}}{dt} + \frac{d\mathbf{L}'}{dt} = \mathbf{M}^{(ext)}$$

Nel riferimento del CM:

$\mathbf{L}_{CM} = 0$

$\rightarrow \mathbf{M}^{(ext)}\,' = \dfrac{d\mathbf{L}'}{dt}$

II Eq. cardinale nel riferimento del CM

$\rightarrow \mathbf{M}^{(ext)}\,' = 0 \rightarrow \mathbf{L}' =$ cost

Conservazione del mom. angolare per sistemi isolati nel riferimento del CM

Esempi:

Sistema solare

Atomo

Complessivamente:

$$\mathbf{F}^{(ext)} = \frac{d\mathbf{P}}{dt} = M\mathbf{a}_{CM}$$

$$\mathbf{M}^{(ext)} = \frac{d\mathbf{L}}{dt}$$ mom. rispetto a riferimento inerziale

oppure

$$\mathbf{M}^{(ext)}{}' = \frac{d\mathbf{L}'}{dt}$$ mom. rispetto a CM

Equazioni cardinali della dinamica dei sistemi

Primo teorema di Konig:

$$\mathbf{L} = \sum_{i=1,N} \mathbf{r}_i \times \mathbf{p}_i$$

$$\rightarrow \mathbf{L} = \mathbf{r}_{CM} \times M\mathbf{v}_{CM} + \mathbf{L}' = \mathbf{L}_{CM} + \mathbf{L}'$$

Scomposizione del mom. angolare totale

Secondo teorema di Konig :

$$E_k = \frac{1}{2} \sum_{i=1,N} m_i v_i^2$$

Teorema del lavoro e dell' energia cinetica:

$$m_i \underbrace{\frac{d\mathbf{v}_i}{dt} \cdot d\mathbf{r}_i}_{d\mathbf{v}_i \cdot \frac{d\mathbf{r}_i}{dt} = \mathbf{v}_i \cdot d\mathbf{v}_i} = \mathbf{F}_i^{(ext)} \cdot d\mathbf{r}_i + \mathbf{F}_i^{(int)} \cdot d\mathbf{r}_i = dW_i^{(ext)} + dW_i^{(int)}$$

$$\rightarrow d\left(\frac{1}{2} m_i v_i^2\right) = dW_i^{(ext)} + dW_i^{(int)} \rightarrow W_i^{(ext)} + W_i^{(int)} = \Delta E_k^i$$

Sommando su tutti i punti:

$W^{(ext)} + W^{(int)} = \Delta E_k$ ($\leftarrow$ III principio non annulla il lavoro interno)

$$E_k = \frac{1}{2} \sum_{i=1,N} m_i v_i^2 = \frac{1}{2} \sum_{i=1,N} m_i \left(\mathbf{v}_{CM} + \mathbf{v}_i'\right)^2$$

$$E_k = \frac{1}{2} \sum_{i=1,N} m_i \mathbf{v}_{CM}^2 + \frac{1}{2} \sum_{i=1,N} m_i \mathbf{v}'^2_i + \underbrace{\sum_{i=1,N} m_i \mathbf{v}_i'}_{=0} \cdot \mathbf{v}_{CM}$$

$$\rightarrow E_k = \frac{1}{2} M\mathbf{v}_{CM}^2 + \frac{1}{2} \sum_{i=1,N} m_i \mathbf{v}'^2_i$$

Scomposizione en. cinetica totale

Applicazione I teorema di Konig

Mom. angolare di una coppia di punti rispetto al loro CM

$$\mathbf{r}_{CM} = \frac{m_1\mathbf{r}_1 + m_2\mathbf{r}_2}{m_1 + m_2}$$

$$\rightarrow \begin{cases} \mathbf{r}_1' = \mathbf{r}_1 - \mathbf{r}_{CM} = \mathbf{r}_1 - \dfrac{m_1\mathbf{r}_1 + m_2\mathbf{r}_2}{m_1 + m_2} = \dfrac{m_2\left(\mathbf{r}_1 - \mathbf{r}_2\right)}{m_1 + m_2} = \dfrac{m_2}{m_1 + m_2}\mathbf{r}_{12} \\ \mathbf{r}_2' = \mathbf{r}_2 - \mathbf{r}_{CM} = \dfrac{m_1\left(\mathbf{r}_2 - \mathbf{r}_1\right)}{m_1 + m_2} = -\dfrac{m_1\left(\mathbf{r}_1 - \mathbf{r}_2\right)}{m_1 + m_2} = -\dfrac{m_1}{m_1 + m_2}\mathbf{r}_{12} \end{cases}$$

$$\rightarrow \begin{cases} \mathbf{v}_1' = \mathbf{v}_1 - \mathbf{v}_{CM} = \dfrac{m_2}{m_1 + m_2}\mathbf{v}_{12} \rightarrow \mathbf{p}_1' = \dfrac{m_1 m_2}{m_1 + m_2}\mathbf{v}_{12} = \mu_{12}\mathbf{v}_{12} \\ \mathbf{v}_2' = \mathbf{v}_2 - \mathbf{v}_{CM} = -\dfrac{m_1}{m_1 + m_2}\mathbf{v}_{12} \rightarrow \mathbf{p}_2' = -\dfrac{m_1 m_2}{m_1 + m_2}\mathbf{v}_{12} = -\mu_{12}\mathbf{v}_{12} = -\mathbf{p}_1' \end{cases}$$

$\mu_{12} = \dfrac{m_1 m_2}{m_1 + m_2}$ massa ridotta di 1 e 2

$$\rightarrow \mathbf{L}' = \mathbf{r}_1' \times \mathbf{p}_1' + \mathbf{r}_2' \times \mathbf{p}_2' = \frac{m_2}{m_1 + m_2}\mathbf{r}_{12} \times \mu_{12}\mathbf{v}_{12} + \left(-\frac{m_1}{m_1 + m_2}\mathbf{r}_{12}\right) \times \left(-\mu_{12}\mathbf{v}_{12}\right)$$

$$\rightarrow \mathbf{L}' = \mathbf{r}_{12} \times \mu_{12}\mathbf{v}_{12}\left(\frac{m_2}{m_1 + m_2} + \frac{m_1}{m_1 + m_2}\right) = \mathbf{r}_{12} \times \mu_{12}\mathbf{v}_{12}$$

Esempi di massa ridotta

Neutrone-Protone $\mu_{np} = \dfrac{m_n m_p}{m_n + m_p} = \dfrac{1.675\ 10^{-27} 1.672\ 10^{-27}}{1.675\ 10^{-27} + 1.672\ 10^{-27}} \approx \dfrac{m_n}{2} \approx \dfrac{m_p}{2}$

Elettrone-Protone $\mu_{ep} = \dfrac{m_e m_p}{m_e + m_p} = \dfrac{9.11\ 10^{-31} 1.67\ 10^{-27}}{9.11\ 10^{-31} + 1.67\ 10^{-27}} \approx m_e$

Applicazione II teorema di Konig

En. cinetica di due particelle rispetto al CM

$$E_k' = \frac{1}{2} m_1 v_1'^2 + \frac{1}{2} m_2 v_2'^2$$

$$\begin{cases} \mathbf{v}_1' = \mathbf{v}_1 - \mathbf{v}_{CM} = \dfrac{m_2}{m_1 + m_2} \mathbf{v}_{12} \\ \mathbf{v}_2' = \mathbf{v}_2 - \mathbf{v}_{CM} = -\dfrac{m_1}{m_1 + m_2} \mathbf{v}_{12} \end{cases}$$

$$\rightarrow E_k' = \frac{1}{2} m_1 \left(\frac{m_2}{m_1 + m_2} \right)^2 v_{12}^2 + \frac{1}{2} m_2 \left(\frac{m_1}{m_1 + m_2} \right)^2 v_{12}^2$$

$$\rightarrow E_k' = \frac{1}{2} \left[m_1 \left(\frac{m_2}{m_1 + m_2} \right)^2 + m_2 \left(\frac{m_1}{m_1 + m_2} \right)^2 \right] v_{12}^2$$

$$\rightarrow E_k' = \frac{1}{2} \left[\frac{m_1 m_2^2}{(m_1 + m_2)^2} + \frac{m_2 m_1^2}{(m_1 + m_2)^2} \right] v_{12}^2 = \frac{1}{2} \left[\frac{m_1 m_2 (m_1 + m_2)}{(m_1 + m_2)^2} \right] v_{12}^2$$

$$\rightarrow E_k' = \frac{1}{2} \frac{m_1 m_2}{(m_1 + m_2)} v_{12}^2 = \frac{1}{2} \mu_{12} v_{12}^2$$