Moto in 3D: vettore posizione

$$\boldsymbol{r} = \boldsymbol{r}(t) \leftrightarrow \begin{cases} x = x(t) \\ y = y(t) \\ z = z(t) \end{cases}$$

Eq. vettoriale equivalente a 3 eq. scalari:

$$\begin{cases} x(t) = \boldsymbol{r}(t) \cdot \hat{\boldsymbol{i}} \\ y(t) = \boldsymbol{r}(t) \cdot \hat{\boldsymbol{j}} \\ z(t) = \boldsymbol{r}(t) \cdot \hat{\boldsymbol{k}} \end{cases}$$ proiezioni = componenti

Vel. istantanea (vettoriale):

$$\boldsymbol{v}(t) = \lim_{\Delta t \to 0} \frac{\boldsymbol{r}(t + \Delta t) - \boldsymbol{r}(t)}{\Delta t} = \frac{d\boldsymbol{r}}{dt} \leftrightarrow \begin{cases} \dfrac{dx}{dt} \\ \dfrac{dy}{dt} \\ \dfrac{dz}{dt} \end{cases}$$

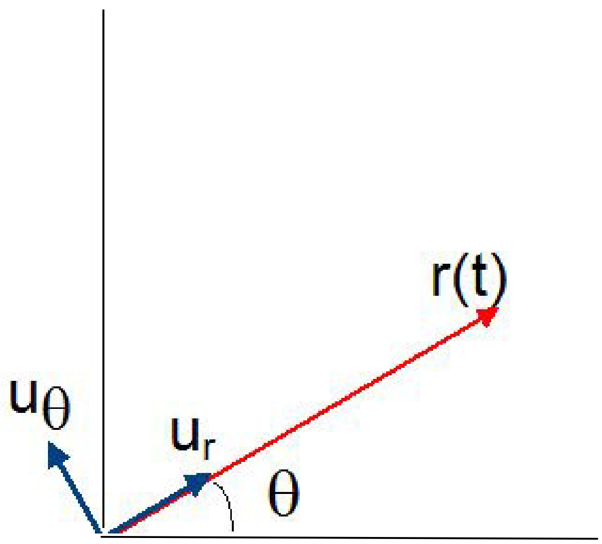
Acc. istantanea (vettoriale):

$$\boldsymbol{a}(t) = \lim_{\Delta t \to 0} \frac{\boldsymbol{v}(t + \Delta t) - \boldsymbol{v}(t)}{\Delta t} = \frac{d\boldsymbol{v}}{dt} \leftrightarrow \begin{cases} \dfrac{dv_x}{dt} = \dfrac{d^2 x}{dt^2} \\ \dfrac{dv_y}{dt} = \dfrac{d^2 y}{dt^2} \\ \dfrac{dv_z}{dt} = \dfrac{d^2 z}{dt^2} \end{cases}$$

Restrizione a moti piani (quasi sempre, ma non sempre lo sono): 2D

Moto piano in coordinate polari

(←utili quando c'e' un centro di simmetria):



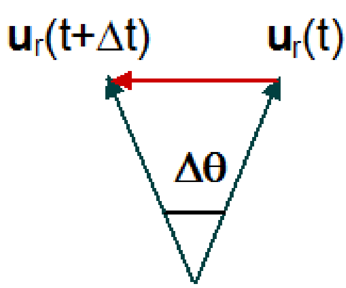
Velocita'

$$\boldsymbol{r} = r\hat{\boldsymbol{u}}_r$$

$$\rightarrow \frac{d\boldsymbol{r}}{dt} = \frac{d\left(r\hat{\boldsymbol{u}}_r\right)}{dt} = \frac{dr}{dt}\hat{\boldsymbol{u}}_r + r\frac{d\hat{\boldsymbol{u}}_r}{dt}$$

$$\frac{d\hat{\boldsymbol{u}}_r}{dt} = ?$$

$$\left|\hat{\boldsymbol{u}}_r\left(t+\Delta t\right) - \hat{\boldsymbol{u}}_r\left(t\right)\right| \approx \left|\hat{\boldsymbol{u}}_r\left(t\right)\right|\Delta\theta$$



$$\hat{\boldsymbol{u}}_r(t+\Delta t)-\hat{\boldsymbol{u}}_r(t)\approx \Delta\theta\hat{\boldsymbol{u}}_\theta, \hat{\boldsymbol{u}}_\theta \perp \hat{\boldsymbol{u}}_r$$

$$\rightarrow \frac{d\hat{\boldsymbol{u}}_r}{dt}=\frac{d\theta}{dt}\hat{\boldsymbol{u}}_\theta$$

Un po' piu' formalmente:

$$\left|\hat{\boldsymbol{u}}_r\right|=1\rightarrow \hat{\boldsymbol{u}}_r\cdot\hat{\boldsymbol{u}}_r=1\rightarrow \frac{d}{dt}\left(\hat{\boldsymbol{u}}_r\cdot\hat{\boldsymbol{u}}_r\right)=0=\frac{d\hat{\boldsymbol{u}}_r}{dt}\cdot\hat{\boldsymbol{u}}_r+\hat{\boldsymbol{u}}_r\cdot\frac{d\hat{\boldsymbol{u}}_r}{dt}$$

$$\rightarrow \hat{\boldsymbol{u}}_r\cdot\frac{d\hat{\boldsymbol{u}}_r}{dt}=0\rightarrow \hat{\boldsymbol{u}}_r\perp\frac{d\hat{\boldsymbol{u}}_r}{dt}\rightarrow\frac{d\hat{\boldsymbol{u}}_r}{dt}\parallel\hat{\boldsymbol{u}}_\theta$$

$$\rightarrow \frac{d\hat{\boldsymbol{u}}_r}{dt}=\frac{d\hat{\boldsymbol{u}}_r}{d\theta}\frac{d\theta}{dt}=\hat{\boldsymbol{u}}_\theta\frac{d\theta}{dt}$$

Infatti:

$$\frac{d\hat{\boldsymbol{u}}_r}{d\theta}=\lim_{\Delta\theta\to 0}\frac{\Delta\hat{\boldsymbol{u}}_r}{\Delta\theta}=\hat{\boldsymbol{u}}_\theta\lim_{\Delta\theta\to 0}\frac{\left|\Delta\hat{\boldsymbol{u}}_r\right|}{\Delta\theta}=\hat{\boldsymbol{u}}_\theta\lim_{\Delta\theta\to 0}\frac{\Delta\theta}{\Delta\theta}=\hat{\boldsymbol{u}}_\theta$$

Quindi:

$$\boldsymbol{v}=\frac{d\boldsymbol{r}}{dt}=\frac{dr}{dt}\hat{\boldsymbol{u}}_r+r\frac{d\hat{\boldsymbol{u}}_r}{dt}$$

$$\rightarrow \boldsymbol{v}=\frac{dr}{dt}\hat{\boldsymbol{u}}_r+r\frac{d\theta}{dt}\hat{\boldsymbol{u}}_\theta$$

Vel. radiale, vel. trasversale

$$v=\frac{ds}{dt}=\sqrt{\left(\frac{dr}{dt}\right)^2+r^2\left(\frac{d\theta}{dt}\right)^2}$$

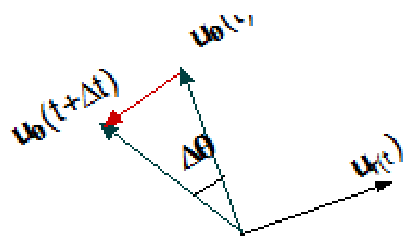
Accelerazione

$$\boldsymbol{v}=\frac{dr}{dt}\hat{\boldsymbol{u}}_r+r\frac{d\theta}{dt}\hat{\boldsymbol{u}}_\theta$$

$$\rightarrow \boldsymbol{a}=\frac{d\boldsymbol{v}}{dt}=\frac{d}{dt}\left(\frac{dr}{dt}\hat{\boldsymbol{u}}_r+r\frac{d\theta}{dt}\hat{\boldsymbol{u}}_\theta\right)$$

$$\boldsymbol{a}=\frac{d}{dt}\left(\frac{dr}{dt}\hat{\boldsymbol{u}}_r+r\frac{d\theta}{dt}\hat{\boldsymbol{u}}_\theta\right)$$

$$\rightarrow \begin{cases} \dfrac{d}{dt}\left(\dfrac{dr}{dt}\hat{\boldsymbol{u}}_r\right)=\dfrac{d^2r}{dt^2}\hat{\boldsymbol{u}}_r+\dfrac{dr}{dt}\dfrac{d\hat{\boldsymbol{u}}_r}{dt}=\dfrac{d^2r}{dt^2}\hat{\boldsymbol{u}}_r+\dfrac{dr}{dt}\dfrac{d\theta}{dt}\hat{\boldsymbol{u}}_\theta \\ \dfrac{d}{dt}\left(r\dfrac{d\theta}{dt}\hat{\boldsymbol{u}}_\theta\right)=\dfrac{dr}{dt}\dfrac{d\theta}{dt}\hat{\boldsymbol{u}}_\theta+r\dfrac{d^2\theta}{dt^2}\hat{\boldsymbol{u}}_\theta+r\dfrac{d\theta}{dt}\dfrac{d\hat{\boldsymbol{u}}_\theta}{dt} \end{cases}$$



Differenza≈–u_r Δθ

$$\hat{\boldsymbol{u}}_\theta \cdot \hat{\boldsymbol{u}}_\theta = 1 \rightarrow \ \hat{\boldsymbol{u}}_\theta \cdot \frac{d\hat{\boldsymbol{u}}_\theta}{dt} = 0 \rightarrow \frac{d\hat{\boldsymbol{u}}_\theta}{dt} \parallel \hat{\boldsymbol{u}}_r$$

$$\frac{d\hat{\boldsymbol{u}}_\theta}{dt} = -\frac{d\theta}{dt}\hat{\boldsymbol{u}}_r$$

$$\rightarrow \boldsymbol{a} = a_r\hat{\boldsymbol{u}}_r + a_\theta\hat{\boldsymbol{u}}_\theta$$

$$a_r = \frac{d^2 r}{dt^2} - r\left(\frac{d\theta}{dt}\right)^2$$ Acc. radiale

$$a_\theta = 2\frac{dr}{dt}\frac{d\theta}{dt} + r\frac{d^2\theta}{dt^2}$$ Acc. trasversa(le)

Componenti polari di velocita' e accelerazione