Caduta da grande altezza con accelerazione *~1/r²*: tempo di caduta

x_0, v_0: posizione e velocita' iniziali

$$\frac{dv}{dt} = -\frac{a}{x^2} \rightarrow \frac{dv}{dx}\frac{dx}{dt} = -\frac{a}{x^2} \rightarrow v\frac{dv}{dx} = -\frac{a}{x^2}$$

$$vdv = -a\frac{dx}{x^2} \rightarrow \frac{1}{2}v^2\Big|_{v_0^2}^{v^2} = \frac{a}{x}\Big|_{x_0}^{x}$$

$$\rightarrow \frac{1}{2}v^2 - \frac{1}{2}v_0^2 = \frac{a}{x} - \frac{a}{x_0}$$

$$\rightarrow v^2 = 2\left(\frac{a}{x} - \frac{a}{x_0}\right) + v_0^2 = 2a\frac{x_0 - x}{x_0 x} + v_0^2$$

$$\rightarrow \frac{dx}{dt} = \sqrt{2a\frac{x_0 - x}{x_0 x} + v_0^2} \rightarrow \frac{dx}{\sqrt{2a\frac{x_0 - x}{x_0 x} + v_0^2}} = dt$$

$$\rightarrow t = \int_{x_0}^{x} \frac{dx}{\sqrt{2a\frac{x_0 - x}{x_0 x} + v_0^2}} = \int_{x_0}^{x} \frac{dx}{\sqrt{\frac{2a(x_0 - x) + v_0^2 x_0 x}{x_0 x}}} = \int_{x_0}^{x} \sqrt{\frac{x_0 x}{2a(x_0 - x) + v_0^2 x_0 x}}\,dx$$

$$\rightarrow t = \int_{x_0}^{0} \sqrt{\frac{x_0 x}{2a(x_0 - x) + v_0^2 x_0 x}}\,dx = \int_{x_0}^{0} \sqrt{\frac{x_0 x}{2ax_0 + (v_0^2 x_0 - 2a)x}}\,dx$$

$$\int \sqrt{\frac{x_0 x}{(-2a + v_0^2 x_0)x + 2ax_0}}\,dx = ?$$

$$y = \sqrt{x} \rightarrow x = y^2 \rightarrow dx = 2ydy$$

$$\rightarrow \int \sqrt{\frac{x_0 x}{(-2a + v_0^2 x_0)x + 2ax_0}}\,dx = 2\sqrt{\frac{x_0}{-2a + v_0^2 x_0}} \int \frac{y^2}{\sqrt{y^2 + \frac{2ax_0}{-2a + v_0^2 x_0}}}\,dy$$

$$u = y \rightarrow du = dy$$

$$dv = \frac{ydy}{\sqrt{y^2 + \frac{2ax_0}{-2a + v_0^2 x_0}}} \rightarrow v = \sqrt{y^2 + \underbrace{\frac{2ax_0}{-2a + v_0^2 x_0}}_{c^2}}$$

$$\rightarrow \int \sqrt{\frac{x_0 x}{(-2a + v_0^2 x_0)x + 2ax_0}}\,dx = c\sqrt{\frac{2}{a}}\left[y\sqrt{y^2 + c^2} - \int \sqrt{y^2 + c^2}\,dy\right]$$

$$\rightarrow t = c\sqrt{\frac{2}{a}}\left[\sqrt{x}\sqrt{x + c^2} - \frac{1}{2}\int \sqrt{\frac{x + c^2}{x}}\,dx\right]_{x_0}^{x}$$

$v_0 = 0 \rightarrow c^2 = -x_0 \rightarrow c = \pm i\sqrt{x_0}$

$\rightarrow t = \sqrt{\frac{2x_0}{a}}\left[\sqrt{x}\sqrt{x_0 - x} - \frac{1}{2}x_0 \int \sqrt{\frac{1-u}{u}}du\right]_1^u$

$\int \sqrt{\frac{1-u}{u}}du = ?$

$u = \cos^2 y \rightarrow y = \arccos\left(\sqrt{u}\right), du = -2\sin y \cos y dy$

$\rightarrow \int \sqrt{\frac{1-u}{u}}du = \int -\frac{\sin y}{\cos y}2\sin y \cos y dy = -2\int \sin^2 y dy = -\left(y - \sin y \cos y\right)$

$\rightarrow \int \sqrt{\frac{1-u}{u}}du = -\left(y - \sin y \cos y\right) = -\left(\arccos\left(\sqrt{u}\right) - \sqrt{u}\sqrt{1-u}\right)$

$\rightarrow t = \sqrt{\frac{2x_0}{a}}\left[\sqrt{x}\sqrt{x_0 - x} + \frac{1}{2}x_0\left[\arccos\left(\sqrt{\frac{x}{x_0}}\right) - \sqrt{\frac{x}{x_0}}\sqrt{1 - \frac{x}{x_0}}\right]\right]$

$\rightarrow t = \sqrt{\frac{2x_0}{a}}x_0\left[\sqrt{\frac{x}{x_0}}\sqrt{1 - \frac{x}{x_0}} + \frac{1}{2}\left[\arccos\left(\sqrt{\frac{x}{x_0}}\right) - \sqrt{\frac{x}{x_0}}\sqrt{1 - \frac{x}{x_0}}\right]\right]$

$\rightarrow t\left(x\right) = \sqrt{\frac{1}{2a}}x_0^{3/2}\left[\sqrt{\frac{x}{x_0}}\sqrt{1 - \frac{x}{x_0}} + \arccos\left(\sqrt{\frac{x}{x_0}}\right)\right]$

$x\left(t\right)$ si puo' ottenere per inversione (esprimendolo come serie di potenze),
o numericamente
Tempo di caduta:

$\rightarrow t\left(0\right) = \sqrt{\frac{1}{2a}}x_0^{3/2}\left[\arccos 0\right] = \sqrt{\frac{1}{2a}}x_0^{3/2}\,\frac{\pi}{2}$

Per campo gravitazionale: $a = MG$
Caduta Luna sulla Terra:

$a = 610^{24} \quad 6.6710^{-11} \sim 40\ 10^{13} \rightarrow 2a = 8\ 10^{14} \rightarrow \sqrt{2a} = 2.83\ 10^7 \rightarrow \frac{1}{\sqrt{2a}} = 3.5\ 10^{-8}$

$x_0 = 3.84\ 10^8\ m \rightarrow x_0^{3/2} = 7.5\ 10^{12}$

$t\left(0\right) = 4{,}1\ 10^5 s$

Caduta Terra sul Sole:

$a = 2\ 10^{30}\ 6.67\ 10^{-11} \sim 13.3\ 10^{19} \rightarrow 2a = 2.6\ 10^{20} \rightarrow \sqrt{2a} = 1.61\ 10^{10} \rightarrow \frac{1}{\sqrt{2a}} = 0.62\ 10^{-10}$

$x_0 = 15\ 10^{10}\quad m \rightarrow x_0^{3/2} = 58.1\ 10^{15}$

$t\left(0\right) = 56.5\ 10^5 s$