Lorentz transformation with arbitrary line of motion

“Sometimes it becomes a matter of natural choice for an observer (A) that he prefers a coordinate system of two-dimensional spatial x–y coordinates from which he observes another observer (B) who is moving at a uniform speed along a line of motion, which is not collinear with A’s chosen x- or y-axis.

It becomes necessary in such cases to develop Lorentz transformations where the line of motion is not aligned with either the x- or the y-axis.”

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Introduction

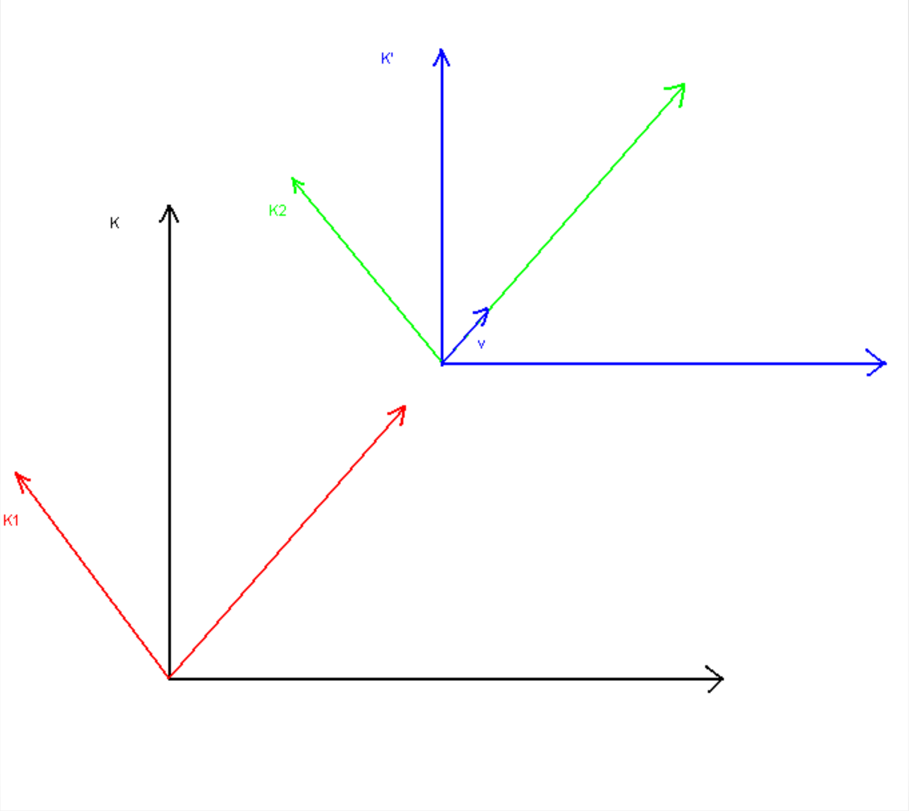
In most treatments on special relativity , the line of motion is aligned with the x-axis.

This is a natural choice because in such a situation the (y, z) coordinates are invariant under the Lorentz transformations.

However, it is of interest to study the case when the line of motion does not coincide with any of the coordinate axes.

Introduction

We will derivate the Lorentz transformation when the line of motion is not aligned with any of the axes in a two-dimensional space.

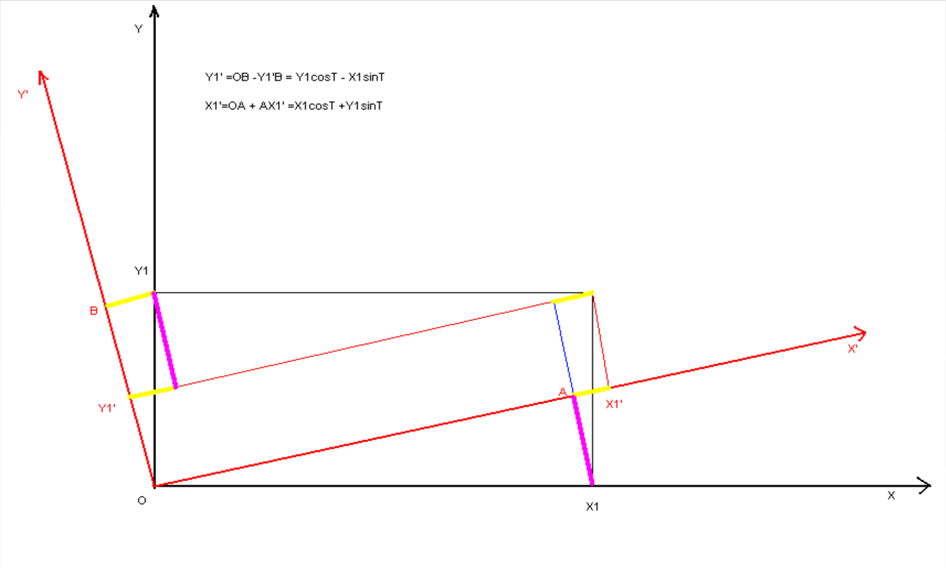


- In order to do this we will:
- Rotate the frames K of an angle ϑ
- apply the Lorentz transformation along the x-axis
- apply a new rotation of an angle $-\vartheta$

Development of generalized 2D transformations

For a rotation by an angle ϑ we have this equations:

- $x' = x \cos \theta + y \sin \theta$ and $y' = -x \sin \theta + y \cos \theta$
- Within an inertial frame, when we add the time coordinate and consider the fact that a spatial rotation has no effect on the time coordinate, we add the equation $t' = t$



Thus the transformation matrix $R(\theta)$ results:

$$\begin{pmatrix} x' \\ y' \\ t' \end{pmatrix} = \begin{pmatrix} \cos\theta & \sin\theta & 0 \\ -\sin\theta & \cos\theta & 0 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} x \\ y \\ t \end{pmatrix}$$

Development of generalized 2D transformations

As we know the Lorentz transformation along the x-axis yields the following matrix in a two-dimensional spatial system:

$$\begin{pmatrix} x' \\ y' \\ t' \end{pmatrix} = \begin{pmatrix} \gamma & 0 & -v\gamma \\ 0 & 1 & 0 \\ -v\gamma/c^2 & 0 & \gamma \end{pmatrix} \begin{pmatrix} x \\ y \\ t \end{pmatrix}$$

Where γ= $1/\sqrt{(1-v^2/c^2)}$

Development of generalized 2D transformations

When the line of motion is inclined at an angle θ with the x-axis, we can use the transformation R(-θ)*Lxv*R(θ) between two frames K and K' which observe each other moving at a velocity of +v and −v, respectively. So we will obtain the matrix A for the transformation of the coordinates from K to K'

$$A = \begin{pmatrix} \gamma\cos^2\theta + \sin^2\theta & \sin\theta.\cos\theta(\gamma-1) & -v\gamma\cos\theta \\ \sin\theta\cdot\cos\theta(\gamma-1) & \gamma\sin^2\theta + \cos^2\theta & -v\gamma\sin\theta \\ \dfrac{-v\gamma\cos\theta}{c^2} & \dfrac{-v\gamma\sin\theta}{c^2} & \gamma \end{pmatrix}$$

Development of generalized 2D transformations

The matrix B =R(-θ)*Lx-v*R(θ) transforms the event coordinates (x' , y' , t') of frame K' to (x, y, t) of frame K, when K and K 'are moving at a proper angle θ

Matrix B is the inverse of matrix A, and can be also obtained by substituting −v for v in matrix A

$$B = \begin{pmatrix} \gamma\cos^2\theta + \sin^2\theta & \sin\theta\cdot\cos\theta(\gamma-1) & v\gamma\cos\theta \\ \sin\theta\cdot\cos\theta(\gamma-1) & \gamma\sin^2\theta+\cos^2\theta & v\gamma\sin\theta \\ \frac{v\gamma\cos\theta}{c^2} & \frac{v\gamma\sin\theta}{c^2} & \gamma \end{pmatrix}$$

Development of generalized 2D transformations

How should an observer in K see the frame K' at t=0 ?

X'-axis y'=0 ->

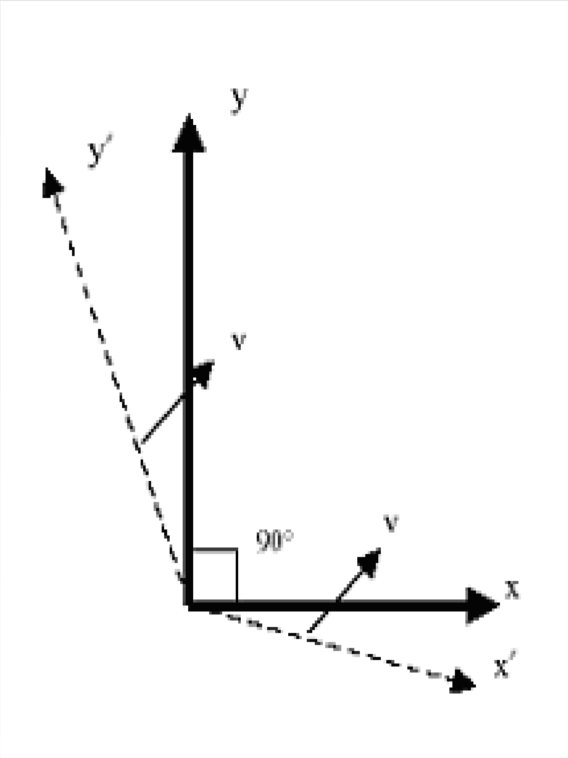
$\sin\theta*\cos\theta*(\gamma-1)*x + (\gamma\sin\theta^2+\cos\theta^2)*y=0$

this line has a negative slope and is not aligned with the x-axis

Y'-axis x'=0 ->

$(\gamma\cos\theta^2+\sin\theta^2)*x + \sin\theta*\cos\theta*(\gamma-1)*y=0$

this line has a negative slope and is not aligned with the y-axis



Application 1. Observer moving at an incline with respect to a rod

What will be the apparent length of the rod as seen by the observer?

In F system,co-moving with the rod, the x-axis is along the length of the rod and one end of the rod has coordinates x = 0; y = 0. The other end of the rod has coordinates x = L ; y = 0

In order to transform from F to O we use the transformation LxvRθ

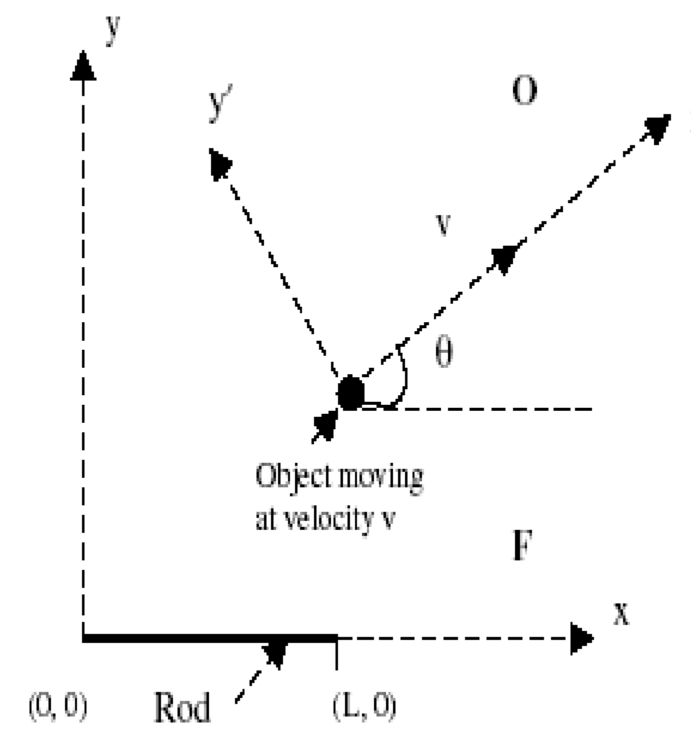


Figure 3. Rod in frame F and object in frame O moving at velocity v.

Application 1. Observer moving at an incline with respect to a rod

The resultant transformation from F to O is:

$$T = L_{xv} R_\theta = \begin{pmatrix} \gamma & 0 & -v\gamma \\ 0 & 1 & 0 \\ -v\gamma/c^2 & 0 & \gamma \end{pmatrix} \begin{pmatrix} \cos\theta & \sin\theta & 0 \\ -\sin\theta & \cos\theta & 0 \\ 0 & 0 & 1 \end{pmatrix} = \begin{pmatrix} \gamma\cos\theta & \gamma\sin\theta & -v\gamma \\ -\sin\theta & \cos\theta & 0 \\ \frac{-v\gamma\cos\theta}{c^2} & \frac{-v\gamma\sin\theta}{c^2} & \gamma \end{pmatrix}$$

One end of the rod at t = 0 in F has the coordinates (0, 0, 0) and it transforms to (0, 0, 0) in O. This corresponds to t' = 0.
In order to observe the other end of the rod from the inertial frame O at t' = 0, we transform the coordinates of the other end of the rod at some instant t in frame F and set t' = 0

$$\begin{pmatrix} x' \\ y' \\ 0 \end{pmatrix} = T \begin{pmatrix} L \\ 0 \\ t \end{pmatrix}$$

Application 1. Observer moving at an incline with respect to a rod

we obtain this equations:

$$x' = L\gamma\cos\theta - v\gamma t$$

$$y' = -L\sin\theta$$

$$0 = -(Lv\gamma\cos\theta)/c^2 + \gamma t.$$

We can solve them and obtain:

$x' = L\cos\theta/\gamma$ and $y' = -L\sin\theta$

.

Thus at the instant $t = 0$ in O, one end of the rod has coordinates $x = 0$; $y = 0$, and the other end has coordinates $x = L\cos\theta/\gamma$ and $y = -L\sin\theta$. Therefore the apparent length of the rod as observed by O is:

$$L' = L * \sqrt{\left(\cos^2\theta/\gamma^2 + \sin^2\theta\right)}$$

Application 2. 'collision of incident rods'

As shown in the figure, the bottom collision is taken as the spacetime origin (0, 0, 0) in both the frames K and K .
The top collision has the coordinates (L, 0, t) in frame K and (L, 0, t') in frame K '.
Using matrix A to transform (L, 0, t) and equating the result to (L, 0, t') we obtain:

$$(\gamma \cos^2\theta + \sin^2\theta)L - (v\gamma\cos\theta)t = L$$

$$L\sin\theta\cos\theta(\gamma - 1) - (v\gamma\sin\theta)t = 0$$

$$-(v\gamma/c^2)L\cos\theta + \gamma t = t'.$$

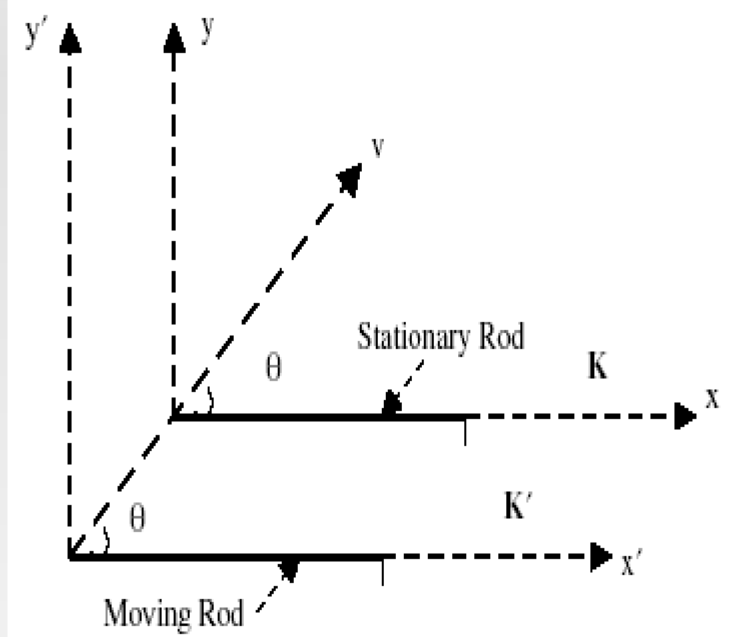


Figure 4. Rod in frame K′ moves towards stationary rod in frame K at velocity v.

Application 2. 'collision of incident rods'

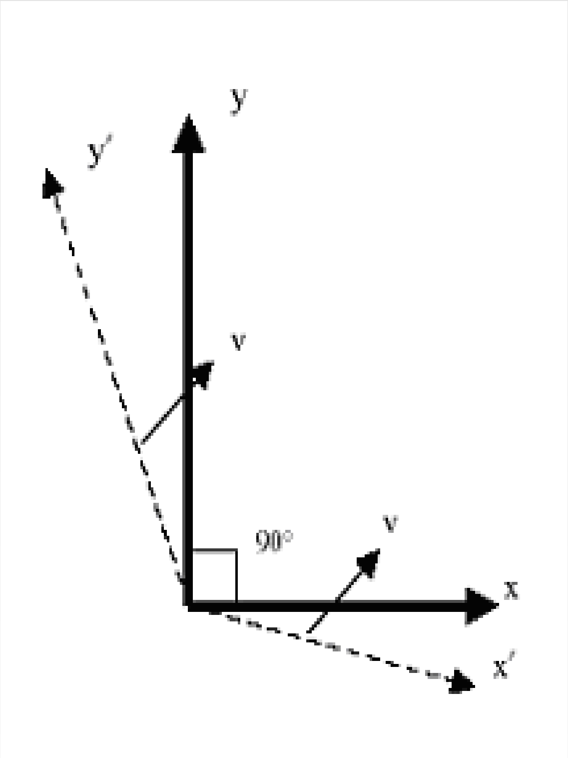
From the first two equations we get the same result, namely

$$t = (L/v) \cos \theta \, [1 - (1/\gamma)]$$

Substituting this into third equation it yields :

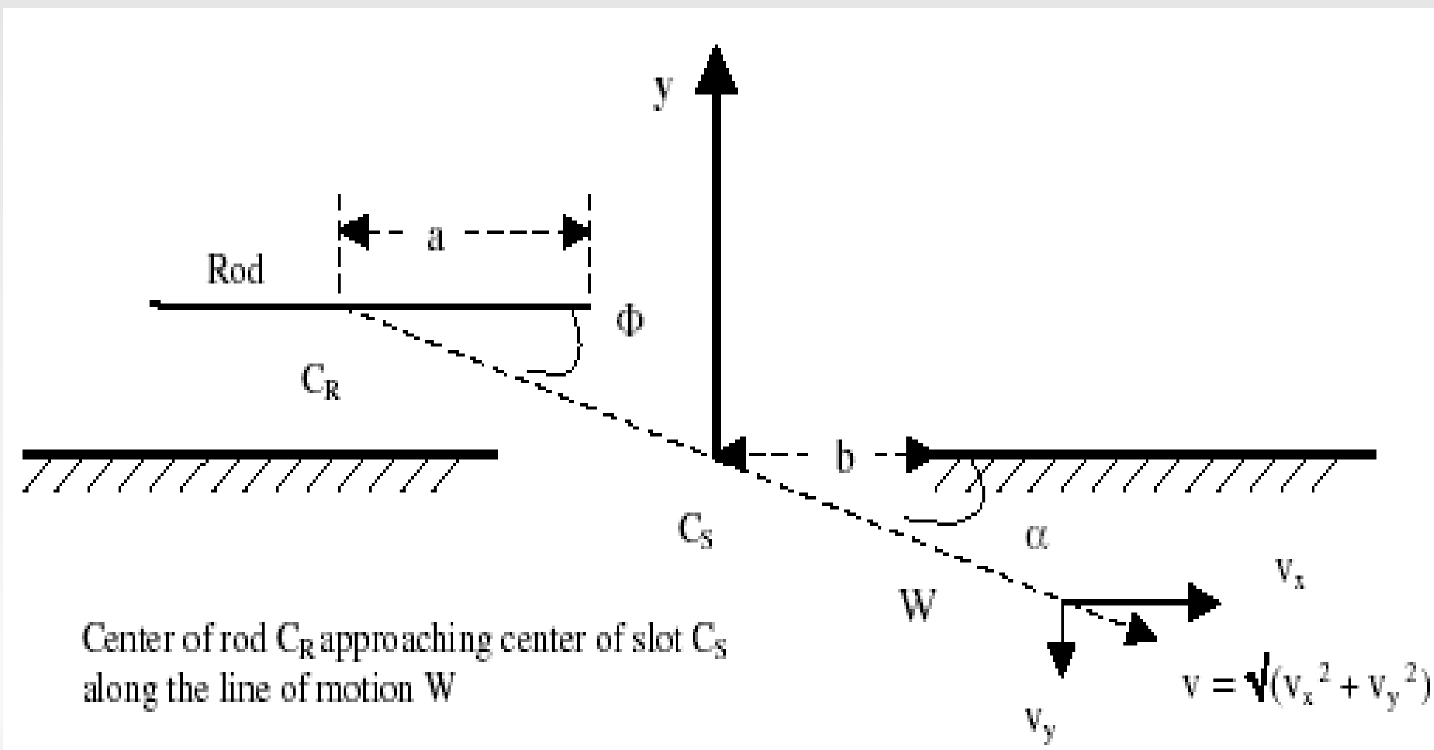
$$t' = (L/v) \cos \theta \, [(1/\gamma) - 1] = -t \,!$$

This confirms the reversal in time order of the top and bottom collisions in the two inertial frames co-moving with each of the two rods respectively.



Application 3.
The rod and slot problem

In this scenario the rod exhibits motion in two directions, but only with constant velocity. Furthermore, the line of motion (that is, the line joining the centres of the rod and slot) is not aligned with either the axis of the rod or the slot, there is no gravity, and thus no stress or propagation of stress. The meeting of the centres of the rod and slot is the spacetime origin with (x, y, t) = (x , y , t) = (0, 0, 0)



Center of rod C_R approaching center of slot C_S along the line of motion W

Application 3.
The rod and slot problem

When we transform from the rod's reference frame to the slot's reference frame, we have to use the following transformations in sequence: $R(-\phi)$, $L_x(-v)$ e $R(\alpha)$.
The overall transformation is $D=R(\alpha)L_x(-v)R(-\phi)$

$$D = \begin{pmatrix} \gamma\cos\phi\cos\alpha + \sin\alpha\sin\phi & -\gamma\sin\phi\cos\alpha + \sin\alpha\cos\phi & v\gamma\cos\alpha \\ -\gamma\cos\phi\sin\alpha + \cos\alpha\sin\phi & \gamma\sin\phi\sin\alpha + \cos\phi\cos\alpha & -v\gamma\sin\alpha \\ \frac{v\gamma\cos\phi}{c^2} & \frac{-v\gamma\sin\phi}{c^2} & \gamma \end{pmatrix}$$

Application 3.
The rod and slot problem

Let a denote half the length of the rod, and b denote half the length of the slot. The leading edge of the rod has the coordinates x = a; y = 0 for any arbitrary time t in the rod’s frame. This can be denoted by the triple (a, 0, t).

Similarly the front edge of the slot has the coordinate (b, 0, t') in the slot's frame for any arbitrary time t'.

In order for the rod to just pass through the slot, (a, 0, t) must transform to (b, 0, t) for some value of t.

$$\begin{pmatrix} b \\ 0 \\ t' \end{pmatrix} = D \begin{pmatrix} a \\ 0 \\ t \end{pmatrix} \longrightarrow$$

$$a\gamma\cos\Phi\cos\alpha + a\sin\alpha\sin\Phi + vt\gamma\cos\alpha = b$$

$$-a\gamma\cos\Phi\sin\alpha + a\cos\alpha\sin\Phi - vt\gamma\sin\alpha = 0$$

$$(av\gamma/c^2)\cos\Phi + \gamma t = t'.$$

Application 3.
The rod and slot problem

From the second equation we get:

$$t = (a\cos\alpha\sin\Phi - a\gamma\cos\Phi\sin\alpha)/(v\gamma\sin\alpha)$$

Substituting for t in the first we get

$$a\gamma\cos\Phi\cos\alpha + a\sin\alpha\sin\Phi + (a\cos\alpha\sin\Phi - a\gamma\cos\Phi\sin\alpha)\cos\alpha/\sin\alpha = b.$$

Multiplying both sides by sin α and simplifying, we obtain:

$$a\sin\Phi = b\sin\alpha.$$

Summary

- We derived a general Lorentz transformation in two-dimensional space with an arbitrary line of motion.
- We applied it to two problems and demonstrated that it leads to the same solution as already established in the literature. (Lorentzian contraction and the reversal in time order)
- In the third problem, we see the merit of using the Lorentz transformations with the line of motion not coinciding with any of the coordinate axes when there is a preferred system.

Next goal:
Lorentz transformation in 3D

Probably it is not so difficult: we have only to add a new rotation ϕ in the x-z plane and work with for rows matrix.

$$\begin{vmatrix} \cos\phi & 0 & \sin\phi & 0 \\ 0 & 1 & 0 & 0 \\ -\sin\phi & 0 & \cos\phi & 0 \\ 0 & 0 & 0 & 1 \end{vmatrix} = R(\phi)$$

$R(\theta)*R(\phi)*L(xv)*R(-\phi)*R(-\theta)$