Chapter 2

STRUCTURE OF LIE ALGEBRAS

2.1 Introduction

In the present and in the following chapter we address the fundamental problem of casting a general Lie algebra into a canonical form. This is a basic instrument to classify all possible algebras. Actually the classification of all Lie algebras turns out to be a too ambitious problem, yet if we restrict our attention to those Lie algebras that are semisimple, then the classification is possible and exhaustive. What semisimple means is precisely what is established in the present chapter by developing some basic lore about linear algebra. Furthermore the basic result of the chapter is the Levi decomposition theorem: it states that the most general Lie algebra is the semidirect product of a *semisimple algebra* with a *solvable ideal*. In a nutshell solvable Lie algebras, for which an exhaustive classification does not exist, are the trivial part, in the sense that all their linear representations are given by triangular matrices. On the contrary, for the semisimple algebras an exhaustive classification is provided by the formalism of roots and Dynkin diagrams discussed in the next chapter. Solvable Lie algebras are anyhow important also in the context of differential geometry. In later chapters, discussing non–compact homogeneous spaces, we show how solvable Lie algebras provide an efficient and privileged way of encoding their local geometry.

2.2 Linear Algebra preliminaries

Let us consider a vector space V constructed on the field of complex numbers $\mathbb{C}$, whose dimension we denote by $\dim V = n$. We name $\mathrm{Hom}(V,V)$ the ring of all linear endomorphisms of V. In other words an element $A \in \mathrm{Hom}(V,V)$ is a linear map:

$$\begin{aligned} & A : quadV \;\to\; V \\ & \quad \forall\, \alpha, \beta \,\in \mathbb{C}\,,\, \forall\, \overrightarrow{v}, \overrightarrow{w} \in V \;:\quad A\,(\alpha\, \overrightarrow{v} + \beta\, \overrightarrow{w}) = \alpha\, A\,(\overrightarrow{v}) + \beta\, A\,(\overrightarrow{w}) \end{aligned} \tag{2.2.1}$$

As we know, if $(\overrightarrow{e}_1, \ldots, \overrightarrow{e}_n)$ is a basis of V, in such a basis the endomorphism A is represented by the matrix A_{ij} determined by the condition:

$$A\,(\overrightarrow{e}_j) = \overrightarrow{e}_i\, A_{ij} \tag{2.2.2}$$

Indeed if $\{v^i\}$ are the components of the vector $\overrightarrow{v}$ in the basis $\{\overrightarrow{e}_i\}$, we have:

$$A\left(v^j\, \overrightarrow{e}_j\right) = \overrightarrow{e}^i\, A_{ij}\, v^j \tag{2.2.3}$$

and hence the components of the vector $\overrightarrow{v}' \equiv A(\overrightarrow{v})$ are given by:

$$v^{j\prime} = A_{ij}\, v^j \tag{2.2.4}$$

The association:

$$A \mapsto A_{ij} \tag{2.2.5}$$

is an isomorphism of $\mathrm{Hom}(V, V)$ onto the ring $M_n(\mathbb{C})$ of $n \times n$ matrices with complex coefficients.

Definition 2.2.1. << A matrix A_{ij} such that $A_{ij} = 0$ if $i > j$ is named **upper triangular**. A matrix such that $A_{ij} = 0$ if $i < j$ is named **lower triangular**. Finally a matrix that is simultaneously upper and lower triangular is named **diagonal** >>

we recall the concept of eigenvalue:

Definition 2.2.2. << Let $A \in \mathrm{Hom}(V, V)$. A complex number $\lambda \in \mathbb{C}$ is named an **eigenvalue** of A if $\exists\, \overrightarrow{v} \in V$ such that:

$$A\,\overrightarrow{v} = \lambda\,\overrightarrow{v} \tag{2.2.6}$$

>>

Definition 2.2.3. << Let λ be an eigenvalue of the endomorphism $A \in \mathrm{Hom}(V, V)$, the set of vectors $\overrightarrow{v} \in V$ such that $A\,\overrightarrow{v} = \lambda\,\overrightarrow{v}$ is named the **eigenspace** $V_\lambda \subset V$ pertaining to the eigenvalue λ. It is obvious that it is a vector subspace. >>

As it is known from elementary courses in Geometry and Algebra the possible eigenvalues of A are the roots of the secular equation:

$$\det\,(\lambda\,\mathbf{1} - \mathcal{A}) = 0 \tag{2.2.7}$$

where $\mathbf{1}$ is the unit matrix and $\mathcal{A}$ is the matrix representing the endomorphism A in an arbitrary basis.

Definition 2.2.4. << An endomorphism $N \in \mathrm{Hom}(V, V)$ is named nilpotent if there exists an integer: $\exists\, k \in \mathbb{N}$ such that:

$$N^k = 0 \tag{2.2.8}$$

>>

Lemma 2.2.1. << A nilpotent endomorphism has always the unique eigenvalue $0 \in \mathbb{C}$ >>

Proof 2.2.1.1. Let λ be an eigenvalue and let $\overrightarrow{v} \in V_\lambda$ be an eigenvector. We have:

$$N^r\,\overrightarrow{v} = \lambda^r\,\overrightarrow{v} \tag{2.2.9}$$

Choosing $r = k$ we obtain $\lambda^k = 0$ which necessarily implies $\lambda = 0$. ∎

Lemma 2.2.2. << Let $N \in \mathrm{Hom}(V, V)$ be a nilpotent endomorphism. In this case one can choose a basis $\{\overrightarrow{e}_i\}$ of V such that in this basis the matrix N_{ij} satisfies the condition $N_{ij} = 0$ for $i \geq j$. >>

Proof 2.2.2.1. Let $\overrightarrow{e}_1$ be a null eigenvector of N, namely $N\,\overrightarrow{e}_1 = 0$ and let E_1 be the subspace of V generated by $\overrightarrow{e}_1$. ¿From N we induce an endomorphism N_1 acting on the space V/E_1, namely the vector space of equivalence classes of vectors in V modulo the relation:

$$\overrightarrow{v} \sim \overrightarrow{w} \Leftrightarrow \overrightarrow{v} - \overrightarrow{w} = m\,\overrightarrow{e}_1 \quad , \quad m \in \mathbb{C} \tag{2.2.10}$$

Also the new endomorphism $N_1 \; : \quad V/E_1 \; \rightarrow \; V/E_1$ is nilpotent. If $\dim V/E_1 \neq 0$, then we can find another vector $e_2 \in V$ such that $(e_2 + E_1) \in V/E_1$ is an eigenvector of N_1. Continuing iteratively this process we obtain a basis $\overrightarrow{e}_1, \dots, \overrightarrow{e}_n$ of V such that:

$$N\, e_1 = 0 \quad ; \quad N\, e_p = 0 \,\mathrm{mod}\, (e_1, \dots, e_{p-1}) \quad ; \quad 2 \leq p \leq n \tag{2.2.11}$$

where $(e_1, \dots, e_{p-1})$ denotes the subspace of V generated by the vectors $e_1, \dots, e_{p-1}$. In this basis the matrix representing N is triangular. Similarly if N_{ij} is triangular with $N_{ij} = 0$ for $i \geq j$, then the corresponding endomorphism is nilpotent ■

Definition 2.2.5. << Let $\mathcal{S} \subset \mathrm{Hom}(V, V)$ be a subset of the ring of endomorphisms and $W \subset V$ a vector subspace. The subspace W is named **invariant** with respect to $\mathcal{S}$ if $\forall S \in \mathcal{S}$ we have $S\, W \subset W$. The space V is named **irreducible** if does not contain invariant subspaces. >>

Definition 2.2.6. << A subset $\mathcal{S} \subset \mathrm{Hom}(V, V)$ is named **semisimple** if every invariant subspace $W \subset V$ admits an orthogonal complement which is also invariant. In that case we can write:

$$V = \bigoplus_{i=1}^{p} W_i \tag{2.2.12}$$

where each subspace W_i is invariant >>

A fundamental and central result in Linear Algebra, essential for the further development of Lie algebra theory is the *Jordan's decomposition theorem* that we quote without proof.

Theorem 2.2.1. << Let $L \in \mathrm{Hom}\,(V, V)$ be an endomorphism of a finite dimensional vector space V. Then there exists and it is unique the following Jordan decomposition:

$$L = S_L + N_L \tag{2.2.13}$$

where S_L is semisimple and N_L is nilpotent. Furthermore, both S_L and N_L can be expressed as polynomials in L >>

2.3 Types of Lie Algebras and Levi's decomposition

In the previous section we have discussed the notion of semi–simplicity and of nilpotency for endomorphisms of vector spaces, namely for matrices. Such notions can now be extended to entire Lie algebras. This is not surprising since Lie algebras admit linear representations where each of their elements is replaced by a matrix. In the present section we discuss *solvable*, *nilpotent* and *semisimple* Lie algebras. Solvable and nilpotent Lie algebras are those for which all linear representations are provided by triangular matrices. Semisimple Lie algebras are those which do not admit any invariant subalgebra (or ideal) which is solvable. The main result in this section will be Levi's theorem which is the counterpart for algebras of Jordan's decomposition theorem 2.2.1 holding true for matrices.

Consider a Lie algebra $\mathbb{G}$ and define:

$$\mathcal{D}\mathbb{G} \;=\; [\mathbb{G}\,,\,\mathbb{G}] \tag{2.3.14}$$

the set of all elements $g \in \mathbb{G}$ that can be written as the Lie bracket of two other elements $g = [g_1\,,\,g_2]$. Clearly $\mathcal{D}\mathbb{G}$ is an ideal in $\mathbb{G}$.

Definition 2.3.1. << The sequence $\mathcal{D}^n\,\mathbb{G} = \left[\mathcal{D}^{n-1}\mathbb{G}\,,\,\mathcal{D}^{n-1}\mathbb{G}\right]$ of ideals:

$$G \supset \mathcal{D}\mathbb{G} \supset \mathcal{D}^2\mathbb{G} \supset \dots \supset \mathcal{D}^n\mathbb{G} \tag{2.3.15}$$

is named the **derivative series** of the Lie algebra $\mathbb{G}$ >>

2.3.1 Solvable Lie Algebras

Definition 2.3.2. << A Lie algebra $\mathbb{G}$ is named **solvable** if there exists an integer $n \in \mathbb{N}$ such that

$$\mathcal{D}^n \mathbb{G} = \{0\} \tag{2.3.16}$$

is named the **derivative series** of the Lie algebra $\mathbb{G}$ >>

Lemma 2.3.1. << A subalgebra $\mathbb{K} \subset \mathbb{G}$ of a sovable Lie algebra is also solvable. >>

Proof 2.3.1.1. Indeed let G be solvable and $\mathbb{K} \subset \mathbb{G}$ be a subalgebra. Clearly $\mathcal{D}\mathbb{K} \subset \mathcal{D}\mathbb{G}$ and hence at every level n we have $\mathcal{D}^n\mathbb{K} \subset \mathcal{D}^n\mathbb{G}$, so that the lemma follows. ■

Definition 2.3.3. << A Lie algebra $\mathbb{G}$ has the **chain property** if and only if, for each ideal $\mathbb{H} \subset \mathbb{G}$ there exists an ideal $\mathbb{H}_1 \subset \mathbb{H}$ of the considered ideal which has codimension one in $\mathbb{H}$. >>

The above definition can be illustrated in the following way. Let $\mathbb{H}$ be the considered ideal in $\mathbb{G}$. If G has the chain property, then $\mathbb{H}$ can be written in the following way:

$$\mathbb{H} = \mathbb{H}_1 \oplus \lambda \overrightarrow{X} \tag{2.3.17}$$

where $\mathbb{H}_1$ is a subspace of dimension:

$$\dim \mathbb{H}_1 = \dim \mathbb{H} - 1 \tag{2.3.18}$$

and $\overrightarrow{X} \in \mathbb{H}$, $\left(\overrightarrow{X} \ni \mathbb{H}_1\right)$is an element that belongs to $\mathbb{H}$ but not to $\mathbb{H}_1$. Furthermore we have:

$$\forall \overrightarrow{Z} \in \mathbb{H}_1 \quad , \quad \left[\overrightarrow{Z} \, , \overrightarrow{X}\right] \in \mathbb{H}_1 \tag{2.3.19}$$

¿From this definition we obtain the following:

Lemma 2.3.2. << A Lie algebra $\mathbb{G}$ is solvable if and only if it admits the chain property >>

Proof 2.3.2.1. Let G be solvable and let us put $\dim \mathbb{G} = n$ and $\dim \mathcal{D}\mathbb{G} = m$. By hypothesis of solvability we have that $\mathcal{D}\mathbb{G} \neq \mathbb{G}$, so that $n - m = p > 0$. Let us choose $p-1$ linear independent elements $\{X_1, \ldots, X_{p-1}\} \in \mathbb{G}$, such that $X_i \ni \mathcal{D}\mathbb{G}$ and define the subspace:

$$H_1 = \mathcal{D}\mathbb{G} + \lambda_1 \overrightarrow{X}_1 + \ldots + \lambda_{p-1} \overrightarrow{X}_{p-1} \quad , (\lambda_i \in \mathbb{C}) \tag{2.3.20}$$

By construction H_1 has codimension one and it is an ideal. This construction can be repeated for each ideal $\mathbb{H} \subset \mathbb{G}$since it is solvable. Hence $\mathbb{G}$ admits the chain property.

Conversely let G be a Lie algebra admitting the chain property. Then we find a sequence of ideals:

$$\mathbb{G} = \mathbb{G}^0 \supset \mathbb{G}^1 \supset \mathbb{G}^2 \supset \ldots \supset \mathbb{G}^n = \{0\} \tag{2.3.21}$$

such that G^r is an ideal in G^{r-1} of codimension one so that $\mathcal{D}^{r-1}\mathbb{G} \subset \mathbb{G}^r$. Hence $\mathbb{G}$ is solvable. ■

We can now state the most relevant property of solvable Lie algebras. This is encoded in the following Levi's theorem and in its corollary.

Theorem 2.3.1. << Let G be a solvable Lie algebra and let V be a finite dimensional vector space on the field $\mathbb{F} = \mathbb{C}$, or $\mathbb{R}$, algebraically closed. Furthermore let:

$$\pi : \quad \mathbb{G} \to \mathrm{Hom}\,(V, V) \tag{2.3.22}$$

be a homomorphism of $\mathbb{G}$ on the algebra of linear endomorphisms of V. Then there exists a vector $\overrightarrow{v} \in V$ such that it is a simultaneous eigenvector for all elements $\pi(g)$, $(\forall g \in \mathbb{G})$. >>

Proof 2.3.1.1. The proof is constructed by induction. If $\dim\mathbb{G} = 1$, then there is just one endomorphism $\pi(g)$ and it necessarily admits an eigenvector. Suppose next that the theorem is true for each solvable algebra $\mathbb{K}$ of dimension

$$\dim \mathbb{K} \, < \, \dim \mathbb{G} \tag{2.3.23}$$

Consider an ideal $\mathbb{H} \subset \mathbb{G}$ of codimension one:

$$\dim \mathbb{G} = \dim \mathbb{H} + 1 \tag{2.3.24}$$

Such an ideal exists because the Lie algebra is solvable and, therefore, admits the chain property. Write:

$$\mathbb{G} = \mathbb{H} + \lambda \overrightarrow{X} \quad (\lambda \in \mathbb{F}) \tag{2.3.25}$$

where $\overrightarrow{X}$ is an element of $\mathbb{G}$ not contained in $\mathbb{H}$. By the induction hypothesis there exists a vector $\overrightarrow{e}_0 \in V$ such that:

$$\forall H \in \mathbb{H} \quad : \quad \pi(H)\, \overrightarrow{e}_0 = \lambda(H)\, \overrightarrow{e}_0 \tag{2.3.26}$$

where $\lambda(H) \in \mathbb{F}$ is an eigenvalue depending on the considered element H. Define next the following vectors:

$$\overrightarrow{e}_p = [\pi\,(X)]^p \; \overrightarrow{e}_0 \qquad p = 1, 2, \ldots \tag{2.3.27}$$

The subspace $W \subset V$ spanned by the vectors $\overrightarrow{e}_p$ $(p \geq 0)$ is clearly invariant with respect to $\pi(X)$. We can also show what follows:

$$\pi\,(H)\;\overrightarrow{e}_p = \lambda\,(H)\;\overrightarrow{e}_p \bmod (\overrightarrow{e}_0, \ldots, \overrightarrow{e}_{p-1}) \quad ; \quad (\forall\, H \,\in \mathbb{H}) \tag{2.3.28}$$

Indeed, eq.(2.3.28) is true for $p = 0$ and assuming true for p we get:

$$\begin{aligned} \pi\,(H)\;\overrightarrow{e}_{p+1} &= \pi\,(H)\;\pi\,(X)\;\overrightarrow{e}_p = \pi\,([H\,,\,X])\;\overrightarrow{e}_p + \pi\,(X)\;\pi\,(H)\;\overrightarrow{e}_p \\ &= \lambda\,([H\,,\,X])\;\overrightarrow{e}_p + \lambda\,(H)\;\overrightarrow{e}_{p+1} + \bmod (\overrightarrow{e}_0, \ldots, \overrightarrow{e}_{p-1}) \end{aligned} \tag{2.3.29}$$

(Note that $[H\,,\,X] \in H$). Hence we find:

$$\pi\,(H)\;\overrightarrow{e}_{p+1} = \lambda(H)\,\overrightarrow{e}_{p+1} + \bmod (\overrightarrow{e}_0, \ldots, \overrightarrow{e}_p) \tag{2.3.30}$$

It follows that the subspace W is invariant with respect to $\pi\,(\mathbb{G})$ and that:

$$\mathrm{Tr}_W\,\pi\,(H) = \lambda\,(H)\;\dim W \tag{2.3.31}$$

On the other hand we have:

$$\mathrm{Tr}_W\;(\pi\,([H\,,\,X])) = 0 \quad \Rightarrow \quad \lambda\,([H\,,\,X]) = 0 \tag{2.3.32}$$

Repeating the argument by induction, from the relation:

$$\pi(H)\ \overrightarrow{e}_{p+1} = \pi([H\,,\,X])\,\overrightarrow{e}_p + \pi(X)\ \pi(H)\ \overrightarrow{e}_p \qquad (2.3.33)$$

and the original definition of the eigenvalue $\lambda(H)$ in eq. (2.3.26) we conclude that:

$$\pi(H)\,\overrightarrow{e}_p = \lambda(H)\,\overrightarrow{e}_p \qquad (p \geq 0) \qquad (2.3.34)$$

This shows that $\forall H \in \mathbb{H}$ we have $\pi(H) = \lambda(H)\ \mathbf{1}$ on the vector subspace W. Choosing a vector $\overrightarrow{e}'_p \in W$ that is eigenvector of $\pi(X)$ we find that is a simultaneous eigenvector for all elements $\pi(g)$, $(\forall g \in \mathbb{G})$.■

Corollary 2.3.1. << Let $\mathbb{G}$ be a solvable Lie algebra and π a linear representation of $\mathbb{G}$ on a finite dimensional vector space V. Then there exists a basis $\{\overrightarrow{e}_1, \ldots, \overrightarrow{e}_n\}$ where every $\pi(X)$, $(\forall X \in \mathbb{G})$ is a triangular matrix. >>

Proof 2.3.1.1. Let e_1 be the simultaneous eigenvector of all $\pi(X)$. The representation π induces a new linear representation on the quotient vector space V/E_1 where $E_1 \equiv \lambda\overrightarrow{e}_1$. Hence applying theorem 2.3.1 to this new representation we conclude that there is a common eigenvector $\overrightarrow{e}_2 + \lambda\overrightarrow{e}_1$ for all $\pi(X)$. Continuing in this way we obtain a basis such that:

$$\pi(X)\ \overrightarrow{e}_i \equiv 0 \bmod (\overrightarrow{e}_1, \to e_2, \ldots, \overrightarrow{e}_i) \qquad (2.3.35)$$

so that $\pi(X)$ is indeed upper triangular. ■

We are now ready to introduce the concept of semisimple Lie algebras and discuss their general properties.

2.3.2 Semisimple Lie Algebras

We introduce some more definitions.

Definition 2.3.4. << Let $\mathbb{G}$ be a Lie algebra. An ideal $\mathbb{H} \subset \mathbb{G}$ is named **maximal** if there is no other ideal $\mathbb{H}' \subset \mathbb{G}$ such that $\mathbb{H}' \supset \mathbb{H}$ except $\mathbb{H}$ itself. >>

Definition 2.3.5. << The maximal solvable ideal of a Lie algebra $\mathbb{G}$ is named the radical of $\mathbb{G}$ and it is denoted Rad$\mathbb{G}$.>>

Definition 2.3.6. << A Lie algebra $\mathbb{G}$ is named semisimple if and only if Rad$\mathbb{G} = 0$.>>

As an immediate consequence of the definition we have the

Theorem 2.3.2. << A Lie algebra $\mathbb{G}$ is semisimple if and only if it does not have any non–trivial abelian ideal. >>

Proof 2.3.2.1. We have to show the equivalence of the following two propositions:

a : $\mathbb{G}$ has a solvable ideal
b : $\mathbb{G}$ has an abelian ideal

i) Let us show that b ⇒ a. Let $\mathcal{I} \subset \mathbb{G}$ be the abelian ideal. By definition we have $[\mathcal{I}\,,\,\mathcal{I}] = \mathcal{D}\mathcal{I} = 0$ Hence $\mathcal{I}$ itself is a solvable ideal and this proves a.

ii) Let us now show that a ⇒ b. To this effect let $\mathcal{I} \subset \mathbb{G}$ be the solvable ideal. By definition, since $\mathcal{I}$ is non trivial $\exists k \in \mathbb{N}$ such that $\mathcal{D}^{k-1}\mathcal{I} \neq 0$ and $\mathcal{D}^k\mathcal{I} = 0$. Then the $\mathcal{D}^{k-1}\mathcal{I}$ is abelian and being the derivative of an ideal is an ideal. Hence b is true and this concludes the proof of the theorem.

■

2.3.3 Levi's decomposition of Lie algebras

We want to proof that any Lie algebra can be seen as the semidirect product of its radical with a semisimple Lie algebra. Such a decomposition is named the Levi decomposition and is what we want to illustrate in the present section. To this effect we need to introduce some preliminary notions. The first is the notion of **Lie algebra cohomology**. It is a further very relevant example of an algebraic construction that realizes the paradigm introduced in section 1.6.3. The second is the equally important notion of semidirect product.

2.3.3.1 Lie algebra Cohomology

Let $\mathbb{G}$ be a Lie algebra and let $\rho : \mathbb{G} \to \mathrm{End}(F)$ be a representation of $\mathbb{G}$ on a complex, finite dimensional vector space F. Let $V^s(\mathbb{G}, \rho)$ be the vector space of all antisymmetric linear maps

$$\theta \quad : \quad \underbrace{\mathbb{G} \times \mathbb{G} \times \ldots \times \mathbb{G}}_{\text{s times}} \to F \tag{2.3.36}$$

The spaces $V^s(\mathbb{G}, \rho)$ are the spaces of s–**cochains**. We can next define a coboundary operator d in the following way. Let $\theta \in V^s(\mathbb{G}, \rho)$ be an s–cochain, the value of the $s+1$–cochain $d\theta$ on any set of $s+1$ elements $X_1, \ldots, X_{s+1}$ of the Lie algebra $\mathbb{G}$ is given by the following expression:

$$\begin{aligned} d\left(X_1, X_2, \ldots X_{s+1}\right) = & \sum_{i=1}^{s+1} (-)^{i+1} \rho\left(X_i\right) \theta\left(X_1, \ldots, X_{i-1}, X_{i+1}, \ldots, X_{s+1}\right) \\ & - \sum_{r=1}^{s+1} \sum_{q<r} (-1)^{r+q} \theta\left(X_1, \ldots, \widehat{X}_q, \ldots, \widehat{X}_r, \ldots, X_{s+1}, [Z_q \,,\, X_r]\right) \end{aligned} \tag{2.3.37}$$

where the hat on top of an X–element means that it is omitted. It is straightforward to verify that by applying a second time the coboundary operator d we obtain identically zero namely that:

$$d^2 = 0 \tag{2.3.38}$$

In particular if we consider the case of 1–cochains $\theta^{[1]}$ that are maps $\mathbb{G} \to F$ from the Lie algebra to the vector space F, by applying the general definition (2.3.37) we obtain:

$$\forall X, Y \in \mathbb{G} \, : \, d\theta^{[1]}(X, Y) = \rho(X)\theta(Y) - \rho(Y)\theta(X) - \theta\left([X \,,\, Y]\right) \tag{2.3.39}$$

Given the coboundary operator d we have the usual definitions of an *elliptic complex*:

(i) The space $C^{(n)}(\mathbb{G}, \rho)$ of n–cocycles is the vector space of all n–chains $\theta^{[n]}$ that are closed $d\theta^{[n]} = 0$

(ii) The space $B^{(n)}(\mathbb{G}, \rho)$ of n–coboundaries is the vector space of all n–chains $\theta^{[n]}$ that are exact, namely that can be written as $\theta^{[n]} = d\phi^{[n-1]}$ for some $n-1$–chain $\phi^{[n-1]}$

(iii) The n–th cohomology group $H^{[n]}(\mathbb{G}, \rho)$ of the Lie algebra $\mathbb{G}$ relative to the linear representation ρ is the quotient:

$$H^{[n]}(\mathbb{G}, \rho) = \frac{C^{(n)}(\mathbb{G}, \rho)}{B^{(n)}(\mathbb{G}, \rho)} \tag{2.3.40}$$

namely it is the vector space whose equivalence classes are the n–cocycles modulo the n–coboundaries.

We have a useful general cohomological property of semisimple Lie algebras that follows from the above definitions but whose proof we omit for brevity

Theorem 2.3.3. << Let $\mathbb{G}$ be a semisimple Lie algebra and ρ a linear representation of $\mathbb{G}$ on a finite dimensional vector space F. Then the first two cohomology groups are trivial:

$$H^{[1]}(\mathbb{G},\rho) = H^{[2]}(\mathbb{G},\rho) = 0 \tag{2.3.41}$$

>>

2.3.3.2 Semidirect product

We begin with two definitions:

Definition 2.3.7. << Let $\mathbb{G}$ be a Lie algebra and let $\sigma : \mathbb{G} \to \mathbb{G}$ be an endomorphism of vector spaces. We say that σ is a **derivation** of the algebra if the following property holds true:

$$\forall a, b \in \mathbb{G} \,:\, \sigma([a\,,\,b]) = [\sigma(a)\,,\,b] + [a\,,\,\sigma(b)] \tag{2.3.42}$$

>>

Definition 2.3.8. Let $\mathbf{q}$ and $\mathbf{m}$ be two Lie algebras and let σ be a linear representation of $\mathbf{m}$ on $\mathbf{q}$ such that $\forall Y \in \mathbf{m}$ the map $\sigma(Y)$ is a derivation of $\mathbf{q}$. Next let X, X' be elements of $\mathbf{q}$ and Y, Y' be elements of $\mathbf{m}$. We define the Lie bracket of the ordered pair (X, Y) with the ordered pair (X', Y') in the following way:

$$[(X,Y)\,,\,(X',Y')] = ([X\,,\,X'] + \sigma(Y)X' - \sigma(Y')X\,,\,[Y\,,\,Y']) \tag{2.3.43}$$

With this definition of the Lie bracket $\mathbf{q} \times_\sigma \mathbf{m}$ becomes a Lie algebra and it is named the smidirect product of $\mathbf{q}$ with $\mathbf{m}$ relative to the representation σ. >>

It is a straightforward exercise to check that the definition of the Lie bracket (2.3.43) is consistent and satisfies Jacobi identity.

Let us now consider a Lie algebra $\mathbb{G}$ and let $\mathcal{Q} \subset \mathbb{G}$ be an ideal and $\mathbb{M} \subset \mathbb{G}$ a subalgebra such that, as vector spaces, we have the following orthogonal decomposition:

$$\mathbb{G} = \mathcal{Q} \oplus \mathbb{M} \quad \Rightarrow \quad \mathcal{Q} \bigcap \mathbb{M} = 0 \tag{2.3.44}$$

Obviously $\mathbb{G}$ can be regarded as the semidirect product of $\mathbb{Q}$ with $\mathbb{M}$. It suffices to use as derivation σ the internal derivation provided by the Lie bracket of $\mathbb{G}$:

$$\forall Y \in \mathbb{M}\,,\, \forall X \in \mathcal{Q} \quad : \quad \sigma(Y)\; X \equiv -[X\,,\,Y] \tag{2.3.45}$$

Definition 2.3.9. << Let $\mathbb{G}$ be a Lie algebra. We say that $\mathbb{G}$ is **decomposed according to Levi** if there exists a subalgebra $\mathbb{L} \subset \mathbb{G}$ such that:

$$\mathbb{G} = \mathbb{L} \times_{[,]} \mathrm{Rad}\,\mathbb{G} \tag{2.3.46}$$

Obviously, since $\mathbb{G}/\mathrm{Rad}\,\mathbb{G}$ is semisimple and since $\mathbb{L} \sim \mathbb{G}/\mathrm{Rad}\,\mathbb{G}$ also $\mathbb{L}$ is semisimple. It is named a **Levi subalgebra**. >>

Relying on this definition we can state the following fundamental theorem:

Theorem 2.3.4. << Let G be a Lie algebra and denote $\mathcal{Q} \equiv \mathrm{Rad}\,\mathbb{G}$. Every $\mathbb{G}$ admits Levi subalgebras. Furthermore if $\mathbb{L} \subset \mathbb{G}$ is a Levi subalgebra of $\mathbb{G}$, it is also a Levi subalgebra of $\mathcal{D}\mathbb{G}$ and

$$\mathcal{D}\mathbb{G} = [\mathcal{Q}\,,\,\mathbb{G}] \oplus \mathbb{L} \tag{2.3.47}$$

is a Levi decomposition of $\mathcal{D}\,\mathbb{G}$. >>

In order to prove theorem 2.3.4 we need the following lemma:

Lemma 2.3.3. << Let $\mathbb{H}$ be a Lie algebra and $\mathcal{Q}$ its radical. If $\mathcal{A} \subset \mathbb{H}$ is an ideal such that $\mathbb{H}/\mathcal{A}$ is semisimple, then $\mathcal{Q} \sqsubseteq \mathcal{A}$. Furthermore, if π is a homomorphism of $\mathbb{H}$ onto an algebra $\mathbb{H}'$, then $\pi\,(\mathcal{Q})$ is the radical of $\mathbb{H}'$ >>

Proof 2.3.3.1. Consider the natural map:

$$\tau : \quad \mathbb{H} \to \mathbb{H}/\mathcal{A} \tag{2.3.48}$$

that to each element $h \in \mathbb{H}$ associates the equivalence class $h + \mathcal{A}$. If $\mathcal{Q} \nsubseteq \mathcal{A}$ we have that $\tau\,(\mathcal{Q})$ is a non zero and solvable ideal of $\mathbb{H}/\mathcal{A}$. Indeed, under the homomorphism τ the ideal $\mathcal{Q}$ flows into an ideal $\tau\,(\mathcal{Q})$. Furthermore under the homomorphism we have $\mathcal{D}\tau\,(\mathcal{Q}) = \tau\,(\mathcal{D}\mathbb{G})$ so that if $\mathcal{Q}$ is solvable, the same is true also for $\tau\,(\mathcal{Q})$. The existence of a solvable ideal is in contradiction with the assumption that $\mathbb{H}/\mathcal{A}$ is semisimple. Hence $\mathcal{Q} \subseteq \mathcal{A}$, necessarily.

Let us come to the second part of the lemma and let $\mathcal{Q}' = \mathrm{Rad}\,\mathbb{H}'$. The homomorphism π induces a homomorphism of $\mathbb{H}/\mathcal{Q}$ in $\mathbb{H}'/\pi\,(Q)$, hence since $\mathbb{H}/\mathcal{Q}$ is semisimple, also $\mathbb{H}/\pi\,(\mathcal{Q})$ is semisimple. Therefore, relying on the previous result $\mathcal{Q}' \subset \pi\,(\mathcal{Q})$. On the other hand $\pi\,(\mathcal{Q})$ is a solvable ideal of $\mathbb{H}$. This implies $\pi\,(\mathcal{Q}) \subset \mathcal{Q}'$. We conclude $\pi\,(\mathcal{Q}) = \mathcal{Q}'$ and the lemma is proved. ■

Let us now come to the proof of the main theorem 2.3.4.

Proof 2.3.4.1. The proof of theorem 2.3.4 is by induction on the dimension of the radical $\dim \mathcal{Q}$. If $\dim \mathcal{Q} = 0$, then $\mathbb{G}$ is semisimple and it is by itself a Levi subalgebra. Let us then assume that $\dim \mathcal{Q} \geq 1$ and that Levi subalgebras do exist for any Lie algebra $\mathbb{G}'$ such that $\dim \mathrm{Rad}\,\mathbb{G}' < \dim \mathrm{Rad}\,\mathbb{G}$. Consider two cases:

1st Case The radical $\mathcal{Q}$ is non abelian, namely ($\mathcal{D}\mathcal{Q} \neq 0$).

As we know $\mathcal{D}\mathcal{Q}$ is by itself an ideal. Hence consider the Lie algebra $\mathbb{G}' \equiv \mathbb{G}/\mathcal{D}\mathcal{Q}$ and let π be the natural map:

$$\pi : \quad \mathbb{G} \to \mathbb{G}' \equiv \mathbb{G}/\mathcal{D}\mathcal{Q} \tag{2.3.49}$$

Relying on the lemma 2.3.3 we have: $\mathcal{Q}' = \mathrm{Rad}\,\mathbb{G}' = \pi\,[\mathcal{Q}]$. Hence:

$$\mathcal{Q}' = \mathcal{Q}/\mathcal{D}\mathcal{Q} \quad \Rightarrow \quad \dim \mathcal{Q}' < \dim\ \mathcal{Q} \tag{2.3.50}$$

By induction hypothesis $\mathbb{G}'$ admits a Levi subalgebra $\mathcal{M}'$ and we have, as vector spaces:

$$\mathbb{G}' = \mathcal{Q}' \oplus \mathcal{M}' \tag{2.3.51}$$

Define $\mathcal{M}_0 = \pi^{-1}\,(\mathcal{M}')$. We obtain:

$$\mathbb{G} = \pi^{-1}\,(\mathbb{G}') = \mathcal{Q} \oplus \mathcal{M}_0 \tag{2.3.52}$$

Furthermore it is true that $\mathcal{DQ} = \mathcal{Q} \bigcap \mathcal{M}_0$ (indeed the common elements of $\mathcal{Q}$ and $\mathcal{M}_0$ must be contained in the kernel of π, namely $\pi^{-1}(0)$). Hence $\mathcal{DQ}$ is a solvable ideal of $\mathcal{M}_0$ and since $\mathcal{M}_0/\mathcal{DQ} \sim \mathcal{M}' =$ semisimple algebra, then $\mathcal{DQ} \subset \mathrm{Rad}\mathcal{M}_0$. Yet in force of lemma 2.3.3 we also have $\mathrm{Rad}\mathcal{M}_0 \subset \mathcal{DQ}$ which implies $\mathrm{Rad}\mathcal{M}_0 = \mathcal{DQ}$. Since $\mathcal{Q}$ is solvable by definition we have: $\dim\mathcal{DQ} < \dim\mathcal{Q}$. Then by induction hypothesis we conclude that $\mathcal{M}_0$ admits a Levi decomposition:

$$\mathcal{M}_0 = \mathbb{L} \oplus \mathcal{DQ} \quad ; \quad \mathbb{L} \bigcap \mathcal{DQ} = 0 \tag{2.3.53}$$

from which we conclude:

$$\mathbb{G} = \mathbb{L} \oplus \mathcal{Q} \tag{2.3.54}$$

and the theorem is proved in this case.

2nd Case The radical $\mathcal{Q}$ is abelian, namely $(\mathcal{DQ} = 0)$.

To prove the theorem in this case we have to use theorem 2.3.3 stating that the second cohomology group of a semisimple Lie algebra vanishes. Define $\mathbb{G}_1 = \mathbb{G}/\mathcal{Q}$ (so that $\mathbb{G}_1$ is semisimple) and let π be the natural map of $\mathbb{G}$ onto $\mathbb{G}_1$. Let μ be any linear map of $\mathbb{G}_1$ into $\mathbb{G}$ such that $\pi \circ \mu = \mathbf{1}$. For each $X_1 \in \mathbb{G}_1$ define $\rho(X_1)$ the endomorphism $\mathrm{ad}(X)|_{\mathcal{Q}}$ where X is such that $\pi(X) = X_1$. Since $\mathcal{Q}$ is abelian this is a well posed definition. Indeed $X_1 = X + \mathcal{Q}$ and $\forall q \in \mathcal{Q}$:

$$\rho(X_1)\, q = [X + \mathcal{Q}\, , q] = [X\, , q] \tag{2.3.55}$$

The map $X_1 \mapsto \rho(X_1)$ is a linear representation of the semisimple Lie algebra $\mathbb{G}_1$ on the vector space $\mathcal{Q}$. Obviously $\rho(X_1) = \mathrm{ad}\, \mu(X_1)|_{\mathbb{G}}$. Next define:

$$\forall X, Y \in \mathbb{G}_1 \quad : \quad \theta(X\, , Y) \equiv [\mu(X)\, , \mu(Y)] - \mu([X\, , Y]) \tag{2.3.56}$$

Since π is a homomorphism and $\pi \circ \mu = \mathrm{id}$ then we have that $\pi(\theta(X\, , Y)) = 0 \Rightarrow \theta(X\, , Y) \in \mathcal{Q}$. This guarantees that $\theta \in V^2(\mathbb{G}_1\, , \rho)$ is a 2-cochain of the Lie Algebra $\mathbb{G}_1$ relative to the representation ρ. By direct calculation and use of Jacobi identity we can immediately verify that θ is actually a 2–cycle, namely $d\theta = 0$. Since the second cohomology group vanishes for semisimple Lie algebra $H^2(\mathbb{G}_1\, , \rho) = 0$, it follows that there exists a linear map

$$\exists \nu : \quad \mathbb{G}_1 \mapsto \mathcal{Q} \tag{2.3.57}$$

such that $d\nu = \theta$, namely:

$$[\mu(X)\, , \mu(Y)] - \mu([X\, , Y]) = [\mu(X)\, , \mu(Y)] - [\mu(X)\, , \nu(Y)] - [\mu(Y)\, , \nu(X)] - \nu([X\, , Y]) \tag{2.3.58}$$

Since $\nu(X) \in \mathcal{Q}$ we have $[\nu(X)\, , \nu(Y)] = 0$. Hence defining:

$$\lambda(X) = \mu(X) - \nu(X) \tag{2.3.59}$$

we see that $\lambda : \mathbb{G}_1 \mapsto \mathbb{G}_1$ is a homomorphism. It is also evident that $\pi \circ \lambda = \mathrm{id}$. So we have found a map $\lambda : \mathbb{G}_1 \mapsto \mathbb{G}$ which is a homomorphism of algebras. It follows that $\lambda(\mathbb{G}_1) \subset \mathbb{G}$ is a subalgebra. Furthermore by construction:

$$\mathbb{G} = \mathcal{Q} \oplus \lambda(\mathbb{G}_1) \quad ; \quad \mathcal{Q} \bigcap \lambda(\mathbb{G}_1) = 0 \tag{2.3.60}$$

Hence $\lambda(\mathbb{G}_1)$ is a Levi subalgebra and we have completed the induction argument also in this case.

The theorem is proved. ■

2.3.4 An illustrative example: the Galilei group

The invariance group of classical non relativistic mechanics is the *Galilei group* which consists of the following transformations on the space–time manifold whose points are labeled by the three space coordinates x^i and by the instant of time t:

$$\begin{pmatrix} x^i \\ t \end{pmatrix} \mapsto \begin{pmatrix} x^{i\prime} \\ t' \end{pmatrix} \tag{2.3.61}$$

where

$$\begin{cases} x^{i\prime} &= R^i{}_j\, x^j + v^i\, t + c^i \\ t' &= t + T \end{cases} \tag{2.3.62}$$

and

$$\begin{array}{lcl} R^i{}_j & = & \text{rotation matrix } RR^T = 1 \\ x^i & \mapsto & x^i + c^i \quad \text{is a translation} \\ x^i & \mapsto & x^i + v^i\, t \quad \text{corresponds to a special Galilei transformation} \\ t & \mapsto & t + T \quad \text{corresponds to a time translation} \end{array} \tag{2.3.63}$$

The total number of parameters is 10 just as for the relativistic Poincaré group. Let us write the corresponding Lie algebra. For the rotations we have the *angular momentum* generators:

$$J_{ij} = x_i\, \partial_j - x_j\, \partial_i \quad \rightarrow \quad J_i = \epsilon_{ijk}\, x_j\, \partial_k \tag{2.3.64}$$

for the space translations we have the *momentum generators*

$$P_i = \partial_i \tag{2.3.65}$$

while the *galileian boosts* are generated by:

$$K_i = t\, \partial_i \tag{2.3.66}$$

Finally the *hamiltonian* generates time translations:

$$H = \partial_t \tag{2.3.67}$$

By explicit evaluation of the commutators we find that the Galilei Lie algebra has the following structure:

$$\begin{array}{rcc c rcc} [J_i\, ,\, J_j] & = & \epsilon_{ijk}\, J_k & ; & [J_i\, ,\, P_j] & = - & \epsilon_{ijk}\, P_k \\ [J_i\, ,\, K_j] & = - & \epsilon_{ijk}\, K_k & ; & [J_i\, ,\, H] & = & 0 \\ [P_i\, ,\, H] & = & 0 & ; & [P_i\, ,\, P_j] & = & 0 \\ [K_i\, ,\, H] & = & -P_i & ; & [K_i\, ,\, K_j] & = & 0 \\ [P_i\, ,\, K_j] & = & 0 & & & & \end{array} \tag{2.3.68}$$

We can ask the question whether the Galilei algebra $\mathbb{G}$ is *semisimple*. The answer is no. Indeed P_i ($i = 1, 2, 3$) generate an abelian ideal since we easily verify that $[P\, ,\, X] \subset P$, $\forall X \in \mathbb{G}$, so that P is an ideal. Next we inquiry whether $\mathbb{G}$ is *solvable*. The derivative algebra $D\mathbb{G}$ is made by J_i, P_i, K_i. We easily verify, however, that $D^2\mathbb{G} = D\mathbb{G}$ so that $\mathbb{G}$ is not solvable. On the other hand if we consider the subalgebra $S^{(0)}$ generated by $\{P, K, H\}$ we see that:

$$DS^{(0)} = S^{(1)} = \{P\} \quad ; \quad DS^{(1)} = \{0\} \tag{2.3.69}$$

so that $S^{(0)}$ is solvable. The algebra generated by J_i is instead semisimple. Hence the Galilei algebra is, according to Levi's theorem, the direct product of a semisimple algebra with a solvable one.

2.4 The Adjoint representation and Cartan's Criteria

Let us now introduce the concept of *adjoint representation* of a Lie algebra $\mathbb{G}$. Given such an algebra, to each element $X \in \mathbb{G}$ we can associate a *linear endomorphism*:

$$\mathrm{ad}_X \quad : \quad \mathbb{G} \to \mathbb{G} \tag{2.4.70}$$

defined by:

$$\forall Y \in \mathbb{G} \quad : \quad \mathrm{ad}_X(Y) \equiv [X\,,\,Y] \tag{2.4.71}$$

If we choose a basis $\{T_A\}$ we immediately get.

$$\left(\mathrm{ad}_X\right)_A{}^B = X^M\, f_{MA}{}^B \tag{2.4.72}$$

where $f_{MA}{}^B$ are the Lie algebra structure constants, defined by:

$$[T_A\,,\,T_B] = f_{AB}{}^C\, T_C \tag{2.4.73}$$

Then we can introduce the bilinear symmetric Killing form of the Lie algebra:

$$\kappa : \mathbb{G} \otimes \mathbb{G} \to \mathbb{K} \tag{2.4.74}$$

defined by:

$$\forall X, Y \in \mathbb{G} \quad : \quad \kappa\,(X,Y) = Tr\ (\mathrm{ad}_X\, \mathrm{ad}_Y) \tag{2.4.75}$$

where $\mathbb{K}$ is the field over which the Lie algebra is constructed, namely $\mathbb{K} = \mathbb{C}$ for complex Lie algebras and $\mathbb{K} = \mathbb{R}$ for real Lie algebras.

In a basis we obtain:

$$\begin{aligned} \kappa\,(X,Y) \;&=\; \left(\mathrm{ad}_X\right)_A{}^B\, \left(\mathrm{ad}_Y\right)_B{}^A \;=\; X^M\, Y^N\, f_{MA}{}^B\, f_{NB}{}^A \\ &= X^M\, Y^N\, g^{(Killing)}_{MN} \end{aligned} \tag{2.4.76}$$

where the symmetric tensor $g^{(Killing)}_{MN} = f_{MA}{}^B\, f_{NB}{}^A$ is named the Killing metric.

2.4.1 Cartan's Criteria

Whether a Lie algebra is solvable or semisimple is fully encoded in the properties of the Killing form which therefore provides a very useful global tool to test the structure of the Lie algebra. That this is the case is established by two simple but very important theorems that go under the name of Cartan's criteria.

The first Cartan's criterion establishes a test of solvability and is provided by the following theorem.

Theorem 2.4.1. << A Lie algebra $\mathbb{G}$ is solvable if and only if

$$\forall X, Y, Z \in \mathbb{G} \qquad \kappa\,(X\,,\,[Y\,,\,Z]) = 0 \tag{2.4.77}$$

>>

Proof 2.4.1.1. piripicchio nil radical ■

The second Cartan's criterion, which uses the first in its own proof is a test of semi–simplicity. It is given by the following theorem.

Theorem 2.4.2. << A Lie algebra $\mathbb{G}$ is semisimple if and only if the Killing form $\kappa\,(\,,\,)$ on $\mathbb{G}$ is non degenerate. >>

Proof 2.4.2.1. We recall that a bilinear form $\kappa\,(\,,\,)$ on a vector space $\mathbb{G}$ is degenerate if $\exists X \in \mathbb{G}$ such that $\forall Y \in \mathbb{G}$ we have $\kappa\,(X\,,\,Y) = 0$. In a basis X_i this implies that the determinant of the matrix $\kappa_{ij} = \kappa\,(X_i\,,\,X_j)$ vanishes $\det\kappa_{ij} = 0$.

To prove the theorem we have to show that the following statements are both true:

a) If $\mathbb{G}$ is semisimple then κ is non degenerate.
b) If κ is non degenerate then $\mathbb{G}$ is semisimple.

Let us begin with the case a and let us assume that κ is degenerate, namely the set:

$$B = \{X \;:\; \kappa(X\,,\,Y) = 0\;,\;\forall Y \in \mathbb{G}\} \tag{2.4.78}$$

is non empty. We can immediately verify that B is an ideal of $\mathbb{G}$. Indeed $\forall X \in B$ and $\forall Z \in \mathbb{G}$ we have $[X\,,\,Y] \in B$ since $\kappa\,([X, Z]\,,\,Y) = 0\ \forall Y \in \mathbb{G}$. This follows from the properties of the Killing form that imply $\kappa\,([X, Z]\,,\,Y) = \kappa\,(X\,,\,[X, Y]) = 0$. Next we can show that:

$$\forall X, X' \in B \quad : \quad \kappa\,(X\,,\,X') = \kappa_B\,(X\,,\,X') \tag{2.4.79}$$

where $\kappa_B\,(,\,)$ denotes the restriction of the Killing form to ideal B. Indeed, given $Z \in \mathbb{G}$ we have:

$$\mathrm{ad}_X\,\mathrm{ad}_{X'}\,Z = [X, [X'\,,\,Z]] \in B \quad \text{since} \quad [X'\,,\,Z] \in B \tag{2.4.80}$$

This means that the image of the linear map $\mathrm{ad}_X\,\mathrm{ad}_{X'}$ is contained in the ideal B which implies that the only contribution to the trace come from its restriction to the subspace B. By our definition of the ideal B we have $\kappa\,(X\,,\,X') = 0$ for all $X, X' \in B$, which by the above argument implies also $\kappa_B\,(X\,,\,X') = 0$. Hence the algebra $\mathbb{G}$ admits an ideal B whose Killing form is identically vanishing. By the first Cartan criterion 2.4.1 it follows that the ideal B is solvable. Yet this contradicts the assumption that the Lie algebra $\mathbb{G}$ was semisimple, so B is necessarily the empty set and the Killing form is non degenerate.

Let us turn to case b). Assume that the Lie algebra $\mathbb{G}$ is not semisimple and let us show that this implies that the Killing form is degenerate. If $\mathbb{G}$ is not semisimple there is a non–trivial solvable ideal $\mathcal{Q}$. By definition $\exists k \in \mathbb{N}$ such that:

$$\mathcal{A} \equiv \mathcal{D}^k\mathcal{Q} \neq 0 \quad ; \quad \mathcal{D}^{k+1}\mathcal{Q} = 0 \tag{2.4.81}$$

The subalgebra $\mathcal{A}$ is a non–trivial abelian ideal. As a next step we show that:

$$\forall X \in \mathcal{A}\,,\,\forall Y \in \mathbb{G} \quad : \quad \kappa\,(X\,,\,Y) = \kappa_{\mathcal{A}}\,(X\,,\,Y) \tag{2.4.82}$$

Indeed, given $Z \in \mathbb{G}$ we have $\mathrm{ad}_X \circ \mathrm{ad}_Y\,(Z) = [X\,,\,[Y\,,\,Z]] \in \mathcal{A}$ since $\mathcal{A}$ is an ideal. Hence the image of $\mathrm{ad}_X \circ \mathrm{ad}_Y$ as Z varies in $\mathbb{G}$ takes values only in $\mathcal{A}$ and therefore its trace takes contributions only from $\mathcal{A}$. This suffices to prove that eq.(2.4.82) is true. Next we observe that:

$$\forall X \in \mathcal{A}\,,\,\forall Y \in \mathbb{G} \quad \text{we have} \quad \kappa_{\mathcal{A}}\,(X, Y) = 0 \tag{2.4.83}$$

Indeed, given $X' \in \mathcal{A}$ we have $\mathrm{ad}_X \circ \mathrm{ad}_Y\,(X') = [X\,,\,[Y\,,\,X']] = 0$ since both $X \in \mathcal{A}\,,\,[Y\,,\,X'] \in \mathcal{A}$ and $\mathcal{A}$ is abelian. Hence there is no contribution to the trace. On the other hand, in force of eq. (2.4.82) we conclude that $\kappa\,(X\,,\,Y) = 0$ for all $Y \in \mathbb{G}$ and all $X \in \mathcal{A}$. This means that the Killing form is degenerate unless $\mathcal{A}$ is empty. So there cannot be any non-trivial solvable ideal and the algebra $\mathbb{G}$ has to be semisimple. This concludes the proof of the theorem. ■