Chapter 5

REPRESENTATIONS OF LIE ALGEBRAS

5.1 Introduction

In this Chapter ...

5.1.1 Relation to group representations

Via the exponential map, a matrix representation of a real section of a Lie algebra $\mathbb{G}$

$$\mathcal{D} \,:\, x \in \mathbb{G} \mapsto \mathcal{D}(x) \in \mathrm{Hom}(V)\,, \tag{5.1.1}$$

where V is a real vector space, gives rise to a representation of the associated simply connected Lie group Gs

$$D \,:\, g = \mathrm{e}^{x} \;\mapsto \mathcal{D}(g) = \mathrm{e}^{\mathcal{D}(x)} \in \mathrm{End}(V)\,. \tag{5.1.2}$$

Indeed, the homomorphicity of the map $\mathcal{D}$ defining the Lie algebra representation, Eq. (3.2.52), implies the homomorphicity, in the group sense, Eq. (1.2.10), of the map $\mathcal{D}$: for any $g = \mathrm{e}^{x}$, $h = \mathrm{e}^{y}$,

$$\begin{aligned} D(g)D(h) &= \mathrm{e}^{\mathcal{D}(x)}\mathrm{e}^{\mathcal{D}(y)} = \exp\left(\mathcal{D}(x) + \mathcal{D}(y) + \frac{1}{2}\left[\mathcal{D}(x), \mathcal{D}(y)\right] + \ldots\right) \\ &= \exp\left[\mathcal{D}\left(x + y + \frac{1}{2}\left[x,y\right] + \ldots\right)\right] = D\left(\mathrm{e}^{x+y+\frac{1}{2}[x,y]+\ldots}\right) = D(gh)\,. \end{aligned} \tag{5.1.3}$$

Namely, the representation of the group is obtained by exponentiating the matrix representatives of the algebra. Notice that the linear operators $\mathcal{D}(x)$ in Eq. (5.1.1) need not be invertible, but the operators $D(g)$ are instead invertible.

5.1.2 Irreducible representations

Just as it happens for group representations, Lie algebra representations can be reducible, fully reducible or irreducible. The definitions (starting from the definition of equivalent representations) are exactly as in Sec. 2.1.6, so we do not repeat them here. In particular, irreducible representations admit no invariant subspaces, and can be used as a buinding block of reducible representations. The latter, in an appropriate basis, are made of block-diagonal matrices, each block (when no further reducible) corresponding to an irrep. So, as we aready argued for group representations, we need to concentrate on irrepses only.

We will use for irreducible represetation of Lie algebras a notation reminiscent to the one adopted for group representations, denoting them as

$$\mathcal{D}_\mu \;:\; x \in \mathbb{G} \mapsto \mathcal{D}_\mu(x) \in \mathrm{Hom}(V)\,. \tag{5.1.4}$$

The dimension of $\mathcal{D}_\mu$, which is the dimension of the carrier space V, is denoted as d_μ.

5.1.2.1 Schur's lemma

The Lie algebraic analogue of Schur's lemma holds. In an irreducible representation $\mathcal{D}_\mu$, a matrix commuting with all matrix representatives can only be proportional to the identity:

$$[A, \mathcal{D}_\mu(x)] = 0 \;\forall x \in \mathbb{G} \;\Rightarrow A \propto \mathbf{1} \;\text{ (or } A = \mathbf{0})\,. \tag{5.1.5}$$

The proof is analogous to that of Schur's lemma for group representations: eigenspaces of A correspond to invariant subspaces of $\mathcal{D}$. Since the representation is irreducible, the only invariant subspaces are the whole carrier space V or the trivial subspace $\{0\}$, corresponding to $A = \mathbf{1}$ and $A = \mathbf{0}$ respectively.

5.1.2.2 Conjugate representations

Consider a Lie algebra $\mathbb{G}$, with generators t_a $(a = 1, \ldots, \dim \mathbb{G})$ satisfying

$$[t_a, t_b] = c_{ab}{}^c\, t_c\,. \tag{5.1.6}$$

Let $T_a = \mathcal{D}(t_a)$ be the representatives of the generators in a representation $\mathcal{D}$ of dimension d; the T_a's are usually referred to simply as "the generators in the representation $\mathcal{D}$".

The Lie algebra $\mathbb{G}$ admits then always another representation $\overline{\mathcal{D}}$, of the same dimension d, in which the generators are taken to be

$$\overline{T}_a \equiv \overline{\mathcal{D}}(t_a) = -\left(\mathcal{D}(t_a)\right)^T = -\left(T_a\right)^T\,. \tag{5.1.7}$$

Indeed, these generators satisfy the same algebra:

$$\left[\overline{T}_a, \overline{T}_b\right] = \left[T_a^T, T_b^T\right] = -\left[T_a, T_b\right]^T = -c_{ab}{}^c\, T_c^T = c_{ab}{}^c\, \overline{T}_c\,. \tag{5.1.8}$$

The representation $\overline{\mathcal{D}}$ is called the conjugate representation of $\mathcal{D}$. The reason is that in a real section of the algebra the generators are typically chosen to be anti-hermitean or hermitean (depending on the compactness or not of the corresponding one-parameter subspace). If, for instance, the generators are antihermitean, $T_a = -T_a^\dagger$, then we have

$$\overline{T}_a = -T_a^T = T_a^*\,, \tag{5.1.9}$$

namely, the generators in the conjugate representation are actually the complex conjugates of the generators.

In general, the operation of complex conjugation does not correspond to a change of basis $T_a \mapsto S^{-1} T_a S$. When it does, the conjugate representation $\overline{\mathcal{D}}$ is equivalent to $\mathcal{D}$. The self-conjugated representation $\mathcal{D}$ is also called a *real* representation.

5.2 Representations of complexified simple Lie algebras

We have discussed in the previous Chapter how general Lie algebras can be obtained from "basic building blocks" corresponding to simple Lie algebras. The complexified simple Lie algebras can be described in the canonical form of Cartan, which is extrmely well suited also for the discussion of their possible

5.2.1 The weight lattice

Let $\mathcal{D}$ be a representation of a (semi-)simple Lie algebra $\mathbb{G}$ of rank r. The *weights* of this representation are a set of vectors $\vec{\lambda}$, labelling the states $|\lambda\rangle$ in the representation, whose r components λ_i are the eigenvalues of the Cartan generators H_i on the states $|\lambda\rangle$:

$$\mathcal{D}(H_i)|\lambda\rangle = \lambda_i|\lambda\rangle\,,\ (i = 1, \ldots, r)\,. \tag{5.2.10}$$

Indeed, the Cartan generators commute with each other and can be simultaneously diagonalized. The basis $\{|\lambda\rangle\}$ is the basis in which this diagonalization is displayed. The weight vectors are vectors in the (Euclidean) r-dimensional space corresponding to the (dual of the) Cartan subalgebra $\mathcal{H}^*$, that is, the same space in which the roots $\vec{\alpha}$ live.

A representation $\mathcal{D}$ is identified by the corresponding set of weights $\Pi_{\mathcal{D}}$. Notice that a certain weight $\vec{\lambda}$ can occur in general with a multeplicity $m(\vec{\lambda}) \geq 1$.

The weight vectors $\vec{\lambda}$ of any representation can be shown to obey a very restrictive *integrality property*:

$$\forall\vec{\alpha} \in \Phi\,,\quad \langle\vec{\lambda}, \vec{\alpha}\rangle \in \mathbb{Z} \tag{5.2.11}$$

of its hook scalar products with the roots of $\mathbb{G}$. The proof is along similar lines to the proof of the integrability property of the hook products of two roots, see Sec. 6.1.1.

The condition Eq. (5.2.11) is tantamount to say that the possible weight vectors are rescricted to a lattice

$$\Lambda_{\mathrm{W}} = \{n_i\vec{\lambda}^i\,,\ n_i \in \mathbb{Z}\,,\ i = 1, \ldots, r\,\} \tag{5.2.12}$$

generatated by the so-called *simple weights* $\vec{\lambda}^i$, which have the properties that

$$\langle\vec{\lambda}^i, \vec{\alpha}_j\rangle = \delta^i{}_j\,, \tag{5.2.13}$$

where $\alpha_j \in \Delta$ are the simple roots of $\mathbb{G}$. Indeed, any root $\vec{\alpha} \in \Phi$ can be written as $\vec{\alpha} = m^i\vec{\alpha}_i$, with $m^i \in \mathbb{Z}$ (and all positive or negative). Then, using Eq. (5.2.12) and Eq. (5.2.13) we have

$$\langle\vec{\lambda}, \vec{\alpha}\rangle = \sum_{i,j} n_i m^j \langle\vec{\lambda}^i, \vec{\alpha}_j\rangle = \sum_i n^i m_i \in \mathbb{Z}\,. \tag{5.2.14}$$

Taking into account the definition Eq. (4.4.99) of the Cartan matrix: $C_{ij} = \langle\vec{\alpha}_i, \vec{\alpha}_j\rangle$, we have from Eq. (5.2.14) that the simple weights can be expanded on the basis provided by the simple roots as follows:

$$\vec{\lambda}^i = (C^{-1})^{ij}\vec{\alpha}_j\,. \tag{5.2.15}$$

Of course, the inverse relation

$$\vec{\alpha}^i = C_{ij}\vec{\lambda}^j \tag{5.2.16}$$

holds as well.

The weight lattice Λ_{W} is the *dual lattice* of the root lattice Λ_{R} generated by the simple roots:

$$\Lambda_{\mathrm{R}} = \{n^i \vec{\alpha}_i\,,\ n^i \in \mathbb{Z}\,,\ i = 1, \ldots, r\,\} \tag{5.2.17}$$

The definition of the dual lattice of a lattice as in Eq. (**??**) is precisely is that it is spanned by basis vectors $\vec{\lambda}^i$ that are dual to the $\vec{\alpha}_i$ in the sense of Eq. (5.2.14).

5.2.1.1 *Highest weights and irreducible representations*

Afinite-dimensional irreducible representation $\mathcal{D}_\mu$ of $\mathbb{G}$ is determined by assigning a single "dominant" weight

$$\vec{\Lambda} = m_i \vec{\lambda}^i\,, \quad (m_i \geq 0\, \forall i = 1, \ldots, r) \tag{5.2.18}$$

as the "highest weight" of the representation, namely as the state that is annihilated by asll "raising" operators E_α, with $\vec{\alpha} \in \Phi^+$:

$$\mathcal{D}_\mu(E_\alpha)|\Lambda\rangle = 0\,, \quad \forall \vec{\alpha} \in \Phi^+\,. \tag{5.2.19}$$

Thus, an irreducible representation may be labeled by the set of non-negative integers $\{m_i\}$ describing the expansion of the highest weight $\vec{\Lambda}$ in the basis of the simple weights, see Eq. (5.2.18). These integers are called the *Dynkin labels* of the irreducible representation. We will use often the notation

$$\vec{\lambda} = \{m_1, m_2, \ldots\} \tag{5.2.20}$$

to denote a weight vector

$$\vec{\lambda} = m_1 \vec{\lambda}^1 + m_2 \vec{\lambda}^2 + \ldots\,. \tag{5.2.21}$$

The other weights in the representation defined by $\vec{\Lambda}$, and their multiplicities, can be determined taking into account the following properties.

i) For every weight $\vec{\lambda}$, i.e., for every state $|\lambda\rangle$ in the representation, the action of the lowering operators can only be as follows:

$$\forall \vec{\alpha} \in \Phi^+\,, \quad \mathcal{D}_\mu(E_{-\alpha)}|\lambda\rangle = \begin{cases} 0\,; \\ |\lambda - \alpha\rangle\,. \end{cases} \tag{5.2.22}$$

This follows from the homomorphicity of the representation:

$$\begin{aligned} \mathcal{D}_\mu(H_i)\mathcal{D}_\mu(E_{-\alpha)}|\lambda\rangle &= \left[\mathcal{D}_\mu(H_i), \mathcal{D}_\mu(E_{-\alpha)}\right]|\lambda\rangle + \mathcal{D}_\mu(E_{-\alpha)}\left[\mathcal{D}_\mu(H_i), |\rangle\right]\lambda \\ &= -\alpha_i \mathcal{D}_\mu(E_{-\alpha)}|\lambda\rangle + \lambda_i \mathcal{D}_\mu(E_{-\alpha)}|\lambda\rangle\,. \end{aligned} \tag{5.2.23}$$

So, if it is not zero, $\mathcal{D}_\mu(E_{-\alpha)}|\lambda\rangle$ corresponds to a weight vector $\vec{\lambda} - \vec{\alpha}$ of the representation.

ii) Moreover, the $\vec{\alpha}$-string through $\vec{\lambda}$,

$$\vec{\lambda} - r\vec{\alpha}\,, \ldots, \vec{\lambda}\,, \ldots, \vec{\lambda} + q\vec{\alpha} \tag{5.2.24}$$

has an asymmetry

$$r - q = \langle \vec{\lambda}, \vec{\alpha} \rangle\,. \tag{5.2.25}$$

This is entirely analogous to the meaning of the hook product between two roots, see Eq. (5.2.17). In particular, from the highest weight $\vec{\Lambda} = m_i \vec{\lambda}^i$ start $\vec{\alpha}_i$-strings of total length

$$\langle \vec{\Lambda}, \vec{\alpha}_i \rangle = m_i + 1 \tag{5.2.26}$$

(since Λ is the highest weight, for all such strings $q = 0$ in Eq. (5.2.25), and the total number of elements in the string is $r + 1$). Considering the weights appearing an these strings, they can still have positive "Dynkin labels", so that they may again give rise to some $\vec{\alpha}_k$ strings, and so on.

iii) There are some rules to decide the multeplicity of weights appearing in several of the $\vec{\alpha}_i$ strings construced as above. Define the *level* of a weight $\vec{\lambda}$ as the number of simple roots that must be subtracted from the h.w. $\vec{\Lambda}$ to get to it. Then
   - the total number of weights (counted with their multiplicities) of the weights in two levels symmetrically placed betwee the highest and lowest weight must be the same;
   - the number of weights at a level k must be non-decreasing with k until half-way towards the lowest weight (after this point, it must of course non-increasing, according to the previous rule).

iv) The set $\Pi_{\mathcal{D}}$ of weights is closed under the action of the Weyl group $\mathcal{W}_{\mathbb{G}}$ of the Lie algebra.

5.2.1.2 *Weights of conjugate representations*

Let $\Pi_{\mathcal{D}} = \{\vec{\lambda}\}$ be the set of weights of a representation $\mathcal{D}$. The set of weights $\Pi_{\overline{\mathcal{D}}}$ of the conjugate representation $\overline{\mathcal{D}}$ contains the negative of the weights of $\mathcal{D}$:

$$\Pi_{\overline{\mathcal{D}}} = \{-\vec{\lambda}\,,\ \vec{\lambda} \in \Pi_{\mathcal{D}}\}\,. \tag{5.2.27}$$

Indeed, the Cartan generators H_i can be realized in $\mathcal{D}$ as diagonal matrices $\mathcal{D}(H_i)$. Therefore the definition Eq. (5.1.9) of the conjugate representation implies that

$$\overline{\mathcal{D}}(H_i) = -\mathcal{D}(H_i)\,. \tag{5.2.28}$$

Hence, from the definition Eq. (5.2.10) of the weigth components λ_i as the eigenvalues of the representatives of the Cartan generators H_i it follows that if to each weight $\vec{\lambda} = \{\lambda_1, \ldots, \lambda_r\}$ of $\mathcal{D}$ corresponds the weight $-\vec{\lambda}$ of $\overline{\mathcal{D}}$.

5.2.1.3 *Weights of the adjoint representation*

Comparing the Cartan canonical form Eq. (5.2.17) of a Lie algebra $\mathbb{G}$ of rank r with the definition Eq. (5.2.10) of the weight vectors, it is clear that the roots $\vec{\alpha}$ of $\mathbb{G}$, plus r null vectors, represent the weights of the adjoint representation. Recall that the adjoint representation acts on the vector space given by $\mathbb{G}$ itself. Denote as $|\alpha\rangle$ the basis state of $\mathbb{G}$ given by the generator E_α, for every root $\vec{\alpha}$, and as $|i\rangle$ $(i = 1, \ldots, r)$ the states given by the Cartan generators. We have then

$$\mathrm{ad}(H_i)|\alpha\rangle \equiv |\,[H_i, E_\alpha]\rangle = \alpha^i|\alpha\rangle\,. \tag{5.2.29}$$

Example: irreducible representations of A_1 The $A_1 \sim \mathrm{sl}(2,\mathbb{C})$ algebra admits as a compact real form the su(2) algebra. We expect therefore the finite-dimensional irreducible representations of A_1 to be characterized by the semi-integer "spin" j, as we are familiar from Quantum Mechanics.

Since A_1 has rank 1, it has a single simple root $\vec{\alpha}_1$, a uni-dimensional vector whose norm we chose to be $\sqrt{2}$. The Cartan matrix has a single entry $C_{11} = 2$, so the single simple weight is given by

$$\vec{\lambda}^1 = \frac{\vec{\alpha}_1}{2}\,, \tag{5.2.30}$$

so its norm is $1/\sqrt{2}$. An irreducible representation is defined by a dominant weight

$$\vec{\Lambda} = n\,\vec{\lambda}^1\,, \tag{5.2.31}$$

i.e. by a single Dynkin label $n \in \mathbb{Z}$ (with $n \geq 0$). From $\vec{\Lambda}$ there departs an $\vec{\alpha}_1$-string of total length $n+1$:

$$\vec{\Lambda} = n\vec{\lambda}^1\,,\ \vec{\Lambda} - \vec{\alpha}_1 = (n-2)\vec{\lambda}^1\,,\ \ldots\,,\ \vec{\Lambda} - n\vec{\alpha}_1 = -n\vec{\lambda}^1\,. \tag{5.2.32}$$

So the representation defined by the Dynkin label n is $n+1$-dimensional. The single components of the weight vectors appearing in Eq. (5.2.35) represent, according to Eq. (5.2.10), the eigenvalue of the single Cartan generator H_1 in this representation. The component of $\vec{\lambda}^1$ is $1/\sqrt{2}$, so the eigenvalues corresponding to Eq. (5.2.35) are

$$\frac{n}{\sqrt{2}}\,,\ \frac{n-2}{\sqrt{2}}\,,\ \ldots\,,\ -\frac{n}{\sqrt{2}}\,. \tag{5.2.33}$$

Notice that the above values are obtained starting from a normalization in which, in the 2×2 representation, H_1 is normalized so that $\mathrm{tr}(H_1)^2 = 1$, see Eq. (5.2.17). The traditional normalization in Physics for su(n) algebras is that in the fundamental $n \times n$ representation $\mathrm{tr_f}\, t_a t_b = \frac{1}{2}\delta_{ab}$. This choice correponds to rescale H_1, and thus its eigenvalues, by $\frac{1}{\sqrt{2}}$. Setting

$$n = 2j\,, \tag{5.2.34}$$

the eigenvalues Eq. (5.2.35) in the $2j+1$-dimensional representation labeled by j are given, in the "physical" normalization, by

$$j\,,\, j-1\,,\, \ldots\,,\, -j\,. \tag{5.2.35}$$

The eigenvalue of the Cartan generator are usually denoted as m, and referred to as the "third component" of the spin, as $H_1 \propto \sigma_3$.

5.2.1.4 Highest weight prepresentations of A_2

The algebra $A_2 \sim \mathrm{sl}(3,\mathbb{R})$ admits as a compact real section the algebra su(3). It has rank 2, and its root system was already described in Sec. 5.2.17. The simple roots can be described, according to the general theory for A_n algebras, in an $\mathbb{R}^3$ space with versors $\vec{e}_i$, as

$$\vec{\alpha}_1 = \vec{e}_1 - \vec{e}_2\,,\quad \vec{\alpha}_2 = \vec{e}_2 - \vec{e}_3\,. \tag{5.2.36}$$

They belong to the hyperplane orthogonal to $\sum_{i=1}^3 \vec{e}_i$. An O.N. basis for this hyperplane is given by

$$\mathbf{i} = \frac{\vec{e}_1 - \vec{e}_2}{\sqrt{2}}\,,\quad \mathbf{j} = \frac{\vec{e}_1 + \vec{e}_2 - 2\vec{e}_3}{\sqrt{6}}\,. \tag{5.2.37}$$

We have then, from Eq. (5.2.36) and Eq. (5.2.37),

$$\vec{\alpha}_1 = \sqrt{2}\mathbf{i}\,,\quad \vec{\alpha}_2 = -\frac{1}{\sqrt{2}}\mathbf{i} + \sqrt{\frac{3}{2}}\mathbf{j}\,. \tag{5.2.38}$$

We will write also simply $\vec{\alpha}_1 = (\sqrt{2}, 0)$ and $\vec{\alpha}_2 = (-1/\sqrt{2}, \sqrt{3/2})$. These are exactly the simple root vectors of Fig. 5.2.17, obtained there by considering their relative angle $2\pi/3$ and their norm $\sqrt{2}$.

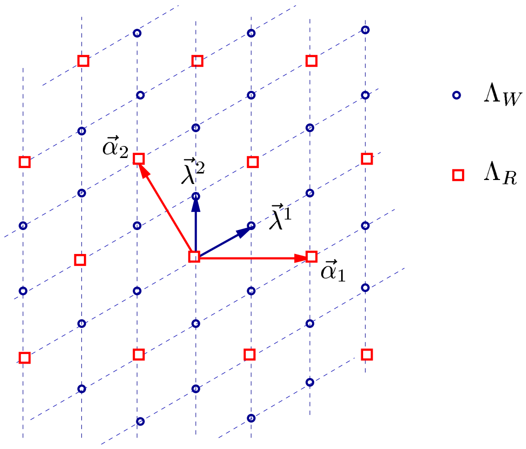

Figure 5.1. The weight and root lattices for A_2.

The Cartan matrix and its inverse are given by

$$C = \begin{pmatrix} 2 & -1 \\ -1 & 2 \end{pmatrix}, \quad C^{-1} = \frac{1}{3}\begin{pmatrix} 2 & 1 \\ 1 & 2 \end{pmatrix}, \tag{5.2.39}$$

so, according to Eq.s (5.2.15,5.2.16), the simple weights $\vec{\lambda}^1$ and $\vec{\lambda}^2$ are related to the simple roots by

$$\begin{cases} \vec{\alpha}_1 &= 2\vec{\lambda}^1 - \vec{\lambda}^2\,, \\ \vec{\alpha}_2 &= -\vec{\lambda}^1 + 2\vec{\lambda}^2\,; \end{cases} \qquad \begin{cases} \vec{\lambda}^1 &= \frac{1}{3}(2\vec{\alpha}_1 + \vec{\alpha}_2) = \frac{1}{\sqrt{2}}\left(1, \frac{1}{\sqrt{3}}\right), \\ \vec{\lambda}^2 &= \frac{1}{3}(\vec{\alpha}_1 + 2\vec{\alpha}_2) = \frac{1}{\sqrt{2}}\left(0, \frac{2}{\sqrt{3}}\right). \end{cases} \tag{5.2.40}$$

The simple weights and simple roots, and the weight and root lattices Λ_{W} and Λ_{R} that they generate, are drawn in Fig. 5.1.

Let us construct some irreducible representations of A_2.

- Let us take the highest weight

$$\vec{\Lambda} = \vec{\lambda}^1 = \{1, 0\}\,, \tag{5.2.41}$$

where in the last term we utilized the handy notation of Eq. (5.2.20). From $\vec{\Lambda}$ departs an $\vec{\alpha}_1$ string of total length 2:

$$\vec{\Lambda} = \{1, 0\} \xrightarrow{-\vec{\alpha}_1} \{-1, 1\}\,, \tag{5.2.42}$$

where we used Eq. (5.2.40) that says that $\vec{\alpha}_1 = \{2, -1\}$. From the new weight $\{-1, 1\} = -\vec{\lambda}^1 + \vec{\lambda}^2$ obtained above starts an $\vec{\alpha}_2$-string of total lenght 2. Indeed, $\langle -\vec{\lambda}^1 + \vec{\lambda}^2, \vec{\alpha}_2\rangle = 1$. So, altogether we have the following structure:

$$\{1, 0\} \xrightarrow{-\vec{\alpha}_1} \{-1, 1\} \xrightarrow{-\vec{\alpha}_2} \{0, -1\}\,. \tag{5.2.43}$$

This set of weights is depicted in Fig. 5.3. The last weight obtained has no more positive coefficients in its expansion in simple weights, so no more $\vec{\alpha}_i$-strings depart from it.
The representation defined by $\vec{\Lambda} = \{1, 0\}$ is therefore 3-dimensional. It is called the *fundamental* representation, and it is usually denoted as **3**.

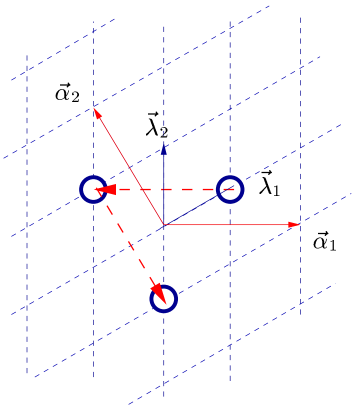

Figure 5.2. The weight diagram of the representation **3** of A_2.

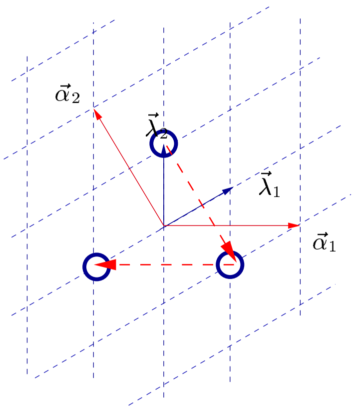

Figure 5.3. The weight diagram of the representation $\overline{\mathbf{3}}$ of A_2.

- Let us take the highest weight
$$\vec{\Lambda} = \vec{\lambda}^2 = \{0,1\}\,. \tag{5.2.44}$$
From $\vec{\Lambda}$ departs an $\vec{\alpha}_2$ string of total length 2:
$$\vec{\Lambda} = \{0,1\} \xrightarrow{-\vec{\alpha}_2} \{1,-1\}\,, \tag{5.2.45}$$
where we used Eq. (5.2.40) that says that $\vec{\alpha}_2 = \{-1,2\}$. From the new weight $\{1,-1\} = \vec{\lambda}^1 - \vec{\lambda}^2$ obtained above starts an $\vec{\alpha}_1$-string of total lenght 2. Indeed, $\langle \vec{\lambda}^1 - \vec{\lambda}^2, \vec{\alpha}_1 \rangle = 1$. So, altogether we have the following structure:
$$\{0,1\} \xrightarrow{-\vec{\alpha}_2} \{1,-1\} \xrightarrow{-\vec{\alpha}_1} \{-1,0\}\,. \tag{5.2.46}$$
This set of weights is depicted in Fig. ??. The last weight obtained has no more positive coefficients in its expansion in simple weights, so no more $\vec{\alpha}_i$-strings depart from it.
The 3-dimensional representation defined by $\vec{\Lambda} = \{0,1\}$ is the conjugate of the fundamental representation **3** described above. Indeed its set of weights contains exactly all the negatives of the 3-dimensional. This representation is called the *anti-fundamental* representation, and it is usually denoted as $\overline{\mathbf{3}}$.
- Let us now consider the irreducible representation with highest weight
$$\vec{\Lambda} = \vec{\lambda}^1 + \vec{\lambda}^2 = \{1,1\}\,. \tag{5.2.47}$$
Notice that, according to Eq. (5.2.40), we have
$$\vec{\Lambda} = \vec{\alpha}_1 + \vec{\alpha}_2\,. \tag{5.2.48}$$
From $\vec{\Lambda}$ start an $\vec{\alpha}_1$- and and $\vec{\alpha}_2$-string, both of total length 2:
$$\begin{aligned} \vec{\Lambda} = \{1,1\} = \vec{\alpha}_1 + \vec{\alpha}_2 &\xrightarrow{-\vec{\alpha}_1} \{-1,2\} = \vec{\alpha}_2\,, \\ &\xrightarrow{-\vec{\alpha}_2} \{2,-1\} = \vec{\alpha}_1\,. \end{aligned} \tag{5.2.49}$$

From the new weight $\{-1,2\} = \vec{\alpha}_2$ starts an $\vec{\alpha}_2$-string of total length 3, which is nothing else than the string containing the root $\vec{\alpha}_2$, $\vec{0}$ and $-\vec{\alpha}_2$:

$$\{-1,2\} = \vec{\alpha}_2 \xrightarrow{-\vec{\alpha}_2} \{0,0\} \xrightarrow{-\vec{\alpha}_2} \{1,-2\} = -\vec{\alpha}_2\,. \tag{5.2.50}$$

Similarly, from the other weight$\{2,-1\} = \vec{\alpha}_1$ starts the $\vec{\alpha}_1$-string

$$\{2,-1\} = \vec{\alpha}_1 \xrightarrow{-\vec{\alpha}_1} \{0,0\} \xrightarrow{-\vec{\alpha}_1} \{-2,1\} = -\vec{\alpha}_1\,. \tag{5.2.51}$$

From the lowest weight $\{1,-2\} = -\vec{\alpha}_2$ in Eq. (5.2.50) departs an $\vec{\alpha}_1$-string of total length 2, and from the lowest weight $\{-2,1\} = -\vec{\alpha}_1$ in Eq. (5.2.53) departs an $\vec{\alpha}_1$-string of total length 2:

$$\{1,-2\} = -\vec{\alpha}_2 \xrightarrow{-\vec{\alpha}_1} \{-1,-1\} = -(\vec{\alpha}_1+\vec{\alpha}_2)\,, \tag{5.2.52}$$

$$\{-2,1\} = -\vec{\alpha}_1 \xrightarrow{-\vec{\alpha}_2} \{-1,-1\} = -(\vec{\alpha}_1+\vec{\alpha}_2)\,. \tag{5.2.53}$$

The null weight $\{0,0\}$, of level 2 is reached subtracting two simple roots from the h.w. in two different ways, see Eq. (5.2.49) and then Eq. (5.2.50) and Eq. (5.2.50). The multiplicity of the weight has to be two, because we have two distinct weights both at level 1 and 3 (so if the multeplicity was 1 it would increase then decrease then increase again, contrary to the general behaviour mentioned in Sec. 5.2.1. Also the lowest weight $\{-1,-1\} = -(\vec{\alpha}_1+\vec{\alpha}_2)$ is reached in different ways, but its multeplicity is 1, as it is symmetric with respect to the unique highest weight.
Thus this representation contains 8 states, namely it has dimension 8. It is usually denoted as **8**. Notice that:

- this is the *adjoint* representation of A_2: indeed its set of weights coincides exactly with the roots of A_2, plus two null weights corresponding to the two Cartan generators.
- It is a real representation (as it is always the case for the adjoint representation): for each weight the opposite weight is also a weight of the representation.

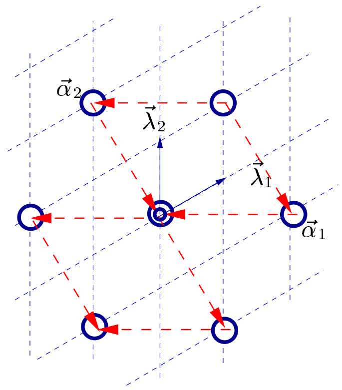

Figure 5.4. The weight diagram of the adjoint representation **8** of A_2.

5.2.2 The Casimir operator in an irreducible representation

The quadratic casimir operator was defined in Eq. (5.2.17) as

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In the Cartan basis, the Killing metric has the form of Eq. (5.2.17). The Casimir operator....

5.2.2.1 Action of the step operators in an irreducible representation

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5.2.2.2 The index of a rapresentation

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