Chapter 1

SOME MATHEMATICAL TOOLS

1.1 Some definitions in algebra

Let us recall "in a nutshell" the definition of some important algebraic structure, increasingly more "refined" than that of group.

Ring A ring $\mathcal{R}$ is a *Abelian group* (for which the additive notation is usually employed: the product is $x+y$, the identity is 0, etc.). Moreover on $\mathcal{R}$ is defined another product (for which the standard multiplicative notation is used: $x \cdot y$ or simply xy when no confusion is possible), which is *distributive* with respect to the sum:

$$\forall x, y, z \in \mathcal{R}\,, \quad x(y+z) = xy + xz\,. \tag{1.1.1}$$

With respect to this second product, $\mathcal{R}$ is a *semigroup*, namely the product is *associative*:

$$\forall x, y, z \in \mathcal{R} \quad x \cdot (y \cdot z) = (x \cdot y) \cdot z\ , \tag{1.1.2}$$

and there is an *identity* element e:

$$x \cdot e = e \cdot x = x\,. \tag{1.1.3}$$

It is instead *not* required the existence of an inverse for each element x. The typical example of a ring is $\mathbb{Z}$ (a part from 1, all other numbers do not admit an inverse in $\mathbb{Z}$).

Field A field is a ring in which all elements *do* admit an inverse with respect to the product, *except* for the neutral element with respect to the addition, 0. Examples of fields are $\mathbb{Q}$ of $\mathbb{R}$.

Vector space A vector space V over a field $\mathbb{F}$ (which for us will always be $\mathbb{R}$ or $\mathbb{C}$) is first of all an Abelian group, whose elements $\vec{v}$ can therefore be summed:

$$\forall \vec{v}, \vec{w} \in V\,, \quad \vec{v} + \vec{w} \in V\,, \tag{1.1.4}$$

there is a neutral element $\vec{0}$ and each element $\vec{v}$ has an inverse $-\vec{v}$. Moreover, a second operation is defined on V, namely the *multiplication by scalars*:

$$\forall \lambda \in \mathbb{F}\,, \forall \vec{v} \in V\,, \quad f\vec{v} \in V\,, \tag{1.1.5}$$

with the following properties:

$$\begin{aligned} \forall\lambda \in \mathbb{F}\,, \forall\vec{v}_1, \vec{v}_2 \in V &: \quad \lambda(\vec{v}_1 + \vec{v}_2) = \lambda\vec{v}_1 + \lambda\vec{v}_2\,; \\ \forall\lambda, \mu \in \mathbb{F}\,, \forall\vec{v} \in V &: \quad (\lambda + \mu)\vec{v} = \lambda\vec{v} + \mu\vec{v}\,; \\ \forall\lambda, \mu \in \mathbb{F}\,, \forall\vec{v} \in V &: \quad (\lambda\mu)\vec{v} = \lambda(\mu\vec{v})\,; \\ \forall\vec{v} \in V &: \quad 1\,\vec{v} = \vec{v}\,. \end{aligned} \tag{1.1.6}$$

An example of a vector space is of course the space of ordinary vectors in, say, three dimensions.

Algebra An Algebra $\mathcal{A}$ is a *vector space* on a filed $\mathbb{F}$ on which an additional operation

$$\cdot\ :\ \mathcal{A} \times \mathcal{A} \to \mathcal{A} \tag{1.1.7}$$

is defined. This additional operation must be *associative* and *distributive* and linear with respect to the vector addition:

$$\begin{aligned} \forall f, g, h \in \mathcal{A}: &\quad f \cdot (g \cdot h) = (f \cdot h) \cdot h\,; \\ \forall f, g, h \in \mathcal{A}\,, \forall\lambda, \mu \in \mathbb{F}: &\quad f \cdot (\lambda g + \mu h) = \lambda\, f \cdot g + \mu\, f \cdot h\,. \end{aligned} \tag{1.1.8}$$

1.2 Vector spaces and linear operators

1.2.1 Vector spaces

General linear groups and basis changes in vector spaces As already recalled, the non-singular $n \times n$ matrices A with real or complex entries, i.e. the elements of $\mathrm{GL}(n, \mathbb{R})$ or $\mathrm{GL}(n, \mathbb{C})$ are *automorphisms* of real or complex vector spaces. Namely, they describe the possible change of basis in a n-dimensional vector space $\mathbb{R}^n$ or $\mathbb{C}^n$: $\vec{e}_i{}' = A_i{}^j \vec{e}_j$ (in matrix notation, $\vec{e}\,' = A\vec{e}$). Two subsequent changes of basis, first with A and then with B, result in a change described by the matrix product BA. The basis elements $\vec{e}_i$ are said to transform *covariantly*. The components v^i of a vector $\vec{v} = v^i\vec{e}_i$ transform *contravariantly*: from $\vec{v} = v^{i\prime}\vec{e}_i{}' = v^i\vec{e}_i$ it follows $v^i = v^{j\prime} A_j{}^i$ (in matrix notation, $v = v'A$, or $v' = A^{-1}v$).

Tensor product spaces Starting from two vector spaces, say V_1 (n-dimensional, with basis $\vec{e}_i$) and V_2 (m-dimensional, with basis $\vec{f}_j$), it is possible to construct bigger vector spaces, for instance by taking the *direct sum* $V_1 \oplus V_2$ of the two spaces or their *direct product* $V_1 \otimes V_2$. The direct product has dimension mn and basis $\vec{e}_i \otimes \vec{f}_j$. A change of basis A in V_1 and B in V_2 induce a change of basis in the direct product space described[1] by $(\vec{e}_i \otimes \vec{f}_j)' = A_i{}^k\, B_j{}^l\, \vec{e}_k \otimes \vec{f}_l$. An element of a direct product vector space is called a *tensor*.

In particular, one can take the direct product of a vector space V of dimension n with itself (in general, m times): $V \otimes \ldots \otimes V$, which we may indicate for shortness as $V^{\otimes m}$. Its n^m basis vectors are $\vec{e}_{i_1} \otimes \ldots \otimes \vec{e}_{i_m}$. An element of $V^{\otimes m}$ is called a *tensor of order* m. An element A of the general linear group G of V (a change of basis in V) induces a change of basis on $V^{\otimes m}$:

$$\vec{e}_{i_1}{}' \otimes \ldots \otimes \vec{e}_{i_m}{}' = A_{i1}^{\ j_1} \ldots A_{im}^{\ jm}\, \vec{e}_{j_1} \otimes \ldots \otimes \vec{e}_{j_m}\ . \tag{1.2.9}$$

Such changes of bases on $V^{\otimes m}$ form a group which is the direct product of G with itself, m times, $G^{\otimes m}$. Its elements are $n^m \times n^m$ matrices, and the association of an element A of G with

[1] The subgroup of all possible changes of basis obtained in this way is the *direct product* $G_1 \otimes G_2$ (a concept we will deal with in the next section) of the general linear groups (groups of basis changes) G_1 and G_2 of V_1 and V_2.

an element of $G^{\otimes m}$ given in Eq. (1.2.9) preserves the product; thus $G^{\otimes m}$ is a representation of G, called an m-th order tensor product representation. This way of constructing representations by means of tensor products is, as we will see, very important.

(Anti)-symmetrized product spaces Within the direct product space $V \otimes V$, we can single out two subspaces spanned respectively from *symmetric* and *antisymmetric* combinations[2] of the basis vectors $\vec{e}_i \otimes \vec{e}_j$:

$$\begin{aligned} \vec{e}_i \vee \vec{e}_j &\equiv \vec{e}_i \otimes \vec{e}_j + \vec{e}_i \otimes \vec{e}_j = (\mathbf{1} + (12))\,(\vec{e}_i \otimes \vec{e}_j)\ , \\ \vec{e}_i \wedge \vec{e}_j &\equiv \vec{e}_i \otimes \vec{e}_j - \vec{e}_i \otimes \vec{e}_j = (\mathbf{1} + (12))\,(\vec{e}_i \otimes \vec{e}_j)\ , \end{aligned} \tag{1.2.10}$$

where in the second equality we denoted by $\mathbf{1}$ and (12) the identity and the exchange element of the symmetric group S_2 acting on the positions of the two elements in the direct product basis. The symmetric subspace has dimension $n(n+1)/2$, the antisymmetric one $n(n-1)/2$.

Similarly, within $V^{\otimes m}$ the *fully symmetric* and *fully antisymmetric* subspaces can easily be defined. Their basis elements can be written as

$$\begin{aligned} \vec{e}_{i_1} \vee \ldots \vee \vec{e}_{i_m} &\equiv \sum_{P \in S_m} P\,(\vec{e}_{i_1} \otimes \ldots \otimes \vec{e}_{i_m})\ , \\ \vec{e}_{i_1} \wedge \ldots \wedge \vec{e}_{i_m} &\equiv \sum_{P \in S_m} (-)^{\sigma(P)} P\,(\vec{e}_{i_1} \otimes \ldots \otimes \vec{e}_{i_m})\ , \end{aligned} \tag{1.2.11}$$

where $\sigma(P)$ is 1 or -1 if the permutation P is even or odd, respectively. The fully symmetric subspace has dimension $n(n+1)\ldots(n+m-1)/m!$, the fully antisymmetric one has dimension $n(n-1)\ldots(n-m+1)/m! = \binom{n}{m}$.

1.2.1.1 Nilpotent and semisimple endomorphisms - Jordan decomposition

Definition 1.2.1. << A matrix A_{ij} such that $A_{ij} = 0$ if $i > j$ is named **upper triangular**. A matrix such that $A_{ij} = 0$ if $i < j$ is named **lower triangular**. Finally a matrix that is simultaneously upper and lower triangular is named **diagonal** >>

Let us recall the concept of eigenvalue:

Definition 1.2.2. << Let $A \in \mathrm{Hom}(V, V)$. A complex number $\lambda \in \mathbb{C}$ is named an **eigenvalue** of A if $\exists\, \overrightarrow{v} \in V$ such that:

$$A\,\overrightarrow{v} \;=\; \lambda\,\overrightarrow{v} \tag{1.2.12}$$

>>

Definition 1.2.3. << Let λ be an eigenvalue of the endomorphism $A \in \mathrm{Hom}(V, V)$, the set of vectors $\overrightarrow{v} \in V$ such that $A\,\overrightarrow{v} = \lambda\,\overrightarrow{v}$ is named the **eigenspace** $V_\lambda \subset V$ pertaining to the eigenvalue λ. It is obvious that it is a vector subspace. >>

As it is known from elementary courses in Geometry and Algebra the possible eigenvalues of A are the roots of the secular equation:

$$\det\,(\lambda\,\mathbf{1} \,-\, \mathcal{A}) \;=\; 0 \tag{1.2.13}$$

[2] The notation $\vec{e}_i \vee \vec{e}_j$, though logical, is not much used. Often, with a bit of abuse, a symmetric combination is simply indicated as $\vec{e}_i \otimes \vec{e}_j$; we will sometimes do so, when no confusion is possible.

where **1** is the unit matrix and $\mathcal{A}$ is the matrix representing the endomorphism A in an arbitrary basis.

Definition 1.2.4. << An endomorphism $N \in \mathrm{Hom}(V, V)$ is named nilpotent if there exists an integer: $\exists\, k \in \mathbb{N}$ such that:

$$N^k = 0 \tag{1.2.14}$$

>>

Lemma 1.2.1. << A nilpotent endomorphism has always the unique eigenvalue $0 \in \mathbb{C}$ >>

Proof 1.2.1.1. Let λ be an eigenvalue and let $\overrightarrow{v} \in V_\lambda$ be an eigenvector. We have:

$$N^r\, \overrightarrow{v} \;=\; \lambda^r\, \overrightarrow{v} \tag{1.2.15}$$

Choosing $r = k$ we obtain $\lambda^k = 0$ which necessarily implies $\lambda = 0$. ■

Lemma 1.2.2. << Let $N \in \mathrm{Hom}(V, V)$ be a nilpotent endomorphism. In this case one can choose a basis $\{\overrightarrow{e}_i\}$ of V such that in this basis the matrix N_{ij} satisfies the condition $N_{ij} = 0$ for $i \geq j$. >>

Proof 1.2.2.1. Let $\overrightarrow{e}_1$ be a null eigenvector of N, namely $N\,\overrightarrow{e}_1 = 0$ and let E_1 be the subspace of V generated by $\overrightarrow{e}_1$. ¿From N we induce an endomorphism N_1 acting on the space V/E_1, namely the vector space of equivalence classes of vectors in V modulo the relation:

$$\overrightarrow{v} \sim \overrightarrow{w} \Leftrightarrow \overrightarrow{v} - \overrightarrow{w} = m\,\overrightarrow{e}_1 \quad , \quad m \in \mathbb{C} \tag{1.2.16}$$

Also the new endomorphism $N_1 \;:\quad V/E_1 \;\to\; V/E_1$ is nilpotent. If $\dim V/E_1 \neq 0$, then we can find another vector $e_2 \in V$ such that $(e_2 + E_1) \in V/E_1$ is an eigenvector of N_1. Continuing iteratively this process we obtain a basis $\overrightarrow{e}_1, \ldots, \overrightarrow{e}_n$ of V such that:

$$N\, e_1 = 0 \quad ; \quad N\, e_p = 0 \,\mathrm{mod}\, (e_1, \ldots, e_{p-1}) \quad ; \quad 2 \leq p \leq n \tag{1.2.17}$$

where $(e_1, \ldots, e_{p-1})$ denotes the subspace of V generated by the vectors $e_1, \ldots, e_{p-1}$. In this basis the matrix representing N is triangular. Similarly if N_{ij} is triangular with $N_{ij} = 0$ for $i \geq j$, then the corresponding endomorphism is nilpotent ■

Definition 1.2.5. << Let $\mathcal{S} \subset \mathrm{Hom}(V, V)$ be a subset of the ring of endomorphisms and $W \subset V$ a vector subspace. The subspace W is named **invariant** with respect to $\mathcal{S}$ if $\forall S \in \mathcal{S}$ we have $S\, W \subset W$. The space V is named **irreducible** if does not contain invariant subspaces. >>

Definition 1.2.6. << A subset $\mathcal{S} \subset \mathrm{Hom}(V, V)$ is named **semisimple** if every invariant subspace $W \subset V$ admits an orthogonal complement which is also invariant. In that case we can write:

$$V = \bigoplus_{i=1}^{p} W_i \tag{1.2.18}$$

where each subspace W_i is invariant >>

A fundamental and central result in Linear Algebra, essential for the further development of Lie algebra theory is the *Jordan's decomposition theorem* that we quote without proof.

Theorem 1.2.1. << Let $L \in \mathrm{Hom}\,(V, V)$ be an endomorphism of a finite dimensional vector space V. Then there exists and it is unique the following Jordan decomposition:

$$L = S_L + N_L \tag{1.2.19}$$

where S_L is semisimple and N_L is nilpotent. Furthermore, both S_L and N_L can be expressed as polynomials in L >>

1.2.2 Some properties of matrices

Let us recall here some definitions and properties of matrices that are important in the following.

It is possible to define a *matrix exponential* function by means of a formal power series expansion: if A is a square matrix,

$$\exp(A) = \sum_{k=0}^{\infty} \frac{1}{k!} A^k \ . \tag{1.2.20}$$

For instance, consider the following 2×2 case:

$$m(\theta) = \theta\,\epsilon = \begin{pmatrix} 0 & \theta \\ -\theta & 0 \end{pmatrix} \quad \Rightarrow \quad \exp[m(\theta)] = \begin{pmatrix} \cos\theta & \sin\theta \\ -\sin\theta & \cos\theta \end{pmatrix} \ , \tag{1.2.21}$$

where θ is a real parameter. The above expression easily follows from Eq. (1.2.20) and from the fact that that the Levi-Civita antisymmetric symbol ϵ_{ij}, seen as a matrix, obeys $\epsilon^2 = -\mathbf{1}$.

This matrix exponentials enjoys various properties of the usual exponential, but not all of them. For instance, with T, S square matrices, one has

$$\exp(T)\exp(S) = \exp(T+S) \quad \Leftrightarrow \quad [T,S] = 0 \ , \tag{1.2.22}$$

i.e. only if the two matrices commute. Otherwise, Eq. (**??**) generalizes to the so-called *Baker-Campbell-Hausdorff* formula

$$\exp(T)\exp(S) = \exp\left(T + S + \frac{1}{2}[T,S] + \frac{1}{12}\left([T,[T,S]] + [[T,S],S]\right) + \dots\right) \ . \tag{1.2.23}$$

Deduce directly the first terms in the expansion above. A property which is immediate to show (check it) is that

$$\exp(U^{-1}TU) = U^{-1}\exp(T)\,U \ . \tag{1.2.24}$$

A very useful property of the determinants is the following:

$$\det\left(\exp(m)\right) = \exp\left(\mathrm{tr}\, m\right) \ , \tag{1.2.25}$$

or equivalently, by taking the logarithm (which for matrices is defined via a formal power series)

$$\det M = \exp\left(\mathrm{tr}(\ln M)\right) \ . \tag{1.2.26}$$

These relations can be proven easily for a diagonalizable matrix; indeed, determinant and trace are invariant under change of basis, so we can think of having diagonalized M. If λ_i are the eigenvalues of M, we have then

$$\det M = \prod_i \lambda_i = \exp\bigl(\sum_i \ln\lambda_i\bigr) = \exp\left(\mathrm{tr}(\ln M)\right) \ . \tag{1.2.27}$$

The result can then be extended to generic matrices as it can be argued that every matrix can be approximated to any chosen accuracy by diagonalizable matrices.

Let us note that on the space of $n \times n$ matrices (with, say, complex entries) one can define a distance by

$$d(M,N) = \bigl(\sum_{i,j=1}^{n} |M_{ij} - N_{ij}|^2\bigr)^{1/2} \ , \tag{1.2.28}$$

where M, M are two matrices. This is nothing else than considering the space of generic $n \times n$ matrices as the $\mathbb{C}^{n^2}$ space parametrized by the n^2 complex entries, and to endow it with the usual topology.

Exponential maps for operators Notice that the definition Eq. (1.2.21) of the exponential map can be generalized to linear *operators* acting on ∞-dimensional spaces of functions, with all the properties described for the matrix case. As a simple example, the exponental of the linear operator $\frac{d}{dx}$ acting of functions of a real variable x was already considered in Eq. (**??**); we found it generates finite translations.

1.3 Manifolds

Let us summarize some definitions and properties regarding differential manifolds that are important in the discussion of Lie groups.

1.3.1 Definition of a manifold

Let us briefly recall the definition of a *manifold*. A manifold $\mathcal{M}$ is a topological space which is *locally homeomorphic* to an $\mathbb{R}^d$ space (d is then called the *dimension* of $\mathcal{M}$). An homeomorphism (i.e. a continuous and invertible map). In Fig.s 1.1,1.2 examples of 1-dimensional spaces that can or cannot be manifolds are depicted.

Being locally homeomorphic with $\mathbb{R}^d$, $\mathcal{M}$ is covered by an *atlas*, i.e. a collections of *charts* $(\mathcal{U}_i, \varphi_i)$, where $\mathcal{U}_i$ is an open neighbourhood of $\mathcal{M}$ and $\phi_i : \mathcal{U}_i \to \mathbb{R}^d$ an omeomorphism. Each chart provides a local system of coordinates. In general, a single chart cannot cover the entire manifold (think of the sphere S^2). Different charts will overlap. In the overlap $\mathcal{U}_i \cap \mathcal{U}_j$ the *transition function* $\psi_{ij} = \phi_j \circ (\phi_i)^{-1}$ allows the comparison of the two coordinate systems. The transition functions satisfy $\psi_{ij} = (\psi_{ji})^{-1}$ and obey the *cocycle condition* that $\psi_{ki} \circ \psi_{jk} \circ \psi_{ij} = \mathrm{id}$, where id is the identical map.

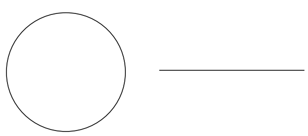

Figure 1.1. 1-dimensional manifolds; around each point, the space looks like $\mathbb{R}$.

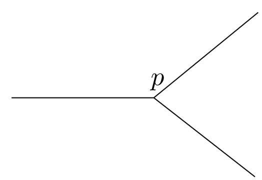

Figure 1.2. This space is not a manifold: around p the space does not look like $\mathbb{R}$.

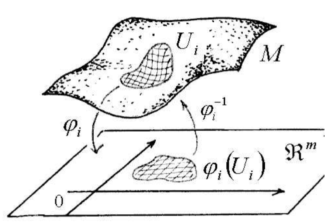

Figure 1.3. An open chart is a homeomorphism of an open subset U_i of the manifold $\mathcal{M}$ onto an open subset of $\mathbb{R}^m$

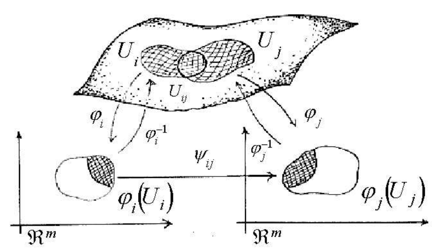

Figure 1.4. A transition function between two open charts is a differentiable map from an open subset of $\mathbb{R}^m$ to another open subset of the same.

The properties of a manifold depend crucially on those of the transition functions. If it is possible to choose all the transition functions to be differentiable, smooth (i.e. infinitely differentiable or "of class $\mathcal{C}^\infty$") or analytic (i.e., expandable in power series) then $\mathcal{M}$ is called a *differentiable*, *smooth* or *analytic* manifold.

On a differentiable manifold, thanks to the local omeomorphism with $\mathbb{R}^d$, the whole machinery of differential geometry can be set up. In particular, one can discuss functions, curves, tangent vectors and vector fields, differential forms. Locally, i.e. in each chart, there is no difference with the differential geometry on an open subset of $\mathbb{R}^d$; however one must then relate the various charts by means of transition functions appropriate for the various quantities.

Although the constructive definition of a differentiable manifold is always in terms of an atlas, in many occurrences we can have other intrinsic global definitions of what $\mathcal{M}$ is and the construction of an atlas of coordinate patches is an a posteriori operation. Typically this happens when the manifold admits a description as an algebraic locus. The prototype example is provided by the $\mathbb{S}^N$ sphere which can be defined as the locus in $\mathbb{R}^{N+1}$ of points $\{x_i\}$ at distance r from the origin, namely the locus such that

$$\sum_{i=1}^{N+1} x_i^2 = r^2 \tag{1.3.29}$$

In particular for $N = 2$ we have the familiar $\mathbb{S}^2$ which is diffeomorphic to the compactified complex plane $\mathbb{C} \cup \{\infty\}$. An atlas of (two) open charts is for instance suggested by the stereographic projection.

1.3.2 Functions on manifolds

A real scalar function on a differentiable manifold $\mathcal{M}$ is a map:

$$f \quad : \quad \mathcal{M} \to \mathbb{R} \tag{1.3.30}$$

that assigns a real number $f(p)$ to every point $p \in \mathcal{M}$ of the manifold. Similarly one can define complex functions; the following discussion is done for real functions, but adaptes readily to the complex case.

The properties of a scalar function are the properties characterizing its local description in the various open charts of an atlas. As shown in Fig. 1.6, the "abstract" function $f : \mathcal{M} \to \mathbf{R}$ (or $\mathbf{C}$) is described locally in each chart $(\mathcal{U}_i, \varphi_i)$ by a function

$$f_{(i)} \equiv f \cdot \varphi_i^{-1} : \mathcal{U}'_i \subset \mathbf{R}^m \to \mathbf{R} \text{ (or } \mathbf{C}) , \tag{1.3.31}$$

which represent a "coordinate presentation" of f. We will say that f is differentiable (smooth, analytic,...) in a point $p \in \mathcal{U}_i$ iff $f_{(i)}$ is differentiable (smooth, analytic,...) in $P = \varphi_i(p) \in \mathcal{U}'_i$. Moreover, f is named differentiable (smooth, analytic,...) if it is differentiable (smooth, analytic,...) in every chart.

We need to glue together the various patches to define globally the function f or to examine its global properties. Gluing the various patches imposes some obvious consistency requirements of the coordinate presentations $f_{(i)}$.

In particular, $\mathcal{U}_i \cup \mathcal{U}_j$ there is the consistency condition on $f_{(i)}$ and $f_{(j)}$ that

$$f_{(j)} = f_{(i)} \cdot \psi_{ji} \ . \tag{1.3.32}$$

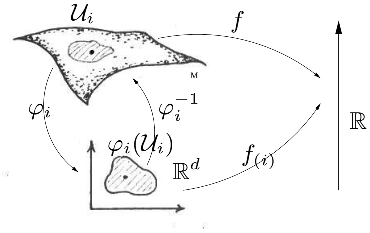

Figure 1.5. Local description of a scalar function on a manifold.

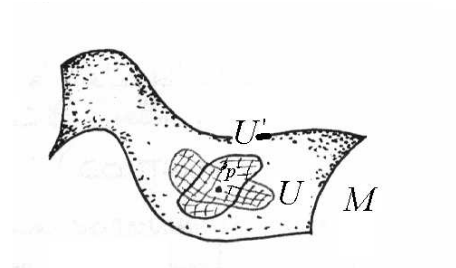

Figure 1.6. A germ of smooth function is the equivalence class of all locally defined function that coincide in some neighborhood of a point.

This expresses the usual transformation of a "scalar" under a coordinate transformation. Indeed, let $x^\mu_{(i)}$ (with $\mu = 1, \ldots \dim \mathcal{M}$) be the coordinates on $\mathcal{U}_i$, and $x^\mu_{(j)}$ those on $\mathcal{U}_i$. The two set of coordinates of a point in $\mathcal{U}_i \cup \mathcal{U}_j$ are related by the transition functions: $x^\mu_{(j)} = \psi_{ij}(x^\mu_{(i)})$. The relation Eq. (1.3.32) reads thus in explicit coordinates

$$f_{(j)}(x^\mu_{(j)}) = f_{(j)}(\psi_{ij}(x^\mu_{(i)})) = (f_{(j)} \cdot \psi_i j)(x^\mu_{(i)}) = f_{(i)}(x^\mu_{(i)})\,. \tag{1.3.33}$$

This is the usual rule for expressing a scalar function $s(x)$ in terms of new coordinates x', namely, the new functional form s' of the scalar is such that $s'(x') = s(x)$.

The algebra of smooth global functions These consistency conditions are rather restrictive. Thus the space of *globally* defined functions of a certain type , e.g., $\mathcal{C}^\infty$ functions, on $\mathcal{M}$ is typically very small if $\mathcal{M}$ is non-trivial, as they must be correctly related on the intersection of *any* two charts. For this very reason, it contains a lot of information about $\mathcal{M}$. It is in some sense possible to replace the "geometric" study of the manifold $\mathcal{M}$ by the "algebraic" study of the algebra of smooth functions on $\mathcal{M}$, which is indicated as $\mathcal{C}^\infty(\mathcal{M})$.

Indeed, $\mathcal{C}^\infty(\mathcal{M})$ is an algebra. It is a vector space over $\mathbb{R}$ (or over $\mathbb{C}$ for complex functions, of course):

$$(\alpha f + \beta g)(p) = \alpha f(p) + \beta g(p), \qquad \text{for } \alpha, \beta \in \mathbb{R}, \text{ and } f, g \in \mathcal{C}^\infty(\mathcal{M})\ , \tag{1.3.34}$$

for every point p in $\mathcal{M}$. Moreover, there is a multiplication of functions that we can define pointwise:

$$(f \cdot g)(p) \equiv f(p) g(p)\ . \tag{1.3.35}$$

This multiplication is associatve and distributive with respect to the sum:

$$f \cdot (\alpha g + \beta h) = \alpha f \cdot g + \beta f \cdot h\ . \tag{1.3.36}$$

Locally defined functions In the following we shall be mostly interested in the local properties of functions. We consider thus the set $\mathcal{C}^\infty_p(\mathcal{U})$ of (real or complex-valued) functions defined in a chart $(\mathcal{U}, \varphi)$ containing a given point p:

$$f_\mathcal{U} \in \mathcal{C}^\infty_p(\mathcal{U}) \to f_\mathcal{U} : \mathcal{U} \times \mathcal{U} \longrightarrow \mathbb{R} \text{ or } \mathbb{C}, \text{ of class } \mathcal{C}^\infty\ . \tag{1.3.37}$$

Of $f_{\mathcal{U}}$ we have of course a coordinate presentation as $(f_{\mathcal{U}} \cdot \varphi^{-1})(x^\mu)$. Of course, $\mathcal{C}_p^\infty(\mathcal{U})$ forms an algebra, exactly as $\mathcal{C}^\infty(\mathcal{M}$ (recall that the product of functions was defined pointwise).

We could of course consider another chart $(\mathcal{U}', \varphi')$, also containing p. Similarly to the principle of analytic continuation, we introduce the following equivalence relation:

$$f_{\mathcal{U}} \sim f_{\mathcal{U}'} \leftrightarrow f_{\mathcal{U}}(p) = f_{\mathcal{U}'}(p) \ , \quad \forall p \in \mathcal{U} \cap \mathcal{U}' \ , \tag{1.3.38}$$

namely, $_{\mathcal{U}}$ and $f_{\mathcal{U}'}$ are equivalent if they coincide in $\mathcal{U} \cap \mathcal{U}'$.

Germs of smooth functions A *germ* of smooth function in a point p is an equivalence class with respect to the above defined equivalence relation, of functions locally defined in p. Thus the set of germs at p, named $\mathcal{C}_p^\infty(\mathcal{M})$, is the set of all smooth functions locally defined in p, having identified all those that coincide in some open neighbourhood of p:

$$\mathcal{C}_p^\infty(\mathcal{M}) = \{f_{\mathcal{U}} \in \mathcal{C}_p^\infty(\mathcal{U}) \text{ for some } \mathcal{U} \text{ such that } p \in \mathcal{U}\}/\sim \ . \tag{1.3.39}$$

Also $\mathcal{C}_p^\infty(\mathcal{M})$ can be naturally given the structure of an algebra.

The concept of germs of functions is useful for the definition of *tangent vectors.*

1.3.3 Tangent vectors and vector fields on a manifold

In elementary geometry the notion of a *tangent* is associated with the notion of a curve. To introduce tangent vectors we begin therefore with the notion of *curves in a manifold.*

Curves on a manifold A curve $\mathcal{C}$ in a manifold $\mathcal{M}$ is a continuous and differentiable map of an interval of the real line (say $[0,1] \subset \mathbb{R}$) into $\mathcal{M}$:

$$\mathcal{C} \ : \ [0,1] \ \mapsto \ \mathcal{M} \tag{1.3.40}$$

In other words a curve is one–dimensional submanifold $\mathcal{C} \subset \mathcal{M}$ (see Fig. 1.7).

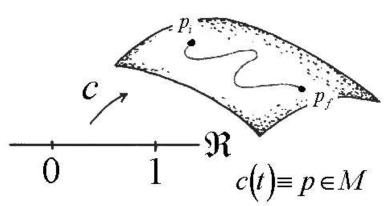

Figure 1.7. A curve in a manifold is a continuous map of an interval of the real line into the manifold itself

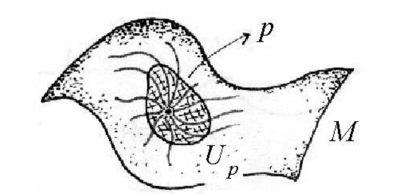

Figure 1.8. In a neighborhood U_p of each point $p \in \mathcal{M}$ we consider the curves that go through p.

Tangent vectors in a point For each point $p \in \mathcal{M}$ let us fix an open neighborhood $U_p \subset \mathcal{M}$ and let us consider all possible curves $\mathcal{C}_p(t)$ that go trough p (see Fig.1.8). Intuitively, the tangent in p to a curve that starts from p is the vector that specifies the curve's *initial* direction. The basic idea is that in an m–dimensional manifold there are as many directions in which the curve can depart as there are vectors in $\mathbb{R}^m$: furthermore for sufficiently small neighborhoods of p

we cannot tell the difference between the manifold $\mathcal{M}$ and the flat vector space $\mathbb{R}^m$. Hence to each point $p \in \mathcal{M}$ of a manifold we can attach an m–dimensional real vector space $T_p\mathcal{M}$, which parametrizes the possible directions in which a curve starting at p can depart. This vector space is named the tangent space to $\mathcal{M}$ at the point p and is, by definition, isomorphic to $\mathbb{R}^m$. Fig. 1.9 depicts the case of the 2-sphere S_2.

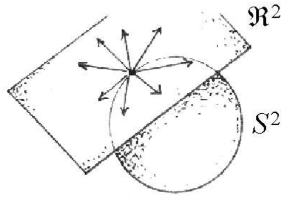

Figure 1.9. The tangent space in a generic point of an S_2 sphere.

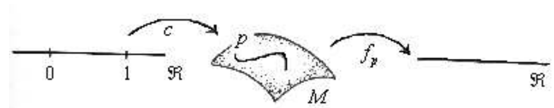

Figure 1.10. The composed map $f_p \circ \mathcal{C}_p$ where f_p is a germ of smooth function in p and $\mathcal{C}_p$ is a curve departing from $p \in \mathcal{M}$.

Let us now make this intuitive notion mathematically precise. Consider a point $p \in \mathcal{M}$ and a germ of smooth function $f \in C_p^\infty(\mathcal{M})$. In any open chart $(U_\alpha, \varphi_\alpha)$ that contains the point p, the germ f is represented by an infinitely differentiable function $f(x^{(\alpha)})$ of m–variables. Let us choose an open curve $\mathcal{C}_p(t)$ that lies in U_α and starts at p (namely, let's for definiteness set $\mathcal{C}_p(0) = p$) and consider the composed map:

$$g_p \equiv f \circ \mathcal{C}_p \quad : \quad [0,1[\, \subset \mathbb{R} \mapsto \mathbb{R} \tag{1.3.41}$$

which is a real function of the real variable t: $g_p(t) \in \mathbb{R}$, see Fig. 1.10. We can calculate its derivative with respect to t in $t = 0$ which in the open chart $(U_\alpha, \varphi_\alpha)$ reads as follows:

$$\frac{d}{dt}\, g_p(t)\, |_{t=0} \;=\; \frac{\partial f}{\partial x^\mu} \cdot \frac{dx^\mu}{dt}\, |_{t=0} \tag{1.3.42}$$

We see from the above formula that the increment of any germ $f \in \mathbb{C}_p^\infty(\mathcal{M})$ along a curve $\mathcal{C}_p(t)$ is defined through the m real coefficients:

$$c^\mu \;\equiv\; \frac{dx^\mu}{dt}\, |_{t=0} \in \mathbb{R} \tag{1.3.43}$$

which can be calculated whenever the parametric form $x^\mu(t)$ of the curve is given. Explicitly we have:

$$\frac{df}{dt} \;=\; c^\mu\, \frac{\partial f}{\partial x^\mu} = \vec{t}_p\, f\,, \tag{1.3.44}$$

where we introduced the differential operator on the space of germs of smooth functions

$$\vec{t}_p \;\equiv\; c^\mu\, \frac{\partial}{\partial x^\mu} \;:\; \mathbb{C}_p^\infty(\mathcal{M}) \;\mapsto\; \mathbb{C}_p^\infty(\mathcal{M})\,, \tag{1.3.45}$$

called a *tangent vector* to the manifold in the point p. Indeed $\vec{t}_p f$ is again a germ of smooth functions. The tangent vector $\vec{t}_p$ associates to any locally defined function f its directional derivative in the initial direction of $\mathcal{C}_p$. Thus the tangent space $T_p\mathcal{M}$ to the manifold $\mathcal{M}$ in the point p can be defined as the vector space of *first order differential operators* on the germs of smooth functions $\mathbb{C}_p^\infty(\mathcal{M})$.

A more abstract definition is based on the concept of *derivation* of an algebra. A derivation $\mathcal{D}$ of an algebra $\mathcal{A}$ is a map: $\mathcal{D} : \mathcal{A} \mapsto \mathcal{A}$ that obeys

$$\begin{cases} \forall\, \alpha, \beta \in \mathbb{R} \quad \forall\, f, g \in \mathcal{A} : \; \mathcal{D}\,(\alpha f + \beta g) = \alpha \mathcal{D} f + \beta \mathcal{D} g & \text{(linearity);} \\ \forall\, f, g \in \mathcal{A} : \; \mathcal{D}(f \cdot g) = \mathcal{D} f \cdot g + f \cdot \mathcal{D} g & \text{(Leibnitz rule).} \end{cases} \tag{1.3.46}$$

The set of derivations $D[\mathcal{A}]$ of an algebra constitutes a real vector space. Indeed, a linear combination of derivations is still a derivation, having set:

$$\forall \alpha, \beta \in \mathbb{R}, \forall \mathcal{D}_1, \mathcal{D}_2 \in D[\mathcal{A}], \forall\, f \in \mathcal{A} : \; (\alpha\, \mathcal{D}_1 + \beta \mathcal{D}_2)\, f = \alpha \mathcal{D}_1 f + \beta \mathcal{D}_2 f\,. \tag{1.3.47}$$

Furthermore, provided with a Lie product given by the commutator in the linear operator sense, it forms a Lie algebra. Indeed, the commutator $[\partial_1, \partial_2]$ of any two derivations of $\mathcal{A}$ is again a derivation, i.e, it satisfies again the two properties in Eq. (1.3.46), linearity and the Leibnitz rule. Linearity is of course no problem, the Leibnitz rule follows because

$$\begin{aligned} [\partial_1, \partial_2]\,(x \cdot y) &= \partial_1\,(\partial_2 x \cdot y + x \cdot \partial_2 y) - (1 \leftrightarrow 2) \\ &= \partial_1 \partial_2 x \cdot y + \partial_2 x \cdot \partial_1 y + \partial_1 x \partial_2 y + x \partial_1 \partial_2 y - (1 \leftrightarrow 2) \\ &= \; [\partial_1, \partial_2]\, x \cdot y + x \cdot [\partial_1, \partial_2]\, y\,. \end{aligned} \tag{1.3.48}$$

A tangent vector $\vec{t}_p$ is clearly a derivation of the algebra $\mathcal{C}_p^\infty$ of locally defined functions: indeed, $\forall \alpha, \beta \in \mathbb{R}$, $\forall f, g \in \mathcal{C}_p^\infty$,

$$\begin{cases} \vec{t}_p\; (\alpha f + \beta g) = \alpha \vec{t}_p f + \beta \vec{t}_p g\,; \\ \vec{t}_p (f \cdot g) = \vec{t}_p f \cdot g + f \cdot \vec{t}_p g\,. \end{cases} \tag{1.3.49}$$

The tangent space $T_p(\mathcal{M}$ can be therefore defined as the vector space of derivations of the algebra of germs of smooth functions in p:

$$T_p \mathcal{M} \;\equiv\; D\left[\mathbb{C}_p^\infty\,(\mathcal{M})\right]\,. \tag{1.3.50}$$

In each coordinate patch a tangent vector is a first order differential operator which is defined independently from the coordinate choice. However, writing it explicitely as in Eq. (1.3.44): $\vec{t}_p = c^\mu \vec{\partial}_\mu$, we have that $\vec{\partial}_\mu$ depends on the coordinate choice, and so does therefore c^μ. In the language of tensor calculus the tangent vector is identified with the m–tuplet of real numbers c^μ which, as we just remarked, that such m–tuplet representing the same tangent vector is different in different coordinate patches. Consider two coordinate patches (U, φ) and (V, ψ) with non vanishing intersection, and name x^μ and y^α the respective coordinates of a point p in the intersection. A tangent vector in p can be expressed in both coordinate systems, and we have:

$$\vec{t}_p = c^\mu\, \frac{\vec{\partial}}{\partial x^\mu} s c^\mu \left(\frac{\partial y^\alpha}{\partial x^\mu}\right) \frac{\vec{\partial}}{\partial y^\alpha} \;=\; c^\alpha\, \frac{\vec{\partial}}{\partial y^\alpha} \tag{1.3.51}$$

so that

$$c^\alpha \equiv c^\mu \left(\frac{\partial y^\alpha}{\partial x^\mu}\right) \tag{1.3.52}$$

Eq.(1.3.52) expresses the transformation rule for the components of a tangent vector from one coordinate patch to another one.

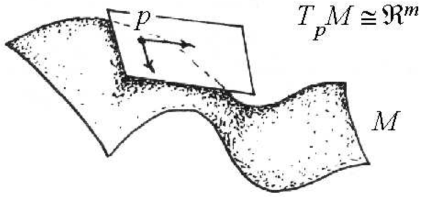

Figure 1.11. The tangent bundle is obtained by gluing together all the tangent spaces.

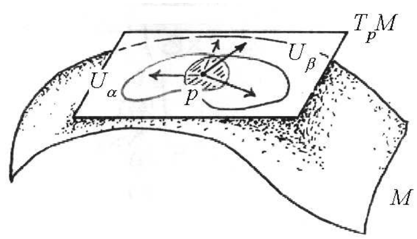

Figure 1.12. Two local charts of the base manifold $\mathcal{M}$ yield two local trivializations of the tangent bundle $T\mathcal{M}$.

Vector fields A *vector field* X is a map

$$T \;:\; p \in \mathcal{M} \mapsto \vec{t}_p \in T_p(\mathcal{M}) \tag{1.3.53}$$

that specifies a tangent vector in any point.

Such a definition poses no problems locally, i.e. in an open neighborhood $\mathcal{U}$ of p. Indeed, even if by definition we have a different vector space $T_p(\mathcal{M})$ in every point, by looking at the coordinate presentation $\vec{t}_p = c^\mu \vec{\partial}_\mu$, valid in the neighbourhood $\mathcal{U}$, we see that we can identify all neighbouring tangent spaces with a single vector space spanned by $\{\vec{\partial}_\mu\}$. That is, locally we can work within a direct product space $\mathcal{U} \times \mathbb{R}^d$, of elements $(p, \vec{t}_p)$ and Eq. (**??**) is well-defined. It consists, in a given coordinate chart $\{x^\mu\}$, in defining *point-dependent tangent vectors*

$$T(x) = T^\mu(x)\vec{\partial}_\mu \,. \tag{1.3.54}$$

The components $T^\mu(x)$ of a vector field transform under coordinate changes exactly as a tangent vector at a specific point, see Eq. (1.3.52), namely, by means of the Jacobiam matrix:

$$X'^\mu(x')\frac{\partial}{\partial x'^\mu} = X^\mu(x)\frac{\partial}{\partial xs^\mu} \quad \Rightarrow X'^\mu(x') = \frac{\partial x'^\mu}{\partial x^\nu} X^\nu(x) \;. \tag{1.3.55}$$

Considering the transformation $x \to x'$ as a coordinate change is a“passive” point of view. One could take an “active” point of view in which the transformation is considered as a mapping $\phi :\, \mathcal{M} \to \mathcal{M}$, sending $x \mapsto x' \equiv \phi(x)$. Such a mapping induces a mapping on the space of vector fields (which is sometimes indicated as $d\phi$, and called *differential* of the map ϕ. Requiring that $X'(x') = X(x)$, i.e., that the transformed vector field at the transformed point equals the original field at x, we find immediately that the transformation Eq. (1.3.55).

The space $\text{Diff}_0(\mathcal{M})$ carries an interesting algebraic structure: it forms a Lie algebra, the Lie product being defined as the commutator of the composed application of vector fields. This aspect is discussed in the main text, in sec. **??**.