Chapter 6

SOME TECHNICAL POINTS

In this chapter we collect some technical points (in particular, proofs or suggestions of prooef of certain theorems) which have been omitted in the main text to concentrate on the main logical points of the theory.

6.1 Regarding chapter 5

6.1.1 An important property of root systems - Sec. 4.4.2

In Sec. 4.4.2 we stated two extremely important property of the root system Φ of a semi-simpe Lie algebra $\mathbb{G}$, namely that $\forall \alpha \beta \in \Phi$,

$$\begin{aligned} \langle \beta , \alpha \rangle &\in \mathbb{Z} \, ; \\ \sigma_\alpha(\beta) \equiv \beta - \langle \alpha , \beta , \alpha \rangle &\in \Phi \, . \end{aligned} \tag{6.1.1}$$

Let us now prove these two assertions.

Let $\alpha, \beta \in \Phi$ be two roots, and let us define the non negative integer $j \in \mathbb{N}$ by means of the following conditions

$$\begin{aligned} \gamma \equiv \beta + j \, \alpha &\in \Phi \\ \gamma + \alpha &\notin \Phi \end{aligned} \tag{6.1.2}$$

In other words j is the maximal integer n for which $\beta + n\alpha$ is a root.

We know that $-\alpha$ is a root and hence we can conclude that:

$$\begin{aligned} \left[E^{-\alpha} \, , \, E^{\gamma} \right] &= \widehat{E}^{\gamma - \alpha} \\ \left[E^{-\alpha} \, , \, \widehat{E}^{\gamma - \alpha} \right] &= \widehat{E}^{\gamma - 2\alpha} \\ &\cdots\cdots\cdots \end{aligned} \tag{6.1.3}$$

where $\widehat{E}^{\gamma - n\alpha}$ denotes some element in the one–dimensional subspace pertaining to the root $\gamma - n\alpha$. Since the number of roots is necessarily finite it follows that there exists some positive integer $g \in \mathbb{N}$ such that:

$$\left[E^{-\alpha} \, , \, \widehat{E}^{\gamma - g\alpha} \right] = \widehat{E}^{\gamma - (g+1)\alpha} = 0 \tag{6.1.4}$$

In general, due to the one–dimensionality of the each root space we can set:

$$\left[E^{\alpha}\, ,\, \widehat{E}^{\gamma-(n+1)\alpha}\right] = \mu_{n+1}\, \widehat{E}^{\gamma-n\alpha} \tag{6.1.5}$$

where μ_{n+1} is some normalization factor. From Jacobi identities we immediately obtain a recursion relation satisfied by these normalization factors. Indeed:

$$\begin{aligned}\left[E^{\alpha}\left[E^{-\alpha}\, ,\, \widehat{E}^{\gamma-n\alpha}\right]\right] &= -\left[E^{-\alpha}\left[\widehat{E}^{\gamma-n\alpha}\, ,\, E^{\alpha}\right]\right] - \left[E^{\gamma-n\alpha}\left[E^{\alpha}\, ,\, E^{-\alpha}\right]\right] \\ &= \mu_n\, \widehat{E}^{\gamma-n\alpha} + \alpha^i \left[H_i\, ,\, \widehat{E}^{\gamma-n\alpha}\right] \\ &= \left(\mu_n + (\gamma\, ,\, \alpha) - n\, (\alpha\, ,\, \alpha)\right)\, \widehat{E}^{\gamma-n\alpha}\end{aligned} \tag{6.1.6}$$

which implies the recursion relation:

$$\mu_{n+1} = \mu_n + (\gamma\, ,\, \alpha) - n\, (\alpha\, ,\, \alpha) \tag{6.1.7}$$

Since by hypothesis $\gamma + \alpha$ is not a root we have

$$[E^{\alpha}\, ,\, E^{\gamma}] = \mu_0\, E^{\gamma+\alpha} = 0 \quad \text{namely} \quad \mu_0 = 0 \tag{6.1.8}$$

This allows to solve the recursion relation explicitly obtaining:

$$\mu_n = n\, (\alpha\, ,\, \gamma) - \tfrac{n(n-1)}{2}\, (\alpha\, ,\, \alpha) \tag{6.1.9}$$

Since, at the other end of the chain, we have assumed that $\gamma - (g+1)\alpha$ is not a root, we conclude that $\mu_{g+1} = 0$ and hence:

$$(g+1)\, \left\{(\alpha\, ,\, \gamma) - \tfrac{g}{2}\, (\alpha\, ,\, \alpha)\right\} = 0 \tag{6.1.10}$$

This implies that $2\frac{(\gamma\, ,\, \alpha)}{(\alpha\, ,\, \alpha)} = g \in \mathbb{N}$ Hence for each pair of roots $\alpha\, \beta$ there exists a non negative integer $j \geq 0$ such that $\gamma = \beta + j\alpha$ is a root and

$$(\alpha\, ,\, \beta) = (\alpha\, ,\, \gamma) - j\, (\alpha\, ,\, \alpha) = \left(\tfrac{g}{2} - j\right)\, (\alpha\, ,\, \alpha) \tag{6.1.11}$$

namely:

$$2\, \frac{(\alpha\, ,\, \beta)}{(\alpha\, ,\alpha)} = g - 2j \in \mathbb{Z} \qquad \text{(positive or negative)} \tag{6.1.12}$$

This concludes the first part of our proof. Let us now consider the string or roots that we have constructed to make the above argument:

$$\begin{aligned}\beta_0 &= \gamma = \beta + j\alpha \\ \beta_1 &= \gamma - \alpha = \beta + (j-1)\alpha \\ \beta_2 &= \gamma - 2\alpha = \beta + (j-2)\alpha \\ &\ldots\ldots\ldots \end{aligned} \tag{6.1.13}$$

$$\beta_g = \gamma - g\alpha = \beta + (j-g)\alpha \tag{6.1.14}$$

Since $2\frac{(\gamma\, ,\, \alpha)}{(\alpha\, ,\, \alpha)} = g$, it is evident by means of the replacement:

$$\beta \mapsto \beta - 2\, \alpha\, \frac{(\beta\, ,\, \alpha)}{(\alpha\, ,\, \alpha)} \tag{6.1.15}$$

the string (6.1.14) is simply reflected into itself $\beta_g \mapsto \beta_0, \beta_{g-1} \mapsto \beta_1, \ldots$. So not only we proved that if β and α are roots then the reflection of β with respect to α is a root but also that the entire string of α through β is invariant under such reflection.

6.1.2 The classification of simple Lie algebras - Sec. 4.4.4

In Sec. 4.4.4 the classification theorem for possible Dynkin diagram, and thus for complex simple Lie algebras, is stated¿ The result is summarized in Fig.s 4.5 and 4.6. Let us now prove the theorem.

Let us consider a Euclidean space E and in E let us consider set of vectors:

$$\mathcal{U} = \{\epsilon_1\,,\,\epsilon_2\,,\,\dots\,,\epsilon_\ell\} \tag{6.1.16}$$

that satisfy the following three conditions:

$$\begin{aligned} (\epsilon_i\,,\,\epsilon_i) &= 1 \\ (\epsilon_i\,,\,\epsilon_j) &\le 0 \quad i \neq j \\ 4\,(\epsilon_i\,\epsilon_j)^2 &= 0,1,2,3 \quad i \neq j \end{aligned} \tag{6.1.17}$$

Such a system of vectors is named *admissible*. It is clear that each admissible system of vectors singles out a coxeter graph Γ. Indeed the vectors ϵ_i correspond to the simple roots α_i divided by their norm:

$$\epsilon_i = \frac{\alpha_i}{\sqrt{|\alpha_i|^2}} \tag{6.1.18}$$

Our task is that of classifying all connected Coxeter graphs.

We proceed through a series of steps.

Step 1 We note that by deleting a subset of vectors ϵ_i in an admissible system those that are left still form an admissible system whose Coxeter graph is obtained from the original one by deleting the corresponding vertices and all the lines that end in these vertices

Step 2 The number of pairs of vertices that are connected by at least one line is strictly less than the number of vectors ϵ_i namely strictly less than ℓ. Indeed let us set $\epsilon = \sum_{i=1}^{\ell} \epsilon_i$ and observe what follows. Since all the ϵ_i are independent we have $\epsilon \neq 0$ Hence

$$0 < (\epsilon\,,\,\epsilon) = \ell + 2\sum_{i<j}\,(\epsilon_i\,,\,\epsilon_j) \tag{6.1.19}$$

If the i-th vertex is joined to the j-th vertex we have $4\,(e_i\,,\,\epsilon_j)^2 = 1,2,3$. Hence we can conclude that, in this case:

$$2\,(\epsilon_i\,,\,\epsilon_j) \le -1 \tag{6.1.20}$$

On the other hand if the i-th vertex is not joined to the j-th vertex we have $2\,(\epsilon_i\,,\,\epsilon_j) = 0$. Naming N_J the number of pairs of vertices joined by at least one line we conclude that:

$$0 < (\epsilon\,,\,\epsilon) < \ell - N_J \quad \Rightarrow \quad N_J \le \ell - 1 \tag{6.1.21}$$

which is what we have asserted

Step 3 The Coxeter graph Γ cannot contain any loop. Indeed if a loop existed this would constitute a subgraph Γ' for which the number of pairs joined by a line N_J would be larger than the number of vertices and this we have shown to be impossible.

Step 4 The number of lines that end up in any vertex can be at most three. Indeed let $\epsilon \in \mathcal{U}$ and let us denote $\eta_1, \eta_2, \dots, \eta_k$ the vectors connected to ϵ by some link. In other words we have $(\epsilon, \eta_i) < 0\,(\forall \eta_i)$. Since there are no loops in the graph it follows that no η_i can be connected to any other η_j, namely $(\eta_i\,,\,\eta - j) = 0 \forall i \neq j$. Since U is a set of linearly independent vectors there must exist a unit vector η_0 in the vector span of $\epsilon, \eta_1, \ldots, \eta_k$ which is orthogonal to $\eta_1, \ldots, \eta_k$. Obviously the projection of such a vector η_0 is non vanishing on ϵ, namely $(\epsilon\,,\,\eta_0) \neq 0$. The set $\eta_0, \eta_1, \ldots, \eta_k$ makes an orthogonal basis for the linear span of the vectors $\epsilon, \eta_1, \ldots, \eta_k$ and we can write:

$$\begin{aligned} \epsilon &= \sum_{i=0}^{k} (\epsilon\,,\,\eta_i)\,\eta_i \\ 1 = (\epsilon\,,\,\epsilon) &= \sum_{i=0}^{k} (\epsilon\,,\,\eta_i)^2 \end{aligned} \tag{6.1.22}$$

This reasoning implies that $\sum_{i=1}^{k} (\epsilon\,,\,\eta_i)^2 < 1$ and hence

$$4 \sum_{i=1}^{k} (\epsilon\,,\,\eta_i)^2 < 4 \tag{6.1.23}$$

On the other hand $4\,(\epsilon\,,\,\eta_i)^2$ is precisely the number of lines that link η_i to ϵ so that eq.(6.1.23) is precisely the statement we wanted to prove in **Step 4**

Step 5 The only connected Coxeter graph that contains a triple line is the G_2 graph of fig.4.3. This immediately follows from **Step 4**.

Step 6 Let $\{\epsilon_1, \ldots, \epsilon_k\} \subset \mathcal{U}$ be a subset of vectors corresponding to a simple line as in fig.6.1. Then the subset $\mathcal{U}' \equiv \{\mathcal{U} - \{\epsilon_1, \ldots, \epsilon_k\}\} \bigcup \{\epsilon\}$ where $\epsilon \equiv \sum_{i=1}^{k} \epsilon_i$ is still an admissible

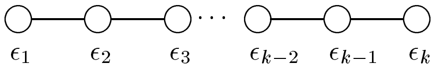

Figure 6.1. A simple line Coxeter graph

system. Graphically the operation of making the transition from the admissible system $\mathcal{U}$ to the admissible system $\mathcal{U}'$ corresponds to collapsing the entire simple line a single vertex. That this statement is true can be proved in the following way. That the vectors composing $\mathcal{U}'$ are linearly independent is obvious. By hypothesis of a simple chain we have:

$$2\,(\epsilon_i\,,\,\epsilon_{i+1}) = -1 \quad 1 \leq i \leq k-1 \tag{6.1.24}$$

so that

$$(\epsilon\,,\,\epsilon) = k + 2 \sum_{i<j} (\epsilon_i\,,\,\epsilon_j) = k - (k-1) = 1 \tag{6.1.25}$$

and hence ϵ is a unit vector. Furthermore each $\eta \in \mathcal{U} - \{\epsilon_1, \ldots, \epsilon_k\}$ can be joined at most to one of the vectors $\epsilon_1, \ldots, \epsilon_k$. Otherwise we would generate a loop. Hence we either have $(\eta\,,\,\epsilon) = 0$ or we have $(\eta\,,\,\epsilon_i)$ for some value of i. In any case we conclude $4\,(\eta, \epsilon_i)^2 = 0, 1, 2, 3$ which is what makes $\mathcal{U}'$ and admissible system.

Step 7 A Coxeter graph cannot contain subgraphs of the form displayed in fig. 6.2 Indeed in

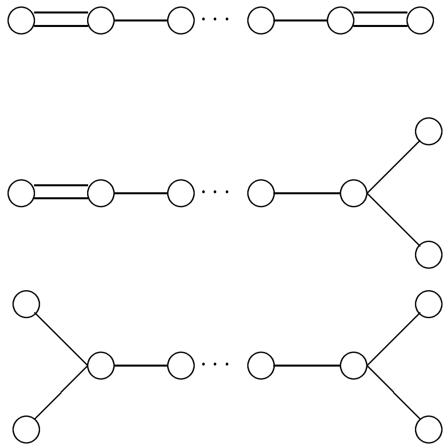

Figure 6.2. Prohibited subgraphs

all these three cases, by using the property shown in **Step 6** and collapsing a simple chain we obtain a graph that contains a vertex where 4 lines converge. This was shown to be forbidden in **Step 4**

Step 8 Relying on the properties we have so far proven we are left with four types of possible Coxeter graphs, namely i) the simple chains of length ℓ corresponding to the A_ℓ Dynkin diagrams of fig.4.5, ii) the G_2 graph of fig.4.3 iii) the graphs of fig.6.3 with a double line and finally iv) the graphs of fig.6.4 with a node.

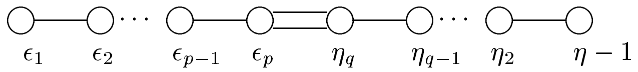

Figure 6.3. Coxeter graph with a double link that is preceded by a simple chain of length p and followed by a simple chain of length q

Step 9 If we consider the graphs of the type shown in fig.6.3 there are only two solutions namely:

$$\begin{array}{lclclcl} p & = & 2 & ; \; q & = & 2 \; \Rightarrow & F_4 \quad \text{Dynkin diagram} \\ p & = & \ell \in \mathbb{N} & ; \; q & = & 1 \; \Rightarrow & B_\ell \; \textit{or} \; C_\ell \quad \text{Dynkin diagrams} \end{array} \tag{6.1.26}$$

Indeed let us set $\epsilon = \sum_{i=1}^{p} i\,\epsilon_i$ and $\eta = \sum_{i=1}^{q} i\,\eta_i$ By hypothesis of simple chains we have $2\,(\epsilon_i\,,\,\epsilon_{i+1}) = -1$, $2\,(\eta_i\,,\,\eta_{i+1}) = -1$ and all the other pairs of vectors are mutually orthogonal. In this way we obtain:

$$\begin{aligned} (\epsilon\,,\,\epsilon) &= \sum_{i=1}^{p} i^2 - \sum_{i=1}^{p-1} i(i-1) = p\,\frac{p-1}{2} \\ (\eta\,,\,\eta) &= \sum_{i=1}^{q} i^2 - \sum_{i=1}^{q-1} i(i-1) = q\,\frac{q-1}{2} \end{aligned} \tag{6.1.27}$$

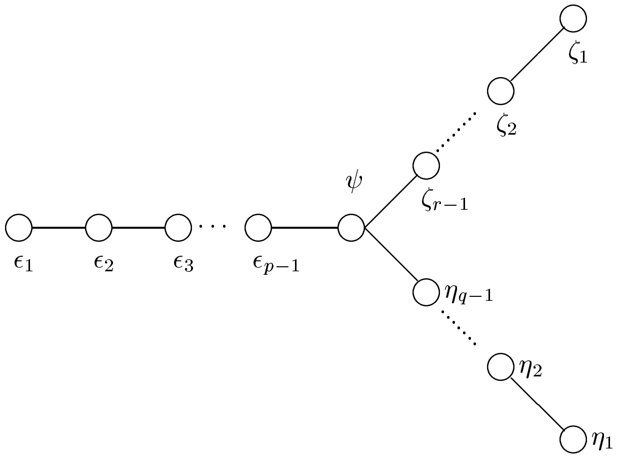

Figure 6.4. Coxeter graph with a node. The unit vector in the node is named ψ while the unit vectors along the three simple lines departing from the node are respectively named $\epsilon_1, \ldots, \epsilon_{p-1}$, $\eta_1, \ldots, \eta_{q-1}$, $\zeta_1, \ldots, \zeta_{r-1}$. The graph is characterized by the three integer numbers p, q, r that denote the lengths of the three simple lines departing from the node

and since by hypothesis of double line we have: $4\left(\epsilon_p\,,\,\eta_q\right)^2 = 2$ it follows that

$$\left(\epsilon\,,\,\eta\right)^2 = p^2\,q^2\,\left(\epsilon_p\,,\,\eta_q\right)^2 = \frac{1}{2}\,p^2\,q^2 \tag{6.1.28}$$

On the other from the triangular Schwarz inequality of Euclidean geometry we have:

$$\begin{array}{c} (\epsilon\,,\,\eta)\;<\;(\epsilon\,,\,\epsilon)\;(\eta\,,\,\eta) \\ \Downarrow \\ (p-1)\;(q-1)\;<2 \end{array} \tag{6.1.29}$$

which for positive integers p, q admits only the two solutions advocated in eq.(6.1.26). The first solution leads to the Dynkin diagram of the exceptional Lie algebra F_4, while the second solution leads to the two infinite series of classical Lie algebras B_ℓ and C_ℓ.

Step 10 Let us finally consider the Coxeter graphs of the type shown in fig.6.4. We claim that the only possible solutions are:

$$(p,q,r) = \left\{\begin{array}{lclllcl} (\ell,1,1) & \Rightarrow & A_\ell & \text{Dynkin diagrams} & \ell & \in & \mathbb{N} \\ (\ell-2,2,2) & \Rightarrow & D_\ell & \text{Dynkin diagrams} & 4 \le \ell & \in & \mathbb{N} \\ (3,3,2) & \Rightarrow & E_6 & \text{Dynkin diagram} & & & \\ (4,3,2) & \Rightarrow & E_7 & \text{Dynkin diagram} & & & \\ (5,3,2) & \Rightarrow & E_8 & \text{Dynkin diagram} & & & \end{array}\right. \tag{6.1.30}$$

To prove this statement we follow a strategy similar to that used in the proof of **Step 9** and we define the following three vectors:

$$\epsilon = \sum_{i=1}^{p-1} i\,\epsilon_i \quad ; \quad \eta = \sum_{i=1}^{q-1} ii\,\eta_i \quad ; \quad \sum_{i=1}^{r-1} i\,\zeta_i \tag{6.1.31}$$

Clearly ϵ, η, ζ are mutually orthogonal and ψ, the vector in the node is not in the subspace generated by ϵ, η, ζ. Hence if in the linear span of $\{\psi, \epsilon, \eta, \zeta\}$ we construct a vector γ that is orthogonal to $\{\epsilon, \eta, \zeta\}$ we obtain that $(\gamma\,,\,\psi) \neq 0$. Normalizing this vector to 1 we can write:

$$\psi = (\psi\,,\,\gamma)\;\gamma + \frac{(\psi\,,\,\epsilon)}{\sqrt{(\epsilon\,,\,\epsilon)}}\,\epsilon + \frac{(\psi\,,\,\eta)}{\sqrt{(\eta\,,\,\eta)}}\,\eta + \frac{(\psi\,,\,\zeta)}{\sqrt{(\zeta\,,\,\zeta)}}\,\zeta \tag{6.1.32}$$

and we obtain:

$$(\psi\,,\,\psi) = 1 = (\psi\,,\,\gamma)^2 + \frac{(\psi\,,\,\epsilon)^2}{(\epsilon\,,\,\epsilon)} + \frac{(\psi\,,\,\eta)^2}{(\eta\,,\,\eta)} + \frac{(\psi\,,\,\zeta)^2}{(\zeta\,,\,\zeta)} \tag{6.1.33}$$

that implies the inequality:

$$1 > \frac{(\psi\,,\,\epsilon)^2}{(\epsilon\,,\,\epsilon)} + \frac{(\psi\,,\,\eta)^2}{(\eta\,,\,\eta)} + \frac{(\psi\,,\,\zeta)^2}{(\zeta\,,\,\zeta)} \tag{6.1.34}$$

By definition of the Coxeter graph in fig.6.4 we have:

$$\begin{aligned}(\psi\,,\,\epsilon) = (p-1)\;(\epsilon_{p-1}\,,\,\psi) \quad \Rightarrow \quad (\psi\,,\,\epsilon)^2 &= \frac{(p-1)^2}{4} \\ (\epsilon\,,\,\epsilon) &= \frac{p(p-1)}{2}\end{aligned} \tag{6.1.35}$$

and similarly for the scalar products associated with the other chains. Inserting these results into the inequality of eq.(6.1.34) we obtain the Diophantine inequality:

$$\frac{1}{p} + \frac{1}{q} + \frac{1}{r} > 1 \tag{6.1.36}$$

whose independent solutions are those displayed in eq.(6.1.30). To this effect it is sufficient to note that eq.(6.1.36) has an obvious permutational symmetry in the three numbers p, q, r. To avoid double counting of solutions we break this symmetry by setting $p \geq q \geq r$ and then we see that the only possibilities are those listed in eq.(6.1.30).

■