Deformations of gauge theories from closed string backgrounds

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This talk is mostly based on...

- M. Billo, M. Frau, I. Pesando, F. Fucito, A. Lerda and A. Liccardo, *"Classical gauge instantons from open strings,"* JHEP **0302** (2003) 045 [arXiv:hep-th/0211250].
- M. Billo, M. Frau, I. Pesando and A. Lerda, *"$N = 1/2$ gauge theory and its instanton moduli space from open strings in R-R background,"* JHEP **0405** (2004) 023 [arXiv:hep-th/0402160].
- M. Billo, M. Frau, F. Lonegro and A. Lerda, *"$N = 1/2$ quiver gauge theories from open strings with R-R fluxes,"* JHEP **0505** (2005) 047 [arXiv:hep-th/0502084].
- M. Billo, M. Frau, F. Fucito, and A. Lerda, *"The D(-1)-D3 system in presence of fluxes and localization deformations,"* in preparation.

Outline

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1. Introduction
2. Gauge instantons from D3/D(-1) systems
3. Deformations of field theories from closed string bkgs
4. Non-anti-commutative deformations from RR backgrounds
5. Localization deformations in D3/D(-1) systems
6. Conclusions and outlook

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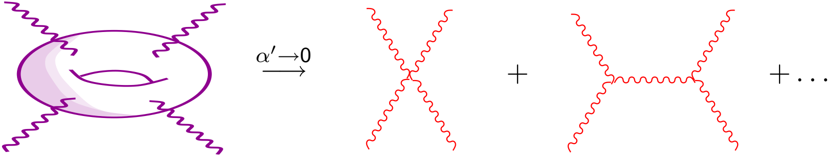
Introduction

String perspective on field theories

- Realizing field theories in a string context is proving itself more and more interesting and useful:
  - perturbative amplitudes (many gluons, ...) via string techniques;
  - construction of "realistic" extensions of Standard model (D-brane worlds)
  - AdS/CFT and its extensions to non-conformal cases;
  - hints about non-perturbative aspects (Matrix models á la Dijkgraaf-Vafa, certain cases of gauge/gravity duality, ...);
  - description of gauge instantons moduli space by means of D3/D(-1) systems.

Field theory from strings

- There is, of course, a "naïve", direct relation: the string spectrum is a collection of fields and the string diagrams encode their interactions.
- The field theory limit $\alpha' \to 0$ selects the lowest (massless) states, corresponding to a finite set of fields.
- A single string scattering amplitude reproduces, for $\alpha' \to 0$, a sum of Feynman diagrams:



- String theory S-matrix elements ⇒ Field theory eff. actions

String amplitudes

- A N-point string amplitude $\mathcal{A}_N$ is schematically given by

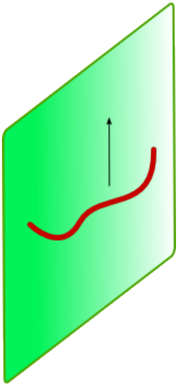
$$\mathcal{A}_N = \int_\Sigma \left\langle V_{\phi_1} \cdots V_{\phi_N} \right\rangle_\Sigma$$

  - V_{ϕ_i} is the vertex for the emission of the field ϕ_i:

$$V_{\phi_i} \equiv \phi_i \, \mathcal{V}_{\phi_i}$$

  - Σ is a Riemann surface of a given topology
  - $\langle \ldots \rangle_\Sigma$ is the v.e.v. in C.F.T. on Σ.

Gauge theories on D-branes



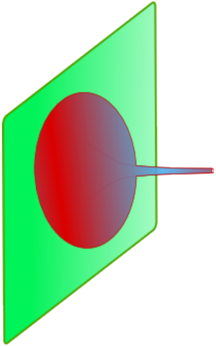
- In the contemporary perspective, we can study gauge theories by considering open strings attached to Dp-branes in a well-suited limit

$$\alpha' \to 0 \text{ with gauge quantities fixed.}$$

- Such strings carry momentum only along the world-volume. In the limit, their massless d.o.f. describe a $p+1$-dimensional gauge + matter theory.

- By placing the branes in different backgrounds, and choosing different confgurations one can construct semi-realistic "brane world" models and find intriguing "geometrical" interpretations of properties of such field theories.

The closed life of D-branes



- The open strings on D-branes unavoidably interact with closed strings, as it is seen in the effective D-brane action:

$$-\tau_p \int d^{p+1}x e^{-\frac{3-p}{2}\phi} \sqrt{-\det\left[G_{\alpha\beta} + e^{-\phi}(B_{\alpha\beta} + F_{\alpha\beta})\right]}$$

$$+\tau_p \int_{V_{p+1}} \sum_n C_n \wedge e^{F+B} ,$$

- The D-branes are sources of closed string fields
- Certain closed fields appear as coupling parameters in the world-volume gauge theory. Yet they depend non-trivially on the transverse directions.
- Relating the transverse distance to the energy scale, this fact is a the hearth of the gauge/gravity correspondence.

The main lines of this talk

- I would like to elaborate on these well-known features of strings and D-branes, in two main directions.

1. Deformed field theories from string backgrounds

Turning on closed string bkg.s (in a computable way), these bkg.s may show up, in the field theory limit, as parameters of novel couplings, i.e., they can induce (and "explain") consistent and possibly interesting deformations of the world-volume field theory.

2. A perturbative handle on non-perturbative effects

D3/D(-1) systems are more than a convenient device to encode the description of instanton moduli space. They really offer a perturbative description of the instanton solutions. Let me introduce the main point by analogy.

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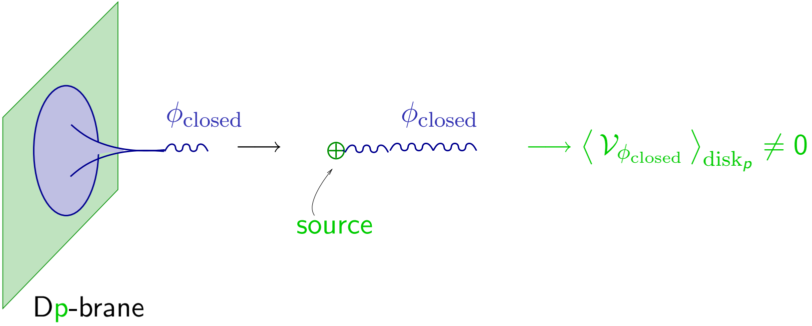
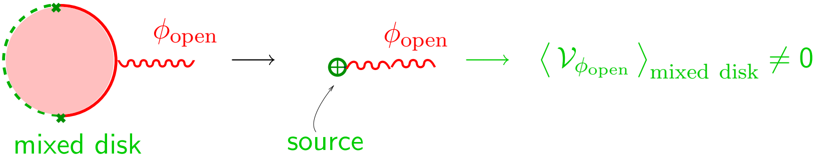
A perturbative handle on non-perturbative effects

- In presence of D-branes, there are non-zero tadpoles of closed string vertices. These emission diagrams provide a perturbative description at large distance of the non-perturbative D-brane solution.



A perturbative handle on non-perturbative effects

- In presence of D(-1) branes, the gauge fields living on D3-branes may acquire non-zero tadpoles on mixed discs:



- Such diagrams encode a perturbative description at large distance of instantons.

[M.B. et al, 2002]

Gauge instantons from D3/D(-1) systems

Instantons and D-instantons

- Consider the Wess-Zumino term of the effective action for a stack of D3 branes:

$$\mathrm{D.B.I.} + \int_{\mathrm{D}_3} \left[C_3 + \frac{1}{2} C_0 \,\mathrm{Tr}\left(F \wedge F \right) \right]$$

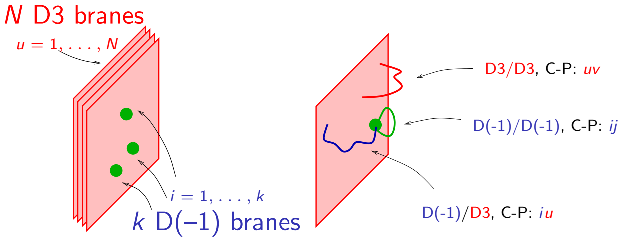
  The topological density of an instantonic configuration corresponds to a localized source for the RR scalar C_0, i.e., to a distribution of D-instantons on the D3's.

- Instanton-charge k solutions of 3+1 dims. $\mathrm{SU}(N)$ gauge theories correspond to k D-instantons inside N D3-branes.

[Witten, 1995, Douglas, 1995, Dorey et al, 1999],...

Stringy description of gauge instantons

	1	2	3	4	5	6	7	8	9	10
D3	–	–	–	–	*	*	*	*	*	*
D(−1)	*	*	*	*	*	*	*	*	*	*



Moduli vertices and instanton parameters

- The strings with at least one end attached to a D-instanton, the D(-1)/D(-1) or the D3/D(-1) strings, carry no momentum.
- The polarizatione of their physical vertices are moduli, rather than fields; they represent the parameters of the instantonic solution.

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- For instance, in the NS sector of the D(-1)/D(-1) strings , we have

$$V_a = a'_\mu \, \psi^\mu \mathrm{e}^{-\phi} \ , \tag{1}$$

$(\mu = 0, \ldots, 3)$: the a'_μ, in the adjoint of $\mathrm{U}(k)$, are associated to the centers of the (multi)-instanton

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- The polarizatione of their physical vertices are moduli, rather than fields; they represent the parameters of the instantonic solution.
- In the NS sector of the D3/D(-1) strings we have

$$V_w(y) = w_{\dot\alpha}\,\Delta(y)\,S^{\dot\alpha}(y)\,\mathrm{e}^{-\phi(y)}\ .$$

where Δ are bosonic twist fields, and $S^{\dot\alpha}$ 4d spin fields. The moduli $w_{\dot\alpha}$ carry Chan-Patons in the bifundamental of $\mathrm{U}(k)\times\mathrm{U}(N)$, and are related to size and orientation in color space.

The ADHM construction from strings

The moduli space of $\mathrm{SU}(N)$ (super-)instantons of top. charge k is described by the so-called (super-) ADHM construction.

- Start from a flat space spanned by the a'_μ ($4k^2$ of them) and the $w_{\dot\alpha}$ ($4kN$), i.e., exactly by the string moduli;
- Take an hyperkähler quotient w.r.t. the action of $\mathrm{U}(k)$;
  - the momentum map equations are the so-called ADHM constraints, the three $k \times k$ matrix equations

$$w_{\dot\alpha} \left(\tau^c\right)^{\dot\alpha}{}_{\dot\beta} \bar{w}^{\dot\beta} + \mathrm{i}\bar\eta^c_{\mu\nu} \left[a'^\mu, a'^\nu\right] = \mathbf{0} \; ,$$

  - The ADHM constraints are retrieved in the string construction from the interactions of the moduli, in the limit $\alpha' \to 0$.
  - Quotienting the constrained hypersurface by $\mathrm{U}(k)$ one remains with the $4kN$-dimensional moduli space.

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Parameter counting

- For the bosonic parameters

	#
a'^{μ}	$4k^2$
$w_{\dot\alpha}, \bar{w}_{\dot\alpha}$	$4kN$
ADHM constraints	$-3k^2$
Global $\mathrm{U}(k)$ inv.	$-k^2$
True moduli	$4kN$

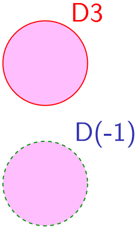
- **After** imposing the constraints, more or less

$$a'^{\mu} \quad \rightsquigarrow \quad \text{multi-center positions, ...}$$

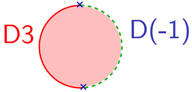
$$w_{\dot\alpha}, \bar{w}_{\dot\alpha} \quad \rightsquigarrow \quad \text{size, orientation inside } \mathrm{SU}(N), \ldots$$

Disk amplitudes and effective actions

Usual disks:



Mixed disks:



Disk amplitudes

$\alpha' \to 0$ field theory limit

Effective actions

D3/D3

D(-1)/D(-1) and mixed

$\mathcal{N} = 4$ SYM action

ADHM measure

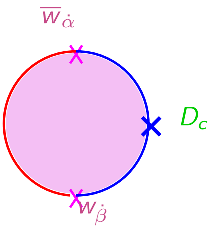
The "field theory" limit

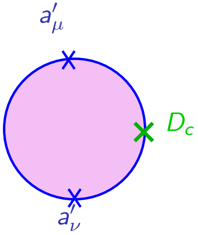
- Normalization for disc diagrams with (part) of their boundary on a D(-1) and coupling for the D(-1) theory:

$$C_0 = \frac{8\pi^2}{g_{\rm YM}^2}\ , \quad g_0 = \frac{g_{\rm YM}}{\sqrt{2\pi\alpha'}}\ .$$

- $g_{\rm YM}$ fixed when $\alpha' \to 0$ to obtain the gauge theory on the D3-branes $\rightsquigarrow$ g_0 blows up.
- The moduli have to be rescaled with powers of g_0 to retain non-trivial interactions; in this way, they acquire the appropriate dimensions to be parameters of an instanton solution

The ADHM constraints from disc diagrams





- These diagrams couple the moduli w or a' to an auxiliary (-1)/(-1) modulus

$$V_D(y) = \frac{1}{2}\, D^-_{\mu\nu}\psi^\nu\psi^\mu(y) = \frac{1}{2} D^-_c\, \bar\eta^c_{\mu\nu}\, \psi^\nu\psi^\mu(y)\ ,$$

disentangling the quartic interactions.

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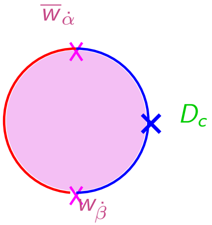

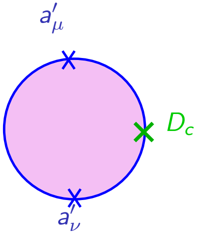

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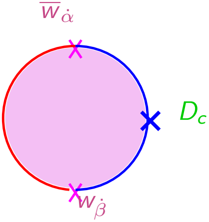
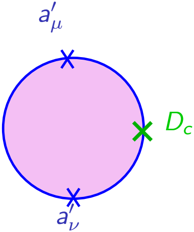
disentangling the quartic interactions.

- In the $\alpha' \to 0$ limit described above, the quadratic terms for D_c drop out $\rightsquigarrow$ Lagrange multiplier: the moduli action contains

$$D_c \left(\underbrace{w_{\dot\alpha}(\tau^c)^{\dot\alpha}{}_{\dot\beta}\bar{w}^{\dot\beta} + \mathrm{i}\bar\eta^c_{\mu\nu}\left[a'^\mu, a'^\nu\right]}_{ADHM\ \ constraint} \right) .$$

The ADHM constraints from disc diagrams





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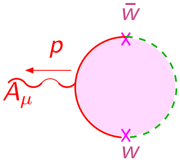
disentangling the quartic interactions.

- The e.o.m. for D_c impose therefore the bosonic ADHM constraint

The moduli measure

- Besides the bosonic ADHM constraint, from the disc diagrams involving bosonic or fermionic moduli we get other contributions, which depend very much on the amount of supersymmetry.
- We will, later, consider the case of instantons in $\mathcal{N} = 2$ gauge theories

The instanton solution from mixed disks



- Mixed disks = sources for gauge theory fields. The amplitude for emitting a gauge field is

$$A_\mu(p) = \langle\, \mathcal{V}_{A_\mu}(-p)\, \rangle_{\rm m.d} = \langle\!\langle\, V_{\bar{w}}\, \mathcal{V}_{A'_\mu}(-p)\, V_w \,\rangle\!\rangle$$
$$= \mathrm{i}\, p^\nu\, \bar{\eta}^c_{\nu\mu} \big(w_{\,\dot{\alpha}}\, (\tau^c)^{\dot{\alpha}}_{\ \dot{\beta}}\, \bar{w}^{\dot{\beta}} \big)\, \mathrm{e}^{-\mathrm{i} p\cdot x_0}\ .$$

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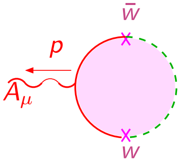

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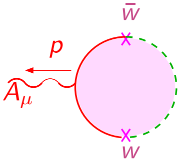
- $\mathcal{V}_{A_\mu}(-p)$: no polariz., outgoing p, 0-picture

$$\mathcal{V}_{A_\mu}(z;-p) = 2\mathrm{i}\, (\partial X_\mu - \mathrm{i} p\cdot \psi\psi_\mu)\, \mathrm{e}^{-\mathrm{i}p\cdot X}(z)$$

The instanton solution from mixed disks



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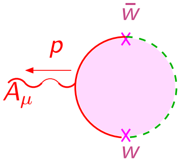
- x_0 = pos. of the D(-1). Broken transl. invariance in the D3 world-volume ⇝ "tadpole"

$$\left\langle \mathrm{e}^{-\mathrm{i} p\cdot X} \right\rangle_{\mathrm{m.d}} \propto \mathrm{e}^{\mathrm{i} p\cdot x_0} \,.$$

The instanton solution from mixed disks



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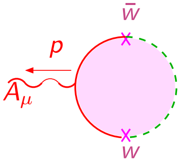
- N.B. From now on we set $k = 1$, i.e. we consider instanton number 1 and, for simplicity, gauge group $\mathrm{SU}(2)$.

The instanton solution from mixed disks



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- One still has to imposing the ADHM constraints. In the $\mathrm{SU}(2)$, $k = 1$ case, a solution is

$$w^u_{\ \dot\alpha} = \rho\, \delta^u_{\ \dot\alpha}$$

Then one has simply

$$A_\mu(p) = -\rho^2 \bar{\sigma}_{\mu\nu}\, p^\nu\, \mathrm{e}^{-\mathrm{i}p\cdot x_0}$$

The classical profile

- The classical profile is obtained by attaching a free propagator and Fourier transforming:

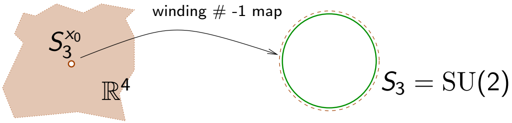
$$A_\mu(x) = \int \frac{d^4 p}{(2\pi)^2} \, A_\mu(p) \, \frac{1}{p^2} \, \mathrm{e}^{\mathrm{i} p\cdot x}$$
$$= -2\mathrm{i}\rho^2 \, \bar{\sigma}_{\mu\nu} \frac{(x-x_0)^\nu}{(x-x_0)^4} \; .$$

- This is exactly the leading term in the large distance approximation $|x - x_0| \gg \rho$ of the $\mathrm{SU}(2)$ instanton connection in the singular gauge:

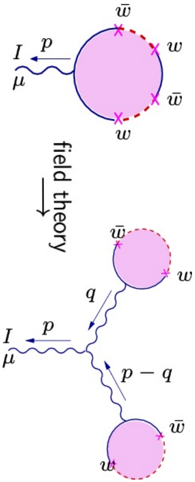
$$A_\mu(x) = 2\mathrm{i}\rho^2 \, \bar{\sigma}_{\mu\nu} \, \frac{(x-x_0)^\nu}{(x-x_0)^2 \left[(x-x_0)^2 + \rho^2\right]}$$
$$= 2\mathrm{i}\rho^2 \, \bar{\sigma}_{\mu\nu} \, \frac{(x-x_0)^\nu}{(x-x_0)^4} \left(1 - \frac{\rho^2}{(x-x_0)^2} + \ldots\right)$$

Why the singular gauge?

- Instanton produced by a point-like source, the D(-1), inside the D3 → singular at the location of the source
- In the singular gauge, rapid fall-off of the fields → e.o.m. reduce to free eq.s at large distance → "perturbative" solution in terms of the source term
- non-trivial properties of the instanton profile from the region near the singularity through the embedding



Additional remarks



- Subleading terms in the long-distance expansion of the solution arise from emission diagrams with more moduli insertions.
- At the field theory level, they correspond to having more source terms.
- The mixed disks emit also other fields, for instance a gaugino $\lambda^\alpha \rightsquigarrow$ account for its leading profile in the super-instanton solution.

Deformations of field theories from closed string bkgs

Deformations by closed string backgrounds

- Open strings interact with closed strings. We can turn on a closed string background and still look at the massless open string d.o.f..
- In this way, deformations of the gauge theory are naturally suggested by their string realization. Such deformations are characterized by
  - new geometry in (super)space-time;
  - new mathematical structures;
  - new types of interactions and couplings.

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Deformations by constant form backgrounds

- The simplest, yet very interesting, effects are obtained considering constant backgrounds for some antisymmetric tensor from the closed string spectrum
- Well known example: non-commutative field theories from open strings in $B^{\mu\nu}$ background

[Chu-Ho, 1999, Seiberg-Witten, 1999],...

$$[x^\mu, x^\nu] = \theta^{\mu\nu}(B)$$

- I will concentrate on two other cases, where the constant background is from the RR sector:
  - Non-anti-commutative (NAC) field theories
  - Nekrasov's ϵ-deformations of the instanton moduli space in $\mathcal{N} = 2$ gauge theories

Non-anti-commutative deformations from RR backgrounds

N.A.C. theories

- In Euclidean space, one can consider a fermionic counterpart to the x-space non-commutativity. For instance, the $\mathcal{N} = 1/2$ algebra in $d = 4$:
$$\{\theta_\alpha, \theta_\beta\} = C_{\alpha\beta} \ , \quad \left\{\theta_\alpha, \theta_{\dot\beta}\right\} = \left\{\theta_{\dot\alpha}, \theta_{\dot\beta}\right\} = 0 \ .$$
- This algebra has been linked to the effect of a constant "graviphoton" bkg from the RR sector via Berkovits' formalism for superstrings on CY.

[Ooguri–Vafa, 2003, de Boer et al, 2003, Seiberg, 2003], ...

- However, derivations of the actual deformed lagrangians have been done via superspace techniques, using the non-anti-commutative $\star$-product between superfields

$$\Psi_1 \star \Psi_2 = \Psi_1 \exp\left(-\frac{C_{\alpha\beta}}{2} \frac{\overleftarrow{\partial}}{\partial\theta^\alpha} \frac{\overrightarrow{\partial}}{\partial\theta^\beta}\right) \Psi_2$$

NAC theories from string diagrams

- It is possible, though, to link $\mathcal{N} = 1/2$ theories to a RR bkg by directly computing the NAC deformed actions from the appropriate field-theory limit of string disc diagrams with the inclusion of RR vertices.

[M.B. et al., 2004; M.B. et al., 2005]

- Recall the expression of a RR vertex in $d = 10$ (e.g., for type IIB):

$$V_F(z,\bar{z}) = F_{AB}(p) S^A(z) \mathrm{e}^{-\frac{\phi}{2}}(z) \tilde{S}^B(\bar{z}) \mathrm{e}^{-\frac{\tilde{\phi}}{2}}(\bar{z}) \mathrm{e}^{\mathrm{i}p\cdot X(z,\bar{z})}$$

  - A, B are 10d Weyl spinor indices, S^A, $\tilde{S}^B$ spin fields, ϕ bosonizes the superghost system

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  - Expanding the bi-spinor $F_{AB}(p)$ over the basis of Γ-matrices yields 1-, 3- and 5-form field strengths

The procedure

1. Choose a tractable geometrical background giving $\mathcal{N} = 2$ SUSY in the bulk, and a configuration of branes supporting $\mathcal{N} = 1$ gauge theories (+ matter)
2. Individuate the specific RR field-strength responsible for the N.A.C. deformation and compute diagrams with insertions of it

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The set-up

- Type IIB string theory on the target space

$$\mathbb{R}^4 \times \mathbb{C}^3/(\mathbb{Z}_2 \times \mathbb{Z}_2)$$

  Decompose $x^M \to (x^\mu, x^a)$, $(\mu = 1, \ldots 4\,, \quad a = 5, \ldots, 10)$.
- The orbifold group generators are
  - g_1: a rotation by π in the 7-8 and by $-\pi$ in the 9-10 plane;
  - g_1: a rotation by π in the 5-6 and by $-\pi$ in the 9-10 plane.
- The origin is a fixed point $\Rightarrow$ the orbifold is a singular, non-compact, Calabi-Yau space.

Residual supersymmetry

- Of the 8 spinor weights of $\mathrm{SO}(6)$, $\vec{\lambda} = (\pm\frac{1}{2}, \pm\frac{1}{2}, \pm\frac{1}{2})$, only

$$\vec{\lambda}^{(+)} = (+\frac{1}{2}, +\frac{1}{2}, +\frac{1}{2}) \ , \qquad \vec{\lambda}^{(-)} = (-\frac{1}{2}, -\frac{1}{2}, -\frac{1}{2})$$

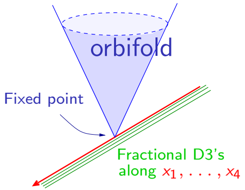
are invariant ones w.r.t. the generators $g_{1,2}$. They are the orbifold realization of the $2 (= 8/4)$ Killing spinors of the CY.

- We remain with $8 (= 32/4)$ real susies in the bulk.
- The internal spin fields organize in irrepses of $\mathbb{Z}_2 \times \mathbb{Z}_2$. E.g.,

$$S^{(\pm\pm\pm)} = \mathrm{e}^{\pm\frac{i}{2}(\varphi_1+\varphi_2+\varphi_3)} \ ,$$

where $\varphi_{1,2,3}$ bosonize the $\mathrm{SO}(6)$ current algebra, is invariant.

Fractional D3-branes



- Place N fractional D3 branes, localized at the orbifold fixed point. The branes preserve $4 = 8/2$ real supercharges.
- The Chan-Patons of open strings attached to fractional branes transform in an irrep of $\mathbb{Z}_2 \times \mathbb{Z}_2$. There are therefore 4 different such branes, labeled by $I = 01, 2, 3$.
- The fractional branes must sit at the orbifold fixed point.

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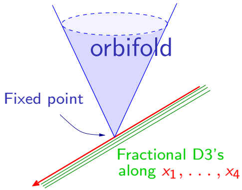

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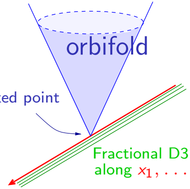

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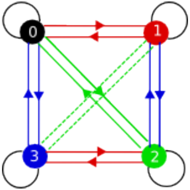
The spectrum: a quiver gauge theory



- The spectrum of massless states from the open strings stretching between $\{N_I\}$ branes of types $\{I\}$ is encoded in a quiver diagram

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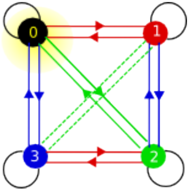

- The spectrum of massless states from the open strings stretching between $\{N_I\}$ branes of types $\{I\}$ is encoded in a quiver diagram

- Each dot corresponds to a $\mathrm{U}(N_I)$ gauge multiplet, with vertices

$$V_A(p) = A_\mu(p)\, \frac{\psi^\mu}{\sqrt{2}}\, \mathrm{e}^{-\phi}\, \mathrm{e}^{\mathrm{i}p\cdot X} \quad ,$$

$$V_\lambda(p) = \mathrm{i}\, \lambda^\alpha(p)\, S_\alpha\, S^{---}\, \mathrm{e}^{-\frac{1}{2}\phi}\, \mathrm{e}^{\mathrm{i}p\cdot X} \quad ,$$

$$V_{\overline{\lambda}}(p) = \overline{\lambda}_{\dot\alpha}(p)\, S^{\dot\alpha}\, S^{+++}\, \mathrm{e}^{-\frac{1}{2}\phi}\, \mathrm{e}^{\mathrm{i}p\cdot X} \quad ,$$

$$V_D(p) = \frac{1}{3}\, D(p)\, \delta_{ij} : \Psi^i\, \overline{\Psi}^j :\ \mathrm{e}^{ip\cdot X}$$

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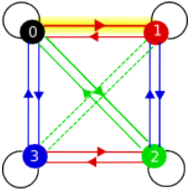

- The spectrum of massless states from the open strings stretching between $\{N_I\}$ branes of types $\{I\}$ is encoded in a quiver diagram

- An oriented link from the I-th to the J-th dot corresponds to a chiral multiplet Φ^{IJ} transforming in the $(N_I, \overline{N}_J)$ representation. For instances, in the (01) case, the string vertices are

$$V_{\varphi^{01}}(p) = \frac{g}{2}\,\varphi^{01}(p)\,\overline{\Psi}^1\,\mathrm{e}^{-\phi}\,\mathrm{e}^{\mathrm{i}p\cdot X}\ ,$$

$$V_{\chi^{01}}(p) = \frac{g}{\sqrt{2}}\,\chi^{01\,\alpha}(p)\,S_\alpha\,S^{-++}\,\mathrm{e}^{-\frac{1}{2}\phi}\,\mathrm{e}^{\mathrm{i}p\cdot X}$$

$$V_{F^{01}}(p) = g\,F^{01}(p)\,\Psi^2\Psi^3\,\mathrm{e}^{\mathrm{i}p\cdot X}\ .$$

Gauge action

- The standard action for the I-th gauge multiplet is retrieved from disc amplitudes in the $\alpha' \to 0$ limit:

$$S = \frac{1}{g_{\mathrm{YM}}^2} \int d^4x \, \mathrm{r} \left(\frac{1}{2} F_{\mu\nu}^2 - 2\bar{\lambda}_{\dot{\alpha}} \bar{\not{D}}^{\dot{\alpha}\beta} \lambda_\beta \right) .$$

- The action can be obtained from cubic diagram only introducing the (anti-selfdual) auxiliary field $H_{\mu\nu} \equiv H_c \bar{\eta}^c_{\mu\nu}$, with (non-BRST-inv) vertex $\frac{1}{2} H_{\mu\nu}(p) : \psi^\nu \psi^\mu : \mathrm{e}^{\mathrm{i}p\cdot X}$:

$$S' = \frac{1}{g_{\mathrm{YM}}^2} \int d^4x \, \mathrm{Tr} \Big\{ (\partial_\mu A_\nu - \partial_\nu A_\mu) \partial^\mu A^\nu + 2\mathrm{i} \, \partial_\mu A_\nu \left[A^\mu, A^\nu \right] - 2\bar{\lambda}_{\dot{\alpha}} \bar{\not{D}}^{\dot{\alpha}\beta} \lambda_\beta + H_c H^c + H_c \, \bar{\eta}^c_{\mu\nu} \left[A^\mu, A^\nu \right] \Big\} \quad ,$$

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- Integrating out H_c gives $H_{\mu\nu} \propto [A_\mu, A_\nu]$ and the usual action

The matter action - kinetic part

- From discs with their boundary on two different types of branes, say 0 and 1, we recover the "kinetic" lagrangian for the chiral multiplet:

$$
\begin{aligned}
\mathcal{L}_{\mathrm{matt}} = \mathrm{Tr}\Big\{ & D_\mu\overline{\varphi}^{10} D_\mu\varphi^{01} - \mathrm{i}\,\overline{\chi}^{10}\bar{\sigma}^\mu D_\mu\chi^{01} + \overline{F}^{10}F^{01} \\
& + \overline{\varphi}^{10}D^0\varphi^{01} - \varphi^{01}D^1\overline{\varphi}^{10} + \sqrt{2}\,\mathrm{i}\left(\overline{\chi}^{10}\overline{\lambda}^0\varphi^{01} - \varphi^{01}\overline{\lambda}^1\overline{\chi}^{10}\right) \\
& + \sqrt{2}\,\mathrm{i}\left(\overline{\varphi}^{10}\lambda^0\chi^{01} - \chi^{01}\lambda^1\overline{\varphi}^{10}\right)\Big\} \quad .
\end{aligned}
$$

The matter action: superpotential

- With at least 3 types of branes, the Chan-Paton structure of the vertices allows for a cubic superpotential.

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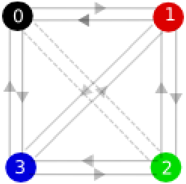
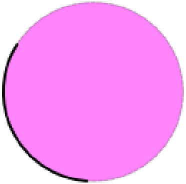
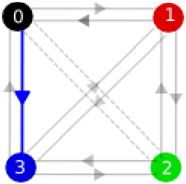
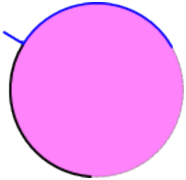




The matter action: superpotential

- With at least 3 types of branes, the Chan-Paton structure of the vertices allows for a cubic superpotential.

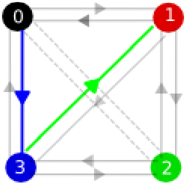


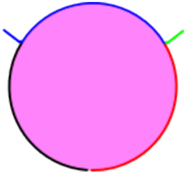


- For instance, insert in a string disc amplitude a vertex for F^{03}. This makes the boundary jump from a brane of type 0 to a 3.

The matter action: superpotential

- With at least 3 types of branes, the Chan-Paton structure of the vertices allows for a cubic superpotential.

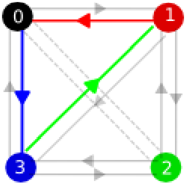


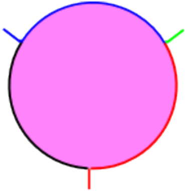


- Then a vertex for φ^{31}. The boundary jumps from type 3 to 1.

The matter action: superpotential

- With at least 3 types of branes, the Chan-Paton structure of the vertices allows for a cubic superpotential.





- Finally, a vertex for φ^{10}. This makes the boundary return from 1 to 0.

The matter action: superpotential

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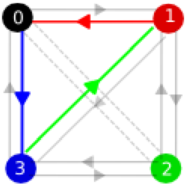
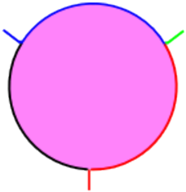


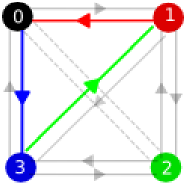


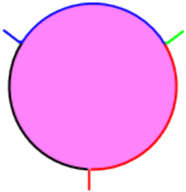
- By explicitly computing the diagram, we find indeed a non-zero coupling of the type

$$g\,\mathrm{Tr}\,\left(F^{03}\varphi^{31}\varphi^{10}\right)$$

The matter action: superpotential

- With at least 3 types of branes, the Chan-Paton structure of the vertices allows for a cubic superpotential.





- There's also the term (related by SUSY)

$$g\mathrm{Tr}\left(\varphi^{03}\chi^{31}\chi^{10}\right)$$

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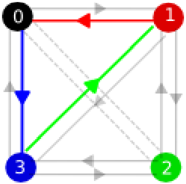

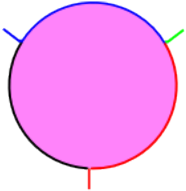

- Of course we can use any "triangle" on the quiver, and pack the two terms in a superfield expression.
- Altogether, we find an holomorphic superpotential of the form

$$W = \frac{g}{3} \sum_{I \neq J \neq K} \mathrm{Tr}\left(\Phi^{IJ}\Phi^{JK}\Phi^{KI}\right)$$

The graviphoton background

- Let us now consider insertions of a RR background to compute the NAC deformation of the above $\mathcal{N} = 1$ theory.
- On $\mathbb{R}^4 \times \frac{\mathbb{R}^6}{\mathbb{Z}_2 \times \mathbb{Z}_2}$, a surviving 4D bispinor vertex is

$$\mathcal{F}_{\alpha\beta}\, S^\alpha S^{(---)} \mathrm{e}^{-\phi/2}(z)\, \tilde{S}^\beta \tilde{S}^{(---)} \mathrm{e}^{-\tilde{\phi}/2}(\bar{z}) \ .$$

  with $\mathcal{F}_{\alpha\beta} = \mathcal{F}_{\beta\alpha}$.
- Decomposing the 5-form along the holom. 3-form of the CY $\rightsquigarrow$ an self-dual 2-form in 4D

$$C_{\mu\nu} \propto \mathcal{F}_{\alpha\beta}(\sigma^{\mu\nu})^{\alpha\beta} \ ,$$

  the graviphoton f.s. of $\mathcal{N} = 1/2$ theories.

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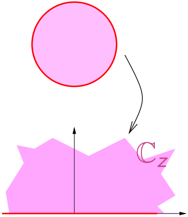
Graviphoton vertices in disc amplitudes



- Conformally mapping the disk to the upper half z-plane, the D3 boundary conditions on spin fields read

$$S^\alpha S^{---}(z) = \tilde{S}^\alpha \tilde{S}^{---}(\bar{z})\Big|_{z=\bar{z}} \ .$$

- When closed string vertices are inserted in a D3 disc,

$$\tilde{S}^\alpha \tilde{S}^{---}(\bar{z}) \longrightarrow S^\alpha S^{---}(\bar{z}) \ .$$

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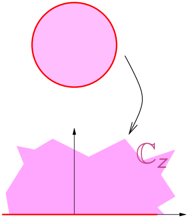

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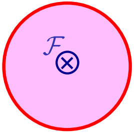
A disc amplitude with a graviphoton

Start inserting a graviphoton vertex on a disc with its boundary all on a single type, say the I-th, of branes:

$$\langle\!\langle\, V_{\bar\lambda}\, V_{\bar\lambda}\, V_A\, V_{\mathcal{F}} \,\rangle\!\rangle$$



where

$$V_{\mathcal{F}}(z,\bar z) = \mathcal{F}_{\alpha\beta}\, S^{\alpha} S^{---} \mathrm{e}^{-\phi/2}(z)\, S^{\beta} S^{---} \mathrm{e}^{-\phi/2}(\bar z)\,.$$

$\rightsquigarrow$ we need two S^{+++} operators to “saturate the charge”

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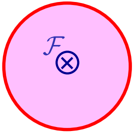

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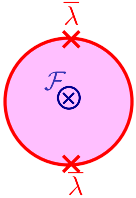


We insert therefore two anti-chiral gauginos:

$$\langle\!\langle\, V_{\overline{\lambda}}\, V_{\overline{\lambda}}\, V_A\, V_{\mathcal{F}} \,\rangle\!\rangle$$

with vertices

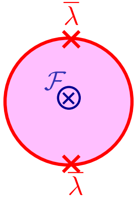
$$V_{\overline{\lambda}}(y;p) = (2\pi\alpha')^{\frac{3}{4}}\, \overline{\lambda}^{\dot\alpha}(p)\, S_{\dot\alpha} S^{+++}\, \mathrm{e}^{-\frac{1}{2}\phi(y)}\, \mathrm{e}^{i\sqrt{2\pi\alpha'}\,p\cdot X(y)}\ .$$

Without other insertions, however,

$$\langle S^\alpha S^\beta S_{\dot\alpha} S_{\dot\beta}\rangle \propto \epsilon^{\alpha\beta}\epsilon_{\dot\alpha\dot\beta}$$

$\rightsquigarrow$ vanishes when contracted with $\mathcal{F}_{\alpha\beta}$.

A disc amplitude with a graviphoton



We insert therefore two anti-chiral gauginos:

$$\langle\!\langle\, V_{\overline{\lambda}}\, V_{\overline{\lambda}}\, V_A\, V_{\mathcal{F}}\,\rangle\!\rangle$$

with vertices

$$V_{\overline{\lambda}}(y;p) = (2\pi\alpha')^{\frac{3}{4}}\,\overline{\lambda}^{\dot\alpha}(p)\,S_{\dot\alpha}S^{+++}\,\mathrm{e}^{-\frac{1}{2}\phi(y)}\,\mathrm{e}^{i\sqrt{2\pi\alpha'}p\cdot X(y)}\,.$$

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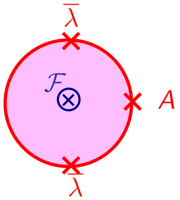
$\rightsquigarrow$ vanishes when contracted with $\mathcal{F}_{\alpha\beta}$.

A disc amplitude with a graviphoton

To cure this problem, insert a gauge field vertex:

$$\langle\!\langle\, V_{\bar\lambda}\, V_{\bar\lambda}\, V_A\, V_{\mathcal{F}} \,\rangle\!\rangle$$

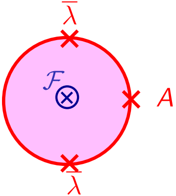


that must be in the 0 picture:

$$V_A(y;p) = 2\mathrm{i}\,(2\pi\alpha')^{\frac{1}{2}}\, A_\mu(p) \Big(\partial X^\mu(y) + \mathrm{i}\,(2\pi\alpha')^{\frac{1}{2}}\, p\cdot\psi\,\psi^\mu(y)\Big) \mathrm{e}^{\mathrm{i}\sqrt{2\pi\alpha'}\,p\cdot X(y)}$$

$\rightsquigarrow$ finally, we may get a non-zero result!

A disc amplitude with a graviphoton



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$\rightsquigarrow$ finally, we may get a non-zero result!

Evaluation of the amplitude

- We have

$$\langle\!\langle\, V_{\bar\lambda}\, V_{\bar\lambda}\, V_A\, V_{\mathcal{F}}\,\rangle\!\rangle \;\equiv\; C_4 \int \frac{\prod_i dy_i dz d\bar z}{dV_{\rm CKG}} \Big\langle V_{\bar\lambda}(y_1;p_1)\, V_{\bar\lambda}(y_2;p_2)\, V_A(y_3;p_3)\, V_{\mathcal{F}}(z,\bar z)\Big\rangle$$

where the normalization for a D3 disk is

$$C_4 = \frac{1}{\pi^2\alpha'^2}\,\frac{1}{g_{\rm YM}^2}$$

and the $\mathrm{SL}(2,\mathbb{R})$-invariant volume is

$$dV_{\rm CGK} = \frac{dy_a\, dy_b\, dy_c}{(y_a-y_b)(y_b-y_c)(y_c-y_a)} \quad .$$

Explicit expression of the amplitude

- Altogether, the explicit expression is

$$
\begin{aligned}
\langle\!\langle\, V_{\bar\lambda}\, V_{\bar\lambda}\, V_A\, V_{\mathcal{F}} \,\rangle\!\rangle = {} & \frac{8}{g_{\rm YM}^2}\,(2\pi\alpha')^{\frac12}\,\mathrm{Tr}\Big(\bar\lambda^{\dot\alpha}(p_1)\,\bar\lambda^{\dot\beta}(p_2)\,p_3^\nu A^\mu(p_3)\Big)\,\mathcal{F}_{\alpha\beta} \\
& \times \int \frac{\prod_i dy_i dz d\bar z}{dV_{\rm CKG}}\,\Big\{\big\langle S_{\dot\alpha}(y_1)\,S_{\dot\beta}(y_2)\;:\psi^\nu\psi^\mu:(y_3)\,S^\alpha(z)\,S^\beta(\bar z)\big\rangle \\
& \times \big\langle S^{+++}(y_1)\,S^{+++}(y_2)\,S^{---}(z)\,S^{---}(\bar z)\big\rangle \\
& \times \big\langle \mathrm{e}^{-\frac12\phi(y_1)}\mathrm{e}^{-\frac12\phi(y_2)}\mathrm{e}^{-\frac12\phi(z)}\mathrm{e}^{-\frac12\phi(\bar z)}\big\rangle \\
& \times \big\langle \mathrm{e}^{\mathrm{i}\sqrt{2\pi\alpha'}p_1\cdot X(y_1)}\mathrm{e}^{\mathrm{i}\sqrt{2\pi\alpha'}p_2\cdot X(y_2)}\mathrm{e}^{\mathrm{i}\sqrt{2\pi\alpha'}p_3\cdot X(y_3)}\big\rangle\Big\}\;.
\end{aligned}
$$

Final result for the amplitude

- Insering the CFT correlators, gauge-fixing $\mathrm{SL}(2,\mathbb{R}$ and performing the remaining integrations, we finally obtain for $\langle\!\langle\, V_{\bar\lambda}\, V_{\bar\lambda}\, V_A\, V_{\mathcal{F}}\,\rangle\!\rangle$ the result

$$\frac{8\pi^2}{g^2_{\rm YM}}\,(2\pi\alpha')^{\frac12}\,\mathrm{Tr}\Big(\bar\lambda(p_1)\cdot\bar\lambda(p_2)\,p_3^\nu A^\mu(p_3)\Big)\,\mathcal{F}_{\alpha\beta}\,(\sigma_{\nu\mu})^{\alpha\beta}\ .$$

- This result is finite for $\alpha'\to 0$ if we keep constant

$$C_{\mu\nu}\equiv 4\pi^2\,(2\pi\alpha')^{\frac12}\,\mathcal{F}_{\alpha\beta}\,(\sigma_{\mu\nu})^{\alpha\beta}$$

- $C_{\mu\nu}$, of dimension (length) will be exactly the one of $\mathcal{N}=1/2$ theory.
- We get an extra term in the gauge theory action:

$$\frac{\mathrm{i}}{g^2_{\rm YM}}\int d^4x\ \mathrm{Tr}\Big(\lambda\cdot\lambda\,\big(\partial^\mu A^\nu-\partial^\nu A^\mu\big)\Big)\,C_{\mu\nu}\ .$$

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$$\frac{8\pi^2}{g^2_{\mathrm{YM}}}\,(2\pi\alpha')^{\frac{1}{2}}\,\mathrm{Tr}\Big(\bar\lambda(p_1)\cdot\bar\lambda(p_2)\,p_3^\nu A^\mu(p_3)\Big)\,\mathcal{F}_{\alpha\beta}\,(\sigma_{\nu\mu})^{\alpha\beta}\quad .$$

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The deformed gauge theory action

- There is also the diagram $\langle\!\langle\, V_\lambda\, V_\lambda\, V_A\, V_{\mathcal{F}}\,\rangle\!\rangle$ involving the auxiliary field H_c

Recall def. of H_c

- Altogether, from disc diagrams with their boundary on a branes of type I only , and with a RR insertion we obtain, in the field theory limit described above, the action

$$\frac{4\mathrm{i}}{g^2}\, C^{\mu\nu}\, \mathrm{Tr}\left\{\left(\partial_\mu A^I_\nu - \frac{\mathrm{i}}{4}\, H^I_{\mu\nu}\right)\bar\lambda^I\bar\lambda^I\right\} \quad .$$

- Adding this to the undeformed Lagrangian and integrating our $H^I_{\mu\nu}$ yelds exactly Seiberg's $\mathcal{N}=1/2$ gauge Lagrangian that follows from the NAC deformation of the superspace:

$$\mathcal{L}_{\rm gauge} = \frac{1}{g^2}\,\mathrm{Tr}\Big\{\frac{1}{2}\,(F^I_{\mu\nu})^2 - 2\mathrm{i}\,\bar\lambda^I\,\bar\sigma^\mu D_\mu\lambda^I - (D^I)^2 + 2\mathrm{i}\, C^{\mu\nu}\, F^I_{\mu\nu}\,\bar\lambda^I\,\bar\lambda^I - 4\,\det C\,\left(\bar\lambda^I\,\bar\lambda^I\right)^2\Big\} - \frac{\mathrm{i}\,\theta_{\rm YM}}{32\pi^2}\,\varepsilon^{\mu\nu\rho\sigma}\,\mathrm{Tr}\,F^I_{\mu\nu}\,F^I_{\rho\sigma}$$

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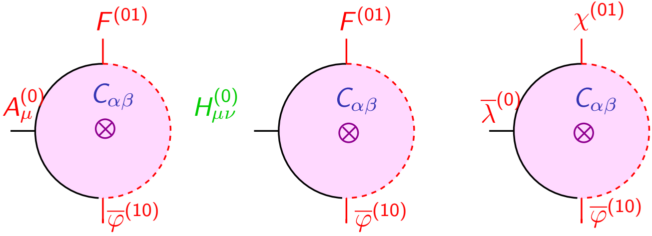
Deformations in the quiver theory

- With different types of branes, i.e., in the quiver gauge theory, there are several other diagrams with graviphoton insertions and involving chiral multiplet fields.
- These arise from discs with portions of their boundary on different types of branes
- Discs attached to two different types of branes contribute to the "kinetic" part of the lagrangian.



Deformations in the "kinetic" quiver lagrangian

- Extra interactions involving $H^I_{\mu\nu}$ $\rightsquigarrow$ its e.o.m. are modified to

$$H^I_{\mu\nu} = -2\left[A^I_\mu, A^I_\mu\right]^{(+)} - 2\, C_{\mu\nu}\left(\bar{\lambda}^I\bar{\lambda}^I + \frac{g^2}{2}\sum_{J\neq I}\left(F^{IJ}\overline{\varphi}^{JI} - \overline{\varphi}^{IJ}F^{JI}\right)\right.$$

Deformations in the "kinetic" quiver lagrangian

- Plugging this back, and taking into account the other diagrams for all possible pairs of boundaries gives the deformation terms for the quiver gauge theory:

$$\frac{1}{g^2}\sum_I \mathrm{Tr}\Big\{ 2\mathrm{i}\, C_{\mu\nu}\, F^I_{\mu\nu}\Big(\bar\lambda^I\bar\lambda^I + \frac{g^2}{2}\sum_{J\neq I}\Big(F^{IJ}\bar\varphi^{JI} - \bar\varphi^{IJ}F^{JI}\Big)\Big)$$

$$+\sqrt{2}\, C^{\mu\nu}\sum_{J\neq I}\mathrm{Tr}\Big\{\big(\bar\lambda^I\bar\sigma_\nu\chi^{IJ} - \chi^{IJ}\sigma_\nu\bar\lambda^J\big)D_\mu\bar\varphi^{JI}\Big\}$$

$$-4\det C\Big(\bar\lambda^I\bar\lambda^I + \frac{g^2}{2}\sum_{J\neq I}\Big(F^{IJ}\bar\varphi^{JI} - \bar\varphi^{IJ}F^{JI}\Big)\Big)^2\Big\}\,.$$

Deformations in the “kinetic” quiver lagrangian

- This coincides with the lagrangian that can be constructed (rather painfully in the quiver case) from deformed super-space techniques...
- ... including, however, terms of the form e.g.

$$-g^2 \det C \, \mathrm{Tr} \, (F^{01}\overline{\varphi}^{10})^2$$

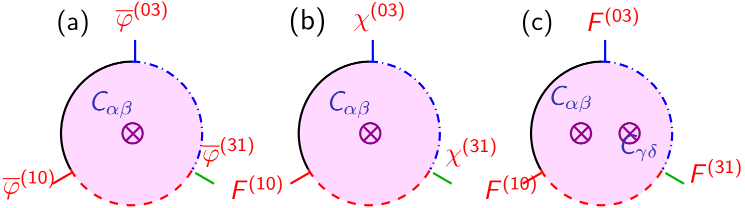
which can be induced with a particular C-dependent “shifts” of the auxiliary fields F^{IJ}. They are inessential at tree level, but they would in any case arise at 1-loop order

[Grisaru et al, 2003, Romagnoni, 2003]

- In our direct string construction they arise naturally.

Deformations of the superpotential

- There can be RR insertions in discs with three different portions of their boundary:



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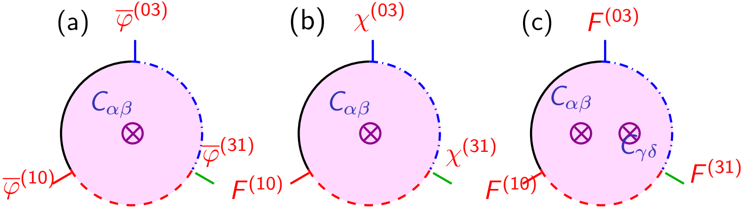
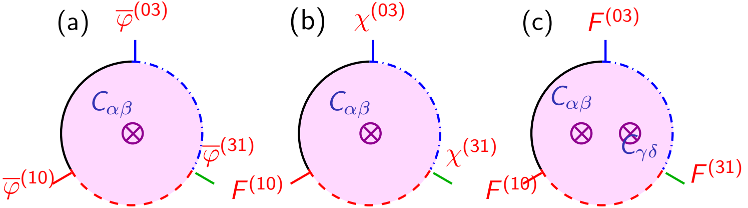


- They reproduce the N.A.C. def.s of the super-potential

$$\mathcal{L}_W + \mathcal{L}_{\overline{W}} = \frac{g}{3} \sum_{I \neq J \neq K} \left[\int d^2\theta \, \mathrm{Tr}\left(\Phi^{IJ} \star \Phi^{JK} \star \Phi^{KI} \right) + \int d^2\bar{\theta} \, \mathrm{Tr}\left(\overline{\Phi}^{IJ} \star \overline{\Phi}^{JK} \star \overline{\Phi}^{KI} \right) \right]$$

Deformations of the superpotential

- There can be RR insertions in discs with three different portions of their boundary:

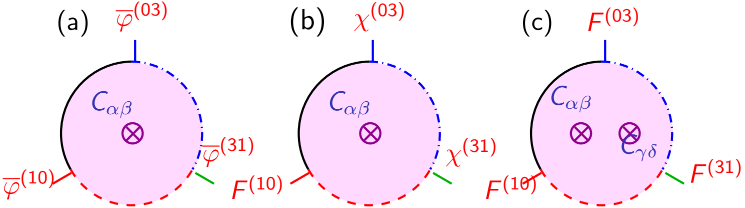


a) $\rightsquigarrow$ deformations of the anti-holomorphic part:

$$2g \sum_{I \neq J \neq K} \mathrm{Tr}\Big(C^{\mu\nu}\, \overline{\varphi}^{IJ} D_\mu \overline{\varphi}^{JK} D_\nu \overline{\varphi}^{KI} \Big)$$

Deformations of the superpotential

- There can be RR insertions in discs with three different portions of their boundary:

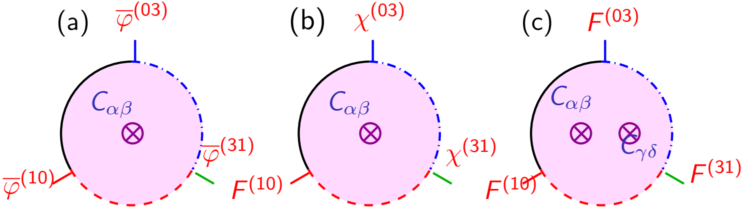


b) $\rightsquigarrow$ deformations of the holomorphic part:

$$\frac{g}{4}\sum_{I\neq J\neq K}\mathrm{Tr}\Big(C^{\mu\nu}\,F^{IJ}\chi^{JK}\sigma^{\mu\nu}\chi^{KI}\Big)$$

Deformations of the superpotential

- There can be RR insertions in discs with three different portions of their boundary:



c) (a bit more difficult to compute: two RR vertices) ⇝ other deformations of the holomorphic part:

$$-\frac{g}{3}\sum_{I\neq J\neq K}\mathrm{Tr}\Big(\det C\, F^{IJ}F^{JK}F^{KI}\Big)$$

Localization deformations in D3/D(-1) systems

Multi-instanton corrections in $\mathcal{N} = 2$ SYM

- The l.e.e.a. for $\mathcal{N} = 2$ gauge theories, say for $\mathrm{SU}(2)$, is determined in terms of a prepotential

$$\mathcal{F}(a;\Lambda) = \mathcal{F}^{\mathrm{pert.}}(a;\Lambda) + \underbrace{\sum_k c_k \left(\frac{\Lambda}{a}\right)^{4n}}_{\textit{instanton contrib.s}}$$

- The exact expression of $\mathcal{F}(a;\Lambda)$ was derived by Seiberg-Witten, based on general holomorphicity requirement plus physical singularity requirements.
- Much work done to check directly the values c_k of the instanton number, success for $k = 1, 2$
- Beyond that, integration on multi-instanton moduli space very difficult.
- Big leap forward: computation at generic k by means of localization techniques applied to the moduli measure, suitably deformed

[Nekrasov, 2002]

Nekrasov's deformation

- Deform the "action" $S_{\rm mod}(a)$ for the moduli of the $\mathcal{N}=2$ super-instanton (in presence of v.e.v.'s a of the complex scalars of the gauge multiplet) to

$$S_{\rm mod}(a,\epsilon)$$

- Here ϵ is the parameter of a space-time $\mathrm{SO}(4) \sim \mathrm{SU}(2)_+ \times \mathrm{SU}(2)_-$ rotation, typically the $\mathrm{U}(1)_- \subset \mathrm{SU}(2)_-$. It appears via the moment maps for this action. The moduli action can then be written as

$$S_{\rm mod}(a,\epsilon) = Q_\epsilon \Sigma(a,\epsilon)$$

where Q_ϵ is a fermionic symmetry, constructed from the SUSY charges upon "topological twist"

[Flume-Poghossian, 2002, Bruzzo et. al, 2002; 2003],...

$$\mathrm{SU}(2)_- \rightarrow \mathrm{diag}\left(\mathrm{SU}(2)_- \times \mathrm{SU}(2)_R\right)$$

The deformed partition function

- This ensures that "localization" theorems can be used to compute efficiently the deformed partition function on the moduli space at any instanton number k

$$\mathcal{Z}_k(a,\epsilon) = \int_{M_k} \mathrm{e}^{-S_{\rm mod}(a,\epsilon)}$$

- Define

$$\sum_k \mathcal{Z}_k(a,\epsilon)\,\Lambda^{4k} = \mathcal{Z}(a,\epsilon;\Lambda) \equiv \exp\left(-\frac{\mathcal{F}(a,\epsilon)}{\epsilon^2}\right)$$

- Then expand

$$\mathcal{F}(a),\epsilon) = \sum_{g=0}^{\infty} \mathcal{F}_g(a)\epsilon^{2g}$$

Nekrasov's results (and conjecture)

- Nekrasov argued that $\mathcal{F}_0(a)$ coincides with the instanton part of the prepotential of the $\mathcal{N} = 2$ theory.
- The terms with $g > 0$ are supposed to appearing in certain gravitational couplings of the theory:

$$\mathcal{F}_g(a)\,(F^+)^{2g-2}R^+$$

with F^+ the self-dual part of a graviphoton field strength, and R^+ the s.d. part of the curvature tensor.

Checked for low g against top. string theory in [klemm et al, 2003]

- The relation to the graviphoton coupling is not explained within the "microscopic" description, but via a "geometrical engineering" of the l.e.e.t.

Nekrasov's deformation from RR background

- In forth-coming paper, we show that the deformed $\mathcal{N} = 2$ moduli action can be obtained from the mixed discs of the D3/D(-1) system
  - placed in the $\mathbb{R}^4 \times \mathbb{C} \times \mathbb{C}^2/\mathbb{Z}_2$ orbifold, to get $\mathcal{N} = 2$ susy on the (fractional) D3'
  - with the insertion of a specific RR background, to be identified with the parameter ϵ.
- We hope this might help to relate more directly the deformed partition function to the graviphoton couplings.

The 3-form RR background

- The constant RR background to be inserted has the vertex

$$V_{\mathcal{M}}(z,\bar{z}) = \mathcal{M}_{\dot{\alpha}\dot{\beta}AB}\, S^{\dot{\alpha}}(z) S^{A}(z) \mathrm{e}^{-\frac{1}{2}\phi(z)}\; \widetilde{S}^{\dot{\beta}}(\bar{z}) \widetilde{S}^{B}(\bar{z}) \mathrm{e}^{-\frac{1}{2}\tilde{\phi}(\bar{z})}$$

- The index A of the internal spin fields is restricted by the orbifold pojection, effectively, to two values
- We choose

$$\mathcal{M}_{\dot{\alpha}\dot{\beta}AB} = \mathcal{N}_{(\dot{\alpha}\dot{\beta})[AB]} + \mathcal{L}_{[\dot{\alpha}\dot{\beta}](AB)}$$

- $\mathcal{N}_{(\dot{\alpha}\dot{\beta})[AB]}$ corresponds to a 3-form of the type $\mathcal{N}_{\mu\nu m}$ (one internal index only), $\mathcal{L}_{[\dot{\alpha}\dot{\beta}](AB)}$ to one of type $\mathcal{L}_{mnp}$.

The moduli action

- Constructing the spectrum of D3/D3 and D3/D(-1) moduli, and computing their tree-level (disc) interactions, in the field theory limit one gets

$$S_{\rm mod} = {\rm tr}\Big\{ -2\big([\chi^\dagger, a_\mu] - a^\nu \mathcal{N}^\dagger_{\mu\nu}\big)\big([\chi, a^\mu] - a_\nu \mathcal{N}^{\mu\nu}\big)$$
$$+ 2\bar{w}^{\dot\alpha}\big(-\varepsilon_{\dot\alpha\dot\beta}\chi^\dagger + 2\mathcal{N}^\dagger_{\dot\alpha\dot\beta}\big)\big(-\varepsilon^{\dot\beta\dot\gamma}\chi + 2\mathcal{N}^{\dot\beta\dot\gamma}\big)w_{\dot\gamma}$$
$$+ {\rm i}\frac{\sqrt{2}}{2} M^{\alpha A}(\varepsilon_{AB}\chi^\dagger - \sqrt{2}\mathcal{L}_{AB})M^B_\alpha + {\rm i}\frac{\sqrt{2}}{2}\bar\mu^A(\varepsilon_{AB}\chi^\dagger - \sqrt{2}\mathcal{L}_{AB})\mu^B$$
$$+ {\rm i}D_c\big(W^c + {\rm i}\bar\eta^c_{\mu\nu}\big[a'^\mu, a'^\nu\big]\big) - {\rm i}\lambda^{\dot\alpha}_A\big(w_{\dot\alpha}\bar\mu^A + \mu^A\bar{w}_{\dot\alpha} + [a'_{\alpha\dot\alpha}, M'^{\alpha A}]\big)\Big\}$$

(for simplicity, written here at zero v.e.v.'s a of scalar fields)

Topological twist and localization

- After having computed the moduli action from the diagrams, we can twist it as described above.
- In practice, we can identify the internal spinor indices A with the space-time spinor indices $\dot{\alpha}$.
- It becomes meaningful to consider the special background

$$\mathcal{L}_{\dot{\alpha}\dot{\beta}} = \sqrt{2}\mathcal{N}_{\dot{\alpha}\dot{\beta}}$$

- The susy charges $Q^{\dot{\alpha}A}$ reorganize as

$$Q^{\dot{\alpha}\dot{\beta}} = Q\,\varepsilon^{\dot{\alpha}\dot{\beta}} + \frac{1}{4}\,Q^{\mu\nu}\,(\sigma_{\mu\nu})^{\dot{\alpha}\dot{\beta}}$$

Final form of the moduli action

- Finally, we have the moduli action

$$
\begin{aligned}
S_{\rm mod} = \mathrm{tr}\Big\{ & -2\big([\chi^\dagger, a_\mu] - a^\nu \mathcal{N}^\dagger_{\mu\nu}\big)\big([\chi, a^\mu] - a_\nu \mathcal{N}^{\mu\nu}\big) \\
& + 2\bar{w}^{\dot\alpha}\big(-\varepsilon_{\dot\alpha\dot\beta}\chi^\dagger + \mathcal{N}^\dagger_{\dot\alpha\dot\beta}\big)\big(-\varepsilon^{\dot\beta\dot\gamma}\chi + \mathcal{N}^{\dot\beta\dot\gamma}\big) w_{\dot\gamma} \\
& + \mathrm{i}\frac{\sqrt{2}}{2} M^{\alpha\dot\alpha}\big(\varepsilon_{\dot\alpha\dot\beta}\chi^\dagger - \mathcal{N}^\dagger_{\dot\alpha\dot\beta}\big) M^{\dot\beta}_{\alpha} + \mathrm{i}\frac{\sqrt{2}}{2}\bar{\mu}^{\dot\alpha}\big(\varepsilon_{\dot\alpha\dot\beta}\chi^\dagger - \mathcal{N}^\dagger_{\dot\alpha\dot\beta}\big)\mu^{\dot\beta} \\
& + \mathrm{i}D_c\big(W^c + \mathrm{i}\bar\eta^c_{\mu\nu}\big[a'^\mu, a'^\nu\big]\big) - \mathrm{i}\lambda^{\dot\alpha}_{\dot\beta}\big(w_{\dot\alpha}\bar\mu^{\dot\beta} + \mu^{\dot\beta}\bar{w}_{\dot\alpha} + [a'_{\alpha\dot\alpha}, M'^{\alpha\dot\beta}]\big)\Big\}
\end{aligned}
$$

- This action is invariant under the action of the scalar fermionic charge Q defined above, and can be written as Q(something). This guarantees the desired localization properties.
- This action indeed coincides with the one used by Nekrasov. choosing

$$
\mathcal{N}_{\dot\alpha\dot\beta} = \epsilon\,(\tau_3)_{\dot\alpha\dot\beta}
$$

Conclusions and outlook

Conclusions

- When gauge (and matter) theories are realized on D-branes, many of their properties are accessible by perturbative world-sheet computations, sometimes unexpectedly. In particular,

- the instantonic sectors of (supersymmetric) YM theories is *really* described by D3/D(-1) systems.
- Disks (partly) attached to the D(-1)'s account, in the $\alpha' \to 0$ field theory limit for
  - the ADHM construction of instanton moduli space;
  - the classical profile of the instanton solution: the mixed disks are the source for it;
  - the "instanton calculus" of correlators.

Conclusions

- When gauge (and matter) theories are realized on D-branes, many of their properties are accessible by perturbative world-sheet computations, sometimes unexpectedly. In particular,

- the open string realization of gauge theories is a very powerful tool, also in discussing possible deformations (induced by closed string backgrounds).
- The deformation of $\mathcal{N} = 1$ gauge theory to $\mathcal{N} = 1/2$ gauge theory is exactly described by the inclusion of a particular RR background.
- Nekrasov's "localization" deformation of the instanton moduli space of $\mathcal{N} = 2$ SYM is also accounted by (a different) RR bkg.

Outlook

- We think it is useful to apply the "perturbative world-sheet" point of view outlined above to other situations as well. For instance,
  - derivation of the instantonic sector of non-commutative gauge theory from the string realization with constant $B_{\mu\nu}$ background

    To appear soon, [M.B. et al]
  - Use of the D3/D(-1) description of the instantonic sectors of $\mathcal{N} = 2$ SYM to search for the gravitational dual description of the Seiberg-Witten solution
  - ...

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