Instanton Calculus In R-R Background And The Topological String

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Foreword

This talk is based on

- M. Billo, M. Frau, F. Fucito and A. Lerda, arXiv:hep-th/0606013 (to appear on JHEP).

It of course builds over a vast literature. The few references scattered on the slides are by no means intended to be exhaustive. I apologize for the many relevant ones which will be missing.

Plan of the talk

Introduction

Stringy instanton calculus for $\mathcal{N} = 2$ SYM

Inclusion of a graviphoton background

Effective action and relation to topological strings

Introduction

General idea

- We consider an explicit example (in a controllable set-up) of a type of computation which is presently attracting some attention:
  - deriving D-instanton-induced interactions in effective actions

- We study D-instanton induced couplings of the chiral and the Weyl multiplet in the $\mathcal{N} = 2$ low energy effective theory
- In this framework, we obtain a natural interpretation of a remarkable conjecture by Nekrasov regarding the $\mathcal{N} = 2$ multi-instanton calculus and its relation to topological string amplitudes on CY's

The quest for the multi-instanton contributions

The semiclassical limit of the exact SW prepotential displays 1-loop plus instanton contributions: ▸ Back

$$\mathcal{F}(a) = \frac{\mathrm{i}}{2\pi} a^2 \log \frac{a^2}{\Lambda^2} + \sum_{k=1}^{\infty} \mathcal{F}^{(k)}(a)$$

- Important task: compute the multi-instanton contributions $\mathcal{F}^{(k)}(a)$ within the "microscopic" description of the non-abelian gauge theory to check them against the SW solution
- Only recently fully accomplished using localization techniques to perform the integration over the moduli space of the ADHM construction of the super-instantons

[Nekrasov 2002, Flume-Poghossian 2002, Nekrasov et al 2003, ...]

The localizing deformation

Introduce a deformation of the ADHM measure on the moduli spaces exploiting the 4d chiral rotations symmetry of ADHM constraints.

- The deformed instanton partition function

$$Z(a,\varepsilon) = \sum_k Z^{(k)}(a,\varepsilon) = \sum_k \int d\widehat{\mathcal{M}}_{(k)} \mathrm{e}^{-\mathcal{S}_{\mathrm{mod}}(a,\varepsilon;\mathcal{M}_{(k)})}$$

can then be computed using localization techniques and the topological twist of its supersymmetries. One has

$$Z(a,\varepsilon) = \exp\left(\frac{\mathcal{F}_{\mathrm{n.p.}}(a;\varepsilon)}{\varepsilon^2}\right)$$

$$\lim_{\varepsilon\to 0} \mathcal{F}_{\mathrm{n.p.}}(a;\varepsilon) = \mathcal{F}_{\mathrm{n.p.}}(a) = \text{non-pert. part of SW prepotential}$$

Multi-instanton calculus and topological strings

What about higher orders in the deformation parameter ε?

- Nekrasov's proposal: terms of order $\varepsilon^{2h} \leftrightarrow$ gravitational F-terms in the $\mathcal{N} = 2$ eff. action involving metric and graviphoton curvatures [Nekrasov 2002, Losev et al 2003, Nekrasov 2005] Back

$$\int d^4x (R^+)^2 (\mathcal{F}^+)^{2h-2}$$

- When the effective $\mathcal{N} = 2$ theory is obtained from type II strings on a "local" CY manifold $\mathfrak{M}$ via geometrical engineering, such terms
  - arise from world-sheets of genus h
  - are computed by the topological string

[Bershadsky et al 1993, Antoniadis et al 1993]

- For the local CY describing the SU(2) theory the proposal has been tested [Klemm et al, 2002]

The aim of this work

- Reproduce the semiclassical instanton expansion of the low energy effective action for the $\mathcal{N} = 2$ SYM theory in the microscopic string realization via (fractional) D3/D(-1) branes

- Show that the inclusion of the graviphoton of the $\mathcal{N} = 2$ bulk sugra, which comes from the RR sector,
  - produces in the effective action the gravitational F-terms which are computed by the topological string on local CY
  - leads exactly to the localization deformation on the instanton moduli space which allows to perform the integration

The aim of this work

- Reproduce the semiclassical instanton expansion of the low energy effective action for the $\mathcal{N} = 2$ SYM theory in the microscopic string realization via (fractional) D3/D(-1) branes

- The situation is therefore as follows:

Microscopic string description

Geometrically engineered string description of l.e.e.t on local CY

deformed multi-instanton computations

topological string amplitudes at genus h

Gravitational F-term interactions

- The two ways to compute the same F-terms must coincide if the two descriptions are equivalent

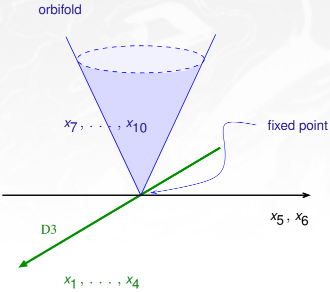
Stringy instanton calculus for $\mathcal{N} = 2$ SYM

SYM from fractional branes

Consider pure SU(N) Yang-Mills in 4 dimensions with $\mathcal{N} = 2$ susy.

- It is realized by the massless d.o.f. of open strings attached to fractional D3-branes in the orbifold background

$$\mathbb{R}^4 \times \mathbb{R}^2 \times \mathbb{R}^4/\mathbb{Z}_2$$



- The orbifold breaks 1/2 SUSY in the bulk, the D3 branes a further 1/2:

$$32 \times \frac{1}{2} \times \frac{1}{2} = 8 \ \text{ real supercharges}$$

Fields and string vertices

- Field content: $\mathcal{N} = 2$ chiral superfield

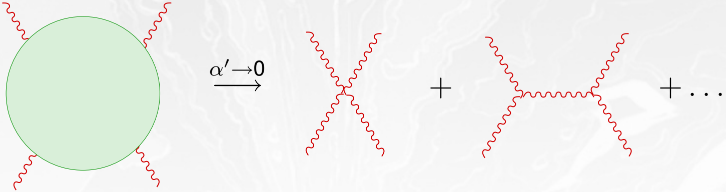
$$\Phi(x,\theta) = \phi(x) + \theta\Lambda(x) + \frac{1}{2}\,\theta\sigma^{\mu\nu}\theta\, F^{+}_{\mu\nu}(x) + \cdots$$

- String vertices:

$$\begin{aligned} V_A(z) &= \frac{A_\mu(p)}{\sqrt{2}}\,\psi^\mu(z)\,\mathrm{e}^{ip\cdot X(z)}\,\mathrm{e}^{-\varphi(z)} \\ V_\Lambda(z) &= \Lambda^{\alpha A}(p)\,S_\alpha(z)S_A(z)\,\mathrm{e}^{ip\cdot X(z)}\,\mathrm{e}^{-\frac{1}{2}\varphi(z)} \\ V_\phi(z) &= \frac{\phi(p)}{\sqrt{2}}\,\overline{\Psi}(z)\,\mathrm{e}^{ip\cdot X(z)}\,\mathrm{e}^{-\varphi(z)} \end{aligned}$$

with all polariz.s in the adjoint of $\mathrm{U}(N)$

Gauge action from disks on fD3's



- String amplitudes on disks attached to the D3 branes in the limit

$$\alpha' \to 0 \text{ with gauge quantities fixed.}$$

give rise to the tree level (microscopic) $\mathcal{N} = 2$ action

$$S_{\text{SYM}} = \int d^4x \, \text{Tr} \Big\{ \frac{1}{2} F_{\mu\nu}^2 + 2\, D_\mu \bar{\phi}\, D^\mu \phi - 2\, \bar{\Lambda}_{\dot{\alpha}A} \bar{D}\!\!\!/^{\dot{\alpha}\beta} \Lambda_\beta^A$$

$$+ \mathrm{i}\sqrt{2}\, g\, \bar{\Lambda}_{\dot{\alpha}A} \epsilon^{AB} \big[\phi, \bar{\Lambda}^{\dot{\alpha}}_B \big] + \mathrm{i}\sqrt{2}\, g\, \Lambda^{\alpha A} \epsilon_{AB} \big[\bar{\phi}, \Lambda^B_\alpha \big] + g^2 \big[\phi, \bar{\phi} \big]^2 \Big\}$$

Scalar v.e.v's and low energy effective action

- We are interested in the l.e.e.a. on the Coulomb branch parametrized by the v.e.v.'s of the adjoint chiral superfields:

$$\langle \Phi_{uv} \rangle \equiv \langle \phi_{uv} \rangle = a_{uv} = a_u \, \delta_{uv} \ , \quad u, v = 1, ..., N \ , \quad \sum_u a_u = 0$$

  breaking SU(N) $\rightarrow$ U(1)$^{N-1}$ [we focus on SU(2)]
- Up to two-derivatives, $\mathcal{N} = 2$ susy forces the effective action for the chiral multiplet Φ in the Cartan direction to be of the form

$$S_{\text{eff}}[\Phi] = \int d^4x \, d^4\theta \, \mathcal{F}(\Phi) + \text{c.c}$$

- We want to discuss the instanton corrections to the prepotential $\mathcal{F}$ ▸ Recall in our string set-up

Instantons and D-instantons

- Consider the Wess-Zumino term of the effective action for a stack of D3 branes:

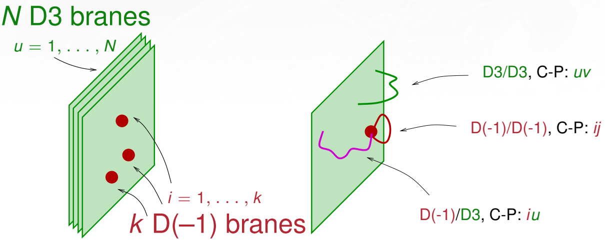
$$\text{D. B. I.} + \int_{\mathrm{D}_3} \left[C_3 + \frac{1}{2} C_0 \,\mathrm{Tr}\left(F \wedge F \right) \right]$$

  The topological density of an instantonic configuration corresponds to a localized source for the RR scalar C_0, i.e., to a distribution of D-instantons on the D3's.
- Instanton-charge k solutions of 3+1 dims. SU(N) gauge theories correspond to k D-instantons inside N D3-branes.

[Witten 1995,Douglas 1995, Dorey 1999, ...]

Stringy description of gauge instantons

	1	2	3	4	5	6	7	8	9	10
D3	—	—	—	—	*	*	*	*	*	*
D(−1)	*	*	*	*	*	*	*	*	*	*



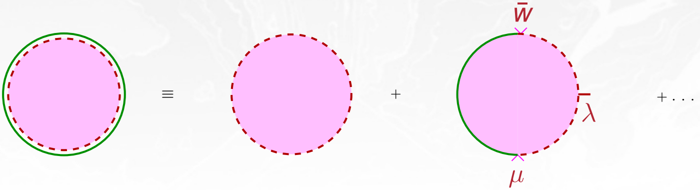
Moduli vertices and instanton parameters

Open strings ending on a D(-1) carry no momentum: moduli (rather than fields) $\leftrightarrow$ parameters of the instanton.

	ADHM	Meaning	Vertex	Chan-Paton
-1/-1 (NS)	a'_μ	*centers*	$\psi^\mu(z)\mathrm{e}^{-\varphi(z)}$	adj. $\mathrm{U}(k)$
	χ	*aux.*	$\overline{\Psi}(z)\mathrm{e}^{-\varphi(z)}$	$\vdots$
(aux. vert.)	D_c	*Lagrange mult.*	$\bar\eta^c_{\mu\nu}\psi^\nu(z)\psi^\mu(z)$	$\vdots$
(R)	$M^{\alpha A}$	*partners*	$S_\alpha(z)S_A(z)\mathrm{e}^{-\frac{1}{2}\varphi(z)}$	$\vdots$
	$\lambda_{\dot\alpha A}$	*Lagrange mult.*	$S^{\dot\alpha}(z)S^A(z)\mathrm{e}^{-\frac{1}{2}\varphi(z)}$	$\vdots$
-1/3 (NS)	$w_{\dot\alpha}$	*sizes*	$\Delta(z)S^{\dot\alpha}(z)\,\mathrm{e}^{-\varphi(z)}$	$k \times \overline{N}$
	$\bar w_{\dot\alpha}$	$\vdots$	$\overline{\Delta}(z)S^{\dot\alpha}(z)\mathrm{e}^{-\varphi(z)}$	$\vdots$
(R)	μ^A	*partners*	$\Delta(z)S_A(z)\mathrm{e}^{-\frac{1}{2}\varphi(z)}$	$\vdots$
	$\bar\mu^A$	$\vdots$	$\overline{\Delta}(z)S_A(z)\mathrm{e}^{-\frac{1}{2}\varphi(z)}$	$\vdots$

Instanton calculus from the string standpoint

Consider disk diagrams involving only moduli $\mathcal{M}_{(k)}$, and no D3/D3 state (these are “vacuum” contributions from the D3 point of view)



$$\overset{\alpha'\to 0}{\simeq} \; -\frac{8\pi^2 k}{g^2} \; - \; \mathcal{S}_{\text{mod}}$$

(the “pure” D(-1) disks yields kC_0 [Polchinski, 1994])

- The combinatorics of boundaries [Polchinski, 1994] is such that these D-instanton diagrams exponentiate

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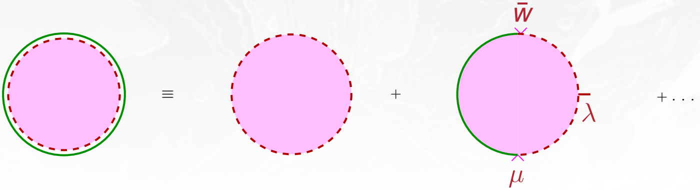


$$\stackrel{\alpha'\to 0}{\simeq} \; -\frac{8\pi^2 k}{g^2} \; - \; \mathcal{S}_{\text{mod}}$$

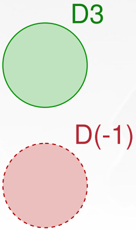
(the “pure” D(-1) disks yields kC_0 [Polchinski, 1994])

- The moduli must be integrated over:

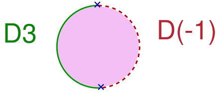
$$Z^{(k)} = \int d\mathcal{M}_{(k)} \, \mathrm{e}^{-\frac{8\pi^2 k}{g^2} - \mathcal{S}_{\text{mod}}}$$

Disk amplitudes and effective actions

Usual disks:



Mixed disks:



Disk amplitudes

$\alpha' \to 0$ field theory limit

Effective actions

D3/D3

D(-1)/D(-1) and mixed

$\mathcal{N} = 2$ SYM action

ADHM measure

The action for the moduli

From disk diagrams with insertion of moduli vertices, in the field theory limit we extract the ADHM moduli action (at fixed k)

$$\mathcal{S}_{\rm mod} = \mathcal{S}_{\rm bos}^{(k)} + \mathcal{S}_{\rm fer}^{(k)} + \mathcal{S}_{\rm c}^{(k)}$$

with Back

$$\mathcal{S}_{\rm bos}^{(k)} = \mathrm{tr}_k\Big\{ -2\,[\chi^\dagger, a'_\mu][\chi, a'^\mu] + \chi^\dagger \bar{w}_{\dot\alpha} w^{\dot\alpha}\chi + \chi \bar{w}_{\dot\alpha} w^{\dot\alpha}\chi^\dagger \Big\}$$

$$\mathcal{S}_{\rm fer}^{(k)} = \mathrm{tr}_k\Big\{ \mathrm{i}\,\frac{\sqrt{2}}{2}\,\bar{\mu}^A \epsilon_{AB}\mu^B\chi^\dagger - \mathrm{i}\,\frac{\sqrt{2}}{4}\,M^{\alpha A}\epsilon_{AB}[\chi^\dagger, M^B_\alpha]\Big\}$$

$$\mathcal{S}_{\rm c}^{(k)} = \mathrm{tr}_k\Big\{ -\mathrm{i}D_c\big(W^c + \mathrm{i}\bar\eta^c_{\mu\nu}\,[a'^\mu, a'^\nu]\big) -\mathrm{i}\lambda^{\dot\alpha}_A\big(\bar\mu^A w_{\dot\alpha} + \bar{w}_{\dot\alpha}\mu^A + [a'_{\alpha\dot\alpha}, M'^{\alpha A}]\big)\Big\}$$

- $\mathcal{S}_{\rm c}^{(k)}$: bosonic and fermionic ADHM constraints

Field-dependent moduli action

Consider correlators of D3/D3 fields, e.g of the scalar ϕ in the Cartan direction, in presence of k D-instantons. It turns out that

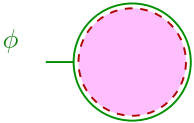
[Green-Gutperle 1997-2000, Billò et al 2002]

- the dominant contribution to $\langle \phi_1 \ldots \phi_n \rangle$ is from n one-point amplitudes on disks with moduli insertions. The result can be encoded in extra moduli-dependent vertices for ϕ's, i.e. in extra terms in the moduli action containing such one-point functions

$$\mathcal{S}_{\rm mod}(\varphi; \mathcal{M}) = \phi(x) J_\phi(\widehat{\mathcal{M}}) + \mathcal{S}_{\rm mod}(\widehat{\mathcal{M}})$$

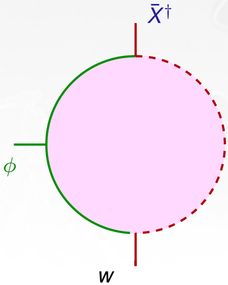
where x is the instanton center and

$$\phi(x) J_\phi(\widehat{\mathcal{M}}) \quad =$$



Moduli action with the unbroken multiplet Φ

To determine $\mathcal{S}_{\text{mod}}(\phi; \mathcal{M})$ we systematically compute mixed disks with a scalar ϕ emitted from the D3 boundary, e.g.



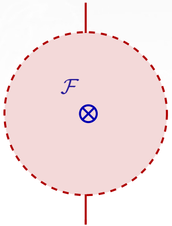
$$\left\langle\!\!\left\langle V_{\bar{X}^\dagger} V_\phi V_w \right\rangle\!\!\right\rangle$$
$$\equiv C_0 \int \frac{\prod_i dz_i}{dV_{\text{CKG}}} \times \langle V_{\bar{X}^\dagger}(z_1) V_w(z_2) V_\phi(z_3) \rangle$$
$$= \ldots = \mathrm{tr}_k \left\{ \bar{X}^\dagger_{\dot{\alpha}}\, \phi(x)\, w^{\dot{\alpha}} \right\}$$

- Other non-zero diagrams couple the components of the gauge supermultiplet to the moduli, related by the Ward identities of the susies broken by the D(-1).
- The superfield-dependent moduli action $\mathcal{S}_{\text{mod}}(\Phi; \mathcal{M})$ is obtained by simply letting $\phi(x) \to \Phi(x, \theta)$

Inclusion of a graviphoton background

Including fields from the closed sector

In the stringy setup, is quite natural to consider also the effect of D-instantons on correlators of fields from the closed string sector.



- The effect can be encoded in a field-dependent moduli action determined from one-point functions of closed string vertices on instanton disks with moduli insertions.
- Our aim is to study interactions in the low energy $\mathcal{N} = 2$ effective action involving the graviphoton ▸ Recall. This is the closed string field we turn now on.

The Weyl multiplet

- The field content of $\mathcal{N} = 2$ sugra:

$$h_{\mu\nu} \text{ (metric)} \, , \quad \psi_\mu^{\alpha A} \text{ (gravitini)} \, , \quad C_\mu \text{ (graviphoton)}$$

can be organized in a chiral Weyl multiplet:

$$W^+_{\mu\nu}(x,\theta) = \mathcal{F}^+_{\mu\nu}(x) + \theta\chi^+_{\mu\nu}(x) + \frac{1}{2}\,\theta\sigma^{\lambda\rho}\theta\, R^+_{\mu\nu\lambda\rho}(x) + \cdots$$

($\chi_{\mu\nu}{}^{\alpha A}$ is the gravitino field strength)

- These fields arise from massless vertices of type IIB strings on $\mathbb{R}^4 \times \mathbb{C} \times \mathbb{C}^2/\mathbb{Z}_2$

Graviphoton vertex

The graviphoton vertex is given by

$$\begin{aligned} V_{\mathcal{F}}(z,\bar{z}) &= \frac{1}{4\pi}\mathcal{F}^{\alpha\beta AB}(p) \\ &\times \left[S_\alpha(z)S_A(z)\mathrm{e}^{-\frac{1}{2}\varphi(z)}S_\beta(\bar{z})S_B(\bar{z})\mathrm{e}^{-\frac{1}{2}\varphi(\bar{z})}\right]\mathrm{e}^{\mathrm{i}p\cdot X(z,\bar{z})} \end{aligned}$$

(Left-right movers identification on disks taken into account)

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- The bi-spinor graviphoton polarization is given by

$$\mathcal{F}^{(\alpha\beta)[AB]} = \frac{\sqrt{2}}{4}\mathcal{F}^{+}_{\mu\nu}\left(\sigma^{\mu\nu}\right)^{\alpha\beta}\epsilon^{AB}$$

Graviphoton vertex

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$$
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\end{aligned}
$$

- A different RR field, with a similar structure, will be useful:

$$
\begin{aligned}
V_{\bar{\mathcal{F}}}(z,\bar{z}) &= \frac{1}{4\pi}\bar{\mathcal{F}}^{\alpha\beta\hat{A}\hat{B}}(p) \\
&\times \left[S_\alpha(z)S_{\hat{A}}(z)\mathrm{e}^{-\frac{1}{2}\varphi(z)}S_\beta(\bar{z})S_{\hat{B}}(\bar{z})\mathrm{e}^{-\frac{1}{2}\varphi(\bar{z})}\right]\mathrm{e}^{\mathrm{i}p\cdot X(z,\bar{z})}
\end{aligned}
$$

$\hat{A}, \hat{B} = 3, 4 \leftrightarrow$ odd “internal” spin fields ▸ Recall

Graviphoton-dependent moduli action

To determine the contribution of the graviphoton to the field-dependent moduli action

- we have to consider disk amplitudes with open string moduli vertices on the boundary and closed string graviphoton vertices in the interior which survive in the field theory limit $\alpha' \to 0$.
- Other diagrams, connected by susy, have the effect of promoting the dependence of the moduli action to the full Weyl multiplet

$$\mathcal{F}^{+}_{\mu\nu} \to W^{+}_{\mu\nu}(x,\theta)$$

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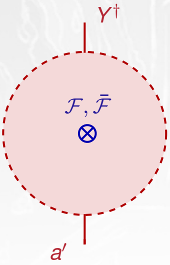
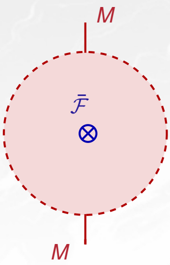
Non-zero diagrams

Very few diagrams contribute.





- Result: (same also with $\bar{\mathcal{F}}^+_{\mu\nu}$)

$$\left\langle\!\!\left\langle V_{Y^\dagger}\, V_{a'}\, V_{\mathcal{F}} \right\rangle\!\!\right\rangle = -4\mathrm{i}\,\mathrm{tr}_k\Big\{ Y^\dagger_\mu\, a'_\nu\, \mathcal{F}^+_{\mu\nu} \Big\}$$

- Moreover, term with fermionic moduli and a $V_{\bar{\mathcal{F}}}$:

$$\left\langle\!\!\left\langle V_M V_M V_{\bar{\mathcal{F}}} \right\rangle\!\!\right\rangle = \frac{1}{4\sqrt{2}}\mathrm{tr}_k\Big\{ M^{\alpha A} M^{\beta B} \bar{\mathcal{F}}^+_{\mu\nu} \Big\} (\sigma^{\mu\nu})_{\alpha\beta}\epsilon_{AB}$$

Effective action and relation to topological strings

Contributions to the prepotential

Integrating over the moduli the interactions described by the field-dependent moduli action $\mathcal{S}_{\text{mod}}(\Phi, W^+; \mathcal{M}(k))$ one gets the effective action for the long-range multiplets Φ and W^+ in the instanton # k sector:

$$S_{\text{eff}}^{(k)}[\Phi, W^+] = \int d^4x\, d^4\theta\, d\widehat{\mathcal{M}}_{(k)}\, \mathrm{e}^{-\frac{8\pi k}{g^2} - \mathcal{S}_{\text{mod}}(\Phi, W^+; \mathcal{M}(k))}$$

The prepotential is thus given by the centred instanton partition function

$$\mathcal{F}^{(k)}(\Phi, W^+) = \int d\widehat{\mathcal{M}}_{(k)}\, \mathrm{e}^{-\frac{8\pi k}{g^2} - \mathcal{S}_{\text{mod}}(\Phi, W^+; \mathcal{M}(k))}$$

- $\Phi(x, \theta)$ and $W^+_{\mu\nu}(x, \theta)$ are constant w.r.t. the integration variables $\widehat{\mathcal{M}}_{(k)}$. We can compute $\mathcal{F}^{(k)}(a; f)$ giving them constant values

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The deformed moduli action

Fixing the multiplets to constant background values

$$\Phi(x,\theta) \to a\,, \qquad W^{+}_{\mu\nu}(x,\theta) \to f_{\mu\nu}$$

one gets a "deformed" moduli action ▸ Recall ▸ Back

$$\mathcal{S}_{\rm mod}(a,\bar a; f,\bar f) =$$
$$-\mathrm{tr}_k\Big\{\big([\chi^\dagger, a'_{\alpha\dot\beta}] + 2\bar f_c(\tau^c a')_{\alpha\dot\beta}\big)\big([\chi, a'^{\dot\beta\alpha}] + 2 f_c(a'\tau^c)^{\dot\beta\alpha}\big)$$
$$-\big(\chi^\dagger \bar w_{\dot\alpha} - \bar w_{\dot\alpha}\,\bar a\big)\big(w^{\dot\alpha}\chi - a\,w^{\dot\alpha}\big) - \big(\chi \bar w_{\dot\alpha} - \bar w_{\dot\alpha}\,a\big)\big(w^{\dot\alpha}\chi^\dagger - \bar a\,w^{\dot\alpha}\big)\Big\}$$
$$+\mathrm{i}\,\frac{\sqrt{2}}{2}\,\mathrm{tr}_k\Big\{\bar\mu^A\epsilon_{AB}\big(\mu^B\chi^\dagger - \bar a\,\mu^B\big)$$
$$-\frac{1}{2}\,M^{\alpha A}\epsilon_{AB}\big([\chi^\dagger, M^B_\alpha] + 2\,\bar f_c\,(\tau^c)_{\alpha\beta}M^{\beta B}\big)\Big\} + S_{\rm c}^{(k)}$$

- The constraint part of the action, $S_{\rm c}^{(k)}$, is not modified

Holomorphicity, Q-exactness

In the action $\mathcal{S}_{\text{mod}}(a, \bar{a}; f, \bar{f})$ the v.e.v.'s a, f and $\bar{a}, \bar{f}$ are not on the same footing: a and f do not appear in the fermionic action.

- The moduli action has the form $\mathcal{S}_{\text{mod}}(a, \bar{a}; f, \bar{f}) = Q\,\Xi$ where Q is the scalar twisted supercharge:

$$Q^{\dot{\alpha}B} \xrightarrow{\text{top. twist}} Q^{\dot{\alpha}\dot{\beta}} \,, \qquad Q \equiv \frac{1}{2}\,\epsilon_{\dot{\alpha}\dot{\beta}}\,Q^{\dot{\alpha}\dot{\beta}}$$

- The parameters $\bar{a}, \bar{f}_c$ appear only in the gauge fermion Ξ
- The instanton partition function

$$Z^{(k)} \equiv \int d\mathcal{M}_{(k)}\, \mathrm{e}^{-\mathcal{S}_{\text{mod}}(a,\bar{a};f,\bar{f})}$$

is independent of $\bar{a}, \bar{f}_c$: variation w.r.t these parameters is Q-exact.

Graviphoton and localization

The moduli action obtained inserting the graviphoton background coincides exactly with the “deformed” action considered in the literature to localize the moduli space integration if we set [Nekrasov 2002, Flume-Poghossian 2002, Nekrasov et al 2003, ...]

$$f_c = \frac{\varepsilon}{2}\,\delta_{3c}\ , \quad \bar{f}_c = \frac{\bar{\varepsilon}}{2}\,\delta_{3c}\ ,$$

and moreover (referring to the notations in the above ref.s)

$$\varepsilon = \bar{\varepsilon}\ , \quad \varepsilon = \epsilon_1 = -\epsilon_2$$

- The localization deformation of the $\mathcal{N} = 2$ ADHM construction is produced, in the type IIB string realization, by a graviphoton background

Expansion of the prepotential

Some properties of the prepotential $\mathcal{F}^{(k)}$:

- from the explicit form of $\mathcal{S}_{\rm mod}(a,0;f,0)$ ▸ Recall it follows that $\mathcal{F}^{(k)}(a;f)$ is invariant under

$$a, f_{\mu\nu} \to -a, -f_{\mu\nu}$$

- Regular for $f \to 0$, to recover the instanton # k contribution to the SW prepotential
- Odd powers of $af_{\mu\nu}$ cannot appear.

Altogether, reinstating the superfields,

$$\mathcal{F}^{(k)}(\Phi, W^+) = \sum_{h=0}^{\infty} c_{k,h}\, \Phi^2 \left(\frac{\Lambda}{\Phi}\right)^{4k} \left(\frac{W^+}{\Phi}\right)^{2h}$$

The non-perturbative prepotential

Sum over the instanton sectors:

$$\mathcal{F}_{\text{n.p.}}(\Phi, W^+) = \sum_{k=1}^{\infty} \mathcal{F}^{(k)}(\Phi, W^+) = \sum_{h=0}^{\infty} C_h(\Lambda, \Phi)(W^+)^{2h}$$

with

$$C_h(\Lambda, \Phi) = \sum_{k=1}^{\infty} c_{k,h} \frac{\Lambda^{4k}}{\Phi^{4k+2h-2}}$$

- Many different terms in the eff. action connected by susy. Saturating the θ integration with four θ's all from W^+

$$\int d^4x \; C_h(\Lambda, \phi)\,(R^+)^2(\mathcal{F}^+)^{2h-2}$$

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Evaluation via localization

To compute $c_{k,h}$, use constant values $\Phi \to a$ and $W^+_{\mu\nu} \to f_{\mu\nu}$

- The localization deformation is obtained for

$$f_{\mu\nu} = \frac{1}{2}\,\varepsilon\,\eta^3_{\mu\nu}\ , \quad \bar{f}_{\mu\nu} = \frac{1}{2}\,\bar{\varepsilon}\,\eta^3_{\mu\nu}$$

  - $Z^{(k)}(a,\varepsilon)$ does not depend on $\bar{\varepsilon}$. However, $\bar{\varepsilon} = 0$ is a limiting case: some care is needed

$\mathcal{F}^{(k)}(a;\varepsilon)$ is well-defined. $S^{(k)}[a;\varepsilon]$ diverges because of the (super)volume integral $\int d^4x\, d^4\theta$. $\bar{\varepsilon}$ regularizes the superspace integration by a Gaussian term. Effective rule:

$$\int d^4x\, d^4\theta \to \frac{1}{\varepsilon^2}$$

- One can then work with the effective action, i.e., the *full* instanton partition function

The deformed partition function vs the prepotential

a and $\varepsilon, \bar{\varepsilon}$ deformations localize completely the integration over moduli space which can be carried out

[Nekrasov 2002, Flume-Poghossian 2002, Nekrasov et al 2003, ...]

- With $\bar{\varepsilon} \neq 0$ (complete localization) a trivial superposition of instantons of charges k_i contributes to the sector $k = \sum k_i$
- Such disconnected configurations do *not* contribute when $\bar{\varepsilon} = 0$. The partition function computed by localization corresponds to the exponential of the non-perturbative prepotential:

$$\begin{aligned} Z(a;\varepsilon) &= \exp\left(\frac{\mathcal{F}_{\text{n.p.}}(a,\varepsilon)}{\varepsilon^2}\right) = \exp\left(\sum_{k=1}^{\infty} \frac{\mathcal{F}^{(k)}(a,\varepsilon)}{\varepsilon^2}\right) \\ &= \exp\left(\sum_{h=0}^{\infty}\sum_{k=1}^{\infty} c_{k,h}\,\frac{\varepsilon^{2h-2}}{a^{2h}}\left(\frac{\Lambda}{a}\right)^{4k}\right) \end{aligned}$$

Summarizing

- The computation via localization techniques of the multi-instanton partition function $Z(a;\varepsilon)$ determines the coefficients $c_{k,h}$ which appear in the gravitational F-terms of the $\mathcal{N}=2$ effective action

$$\int d^4x\ C_h(\Lambda,\phi)\,(R^+)^2(\mathcal{F}^+)^{2h-2}$$

via the relation

$$C_h(\Lambda,\phi)=\sum_{k=1}^{\infty}c_{k,h}\,\frac{\Lambda^{4k}}{\phi^{4k+2h-2}}$$

- The very same gravitational F-terms can been extracted in a completely different way: topological string amplitudes on suitable Calabi-Yau manifolds

Summarizing

- The computation via localization techniques of the multi-instanton partition function $Z(a;\varepsilon)$ determines the coefficients $c_{k,h}$ which appear in the gravitational F-terms of the $\mathcal{N}=2$ effective action

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- The very same gravitational F-terms can been extracted in a completely different way: topological string amplitudes on suitable Calabi-Yau manifolds

Geometrical engineering and topological strings

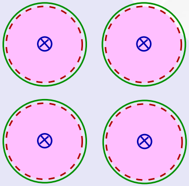
- SW: low energy $\mathcal{N} = 2 \leftrightarrow$ (auxiliary) Riemann surface
- Geometrical engineering: embed directly the low energy theory into string theory as type IIB on a suitable local CY manifold $\mathfrak{M}$ [Kachru et al 1995, Klemm et al 1996-97]
  - geometric moduli of $\mathfrak{M} \leftrightarrow$ gauge theory data (Λ, a);
  - The coefficients C_h in the l.e.e.a. gravitational F-terms

  $$C_h \, (R^+)^2 (\mathcal{F}^+)^{2h-2}$$

  are given by topological string amplitudes at genus h
  [Bershadsky et al 1993-94, Antoniadis et al 1993]
- For the local CY $\mathfrak{M}_{\mathrm{SU(2)}}$ the couplings C_h were checked to coincide with those given by the deformed multi-instanton calculus as proposed by Nekrasov [Klemm et al 2002]

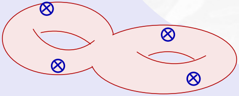
Microscopic vs effective string description

Orbifold space
with D3/D(-1) system
Moduli action depends on
gauge theory data Λ, a
open and closed strings



$2h$ disks
connected by
integration over
moduli

Local CY manifold
Geometric moduli determined
from gauge theory data Λ, a
No branes - closed strings only



genus h Riemann surface

$\chi = 2h - 2$

Same gravitational F-term interactions

$$C_h(\Lambda, a)\,(R^+)^2(\mathcal{F}^+)^{2h-2}$$

Perspectives

Some interesting directions to go...

- Explicit computations of D-instanton and wrapped euclidean branes effects in $\mathcal{N} = 1$ contexts. Very recently considered for
  - neutrino masses [Blumenhagen et al 0609191, Ibanez-Uranga 0609213]
  - susy breaking [Haack et al 0609211, Florea et al 0610003]
- Study of D3's along a CY orbifold to derive BH partition functions in $\mathcal{N} = 2$ sugra (which OSV relates to $|Z_{\text{top}}|^2$)
- Study of the instanton corrections to $\mathcal{N} = 2$ eff. theory in the gauge/gravity context: modifications of the classical solution of fD3's
- ...

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Some notations

String fields in the orbifold space

- In the six directions transverse to the brane,

$$Z \equiv (X^5 + iX^6)/\sqrt{2}\,, \quad Z^1 \equiv (X^7 + iX^8)/\sqrt{2}\,, \quad Z^2 \equiv (X^9 + iX^{10})/\sqrt{2}\,,$$
$$\Psi \equiv (\psi^5 + i\psi^6)/\sqrt{2}\,, \quad \Psi^1 \equiv (\psi^7 + i\psi^8)/\sqrt{2}\,, \quad \Psi^2 \equiv (\psi^9 + i\psi^{10})/\sqrt{2}$$

  the $\mathbb{Z}_2$ orbifold generator h acts by

$$(Z^1, Z^2) \to (-Z^1, -Z^2)\,, \quad (\Psi^1, \Psi^2) \to (-\Psi^1, -\Psi^2)$$

- Under the SO(10) $\to$ SO(4) $\times$ SO(6) induced by D3's, $S^{\dot{\mathcal{A}}} \to (S_\alpha S_{A'}, S^{\dot\alpha} S^{A'})$
- Under SO(6) $\to$ SO(2) $\times$ SO(4) induced by the orbifold, ▸ Back

$S^{A'}$	notat.	SO(2)	SO(4)	$S_{A'}$	notat.	SO(2)	SO(4)	h
S^{+++} S^{+--}	S^A $A=1,2$	$\frac{1}{2}$	$(\mathbf{2},\mathbf{1})$	S_{---} S_{-++}	S_A $A=1,2$	$-\frac{1}{2}$	$(\mathbf{2},\mathbf{1})$	$+1$
S^{-+-} S^{--+}	$S^{\hat{A}}$ $\hat{A}=3,4$	$-\frac{1}{2}$	$(\mathbf{1},\mathbf{2})$	S_{+-+} S_{++-}	$S_{\hat{A}}$ $\hat{A}=3,4$	$\frac{1}{2}$	$(\mathbf{1},\mathbf{2})$	-1