D-instanton effects in RR background and gravitational F-terms

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Foreword

This talk is based on

M. Billo, M. Frau, F. Fucito and A. Lerda, arXiv:hep-th/0606013.

It of course builds over a vast literature. The few references scattered on the slides are by no means intended to be exhaustive. I apologize for the many relevant ones which will be missing.

Plan of the talk

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1. Introduction
2. Microscopic string description of $\mathcal{N} = 2$ SYM
3. Instanton calculus by mixed string diagrams
4. Deformation from a graviphoton background
5. Relation to topological strings on CY

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Introduction

$\mathcal{N} = 2$ SYM and Seiberg-Witten solution

$\mathcal{N} = 2$ SYM theories in $d = 4$: an important test-bed for non-perturbative physics

- Seiberg-Witten: exact expression of the prepotential $\mathcal{F}(a)$ governing the low energy dynamics on the Coulomb branch using duality and monodromy properties; this involves an auxiliary Riemann surface
- "Geometrical engineering" construction [Kachru et al 1995, Katz et al 1996]: SW solution $\hookrightarrow$ Type IIB string theory on a "local" CY manifold $\mathfrak{M}$ whose geometric moduli are suitably related to the gauge theory quantities (Λ, a, ...)

The quest for the multi-instanton contributions

Semi-classical limit: 1-loop plus instanton contributions ▸ Back

$$\mathcal{F}(a) = \frac{\mathrm{i}}{2\pi} a^2 \log \frac{a^2}{\Lambda^2} + \sum_{k=1}^{\infty} \mathcal{F}^{(k)}(a)$$

- Important task: compute the multi-instanton contributions $\mathcal{F}^{(k)}(a)$ within the "microscopic" description of the non-abelian gauge theory to check them against the SW solution
- Only recently fully accomplished using localization techniques to perform the integration over the moduli space of the ADHM construction of the super-instantons

[Nekrasov 2002, Flume-Poghossian 2002, Nekrasov et al 2003, ...]

The localizing deformation

Introduce a deformation of the ADHM measure on the moduli spaces exploiting the 4d chiral rotations symmetry of ADHM constraints.

- The deformed instanton partition function

$$Z(a,\varepsilon) = \sum_k Z^{(k)}(a,\varepsilon) = \sum_k \int d\widehat{\mathcal{M}}_{(k)} \mathrm{e}^{-\mathcal{S}_{\mathrm{mod}}(a,\varepsilon;\mathcal{M}_{(k)})}$$

  can then be computed using localization techniques using the topological twist of its supersymmetries. One has

$$Z(a,\varepsilon) = \exp\left(\frac{\mathcal{F}_{\mathrm{n.p.}}(a;\varepsilon)}{\varepsilon^2}\right)$$

$$\lim_{\varepsilon\to 0} \mathcal{F}_{\mathrm{n.p.}}(a;\varepsilon) = \mathcal{F}_{\mathrm{n.p.}}(a) = \text{non-pert. part of SW prepotential}$$

Multi-instanton calculus and topological strings

What about higher orders in the deformation parameter ε?

- Nekrasov's proposal: terms of order $\varepsilon^{2h} \leftrightarrow$ gravitational F-terms in the $\mathcal{N} = 2$ eff. action involving metric and graviphoton curvatures

  [Nekrasov 2002, Losev et al 2003, Nekrasov 2005]

$$\int d^4x (R^+)^2 (\mathcal{F}^+)^{2h-2}$$

- When the effective $\mathcal{N} = 2$ theory is obtained from type II strings on CY via geometrical engineering, such terms
  - arise from world-sheets of genus h
  - are computed by the topological string [Bershadsky et al 1993, Antoniadis et al 1993]
- For the local CY describing the SU(2) theory the proposal has been tested [Klemm et al, 2002]

"Microscopic" string description of the gauge theory

The "semiclassical" approach to the low energy $\mathcal{N} = 2$ effective action can be given a simple (i.e., calculable) string theory realization

- The non-abelian gauge theory d.o.f. are realized by open strings attached to (fractional) D3 branes. In the $\mathcal{N} = 2$ case, the perturbative 1-loop contributions to the prepotential are easily retrieved. [Douglas-Li 1996, Lawrence et al 1998, ...]
- The instantonic sectors of gauge theories can be realized by including D(-1) branes (a.k.a. D-instantons)
  - The spectrum of the D3/D(-1) systems encodes the quantities of the mathematical ADHM construction of the instanton moduli spaces

    [Witten 1995, Douglas 1995, ... ; Dorey et al, 2002 (review)]

Instantonic effects and gravitational backgrounds

The description of gauge instantons via D(-1)-branes is more than a book-keeping device for the ADHM construction.

- The D(-1)'s act as sources that produce the actual profile of the gauge instanton solution [Billò et al 2002]
- The prescriptions of the "instantonic calculus" of correlators arise naturally [Polchinski 1994, Green-Gutperle 1997-1998, Billò et al 2002]

One can include closed string backgrounds producing interesting deformations of the gauge theory. For instance

- non-commutative theories from NSNS background $B_{\mu\nu}$
- non-anticommutative theories from specific RR backgrounds

[Billò et al 2004-2005,...]

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The aim of this work

- Reproduce the semiclassical instanton expansion of the low energy effective action for the $\mathcal{N} = 2$ SYM theory in the microscopic string realization via (fractional) D3/D(-1) branes
- Show that the inclusion of the graviphoton of the $\mathcal{N} = 2$ bulk sugra, which comes from the RR sector,
  - leads exactly to the localization deformation on the instanton moduli space which allows to perform the integration
  - produces in the effective action the gravitational F-terms which are computed by the topological string on local CY

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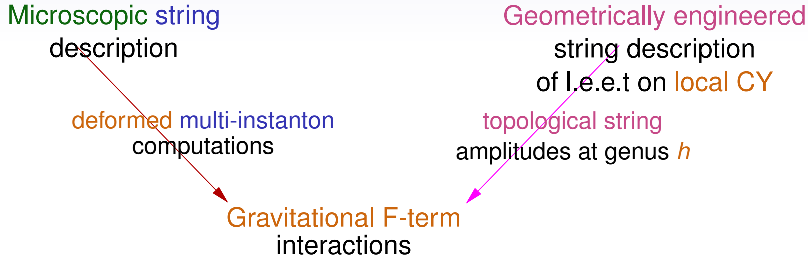
- Reproduce the semiclassical instanton expansion of the low energy effective action for the $\mathcal{N} = 2$ SYM theory in the microscopic string realization via (fractional) D3/D(-1) branes

- The situation is therefore as follows:



- The two ways to compute the same F-terms must coincide if the two descriptions are equivalent

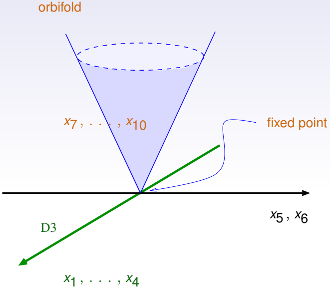
Microscopic string description of $\mathcal{N} = 2$ SYM

SYM from fractional branes

Consider pure SU(N) Yang-Mills in 4 dimensions with $\mathcal{N} = 2$ susy.

- It is realized by the massless d.o.f. of open strings attached to fractional D3-branes in the orbifold background

$$\mathbb{R}^4 \times \mathbb{R}^2 \times \mathbb{R}^4/\mathbb{Z}_2$$



- The orbifold breaks 1/2 SUSY in the bulk, the D3 branes a further 1/2:

$$32 \times \frac{1}{2} \times \frac{1}{2} = 8 \;\; \text{real supercharges}$$

Fields and string vertices

- Field content: $\mathcal{N}=2$ chiral superfield

$$\Phi(x,\theta) = \phi(x) + \theta\Lambda(x) + \frac{1}{2}\,\theta\sigma^{\mu\nu}\theta\, F^{+}_{\mu\nu}(x) + \cdots$$

- String vertices:

$$V_A(z) = \frac{A_\mu(p)}{\sqrt{2}}\,\psi^\mu(z)\,e^{ip\cdot X(z)}\,e^{-\varphi(z)}$$

$$V_\Lambda(z) = \Lambda^{\alpha A}(p)\,S_\alpha(z)S_A(z)\,e^{ip\cdot X(z)}\,e^{-\frac{1}{2}\varphi(z)}$$

$$V_\phi(z) = \frac{\phi(p)}{\sqrt{2}}\,\overline{\Psi}(z)\,e^{ip\cdot X(z)}\,e^{-\varphi(z)}$$

with all polariz.s in the adjoint of U(N)

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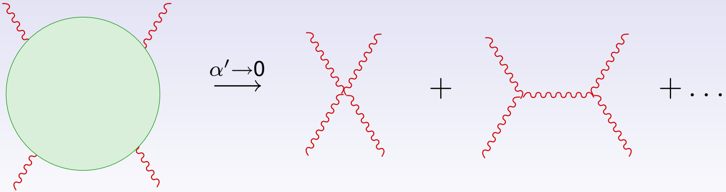
Gauge action from disks on fD3's



- String amplitudes on disks attached to the D3 branes in the limit

  $\alpha' \to 0$ with gauge quantities fixed.

  give rise to the tree level (microscopic) $\mathcal{N} = 2$ action

$$S_{\rm SYM} = \int d^4x \, \mathrm{Tr} \Big\{ \frac{1}{2} F_{\mu\nu}^2 + 2\, D_\mu \bar\phi \, D^\mu \phi - 2\, \bar\Lambda_{\dot\alpha A} \bar{\not{D}}^{\dot\alpha\beta} \Lambda_\beta^A$$
$$+ \mathrm{i}\sqrt{2}\, g\, \bar\Lambda_{\dot\alpha A} \epsilon^{AB} \big[\phi, \bar\Lambda^{\dot\alpha}_B \big] + \mathrm{i}\sqrt{2}\, g\, \Lambda^{\alpha A} \epsilon_{AB} \big[\bar\phi, \Lambda^B_\alpha \big] + g^2 \big[\phi, \bar\phi \big]^2 \Big\}$$

Scalar v.e.v's and low energy effective action

- We are interested in the l.e.e.a. on the Coulomb branch parametrized by the v.e.v.'s of the adjoint chiral superfields:

$$\langle \Phi_{uv} \rangle \equiv \langle \phi_{uv} \rangle = a_{uv} = a_u \, \delta_{uv} \ , \quad u, v = 1, ..., N \ , \quad \sum_u a_u = 0$$

  breaking $SU(N) \to U(1)^{N-1}$ [we focus for simplicity on SU(2)]
- Up to two-derivatives, $\mathcal{N} = 2$ susy forces the effective action for the chiral multiplet Φ in the Cartan direction to be of the form

$$S_{\text{eff}}[\Phi] = \int d^4x \, d^4\theta \, \mathcal{F}(\Phi) + \text{c.c}$$

- We want to discuss the instanton corrections to the prepotential $\mathcal{F}$ Recall in our string set-up

Instanton calculus by mixed string diagrams

Instantons and D-instantons

- Consider the Wess-Zumino term of the effective action for a stack of D3 branes:

$$\text{D. B. I.} + \int_{\mathrm{D}_3} \left[C_3 + \frac{1}{2} C_0 \,\mathrm{Tr}\big(F \wedge F \big) \right]$$

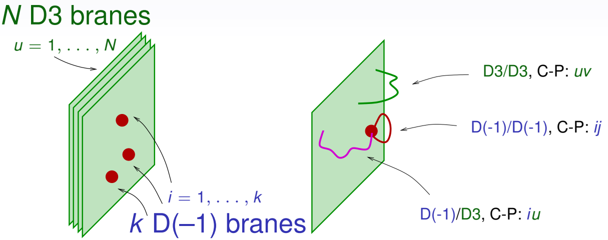
The topological density of an instantonic configuration corresponds to a localized source for the RR scalar C_0, i.e., to a distribution of D-instantons on the D3's.

- Instanton-charge k solutions of 3+1 dims. SU(N) gauge theories correspond to k D-instantons inside N D3-branes.

[Witten 1995,Douglas 1995, Dorey 1999, ...]

Stringy description of gauge instantons

	1	2	3	4	5	6	7	8	9	10
D3	—	—	—	—	*	*	*	*	*	*
D(−1)	*	*	*	*	*	*	*	*	*	*



Moduli vertices and instanton parameters

Open strings with at least one end on a D(-1) carry no momentum: they are moduli, rather than fields ↔ parameters of the instanton.

	ADHM	Meaning	Vertex	Chan-Paton
-1/-1 (NS)	a'_μ	*centers*	$\psi^\mu(z) e^{-\varphi(z)}$	adj. $U(k)$
	χ	*aux.*	$\overline{\Psi}(z) e^{-\varphi(z)}$	⋮
(aux. vert.)	D_c	*Lagrange mult.*	$\bar\eta^c_{\mu\nu}\psi^\nu(z)\psi^\mu(z)$	⋮
(R)	$M^{\alpha A}$	*partners*	$S_\alpha(z) S_A(z) e^{-\frac{1}{2}\varphi(z)}$	⋮
	$\lambda_{\dot\alpha A}$	*Lagrange mult.*	$S^{\dot\alpha}(z) S^A(z) e^{-\frac{1}{2}\varphi(z)}$	⋮
-1/3 (NS)	$w_{\dot\alpha}$	*sizes*	$\Delta(z) S^{\dot\alpha}(z)\, e^{-\varphi(z)}$	$k \times \overline{N}$
	$\bar w_{\dot\alpha}$	⋮	$\overline{\Delta}(z) S^{\dot\alpha}(z) e^{-\varphi(z)}$	⋮
(R)	μ^A	*partners*	$\Delta(z) S_A(z) e^{-\frac{1}{2}\varphi(z)}$	⋮
	$\bar\mu^A$	⋮	$\overline{\Delta}(z) S_A(z) e^{-\frac{1}{2}\varphi(z)}$	⋮

Super-coordinates and centred moduli

- Among the -1/-1 moduli we we can single out the center x_0^μ and its super-partners $\theta^{\alpha A}$: Back

$$
\begin{aligned}
a'^{\mu} &= x_0^\mu \, \mathbb{1}_{k\times k} + y_c^\mu \, T^c \quad (T^c = \text{gen.s of SU}(k)) \\
M^{\alpha A} &= \theta^{\alpha A} \, \mathbb{1}_{k\times k} + \zeta_c^{\alpha A} \, T^c
\end{aligned}
$$

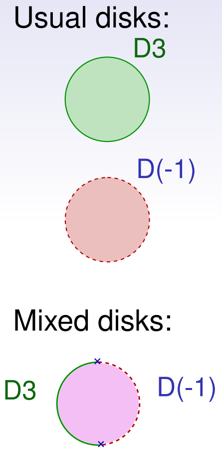
  The moduli x_0^μ and $\theta^{\alpha A}$ decouple from many interactions and play the rôle of superspace coords

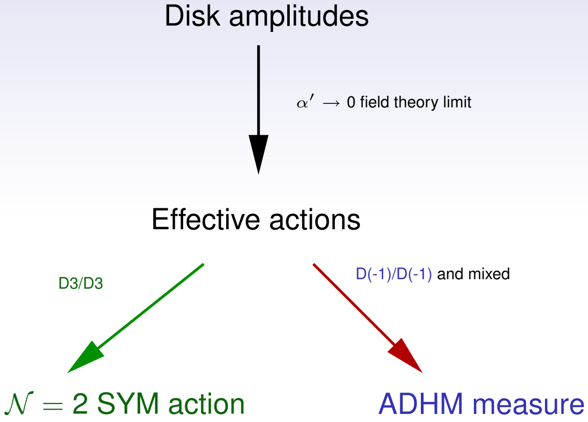
- We will distinguish the moduli $\mathcal{M}_{(k)}$ into

$$
\mathcal{M}_{(k)} \to \left(x_0, \theta \, ; \, \widehat{\mathcal{M}}_{(k)} \right)
$$

  $\widehat{\mathcal{M}}_{(k)}$ are the centred moduli

Disk amplitudes and effective actions





The action for the moduli

From disk diagrams with insertion of moduli vertices, in the field theory limit we extract the ADHM moduli action (at fixed k)

$$\mathcal{S}_{\rm mod} = \mathcal{S}_{\rm bos}^{(k)} + \mathcal{S}_{\rm fer}^{(k)} + \mathcal{S}_{\rm c}^{(k)}$$

with

$$\begin{aligned}
\mathcal{S}_{\rm bos}^{(k)} &= \mathrm{tr}_k\Big\{ -2\,[\chi^\dagger, a'_\mu][\chi, a'^\mu] + \chi^\dagger \bar{w}_{\dot\alpha} w^{\dot\alpha}\chi + \chi \bar{w}_{\dot\alpha} w^{\dot\alpha}\chi^\dagger \Big\} \\
\mathcal{S}_{\rm fer}^{(k)} &= \mathrm{tr}_k\Big\{ \mathrm{i}\,\frac{\sqrt{2}}{2}\,\bar{\mu}^A \epsilon_{AB}\mu^B \chi^\dagger - \mathrm{i}\,\frac{\sqrt{2}}{4}\, M^{\alpha A}\epsilon_{AB}[\chi^\dagger, M^B_\alpha] \Big\} \\
\mathcal{S}_{\rm c}^{(k)} &= \mathrm{tr}_k\Big\{ -\mathrm{i}D_c\big(W^c + \mathrm{i}\bar{\eta}^c_{\mu\nu}\,[a'^\mu, a'^\nu]\big) \\
&\qquad -\mathrm{i}\lambda^{\dot\alpha}_A\big(\bar{\mu}^A w_{\dot\alpha} + \bar{w}_{\dot\alpha}\mu^A + [a'_{\alpha\dot\alpha}, M'^{\alpha A}]\big)\Big\}
\end{aligned}$$

- In $\mathcal{S}_{\rm c}^{(k)}$ the bosonic and fermionic ADHM constraints appear (string moduli span the unconstrained parameter space)

Auxiliary moduli

The quartic interactions in $\mathcal{S}_{\text{bos}}^{(k)}$ can be disentangled using auxiliary moduli Y_μ, $X_{\dot\alpha}$ and $\bar{X}_{\dot\alpha}$:

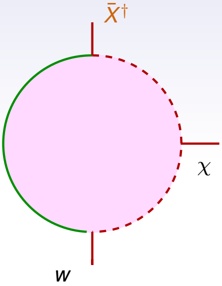
$$\begin{aligned}\mathcal{S}_{\text{bos}}^{\prime(k)} &= \text{tr}_k\Big\{2\,Y_\mu^\dagger Y^\mu + 2\,Y_\mu^\dagger\,[a'^\mu,\chi] + 2\,Y_\mu\,[a'^\mu,\chi^\dagger] \\ &+ \bar{X}_{\dot\alpha}^\dagger X^{\dot\alpha} + \bar{X}_{\dot\alpha} X^{\dagger\,\dot\alpha} + \bar{X}_{\dot\alpha}^\dagger w^{\dot\alpha}\chi + \bar{X}_{\dot\alpha} w^{\dot\alpha}\chi^\dagger - \chi\bar{w}_{\dot\alpha} X^{\dagger\,\dot\alpha} - \chi^\dagger\bar{w}_{\dot\alpha} X^{\dot\alpha}\Big\}\end{aligned}$$

- The corresponding auxiliary vertices are

$$\begin{aligned}V_Y(z) &= \sqrt{2}g_0\,Y_\mu\,\overline{\Psi}(z)\,\psi^\mu(z)\\ V_X(z) &= g_0\,X_{\dot\alpha}\,\Delta(z)S^{\dot\alpha}(z)\,\overline{\Psi}(z)\\ V_{\bar{X}}(z) &= g_0\,\bar{X}_{\dot\alpha}\,\overline{\Delta}(z)S^{\dot\alpha}(z)\,\overline{\Psi}(z)\end{aligned}$$

An example

One of the terms in $\mathcal{S}'^{(k)}_{\text{bos}}$ (involving the auxiliary moduli):

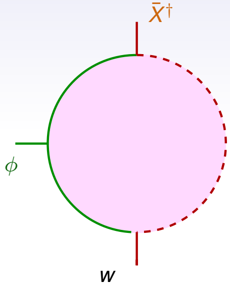


$$\begin{aligned}&\left\langle\!\!\left\langle V_{\bar{X}^\dagger} V_w V_\chi \right\rangle\!\!\right\rangle \\ &\equiv C_0 \int \frac{\prod_i dz_i}{dV_{\text{CKG}}} \times \langle V_{\bar{X}^\dagger}(z_1) V_w(z_2) V_\chi(z_3) \rangle \\ &= \ldots = \mathrm{tr}_k \left\{ \bar{X}^\dagger_{\dot\alpha}\, w^{\dot\alpha}\, \chi \right\}\end{aligned}$$

- Here $C_0 = 8\pi^2/g^2$ is the normalization of $D(-1)$ disks.

Introducing scalar v.e.v.'s

To evaluate the effect of a v.e.v. $\langle \Phi_{uv} \rangle = a_{uv} = a_u \, \delta_{uv}$ compute systematically mixed disk diagrams with a constant scalar ϕ emitted from the D3 boundary. Example: Back



$$\left\langle\!\left\langle V_{\bar{X}^\dagger} \, V_{\phi=a} V_w \right\rangle\!\right\rangle = \ldots = \mathrm{tr}_k \left\{ \bar{X}^\dagger_{\dot\alpha} \, a \, w^{\dot\alpha} \right\}$$

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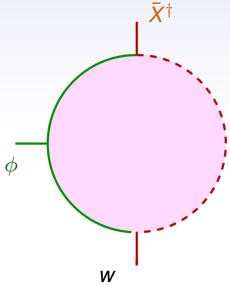

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- The resulting moduli action is obtained with the shifts Back

$$\chi_{ij} \, \delta_{uv} \longrightarrow \chi_{ij} \, \delta_{uv} - \delta_{ij} \, a_{uv} \; , \quad \chi^\dagger_{ij} \, \delta_{uv} \longrightarrow \chi^\dagger_{ij} \, \delta_{uv} - \delta_{ij} \, \bar{a}_{uv}$$

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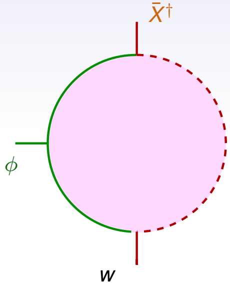

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- The action does not depend on the center super-coordinates x_0^μ and $\theta^{\alpha A}$ Recall

Holomorphicity, Q-exactness

In the action $\mathcal{S}_{\text{mod}}(a, \bar{a}; \mathcal{M}_{(k)})$ the v.e.v.'s a and $\bar{a}$ are not on the same footing: a does not appear in the fermionic action.

- The moduli action has the form

$$\mathcal{S}_{\text{mod}}(a, \bar{a}) = Q\,\Xi$$

where Q is the scalar twisted supercharge:

$$Q^{\dot{\alpha}B} \xrightarrow{\text{top. twist}} Q^{\dot{\alpha}\dot{\beta}}\,, \quad Q \equiv \frac{1}{2}\,\epsilon_{\dot{\alpha}\dot{\beta}}\,Q^{\dot{\alpha}\dot{\beta}}$$

- The parameter $\bar{a}$ appears only in the gauge fermion Ξ
- The instanton partition function

$$Z^{(k)}(a) \equiv \int d\mathcal{M}_{(k)}\, e^{-\mathcal{S}_{\text{mod}}(a,\bar{a})}$$

is independent of $\bar{a}$: variation w.r.t this parameter is Q-exact.

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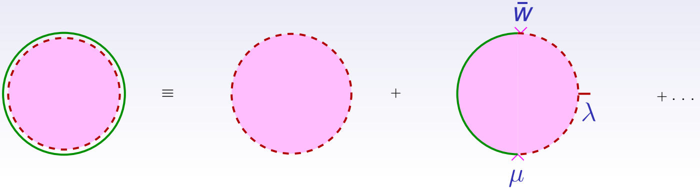
Instanton calculus from the string standpoint

Consider disk diagrams involving only moduli $\mathcal{M}_{(k)}$, and no D3/D3 state (these are "vacuum" contributions from the D3 point of view)



$$\stackrel{\alpha'\to 0}{\simeq} \; -\frac{8\pi^2 k}{g^2} \qquad - \qquad \mathcal{S}_{\text{mod}}$$

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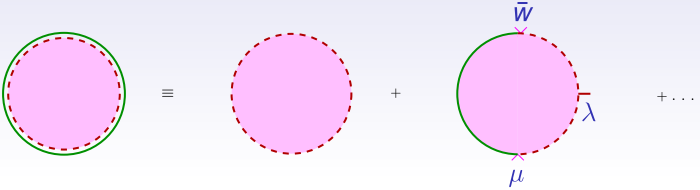

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- The combinatorics of boundaries [Polchinski, 1994] is such that these D-instanton diagrams exponentiate

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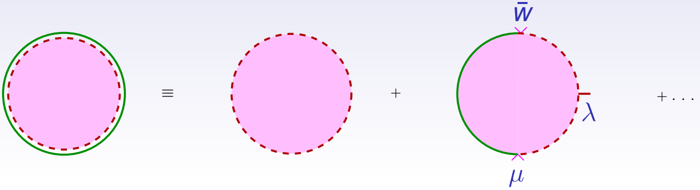

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(the "pure" D(-1) disks yields kC_0 [Polchinski, 1994])

- The moduli must be integrated over (path integral → ordinary integral):

$$Z^{(k)} = \int d\mathcal{M}_{(k)} \mathrm{e}^{-\frac{8\pi^2 k}{g^2} - \mathcal{S}_{\text{mod}}}$$

Field-dependent moduli action

Consider correlators of D3/D3 fields, e.g of the scalar ϕ in the Cartan direction, in presence of k D-instantons. It turns out that

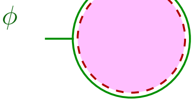
[Green-Gutperle 2000, Billò et al 2002]

- the dominant contribution to $\langle \phi_1 \dots \phi_n \rangle$ is from n one-point amplitudes on disks with moduli insertions. The result can be encoded in extra moduli-dependent vertices for ϕ's, i.e. in extra terms in the moduli action containing such one-point functions

$$\mathcal{S}_{\text{mod}}(\varphi; \mathcal{M}) = \phi(x_0) J_\phi(\widehat{\mathcal{M}}) + \mathcal{S}_{\text{mod}}(\widehat{\mathcal{M}})$$

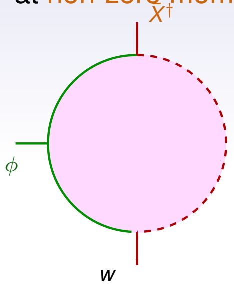
with

$$\phi(x_0) J_\phi(\widehat{\mathcal{M}}) \quad =$$



Moduli action with the unbroken scalar ϕ

The relevant one-point diagrams are those already computed to describe the dependence on the v.e.v. a. We just insert the vertex V_ϕ at non-zero momentum. Recall



$$\left\langle\!\!\left\langle V_{\bar{X}^\dagger} V_\phi V_w \right\rangle\!\!\right\rangle = \ldots = \mathrm{tr}_k \left\{ \bar{X}^\dagger_{\dot\alpha}\, \phi(x_0)\, w^{\dot\alpha} \right\}$$

We get a dependence on the instanton location x_0 (x from now on)

- The field-dependent action $\mathcal{S}_{\rm mod}(\phi; \mathcal{M})$ is thus simply obtained by

$$a \to \phi(x)$$

Effective action for the unbroken multiplet

Other non-zero diagrams couple the components of the gauge supermultiplet to the moduli, related by the Ward identities of the susies broken by the D(-1).

- Example:

$$\left\langle\!\!\left\langle V_{\bar{X}^\dagger} V_{\delta\phi} V_w \right\rangle\!\!\right\rangle = \left\langle\!\!\left\langle V_{\bar{X}^\dagger} \left[\theta^{\alpha A} Q_{\alpha A}, V_\Lambda\right] V_w \right\rangle\!\!\right\rangle = -\left\langle\!\!\left\langle V_{\bar{X}^\dagger} V_\Lambda V_w \int V_\theta \right\rangle\!\!\right\rangle$$

  The contribution of the last diagram can be obtained simply by letting $\phi \to \theta\Lambda$

- This iterates: further couplings with higher components of Φ and more θ-insertions

Effective action for the unbroken multiplet

Other non-zero diagrams couple the components of the gauge supermultiplet to the moduli, related by the Ward identities of the susies broken by the D(-1).

- The superfield-dependent moduli action $\mathcal{S}_{\text{mod}}(\Phi; \mathcal{M})$ is obtained by simply letting

$$a \longrightarrow \Phi(x, \theta)$$

Contributions to the prepotential

Integrating over the moduli the interactions described by $\mathcal{S}_{\text{mod}}(\Phi; \mathcal{M}(k))$ one gets the effective action for the long-range multiplet Φ induced by the k-th instanton sector:

$$S_{\text{eff}}^{(k)}[\Phi] = \int d^4x\, d^4\theta\, d\widehat{\mathcal{M}}_{(k)}\, \mathrm{e}^{-\frac{8\pi k}{g^2} - \mathcal{S}_{\text{mod}}(\Phi;\mathcal{M}(k))}$$

Correspondingly, the prepotential for the low energy $\mathcal{N} = 2$ theory is given by the centred instanton partition function

$$\mathcal{F}^{(k)}(\Phi) = \int d\widehat{\mathcal{M}}_{(k)}\, \mathrm{e}^{-\frac{8\pi k}{g^2} - \mathcal{S}_{\text{mod}}(\Phi;\mathcal{M}(k))}$$

- The superfield $\Phi(x, \theta)$ is a constant w.r.t. $\widehat{\mathcal{M}}_{(k)}$. We can compute $\mathcal{F}^{(k)}$ fixing $\Phi(x, \theta) \rightarrow a$ and using the results of the literature

[see e.g. Dorey et al, 2002]

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Contributions to the prepotential (2)

One finds (Λ is the dynamical scale)

$$\mathcal{F}^{(k)}(\Phi) = c_k\,\Phi^2\left(\frac{\Lambda}{\Phi}\right)^{4k} \quad .$$

- Λ^{4k} stems from the term $\exp(-8\pi k/g^2)$, using the β-function of the $\mathcal{N}=2$, SU(2) theory
- The coefficients c_k (the hard part!) were finally determined using a deformation of the moduli action which localizes the integration

[Nekrasov 2002, Flume-Poghossian 2002, Nekrasov et al 2003, ...]

We will now embed this localization deformation in our string set-up, recognizing it as the effect of a graviphoton background and deriving gravitational corrections to the non-perturbative prepotential

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Deformation from a graviphoton background

The Weyl multiplet

- The field content of $\mathcal{N} = 2$ sugra:

$$h_{\mu\nu} \text{ (metric)} , \quad \psi_\mu^{\alpha A} \text{ (gravitini)} , \quad C_\mu \text{ (graviphoton)}$$

  can be organized in a chiral Weyl multiplet:

$$W^+_{\mu\nu}(x,\theta) = \mathcal{F}^+_{\mu\nu}(x) + \theta\chi^+_{\mu\nu}(x) + \frac{1}{2}\,\theta\sigma^{\lambda\rho}\theta\, R^+_{\mu\nu\lambda\rho}(x) + \cdots$$

  (${\chi_{\mu\nu}}^{\alpha A}$ is the gravitino field strength)
- These fields arise from massless vertices of type IIB strings on $\mathbb{R}^4 \times \mathbb{C} \times \mathbb{C}^2/\mathbb{Z}_2$

Graviphoton vertex

The graviphoton vertex, connected by the supercharges to the other fields in the Weyl multiplet, is given by

$$\begin{aligned} V_{\mathcal{F}}(z,\bar{z}) &= \frac{1}{4\pi}\mathcal{F}^{\alpha\beta AB}(p) \\ &\times \left[S_\alpha(z)S_A(z)\mathrm{e}^{-\frac{1}{2}\varphi(z)}S_\beta(\bar{z})S_B(\bar{z})\mathrm{e}^{-\frac{1}{2}\varphi(\bar{z})}\right]\mathrm{e}^{\mathrm{i}p\cdot X(z,\bar{z})} \end{aligned}$$

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- We will insert the closed string vertices in disk diagrams bounded by the branes → suitable identifications between left- and right-movers taken into account

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- The bi-spinor graviphoton polarization is given by

$$\mathcal{F}^{(\alpha\beta)[AB]} = \frac{\sqrt{2}}{4}\,\mathcal{F}^{+}_{\mu\nu}\left(\sigma^{\mu\nu}\right)^{\alpha\beta}\,\epsilon^{AB}$$

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- A different RR field, with a similar structure, will be useful:

$$V_{\bar{\mathcal{F}}}(z,\bar{z}) = \frac{1}{4\pi}\bar{\mathcal{F}}^{\alpha\beta\hat{A}\hat{B}}(p) \times \left[S_\alpha(z)S_{\hat{A}}(z)\mathrm{e}^{-\frac{1}{2}\varphi(z)}S_\beta(\bar{z})S_{\hat{B}}(\bar{z})\mathrm{e}^{-\frac{1}{2}\varphi(\bar{z})}\right]\mathrm{e}^{\mathrm{i}p\cdot X(z,\bar{z})}$$

$\hat{A},\hat{B} = 3,4 \leftrightarrow$ odd “internal” spin fields Recall

Effect of the graviphoton on the instanton measure

Let us investigate the effect of a graviphoton v.e.v.

$$\langle W^+_{\mu\nu} \rangle \equiv \langle \mathcal{F}^+_{\mu\nu} \rangle \equiv f_{\mu\nu}$$

on the moduli measure.

- We have to consider disk amplitudes with open string moduli vertices on the boundary and closed string graviphoton vertices in the interior which survive in the field theory limit $\alpha' \to 0$.
- We will consider also insertions of vertices of type $V_{\bar{\mathcal{F}}}$, with constant polarization $\bar{\mathcal{F}}^+_{\mu\nu} = \bar{f}_{\mu\nu}$

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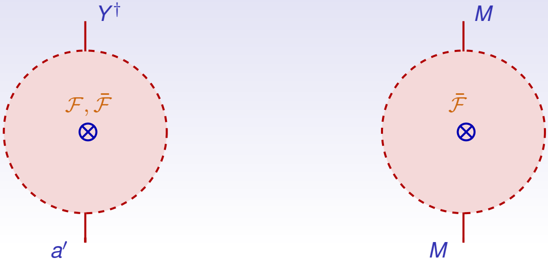
Non-zero diagrams

Very few diagrams contribute.



- The only one involving the true graviphoton is

$$\left\langle\!\!\left\langle V_{Y^\dagger}\, V_{a'}\, V_{\mathcal{F}} \right\rangle\!\!\right\rangle \equiv C_0 \int \frac{dz_1\, dz_2\, dw d\bar{w}}{dV_{\mathrm{CKG}}} \langle V_{Y^\dagger}(z_1)\, V_{a'}(z_2)\, V_{\mathcal{F}}(w, \bar{w})\rangle$$

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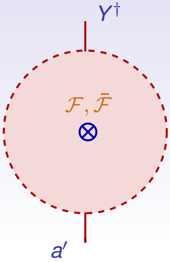

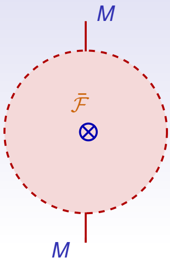

- More explicitly,

$$\left\langle\!\!\left\langle V_{Y^\dagger}\, V_{a'}\, V_{\mathcal{F}} \right\rangle\!\!\right\rangle = \frac{1}{4\pi}\mathrm{tr}_k\left\{ Y^\dagger_\mu\, a'_\nu f_{\lambda\rho}\right\}\left(\sigma^{\lambda\rho}\right)^{\alpha\beta}\epsilon^{AB}\int \frac{dz_1\, dz_2 dw d\bar{w}}{dV_{\mathrm{CKG}}} \times$$
$$\langle \mathrm{e}^{-\varphi(z_2)}\mathrm{e}^{-\frac{1}{2}\varphi(w)}\mathrm{e}^{-\frac{1}{2}\varphi(\bar{w})}\rangle\langle\Psi(z_1)S_A(w)S_B(\bar{w})\rangle$$
$$\langle\psi^\mu(z_1)\psi^\nu(z_2)S_\alpha(w)S_\beta(\bar{w})\rangle$$

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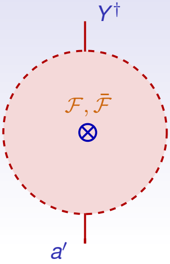

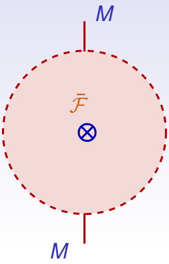

- Result: (same also with $\bar{f}^{\mu\nu}$)

$$\left\langle\!\!\left\langle V_{Y^\dagger}\, V_{a'}\, V_{\mathcal{F}} \right\rangle\!\!\right\rangle = -4\mathrm{i}\,\mathrm{tr}_k\left\{ Y^\dagger_\mu\, a'_\nu\, f^{\mu\nu} \right\}$$

- Moreover, term with fermionic moduli and a $V_{\bar{\mathcal{F}}}$:

$$\left\langle\!\!\left\langle V_M\, V_M\, V_{\bar{\mathcal{F}}} \right\rangle\!\!\right\rangle = \frac{1}{4\sqrt{2}}\mathrm{tr}_k\left\{ M^{\alpha A} M^{\beta B} \bar{f}_{\mu\nu} \right\} (\sigma^{\mu\nu})_{\alpha\beta}\epsilon_{AB}$$

The deformed moduli action

Including the backgrounds $f, \bar{f}$ besides the chiral v.e.v.'s $a, \bar{a}$: Back

$$
\begin{aligned}
&\mathcal{S}_{\rm mod}(a,\bar{a};f,\bar{f}) = \\
&-\mathrm{tr}_k\Big\{\big([\chi^\dagger, a'_{\alpha\dot\beta}] + 2\bar{f}_c(\tau^c a')_{\alpha\dot\beta}\big)\big([\chi, a'^{\dot\beta\alpha}] + 2 f_c(a'\tau^c)^{\dot\beta\alpha}\big) \\
&-\big(\chi^\dagger \bar{w}_{\dot\alpha} - \bar{w}_{\dot\alpha}\,\bar{a}\big)\big(w^{\dot\alpha}\chi - a\, w^{\dot\alpha}\big) - \big(\chi \bar{w}_{\dot\alpha} - \bar{w}_{\dot\alpha}\, a\big)\big(w^{\dot\alpha}\chi^\dagger - \bar{a}\, w^{\dot\alpha}\big)\Big\} \\
&+\mathrm{i}\,\frac{\sqrt{2}}{2}\,\mathrm{tr}_k\Big\{\bar\mu^A \epsilon_{AB}\big(\mu^B\chi^\dagger - \bar{a}\,\mu^B\big) \\
&-\frac{1}{2}\,M^{\alpha A}\epsilon_{AB}\big([\chi^\dagger, M^B_\alpha] + 2\,\bar{f}_c\,(\tau^c)_{\alpha\beta}M^{\beta B}\big)\Big\} + S_{\rm c}^{(k)}
\end{aligned}
$$

- The constraint part of the action, $S_{\rm c}^{(k)}$, is not modified

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\end{aligned}
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- The effect of the $f, \bar{f}$ background amounts to the shift ‣ Recall

$$
[\chi, (\bullet)_\alpha] \to [\chi, (\bullet)_\alpha] + 2 f_c(\tau^c\,\bullet)_\alpha\ , \quad [\chi^\dagger, (\bullet)_\alpha] \to [\chi^\dagger, (\bullet)_\alpha] + 2\bar{f}_c(\tau^c\,\bullet)_\alpha
$$

Holomorphicity, Q-exactness

Also the deformed moduli action has the form

$$\mathcal{S}_{\text{mod}}(a, \bar{a}; f, \bar{f}) = Q\,\Xi$$

where Q is the scalar twisted supercharge.

- The parameters $\bar{a}$,$\bar{f}_c$ appear only in the gauge fermion Ξ
- The instanton partition function

$$Z^{(k)} \equiv \int d\mathcal{M}_{(k)}\, \mathrm{e}^{-\mathcal{S}_{\text{mod}}(a,\bar{a};f,\bar{f})}$$

is independent of $\bar{a}$,$\bar{f}_c$: variation w.r.t these parameters is Q-exact.

Graviphoton and localization

The moduli action obtained inserting the graviphoton background coincides exactly with the "deformed" action considered in the literature to localize the moduli space integration if we set

[Nekrasov 2002, Flume-Poghossian 2002, Nekrasov et al 2003, ...]

$$f_c = \frac{\varepsilon}{2}\,\delta_{3c}\;, \quad \bar{f}_c = \frac{\bar{\varepsilon}}{2}\,\delta_{3c}\;,$$

and moreover (referring to the notations in the above ref.s)

$$\varepsilon = \bar{\varepsilon}\;, \quad \varepsilon = \epsilon_1 = -\epsilon_2$$

- The "shift" rule which yields the deformed action was interpreted as "gauging" the chiral rotations in 4d Euclidean space which are symmetries of the ADHM constraints

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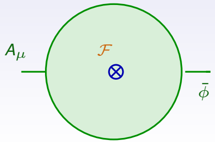
The localization deformation of the $\mathcal{N} = 2$ ADHM construction is produced, in the type IIB string realization, by a graviphoton background

Deformation of the gauge action



- The graviphoton background can be inserted also in D3 disks, producing extra terms in the gauge theory action on the D3 branes
- Collecting all diagrams, one gets

$$S_{\rm SYM} + \int d^4x \mathrm{Tr}\Big\{ -2\,\mathrm{i}\,g\, F_{\mu\nu}\bar{\phi} f^{\mu\nu} - g^2\big(\bar{\phi} f^{\mu\nu}\big)^2 \Big\}$$

(in agreement with couplings between gauge and Weyl multiplets in $\mathcal{N} = 2$ sugra)

- At linear order in g, field eq.s for ϕ

$$D^2\phi = -\mathrm{i}\sqrt{2} g \epsilon_{AB} \Lambda^{\alpha A}\Lambda_\alpha^{\ B} - 2\mathrm{i}\, g\, f_{\mu\nu} F^{\mu\nu}$$

agree with the one implied by the deformed ADHM construction

Weyl multiplet dependence of the effective prepotential

- Just as for the case of the scalar v.e.v's only, we can compute

$$\mathcal{S}_{\text{mod}}(\Phi, W^+; \mathcal{M}(k))$$

containing the one-point couplings of the fields in the gauge and Weyl multiplets to the moduli by simply promoting

$$a \to \Phi(x;\theta) \ , \quad f_{\mu\nu} \to W_{\mu\nu}(x,\theta)$$

- The l.e.e.a. for Φ and W^+ in the instanton # k sector is

$$S_{\text{eff}}^{(k)}[\Phi, W^+] = \int d^4x\, d^4\theta\, d\widehat{\mathcal{M}}_{(k)}\, \mathrm{e}^{-\frac{8\pi k}{g^2} - \mathcal{S}_{\text{mod}}(\Phi, W^+; \mathcal{M}(k))}$$

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Expansion of the prepotential

$\Phi(x,\theta)$ and $W^{+}_{\mu\nu}(x,\theta)$ are constant w.r.t. the integration variables $\widehat{\mathcal{M}}_{(k)}$.
We can compute $\mathcal{F}^{(k)}(a; f)$ and reinstate the full multiplets in the result.

- From the explicit form of $\mathcal{S}_{\text{mod}}(a, 0; f, 0)$ Recall it follows that partition function $\mathcal{F}^{(k)}(a; f)$ is invariant under

$$a, f_{\mu\nu} \to -a, -f_{\mu\nu}$$

- We need a regular expansion for $f \to 0$, and no odd powers of $af_{\mu\nu}$, Altogether, reinstating the superfields,

$$\mathcal{F}^{(k)}(\Phi, W^{+}) = \sum_{h=0}^{\infty} c_{k,h}\, \Phi^2 \left(\frac{\Lambda}{\Phi}\right)^{4k} \left(\frac{W^{+}}{\Phi}\right)^{2h}$$

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The non-perturbative prepotential

Sum over the instanton sectors:

$$\mathcal{F}_{\text{n.p.}}(\Phi, W^+) = \sum_{k=1}^{\infty} \mathcal{F}^{(k)}(\Phi, W^+) = \sum_{h=0}^{\infty} C_h(\Lambda, \Phi)(W^+)^{2h}$$

with

$$C_h(\Lambda, \Phi) = \sum_{k=1}^{\infty} c_{k,h} \frac{\Lambda^{4k}}{\Phi^{4k+2h-2}}$$

- Many different terms in the eff. action connected by susy. Saturating the θ integration with four θ's all from W^+

$$\int d^4x \; C_h(\Lambda, \phi) \, (R^+)^2 (\mathcal{F}^+)^{2h-2}$$

Freezing $\phi \to a$, this is a purely gravitational F-term

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Evaluation via localization

- To determine the coefficients $c_{k,h}$, constant bkg values $\Phi \to a$ and $W^+_{\mu\nu} \to f_{\mu\nu}$ are enough.
- The localization deformation of the multi-instanton partition function $Z^{(k)}(a, \varepsilon)$ is obtained for

$$f_{\mu\nu} = \frac{1}{2}\,\varepsilon\,\eta^3_{\mu\nu}\,, \quad \bar{f}_{\mu\nu} = \frac{1}{2}\,\bar{\varepsilon}\,\eta^3_{\mu\nu}$$

  - (Holomorphicity:) $Z^{(k)}(a, \varepsilon)$ does not smoothly depend on $\bar{\varepsilon}$
  - However, $\bar{\varepsilon} = 0$ is a limiting case: some care is needed
- $\mathcal{F}^{(k)}(a; \varepsilon)$ is well-defined. $S^{(k)}[a; \varepsilon]$ diverges because of the (super)volume integral $\int d^4x\, d^4\theta$
  - $\bar{\varepsilon}$ regularizes the superspace integration by a Gaussian term. One can then work with the effective action, i.e., the *full* instanton partition function

Evaluation via localization

- To determine the coefficients $c_{k,h}$, constant bkg values $\Phi \to a$ and $W^{+}_{\mu\nu} \to f_{\mu\nu}$ are enough.
- The localization deformation of the multi-instanton partition function $Z^{(k)}(a, \varepsilon)$ is obtained for

$$f_{\mu\nu} = \frac{1}{2}\,\varepsilon\,\eta^{3}_{\mu\nu}\ , \quad \bar{f}_{\mu\nu} = \frac{1}{2}\,\bar{\varepsilon}\,\eta^{3}_{\mu\nu}$$

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Example: the case $k = 1$

The moduli action in the $k = 1$ sector (at $\bar{a} = 0$) is

$$\mathcal{S}_{\text{mod}}^{(k=1)} = -2\bar{\varepsilon}\varepsilon\, x^2 - \frac{\bar{\varepsilon}}{2}\,\theta^{\alpha A}\epsilon_{AB}(\tau_3)_{\alpha\beta}\theta^{\beta B} + \widehat{\mathcal{S}}_{\text{mod}}^{(k=1)}(a)$$

The last term does not depend on $\varepsilon, \bar{\varepsilon}$

- Using the $\varepsilon, \bar{\varepsilon}$-independence of the centred partition function

$$\mathcal{F}^{(k=1)}(a) = \int d\widehat{\mathcal{M}}^{(k=1)} e^{-\frac{8\pi^2}{g^2} - \widehat{\mathcal{S}}_{\text{mod}}^{(k=1)}(a)}$$

we have for the $k = 1$ instanton partition function

$$Z^{(k=1)}(a, \varepsilon) = \int d^4x\, d^4\theta e^{-2\bar{\varepsilon}\varepsilon x^2 - \frac{1}{2}\bar{\varepsilon}\,\theta\cdot\theta}\, \mathcal{F}^{(k=1)}(a) = \frac{1}{\varepsilon^2}\mathcal{F}^{(k=1)}(a)$$

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The deformed partition function vs the prepotential

- With $\varepsilon, \bar{\varepsilon}$, the partition function $Z(a;\varepsilon)$ can be computed:
  - (super)volume divergences $\rightarrow \varepsilon$ singularities
  - a and $\varepsilon, \bar{\varepsilon}$ deformations localize completely the integration over moduli space which can be carried out

[Nekrasov 2002, Flume-Poghossian 2002, Nekrasov et al 2003, ...]

- With $\bar{\varepsilon} \neq 0$ (complete localization) a trivial superposition of instantons of charges k_i contributes to the sector $k = \sum k_i$
- Such disconnected configurations do *not* contribute when $\bar{\varepsilon} = 0$
- The partition function computed by localization corresponds to the exponential of the non-perturbative prepotential:

$$
\begin{aligned}
Z(a;\varepsilon) &= \exp\left(\frac{\mathcal{F}_{\text{n.p.}}(a,\varepsilon)}{\varepsilon^2}\right) = \exp\left(\sum_{k=1}^{\infty}\frac{\mathcal{F}^{(k)}(a,\varepsilon)}{\varepsilon^2}\right) \\
&= \exp\left(\sum_{h=0}^{\infty}\sum_{k=1}^{\infty} c_{k,h}\frac{\varepsilon^{2h-2}}{a^{2h}}\left(\frac{\Lambda}{a}\right)^{4k}\right)
\end{aligned}
$$

Summarizing

- The computation via localization techniques of the multi-instanton partition function $Z(a;\varepsilon)$ determines the coefficients $c_{k,h}$ which appear in the gravitational F-terms of the $\mathcal{N}=2$ effective action

$$\int d^4x\; C_h(\Lambda,\phi)\,(R^+)^2(\mathcal{F}^+)^{2h-2}$$

via the relation

$$C_h(\Lambda,\phi)=\sum_{k=1}^{\infty} c_{k,h}\,\frac{\Lambda^{4k}}{\phi^{4k+2h-2}}$$

- The very same gravitational F-terms can been extracted in a completely different way: topological string amplitudes on suitable Calabi-Yau manifolds

Summarizing

- The computation via localization techniques of the multi-instanton partition function $Z(a;\varepsilon)$ determines the coefficients $c_{k,h}$ which appear in the gravitational F-terms of the $\mathcal{N}=2$ effective action

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Relation to topological strings on CY

Geometrical engineering and topological strings

- SW: low energy $\mathcal{N} = 2 \leftrightarrow$ (auxiliary) Riemann surface
- Geometrical engineering: embed directly the low energy theory into string theory as type IIB on a suitable local CY manifold $\mathfrak{M}$

  [Kachru et al 1995, Klemm et al 1996-97]

  - geometric moduli of $\mathfrak{M} \leftrightarrow$ gauge theory data (Λ, a);
  - The coefficients C_h in the l.e.e.a. gravitational F-terms

$$C_h \, (R^+)^2 (\mathcal{F}^+)^{2h-2}$$

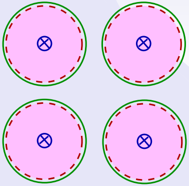
  are given by topological string amplitudes at genus h

  [Bershadsky et al 1993-94, Antoniadis et al 1993]

- For the local CY $\mathfrak{M}_{\mathrm{SU}(2)}$ the couplings C_h were checked to coincide with those given by the deformed multi-instanton calculus as proposed by Nekrasov [Klemm et al 2002]

Microscopic vs effective string description

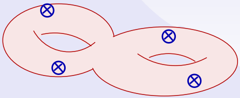
Orbifold space
with D3/D(-1) system
Moduli action depends on
gauge theory data Λ, a
open and closed strings



2h disks
connected by
integration over
moduli

$\chi = 2h - 2$

Local CY manifold
Geometric moduli determined
from gauge theory data Λ, a
No branes - closed strings only



genus h Riemann surface

$\chi = 2h - 2$

Same gravitational F-term interactions

$$C_h(\Lambda, a)\,(R^+)^2(\mathcal{F}^+)^{2h-2}$$

Perspectives

Some interesting directions to go...

- Study of the instanton corrections to $\mathcal{N} = 2$ eff. theory in the gauge/gravity context: modifications of the classical solution of fD3's
- Application of similar techniques to (euclidean) D3's along a CY orbifold to derive BH partition functions in $\mathcal{N} = 2$ sugra (which OSV conjecture relates to $|Z_{\text{top}}|^2$)
- Non-perturbative corrections to $\mathcal{N} = 1$ superpotentials by Euclidean D3's along orbifold directions
- ...

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Some notations

String fields in the orbifold space

- In the six directions transverse to the brane,

$$Z \equiv (X^5 + \mathrm{i}X^6)/\sqrt{2}\,, \quad Z^1 \equiv (X^7 + \mathrm{i}X^8)/\sqrt{2}\,, \quad Z^2 \equiv (X^9 + \mathrm{i}X^{10})/\sqrt{2}\,,$$
$$\Psi \equiv (\psi^5 + \mathrm{i}\psi^6)/\sqrt{2}\,, \quad \Psi^1 \equiv (\psi^7 + \mathrm{i}\psi^8)/\sqrt{2}\,, \quad \Psi^2 \equiv (\psi^9 + \mathrm{i}\psi^{10})/\sqrt{2}$$

  the $\mathbb{Z}_2$ orbifold generator h acts by

$$(Z^1, Z^2) \to (-Z^1, -Z^2)\,, \quad (\Psi^1, \Psi^2) \to (-\Psi^1, -\Psi^2)$$

- Under the SO(10) $\to$ SO(4) $\times$ SO(6) induced by D3's, $S^{\dot{\mathcal{A}}} \to (S_\alpha S_{A'}, S^{\dot\alpha} S^{A'})$
- Under SO(6) $\to$ SO(2) $\times$ SO(4) induced by the orbifold, Back

$S^{A'}$	notat.	SO(2)	SO(4)	$S_{A'}$	notat.	SO(2)	SO(4)	h
S^{+++} S^{+--}	S^A $A=1,2$	$\frac{1}{2}$	$(\mathbf{2},\mathbf{1})$	S_{---} S_{-++}	S_A $A=1,2$	$-\frac{1}{2}$	$(\mathbf{2},\mathbf{1})$	$+1$
S^{-+-} S^{--+}	$S^{\hat{A}}$ $\hat{A}=3,4$	$-\frac{1}{2}$	$(\mathbf{1},\mathbf{2})$	S_{+-+} S_{++-}	$S_{\hat{A}}$ $\hat{A}=3,4$	$\frac{1}{2}$	$(\mathbf{1},\mathbf{2})$	-1