"Remarks on the ε-deformed prepotential of $\mathcal{N}=2^*$ theory"

- Based mostly on 1302.0686 and 1307.6648 by M.B., M. Frau, L. Gallot, I. ~~Pesando~~ A. Lerda and I. Pesando

- Relies on a large literature, some very recent, other less so. I will sparsely quote here some of the references I actually used most; there'll be no pretence of completeness or attribution of priorities, etc. I beg pardon for the many important papers I'll not mention.

- I will discuss mass-deformed and ε-deformed $\mathcal{N}=2$ SCFT's in $d=4$, of which a standard example is the so-called $\mathcal{N}=2^*$ theory, on which I'll focus. This theory comprises a gauge multiplet and an adjoint multiplet.

- For simplicity we consider the rank-one case ($SU(2)$ gauge group), but work is in progress regarding higher rank cases; in the papers we also parallely considered the $SU(2)$ with $N_f=4$ fundamental hypers case, which also has vanishing β-function.

- These $\mathcal{N}=2$ theories have recently received a lot of attention because they sit at the crossroads of many approaches to the computation of exact non-perturbative low-energy dynamics.

- Let me try to (partially) describe such connections in a scheme

Open strings on D3branes
(e.g fD3's on orbifold)

Microscopic $\mathcal{N}=2^*$ theory

RR background
or Ω-background
(dual?)
MB et al 060613
Hellerman, Orlando, Reffert 1204.4192
1106.0279

pert. + non-pert. corrections

$\langle\phi\rangle=a$

l.e.e.a. on Coulomb branch
prepotential $\mathcal{F}(a)$

Kachru et al 9508155
Klemm et al
9604034

"Geometric engineering": closed strings
on local CY (abelian). $\mathcal{F}\sim$ genus 0
top string amplitude

Effective geometric
description: SW curve

SW

$a, a_D\sim$ periods

$a_D\sim\frac{\partial\mathcal{F}}{\partial a}$

BCOV 9302103
9309140
Witten 9306122
Aganagic et al
0607100

holomorphic
anomaly
- modular
anomaly
~~Krefl~~
Krefl-Walcher 1007.0263
Huang-Klemm 1009.1126

"Refined" topological
string amplitudes $\mathcal{F}^{(n,g)}_{top}$

$\epsilon_1\epsilon_2\sim g_s^2$

matrix
models

such amplitudes
are captured by
Antoniadis et al 9307158
〃 1003.2832
1302.6993

ε-deformation
needed to
- regularize integrals on moduli
- fully localize
→ compute the partition function
$Z(a,\epsilon)$

Nekrasov, 0206161
0306211

Nekrasov
Okounkov 0306238

$\epsilon\to0$

ε-deformed $\mathcal{N}=2^*$ theory

$$F(a,\epsilon):=-\epsilon_1\epsilon_2\log Z(a,\epsilon)$$
$$=F_{cl}(a)+\sum_{n,g=0}^{\infty}(\epsilon_1+\epsilon_2)^{2n}(\epsilon_1\epsilon_2)^{g}F^{(n,g)}(a)$$

Physical meaning of $F^{(n,g)}$: amplitudes

$\sim F^{(n,g)}\,Y^{2n}\,W^{2g}$

v.m.

graviphoton

Nekrasov Okounkov 0306238
& Antoniadis 1003.2832

AGT 0906.3219
Gaiotto 0904.2715
Kashani-Poor, Troost 1212.0722, 1305.7408

2d Liouville conformal blocks
- surface and # of vertices det'd by gauge&matter content
(1-point on torus for $\mathcal{N}=2^*$ SU(2))
- c, Δ related to ϵ_1, ϵ_2

- Recall that in a SCFT the β function vanishes: there's no dimensional transmutation and the low-energy theory, in particular the prepotential, depends on the tree-level coupling $\tau_0 = \frac{\theta}{2\pi} + \frac{4\pi i}{g^2}$, even if the conformal symmetry is broken by mass terms.

- The various approaches depicted in the scheme above yield complementary techniques to search for an expression of the deformed prepotential exact in τ_0, i.e. from the microscopic point of view, incorporating all non-perturbative corrections.

- Exact (maybe partial) expressions could shed light on how duality properties are realized in the deformed effective theory.

- In particular, S-duality acts at the tree level as $\tau_0 \to -1/\tau_0$. How is it implemented at the quantum effective level? Gaiotto 0904.2715
Mironov, Morozov, Shakirov 0911.5721

- I try now to illustrate briefly the route by which we derived some (I hope interesting) results in this direction.

Instanton corrections

- On the one side, it is possible to explicitly perform the microscopic computation of instanton effects by means of localization techniques, getting explicit results at low instanton numbers. Nekrasov 0206161
Nekrasov Okounkov 0306211, ...

- These results can, for instance, be described by expanding the prepotential for large a (i.e, in the semiclassical regime) as

$$F = 2\pi i\, \tau_0\, a^2 + h_0^{(m,\varepsilon)} \log \frac{2a}{\mu} - \sum_{\ell=1}^{\infty} \frac{h_\ell(m,\varepsilon)}{2^{\ell+1}} \frac{1}{a^{2\ell}}$$

classical

perturbative

perturbative + instanton;

nontrivial, as a funct. of n.p. [illegible] Okounkov . . .

. One gets (only write ~~very~~ very few first terms; shorthand: $s=\varepsilon_1+\varepsilon_2$, $p=\varepsilon_1\varepsilon_2$, $q=e^{i\pi\tau_0}$)

$$h_0 = \frac{1}{4}(4m^2-s^2)$$

$$h_1 = (4m^2-s^2)(4m^2-s^2+4p)\left(\frac{1}{192} - \frac{1}{8}q^2 - \frac{3}{8}q^4 - \frac{1}{2}q^6 \dots\right)$$

$$h_2 = (4m^2-s^2)(4m^2-s^2+4p)\left(\frac{4m^2-3s^2+6p}{3840} - \frac{s^2}{8}q^2 + \frac{12m^2+18p-21s^2}{16}q^4 + \dots\right)$$

... [MB et al 1302.0686, previous [illegible] techniques in $m=0$, or $\varepsilon_1+\varepsilon_2=0$ limits Huang, Kashani-Poor, Klemm 1109.5728
Huang 1205.3652
Kashani-Poor, Troost 1212.0722

Comparison with SW curve (undeformed)

. In the $\varepsilon\to 0$ limit ($s,p\to 0$ above) these results checks against the SW curve description ~~[illegible]~~ which is available also in the conformal case. In fact, from the curve it's possible to get exact expressions in τ_0 which (when expanded for large a) correspond to (Minahan-Nemeschansky-Warner 9710146 MB et al 1107.3691)

$$h_0 = m^2$$

$$h_1 = \frac{m^4}{12} E_2(\tau_0)$$ ← Eisenstein series

$$h_2 = \frac{m^6}{72}\left(E_2^2 + \frac{1}{5}E_4\right)$$

$$h_3 = \frac{m^8}{1728}\left(3E_2^3 + \frac{12}{5}E_2E_4 + \frac{11}{35}E_6\right)$$

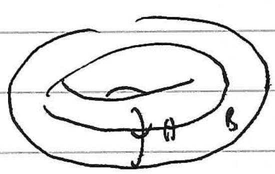
...

. Recall that $E_{2k}(-1/\tau_0) = \tau_0^{2k}\, E_{2k}(\tau_0)$ $(k>1)$ modular of weight $2k$

$E_2(-1/\tau_0) = \tau_0^2\, E_2(\tau_0) + \frac{6}{i\pi}\tau_0$ "quasi"-modular of weight 2

- The action of S-duality on the h_ℓ's (and thus on $\mathcal{F}(a,$ ~~$\mathcal{F}($~~$\epsilon,g)$) follows from these modularity properties. It is consistent with the SW implementation of the duality, the SW curve of $N=2^*$ is a torus

see original 2nd paper



$$a \sim \int_A \lambda \leftarrow \text{SW differential}$$

; $\mathcal{F}$ defined by $a_D \sim \frac{\partial \mathcal{F}}{\partial a}$

$$a_D \sim \int_B \lambda$$

S-duality acts geometrically $S: \begin{pmatrix} a \\ a_D \end{pmatrix} \to \begin{pmatrix} a_D \\ -a \end{pmatrix}$, which on the prepotential corresponds to a Legendre transform: $(\tilde{a} = a_D = S(a))$

$$S: \mathcal{F}(a) \longrightarrow \tilde{\mathcal{F}}(\tilde{a}) = \mathcal{F}(a) - 2\pi i \tilde{a} a$$

where on the r.h.s. $a = a(\tilde{a})$ is obtained by inverting $\tilde{a} = \frac{\partial \mathcal{F}}{\partial a}$.

Exact expressions of the h_ℓ's for $\epsilon \neq 0$

- For $\epsilon \neq 0$ and keeping also m generic, one can make the assumption that $h_\ell(m,\epsilon)$ are still given by (quasi)-modular forms of weight 2ℓ. (Note: we assume, as pointed out in the literature, that $S(\epsilon_1\epsilon_2) = \epsilon_1\epsilon_2$, $S(\epsilon_1+\epsilon_2) = \epsilon_1+\epsilon_2$).

- The first perturb. and n.p. coefficients fix then the combination of Eisenstein series that appears. We get

$$h_0 = \frac{1}{4}(4m^2 - (\epsilon_1+\epsilon_2)^2)$$

$$h_1 = \frac{1}{12} h_0 (h_0 + \epsilon_1\epsilon_2) E_2$$

$$h_2 = \frac{1}{144} h_0 (h_0 + \epsilon_1\epsilon_2) \left[(2h_0 + 3\epsilon_1\epsilon_2) E_2^2 + \frac{1}{5}(2h_0 + 3\epsilon_1\epsilon_2 - 6(\epsilon_1+\epsilon_2)^2) E_4 \right]$$

...

- One "experimentally" finds that the coefficients h_ℓ satisfy a "recursion relation"

$$\frac{\partial h_\ell}{\partial E_2} = \frac{\ell}{12}\sum_{k=0}^{\ell-1} h_k\, h_{\ell-k-1} + \frac{\ell(2\ell-1)}{12}\,\epsilon_1\epsilon_2\, h_{\ell-1}$$

- This recursion relation can be equivalently expressed in terms of the $F^{(n,g)}$ quantities:

$$\partial_{E_2} F^{(n,g)} = -\frac{1}{48}\sum_{n_1=0}^{n}\sum_{g_1=0}^{g} \partial_a F^{(n_1,g_1)}\, \partial_a F^{(n-n_1,g-g_1)} + \frac{1}{48}\partial_a^2 F^{(n,g-1)}$$

Huang, Kashani-Poor, Klemm 1109.5728, Huang 1205.3652, 1302.6095

- This "modular anomaly equation" has been put forward in a series of papers (Klemm et al, ...
based on the interpretation of $F^{(n,g)}$ as "refined" topological string amplitudes.

Connection to topological strings features

- "Usual" topological string amplitudes F^g_{top} (corresponding to $F^{(0,g)}$ in the local case) on CY satisfy the holomorphic anomaly equation of BCOV: 9309140 02103

  - $F^g_{top}(t)$ are apparently holomorphic in the CY moduli t, but they actually acquire a dependence on $\bar t$ via boundary effects in genus g moduli space. The $\bar t$-dependence is linked to lower-genera amplitudes in a non-linear equation.

- The holomorphic anomaly can be expressed as a linear equation (of the heat conduction type) on the so-called "topological wave-function"

$$Z_{top} \sim \exp\Big(-\sum_{g=0}^{\infty} (g_s)^{2g-2} F^{(g)}_{top}(t,\bar t)\Big)$$

Witten, 9306122
(~~Witten~~, ..., ~~Aganagic~~ ... ! We'll come back later to the interpre-
Aganagic, Bouchard, Klemm 0607100 Gunaydin, Neitzke, Pioline 0607200

tation as a "wave function".

- This equation has been analyzed (Aganagic et al, 0607100, ...) for $\varepsilon_1+\varepsilon_2=0$, also ~~in~~ for local CY manifolds that geometrically engineer $N=2$ SYM effective theories. In this case the top. wave function corresponds to the non-pert. ~~wave fu~~ partition function

$$Z_{top}(t,\bar{t}) \longleftrightarrow Z(a,\varepsilon)$$

- It has been shown (Aganagic et al, ...) that changing "polarization" from the basis of moduli $t,\bar{t}$ to that of periods a, a_D the failure of holomorphicity turns into a failure of modularity: BCOV equation $\longrightarrow$ modular anomaly equation.

- (Klemm et al, 1109.5728, ...) proposed an extension of this equation to the ~~generalized~~ refined topological amplitudes the one I wrote before. As we saw, this proposal checks against the explicit expressions obtained via localization techniques.

- Thus we assume now this modular anomaly equation and exploit it as well as we can.

Heat Kernel expression of the deformed prepotential

- Introducing

$$\varphi_0 = F_{cl} - F = -\sum_{n,g=0}^{\infty} (\varepsilon_1+\varepsilon_2)^{2n} (\varepsilon_1\varepsilon_2)^{g} F^{(n,g)}$$

the modular anomaly equation ~~reads~~ can be written as

$$\partial_{\varepsilon_2}\varphi_0 = \frac{1}{48}(\partial_a\varphi_0)^2 + \frac{\varepsilon_1\varepsilon_2}{48}\partial_a^2\varphi_0$$

(the KPZ (Kardar-Parisi-Zhang 1986) eq. in $d=1$)

- For notational simplicity, introduce $t \equiv \bar{E}_2/24$

- Further introduce
$$\Psi = e^{\frac{\varphi_0}{\varepsilon_1\varepsilon_2}} = e^{-\frac{F-F_{cl}}{\varepsilon_1\varepsilon_2}}$$

i.e. the non-classical part of the partition function).

also
(see Wernum et al.)

- The equation turns into the heat conduction equation:
$$\partial_t \Psi - \frac{\varepsilon_1\varepsilon_2}{2}\partial_a^2 \Psi = 0$$

and can thus be ~~fo~~ solved by convolution with the gaussian heat kernel:
$$\Psi(a,t) = \frac{1}{\sqrt{2\pi\varepsilon_1\varepsilon_2 t}}\int_{-\infty}^{\infty} dy \; e^{-\frac{(a-y)^2}{2\varepsilon_1\varepsilon_2 t}} \; \Psi(a,0)$$

$\uparrow$
Initial condition: results at $E_2 = 0$

$$= (G * \Psi_0)(a,t)$$

- This expression can be used to reconstruct the dependence on $\bar{E}_2$, and then all the dependence from $\bar{E}_4$ and $\bar{E}_6$ as well, to a very high order in $1/a^2$ starting from the perturbative results and the very first instanton effects:

  - knowledge of the one-loop prepotential allows to reconstruct the exact form of all the h_ℓ with $\ell \leq 5$

  - one-loop + one-instanton allow to get the h_ℓ up to $\ell = 11$

  - ...

The S-duality action on the ε-deformed prepotential

- The heat Kernel expression can be used to determine the S-duality action on $F(a,\varepsilon)$, a question recently ~~[illegible]~~ discussed in the literature (Galakhov, Mironov, Morozov 1205.4998
Nemkov 1307.0773

- So we start from $\psi(a,t) = (G * \psi_0)(a,t)$, i.e.

$$e^{\frac{\varphi_0(a,t)}{\varepsilon_1\varepsilon_2}} = \frac{1}{\sqrt{2\pi\varepsilon_1\varepsilon_2 t}} \int_{-\infty}^{\infty} dy \; e^{-\frac{(a-y)^2}{2\varepsilon_1\varepsilon_2 t} + \frac{\varphi_0(y,0)}{\varepsilon_1\varepsilon_2}} \qquad (\star)$$

- Apply now S-duality to both sides, defining $S(a) = \tilde{a}$.

- Recall that $t = E_2/24$ and E_2 is quasi-modular of weight 2

$$\Rightarrow \quad S(t) = \tilde{t} = \tau_0^2\left(t + \frac{1}{2\pi i \tau_0}\right)$$

- $\underset{\substack{\uparrow \\ \text{no } E_2!}}{\varphi_0(y,0)}$ is made of terms $\sim \dfrac{m^{2\ell+2}}{y^{2\ell}} f^{(\ell)}(\tau_0)$ ← modular of weight 2ℓ

$$\Rightarrow \quad S: \; \varphi_0(y,0) \rightarrow \tilde{\varphi}_0(\tilde{y},0) = \varphi_0\left(\frac{\tilde{y}}{\tau_0}, 0\right)$$

- We find thus

$$e^{\frac{\tilde{\varphi}_0(\tilde{a},\tilde{t})}{\varepsilon_1\varepsilon_2}} = \frac{1}{\sqrt{2\pi\varepsilon_1\varepsilon_2\tilde{t}}} \int d\tilde{y} \; e^{-\frac{(\tilde{a}-\tilde{y})^2}{2\varepsilon_1\varepsilon_2\tilde{t}} + \varphi_0\left(\frac{\tilde{y}}{\tau_0},0\right)}$$

↑ change integration variable to $y = \tilde{y}/\tau_0$ so that the "initial condition" becomes identical to $(\star)$

Thus we get

$$e^{\frac{\tilde{\varphi}_0(\tilde{a},\tilde{t})}{\varepsilon_1\varepsilon_2}} = \frac{1}{\sqrt{2\pi\varepsilon_1\varepsilon_2\tilde{t}/\tau_0^2}}\int dy\, e^{-\frac{\left(\frac{\tilde{a}}{\tau_0}-y\right)^2}{2\varepsilon_1\varepsilon_2\tilde{t}/\tau_0^2}+\frac{\varphi_0(y,0)}{\varepsilon_1\varepsilon_2}}$$

$$|||$$

$$\tilde{\Psi}\left(\tfrac{\tilde{a}}{\tau_0},\tilde{t}\right) = \left(\tilde{G} * \varphi_0\right)\left(\tfrac{\tilde{a}}{\tau_0};\tilde{t}\right) \qquad (\bigstar\bigstar)$$

i.e: convolution with a modified gaussian kernel of the same φ_0.

- Take now the Fourier transform which maps convolutions to products: with these gaussian kernel we get from $\bigstar$ and $(\bigstar\bigstar)$

$$\begin{cases} \mathcal{F}[\Psi](k) = e^{-2\pi^2\varepsilon_1\varepsilon_2 t\, k^2}\, \mathcal{F}[\varphi_0](k) \\ \mathcal{F}[\tilde{\Psi}](k) = e^{-2\pi^2\varepsilon_1\varepsilon_2\underbrace{\left(t+\frac{1}{2\pi i\tau_0}\right)}_{\text{this is } \tilde{t}/\tau_0^2} k^2} \cdot \mathcal{F}[\varphi_0](k) \end{cases}$$

- Dividing member by member we eliminate the initial condition, getting

$$\mathcal{F}[\tilde{\Psi}](k) = e^{\frac{i\pi\varepsilon_1\varepsilon_2}{\tau_0}k^2}\, \mathcal{F}[\Psi](k)$$

- Now we go back with the inverse Fourier transform, ending up with

$$\tilde{\Psi}(\tilde{x},\tilde{t}) = \ldots = \sqrt{\frac{i\tau_0}{\varepsilon_1\varepsilon_2}}\; e^{-\frac{i\pi\tau_0}{\varepsilon_1\varepsilon_2}\tilde{x}^2}\int dx\, e^{\frac{2\pi i\tau_0\tilde{x}x - \pi i\tau_0 x^2 + \varphi_0(x,t)}{\varepsilon_1\varepsilon_2}}$$

that is,

$$e^{\frac{\tilde{\varphi}(\tilde{a},\tilde{t})}{\varepsilon_1\varepsilon_2}} = \tilde{\Psi}\left(\frac{\tilde{a}}{\tau_0},\tilde{t}\right) = \sqrt{\frac{i\tau_0}{\varepsilon_1\varepsilon_2}}\; e^{\frac{-i\pi\,\tilde{a}^2}{\tau_0\varepsilon_1\varepsilon_2}} \int dx\; e^{\frac{2\pi i\tilde{a}x \underbrace{-\pi i\tau_0 x^2}_{\text{classical part}} + \underbrace{\varphi_0(x,t)}_{\text{complete prepotential}}}{\varepsilon_1\varepsilon_2}}$$

bring it on the other side
to reconstruct the S-dual
of the complete prepotential

- Altogether we find thus

$$e^{-\frac{\tilde{F}(\tilde{a})}{\varepsilon_1\varepsilon_2}} = \sqrt{\frac{i\tau_0}{\varepsilon_1\varepsilon_2}} \int_{-\infty}^{\infty} dx\; e^{\frac{2\pi i\tilde{a}x - F(x)}{\varepsilon_1\varepsilon_2}}$$

The S-duality is realized as a Fourier transform, as proposed in Nemkov (1307.0773) while Galakhov et al, 1205.4998 suggested some modified F.T. (see Ponsot-Teschner 9911110 for earlier attempt)

- This results is perfectly consistent with the interperatation of $a, \tilde{a}$ as a pair of canonically conjugated variables, of S-duality as a canonical transformation and of

$$e^{-\frac{F(a)}{\varepsilon_1\varepsilon_2}} = Z(a,\varepsilon) \qquad \text{as a wavefunction}$$

in a quantization of this phase space, with $\varepsilon_1\varepsilon_2 = g_s^2$ playing the rôle of $\hbar$

- This exactly the interpretation of the topological wave function Z_{top} for "usual" top. amplitudes on CY by Witten (Background independence ...) ^{9306122}, see Aganagic et al. ^{0607100}

$\Rightarrow$ this interpretation extends to the present cases (conformal $\mathcal{N}=2$ with $\varepsilon_1+\varepsilon_2 \neq 0$)

Saddle-point evaluation of the S-dual prepotential

- Set now $F = F_0 + \hbar F_1 + \hbar^2 F_2 + \dots$ (Recall: $\hbar = \epsilon_1 \epsilon_2 = g_s^2$)

- The S-duality is thus described by

$$e^{-\frac{\tilde F(\tilde a)}{\hbar}} = \sqrt{\frac{i\tau_0}{\hbar}} \int dx \; e^{\frac{2\pi i \tilde a x - F_0}{\hbar} + F_1 + \hbar F_2 + \dots}$$

- For $\hbar \to 0$ the integral is dominated by the classical saddle point $x = a_0$ obtained extremizing the first term in the exponent, i.e. such that

$$2\pi i \tilde a = \partial F_0(a_0)$$

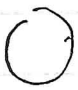
[N.B. By ∂F we mean $\frac{\partial F}{\partial a}$]

- With standard techniques one can compute the expansion around this saddle point, obtaining

$$\tilde F(\tilde a) = F(a_0) + \hbar W_1 + \dots - \partial F_0(a_0)\, a_0$$

with

$$W_1 = \frac{1}{2} \log \frac{\partial^2 F_0}{2\pi i \tau_0}$$



$\frac{1}{\partial^2 F_0}$ ~ propagator

$$W_2 = \frac{1}{2} \frac{\partial^2 F_1}{\partial^2 F_0} + \frac{1}{8} \frac{\partial^4 F_0}{(\partial^2 F_0)^2} + \dots$$

multiple derivatives ~ vertices

- At the lowest order,

$$\tilde{F}(\tilde{a}) = F_0(a_0) - \partial_0 F_0(a_0) \cdot a_0 = F_0(a_0) - 2\pi i \tilde{a} a$$

i.e. $\tilde{F}(\tilde{a})$ is obtained by a Legendre transform, as in the SW case.
This is spoiled by $\hbar$-corrections

S-duality as a Legedre transform also for $\hbar \neq 0$ (i.e $g_s \neq 0$)

- In 1307.6648 we propose that, by redefining the prepotential, ~~one~~ one can implement S-duality as a Legendre transform also for $\hbar \neq 0$.
That is we show that one can introduce

$$\hat{F}(a) = F + \hbar \Delta_1 + \hbar^2 \Delta_2 + \ldots = F_0 + \hbar (F_1 + \Delta_1) + \ldots$$

in such a way that, at any order in $\hbar$,

$$S[\hat{F}](\tilde{a}) = \hat{F}(a) - 2\pi i \tilde{a} a \qquad \text{with } a \text{ such that } 2\pi i \tilde{a} = \partial \hat{F}(a)$$

- Note that the saddle point a gets shifted w.r.t. the value a_0 we had before:

$$a = a_0 + \hbar \delta a_1 + \hbar^2 \delta a_2 + \ldots$$

- For instance, at order $O(\hbar)$, we start from

$$\begin{cases} S[F](\tilde{a}) = F(a_0) + \hbar W_1(a_0) - \partial F_0(a_0) \cdot a_0 & (\boxtimes) \\ W_1 = \frac{1}{2} \log \frac{\partial^2 F_0}{2\pi i \tau_0} \end{cases}$$

~~and~~ while, ~~inserting~~ from the redefinitions of F, we have

$$S[\hat{F}](\tilde{a}) = S[F + \hbar \Delta_1](\tilde{a}) = S[F](\tilde{a}) + \hbar S[\Delta_1](\tilde{a})$$

so that inserting [illegible] and developing the r.h.s. around a instead of around a_0, to order $\hbar$, after a straightforward expansion we end up with

$$S[\hat{F}](\tilde{a}) = \hat{F}(a) - \partial\hat{F}(a)\cdot a + \hbar\left(W_1 - \Delta_1 + S[\Delta_1]\right)$$

$$+\hbar\left(\partial^2 F_0\cdot\delta a_1 + \partial F_1 + \partial\Delta_1\right)\cdot a$$

. This has the form of a Legendre transform iff the last two brackets in the r.h.s. vanish:

$$\begin{cases} W_1 - \Delta_1 + S[\Delta_1] = 0 & \rightsquigarrow \text{find a solution of this to determine } \Delta_1 \\ \partial^2 F_0\cdot\delta a_1 + \partial F_1 + \partial\Delta_1 = 0 & \to \text{then this fixes } \delta a_1 \end{cases}$$

. The solution of the first request is obtained by setting Δ_1 to be just <u>one-half</u> of the quantum correction:

$$\Delta_1 = \frac{1}{2}W_1 = \frac{1}{4}\log\frac{\partial^2 F_0}{2\pi i\tau_0}$$

. Indeed it is possible to show starting from the expression of $S(F_0)$ as a Legendre transform, and from $S(\tau_0) = -1/\tau_0$, that

$$S(\partial^2 F_0) = -\frac{(2\pi i)^2}{\partial^2 F_0} \qquad \text{which then implies} \qquad S[W_1] = -W_1$$

Thus we have

$$\Delta_1 - S[\Delta_1] = \frac{1}{2}W_1 - \frac{1}{2}S[W_1] = \frac{1}{2}W_1 + \frac{1}{2}W_1 = W_1$$

. This procedure can be carried out at all subsequent orders in $\hbar$.

- We think that this is an intriguing ~~possibility~~ result, even if it is not clear to us what the physical meaning of the modified deformed prepotential $\hat{F}$ might be.

- The pattern we've followed to define $\hat{F}$ is unusual, if compared to the procedure one usually employs to define the quantum effective action in QFT

- The analogy is clear:

$$e^{-\frac{\tilde{F}(\tilde{a})}{\cancel{\epsilon_1\epsilon_2}\hbar}} = \sqrt{\frac{i\tau_0}{\hbar}} \int dx \; e^{\frac{2\pi i \tilde{a} x - F(x)}{\hbar}}$$

$$\updownarrow$$

$$e^{-\frac{W[j]}{\hbar}} \sim \int D\Phi \; e^{-\frac{S[\Phi] + j\cdot\Phi}{\hbar}}$$

$x \Leftrightarrow \Phi$

$F(x) \longleftrightarrow S[\Phi]$ tree level action

$2\pi i \tilde{a} \Leftrightarrow j$ current

$\tilde{F}(\tilde{a}) \longleftrightarrow W[j]$

generating functional of connected diagrams

- Usually one introduces an effective action $\Gamma[\Phi]$ related to $W[j]$ by a Legendre transform. Seen the other way round $W[j]$ is given by the inverse Legendre transform of a different quantity $\Gamma[\Phi]$, which incorporates all quantum corrections.

- The analogue of this in our situation would be to determine a new quantity $F_{eff}(a)$ such that

$$\tilde{F}(a) \equiv S[F](a) = F_{eff}(a) - 2\pi i \tilde{a} a \qquad \text{with} \qquad 2\pi i \tilde{a} = \frac{\partial F_{eff}}{\partial a}$$

this is a L.T.

but this is the S-dual of a different object!

- By distributing "half" the quantum corrections into $\hat{F}$ and half in its S-dual, we ~~can~~ could instead reach the situation in which

$$S[\hat{F}](\tilde{a}) = \hat{F}(a) - 2\pi i \tilde{a} a$$

- Some reasons why such manipulations might not be so crazy as they seem:

  - We carried out the perturbative expansion with the "action" $F(a)$: ~~[illegible]~~

$$F(a) = F_0(a) + \hbar F_1 + \hbar^2 F_2 + \dots$$

interpreting:

$$\underbrace{F_0(a)}_{\text{free action}} \qquad \underbrace{\hbar F_1 + \hbar^2 F_2}_{\text{interaction terms}}$$

But note that the couplings in front of interactions are not independent quantities, they are $\hbar$-powers. It's like quantizing an action which already contains quantum effects

  - the choice of F_0 as the free action is somewhat ambiguous, and indeed one can reorganize the expansion around a 'shifted' saddle point and retrieve in the end the same results

  - Attributing some quantum corrections also to F and not all of them to the "effective action" is not inconceivable since F already contains quantum terms from the start