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Small-scale statistics of viscoelastic turbulence

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Abstract. – The small-scale statistics of homogeneous isotropic turbulence of dilute polymer solutions is investigated by means of direct numerical simulations of a simplified viscoelastic fluid model. It is found that polymers only partially suppress the turbulent cascade below the Lumley scale, leaving a remnant energy flux even for large elasticity. As a consequence, fluid acceleration in viscoelastic flows is reduced with respect to Newtonian turbulence, whereas its rescaled probability density is left unchanged. At large scales the velocity field is found to be unaffected by the presence of polymers.

The addition of small amounts of long-chain polymers produces dramatic effects on flowing fluids, the most renowned being the reduction of friction drag in turbulent flow at high Reynolds numbers [1, 2]. Most studies focused on dilute polymer solutions in channel or pipe geometry, where boundary effects are important [3], but recent experimental [4–6] and numerical [7–9] works have shown that polymers affect the turbulent flow even far from (or in the absence of) boundaries. In particular, ref. [8] studied the modification of the turbulent cascade induced by polymers in numerical simulations of homogeneous, isotropic turbulence. In this paper we investigate the effects of polymer addition to the small-scale statistics in fully developed homogeneous-isotropic turbulence by means of direct numerical simulations of a simplified viscoelastic model. We show that, by increasing the elasticity of polymers, the energy flux in the turbulent cascade is partially suppressed and transferred to the elastic degrees of freedom. This suppression remains partial even for large values of elasticity: as a consequence the energy flux to small scales remains finite and the small-scale statistics, such as acceleration probability density function (pdf), retain some characteristics of Newtonian flows.

The mechanism by which dilute polymer solutions can influence turbulent flows is the extreme extensibility of polymers. Polymers, typically composed by a large number of monomers, at equilibrium are coiled in a ball of radius R_0. In the presence of a nonhomogeneous flow, the molecule is deformed in an elongated structure characterized by its end-to-end distance R which can be significantly larger than R_0. The deformation of molecules is the result of the

competition between the stretching induced by differences of velocities and the entropic relaxation of polymers to their equilibrium configuration. Experiments with DNA molecules [10] show that this relaxation is linear, provided that the elongation is small compared with the maximal extension $R \ll R_{max}$, and can be characterized by a typical relaxation time τ [11].

These ingredients lead to the simplest model which describes the behavior of a polymer in a flow, the dumbbell model. Since in applications the typical size of polymers is smaller than the viscous scale of turbulence, stretching is due to velocity gradients and the end-to-end distance evolves according to

$$\frac{\mathrm{d}\boldsymbol{R}}{\mathrm{d}t} = (\boldsymbol{\nabla u})^T \boldsymbol{R} - \frac{1}{\tau}\boldsymbol{R} + \sqrt{\frac{2R_0^2}{\tau}}\boldsymbol{\xi}, \tag{1}$$

where $\boldsymbol{\xi}$ is a Brownian process with correlation $\langle \xi_i(t)\xi_j(t')\rangle = \delta_{ij}\delta(t-t')$.

The relative importance between polymer relaxation and stretching is measured by the Weissenberg number Wi, defined as the product of τ and the characteristic velocity gradient. When $Wi \ll 1$, relaxation is fast compared to the stretching time and polymers remain in the coiled state. For $Wi \gg 1$, on the contrary, polymers are substantially elongated. The transition point is called the coil-stretch transition and occurs at $Wi = \mathrm{O}(1)$.

In the case of dilute solutions, for which the polymer concentration n satisfies $nR_0^3 \ll 1$, the influence of polymers in the coiled state on the fluid is negligible. Above the coil-stretch transition, polymers start to affect the flow. This regime is characterized by large elongations $R \gg R_0$, which allow to disregard the thermal noise in (1). Polymer solutions at macroscopic scales, *i.e.* at scales much larger than typical interpolymer distances, can be described by a local elongation field $\boldsymbol{R}(\boldsymbol{x},t)$ which evolves according to

$$\frac{\partial \boldsymbol{R}}{\partial t} + \boldsymbol{u}\cdot\boldsymbol{\nabla R} = (\boldsymbol{\nabla u})^T \cdot \boldsymbol{R} - \frac{\boldsymbol{R}}{\tau}. \tag{2}$$

Taking the divergence of (2) one easily sees that $\boldsymbol{\nabla}\cdot\boldsymbol{R}$ decays in time and thus we can take $\boldsymbol{R}$ solenoidal.

The effect of polymers on the fluid is in the modification of the stress tensor through an additional elastic component $\boldsymbol{\Pi}^P$ which takes into account the elastic forces of polymers $\boldsymbol{\Pi}^P_{ij} = (2\nu\eta/\tau)(R_iR_j/R_0^2)$, where ν is the solvent viscosity and η (proportional to polymer concentration) represents the zero-shear contribution of polymers to the total solution viscosity $\nu(1+\eta)$. The Navier-Stokes equation for the incompressible velocity field $\boldsymbol{u}(\boldsymbol{x},t)$ thus becomes

$$\frac{\partial \boldsymbol{u}}{\partial t} + \boldsymbol{u}\cdot\boldsymbol{\nabla u} = -\boldsymbol{\nabla}p + \nu\triangle\boldsymbol{u} + \frac{2\nu\eta}{\tau}\,\frac{\boldsymbol{R}\cdot\boldsymbol{\nabla R}}{R_0^2}. \tag{3}$$

Equations (2) and (3) are the so-called uniaxial model for viscoelastic flows which can be also obtained starting from the linear Oldroyd-B model [12] by taking the limit of large elongations [13, 14]. Observe that by introducing the rescaled variable $\boldsymbol{B} = \sqrt{2\nu\eta/\tau}(\boldsymbol{R}/R_0)$, eqs. (2)-(3) formally become the MHD equations for a plasma at zero resistivity with a linear damping $-\boldsymbol{B}/\tau$ [15, 16]. In this representation the coefficient η disappears and thus the dynamics of the uniaxial model is independent of the concentration (which is physically consistent with the assumption of linearity and strong elongation). In the following we will thus take $\eta = 1$ and $\boldsymbol{R}$ will be made dimensionless by rescaling with R_0.

The total energy (kinetic plus elastic) $E_T = (\langle u^2\rangle + \langle B^2\rangle)/2$ of the flow is dissipated at a rate

$$\frac{\mathrm{d}E_T}{\mathrm{d}t} = -\nu\langle|\boldsymbol{\nabla}\times\boldsymbol{u}|^2\rangle - \frac{1}{\tau}\langle B^2\rangle = -\varepsilon_\nu - \varepsilon_\tau, \tag{4}$$

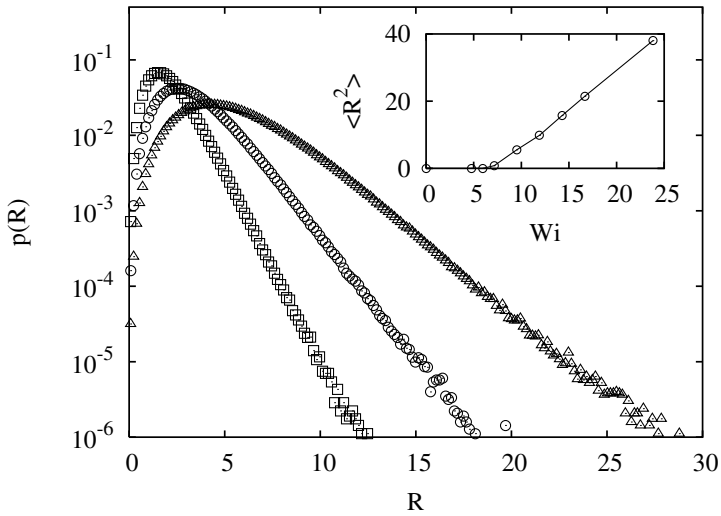

Fig. 1 – Probability density functions of polymer dimensionless elongations above the coil-stretch transition at $Wi = 9.5$ (squares), $Wi = 12$ (circles) and $Wi = 24$ (triangles). For $Wi < Wi^* = 6.5$ the distribution is $p(R) = \delta(R)$. In the inset we show the growth of $\langle R^2 \rangle$ *vs.* Wi.

where ε_ν is the viscous dissipation while the second term ε_τ represents the additional dissipation due to the relaxation of polymers to their equilibrium configuration.

In the following we will consider the statistics of stationary turbulent solutions of the viscoelastic model (2)-(3) integrated in a periodic box of size $L = 2\pi$ at resolution 128^3 by means of a standard pseudo-spectral code for different values of the relaxation time τ. For each τ, a statistically stationary state is obtained by adding to (3) an external forcing term which acts on the largest scales by keeping their energy constant [17]. In stationary conditions the forcing injects energy with a mean rate ε_I which balances the dissipation (4), $\varepsilon_I = \varepsilon_\nu + \varepsilon_\tau$. The turbulent regime for a viscoelastic flow is controlled by two dimensionless parameters, the Reynolds number $R_\lambda = u_{rms}\lambda/\nu$ (where $\lambda = u_{rms}/\langle(\partial_x u_x)^2\rangle^{1/2}$ is the Taylor microscale) and the Weissenberg number which is defined here as $Wi = \tau/\tau_K$, where $\tau_K = (\nu/\varepsilon_\nu)^{1/2}$ is the Kolmogorov time. As a reference run, we integrated the standard Navier-Stokes equations (3) with $\eta = 0$, for which we have $R_\lambda \simeq 87$. In this Newtonian limit, we have also computed the Lagrangian Lyapunov exponent λ_L which is a measure of the mean stretching rate. The dimensionless number $\lambda_L \tau_\eta \simeq 0.13$ is consistent with known simulations [18]. The viscoelastic runs are performed for different values of the relaxation times corresponding to a Weissenberg number in the range $4.8 \leq Wi \leq 24$ (*i.e.* $0.63 \leq \lambda_L \tau \leq 3.14$).

Although the uniaxial model is derived in the limit of strong elongation, it displays a clear coil-stretch transition as a function of Wi [19]. At small Wi the model (2) is characterized by a coiled state with $\boldsymbol{R} = 0$ and the pdf of elongation $p(R) = \delta(R)$. This state persists until the mean stretching rate is comparable with the inverse relaxation time, $\lambda_L \tau \sim 1$ [20, 21] where polymers start to be elongated. Figure 1 shows the pdf of elongations $p(R)$ at different values of Wi above the coil-stretch transition together with the mean-square elongation from which the transition at $Wi^* = 6.5 \pm 0.5$ (corresponding to $\lambda_L \tau = 0.86$) is evident. We remark that because in the linear polymer model (2) there is no maximal elongation [22], the stationarity of the distribution $p(R)$ (and its exponential tail) is entirely due to the feedback on the velocity field in (3).

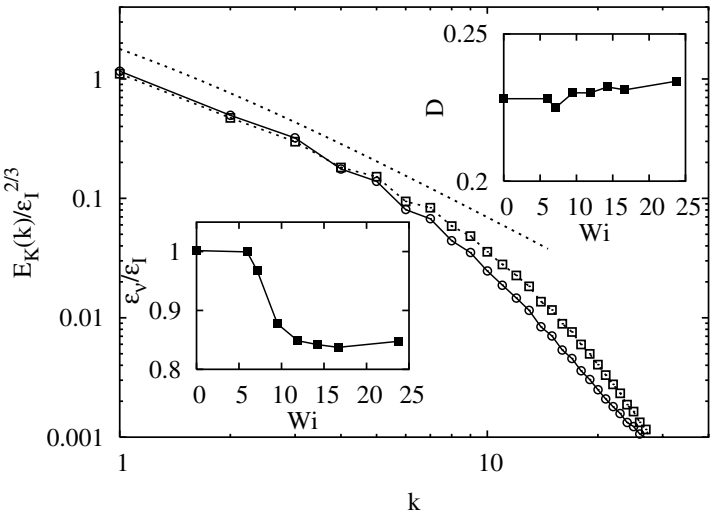

Fig. 2 – Newtonian (squares) and viscoelastic ($Wi = 14.3$, circles) spectra of kinetic energy, normalized with $\varepsilon_I^{2/3}$. The dotted line corresponds to Kolmogorov K41 scaling $E_K(k) \sim k^{-5/3}$. Lower inset: viscous dissipation ε_ν normalized to the energy input ε_I as a function of Wi; the point size is of the order of the statistical uncertainty. Upper inset: drag coefficient $D = \varepsilon_I L/E_K^{3/2}$ *vs.* Wi.

According to the Lumley criterion [19], as Wi increases above Wi^*, polymers start to affect the dynamics of the turbulent cascade at the scale ℓ_L at which the eddy turnover time is of the same order of the relaxation time, *i.e.* $\ell_L \sim (\varepsilon\tau^3)^{1/2}$. For scales $\ell > \ell_L$ we expect that the turbulent cascade is unaffected by polymers, *i.e.* with constant energy flux equal to the energy input ε_I, while for $\ell < \ell_L$ we expect a reduced energy flux $\varepsilon = \varepsilon_\nu < \varepsilon_I$ because below the Lumley scale part of the flux is removed by elastic dissipation.

By increasing Wi above Wi^* two different scenarios are possible: the first is that elastic dissipation in (4) increases with Wi and energy flux at scales $\ell < \ell_L$ vanishes (*i.e.* the Lumley scale ℓ_L becomes the new dissipative scale). A second alternative is that elastic dissipation removes only a fixed fraction of the flux. In this case ε_ν becomes independent of Wi and thus the energy cascade proceeds below ℓ_L (with a reduced flux). This latter scenario has been observed in shell models of viscoelastic fluids [23].

Our numerical simulations at increasing values of Wi indicate that the second scenario occurs. The inset of fig. 2 shows that the ratio $\varepsilon_\nu/\varepsilon_I$, which is by definition unity for $Wi \leq Wi^*$, decreases for $Wi > Wi^*$ but already at $Wi \simeq 15 \simeq 2Wi^*$ saturates to a new value ~ 0.85. These results are in agreement with those reported in [5], where the reduction of vorticity in a viscoelastic solution was experimentally measured, and the Taylor microscale λ was observed to be practically unaffected by the presence of polymers.

The above picture is supported by the comparison of the kinetic energy spectrum of a viscoelastic flow above the coil-stretch transition and the spectrum of the reference Newtonian flow. Figure 2 shows that, although the energy content at small scales is reduced by polymers, a power law spectrum, characteristic of a turbulent cascade *à la* Kolmogorov, is present in the viscoelastic case as well. Moreover the effect of polymers on turbulence is *local* in scales, *i.e.* scales larger than ℓ_L are essentially not affected by the presence of polymers. Since the total kinetic energy E_K is dominated by large scales, we do not observe a significant variation of the "drag" coefficient, here defined as $D = \varepsilon_I L/E_K^{3/2}$ (see the inset in fig. 2 where it is shown

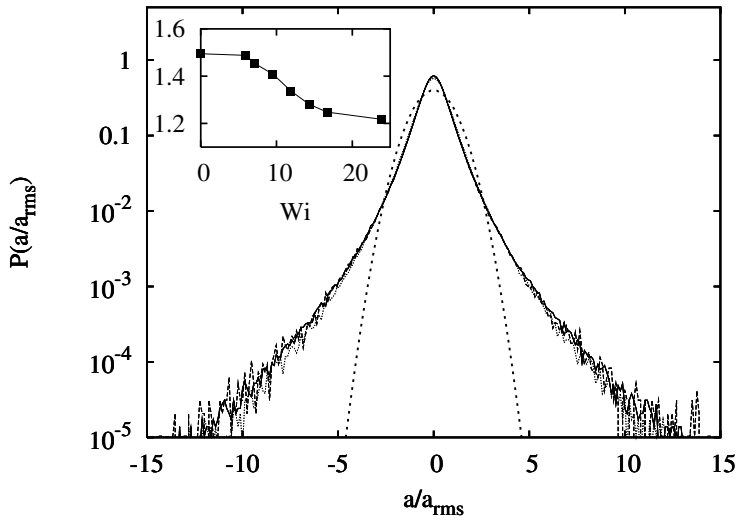

Fig. 3 – Acceleration probability distributions for $Wi = 0, 12, 24$ (continuous, dashed and dotted lines, respectively); the inner dashed curve shows a Gaussian for comparison. Inset: total a_{rms}, normalized with $\varepsilon_I^{3/4}\nu^{-1/4}$, *vs.* Wi.

that from $Wi = 0$ to $Wi = 24$ we measure fluctuations of the drag of about 2% which are within the statistical uncertainty). This is at variance with the results reported in [8], where, however, it must be noticed that a different viscoelastic model was used, with a different large-scale forcing mechanism.

Since the turbulent cascade survives at scales smaller than ℓ_L, we expect to observe features typical of Newtonian turbulence in viscoelastic flows. One of the usual characteristics of small-scale turbulence is the extremely intermittent acceleration which displays fluctuations much larger than the rms value [24, 25]. In fig. 3 we report the statistics for the acceleration in viscoelastic turbulent flows at different values of elasticity above Wi^* compared with the Newtonian case. In the presence of polymers, the rms value a_{rms} is reduced with respect to the Newtonian case, again in agreement with experimental observations [6]. For sufficiently large values of Wi, a_{rms} becomes almost independent of Wi (and, at the largest value $Wi = 24$, is about 80% of the Newtonian case). This is consistent with a reduced value of energy flux at small scales shown in fig. 2: indeed by compensating a_{rms} with the dimensional estimation $\varepsilon_\nu^{3/4}\nu^{-1/4}$ this becomes virtually independent of Wi.

The probability density functions shown in fig. 3 indicate that relative fluctuations of turbulent acceleration are not affected by the presence of polymers. Indeed, the pdf of the rescaled quantity a/a_{rms} is found to be Wi-independent in all the range of Wi investigated. This is a remarkable result which, besides its intrinsic interest, has an important practical consequence: the fact that the *shape* of the acceleration pdf is not affected by polymers suggests the possibility to model small-scale statistics in viscoelastic turbulence below the Lumley scale by means of the same models used for Newtonian fluids, by simply changing global quantities such as the energy flux.

The acceleration $\boldsymbol{a} = \mathrm{d}\boldsymbol{u}/\mathrm{d}t$ in a viscoelastic flow has three different contributions from the rhs of (3): pressure gradients, viscous and elastic contributions. In fully developed turbulence the viscous contribution is negligible and one may ask which is the dominant contribution for $Wi > Wi^*$. Figure 4 (inset) shows that the pressure gradient contribution $a_p = \langle(\boldsymbol{\nabla} p)^2\rangle^{1/2}$

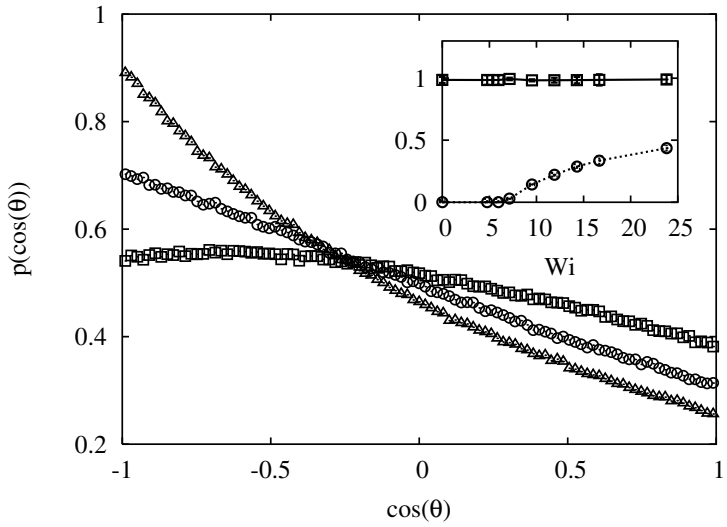

Fig. 4 – Probability density functions of the cosine of the angle θ between the vectors $-\boldsymbol{\nabla} p$ and $\boldsymbol{R}\cdot\boldsymbol{\nabla}\boldsymbol{R}$ for $Wi = 9.5$ (squares), $Wi = 14.3$ (circles) and $Wi = 24.0$ (triangles). Inset: leading contributions to the acceleration, normalized with the total rms acceleration, as a function of Wi: pressure gradient a_p/a_{rms} (squares) and elastic stress contribution a_{el}/a_{rms} (circles).

(which is the only term in the Newtonian limit $Wi = 0$) is always dominant in the range of Wi investigated. Nevertheless, the contribution of polymers is not negligible: at the largest $Wi = 24$ the rms value of the elastic acceleration $a_{el} = (2\nu/\tau)\langle(\boldsymbol{R}\cdot\boldsymbol{\nabla}\boldsymbol{R})^2\rangle^{1/2}$ is almost 50% of the total acceleration a_{rms}.

Since $\boldsymbol{a} \simeq -\boldsymbol{\nabla} p + (2\nu/\tau)\boldsymbol{R}\cdot\boldsymbol{\nabla}\boldsymbol{R}$, the fact that $a_p \simeq a_{rms}$ means that, increasing Wi, the flow develops strong correlations between the pressure gradient and the elastic component in (3). As a measure of this correlation we have computed the angle θ between the vectors $-\boldsymbol{\nabla} p$ and $\boldsymbol{R}\cdot\boldsymbol{\nabla}\boldsymbol{R}$. Figure 4 displays the pdf of $\cos(\theta)$ and shows that, indeed, increasing Wi the two vectors tend to be anticorrelated (*i.e.* $\cos(\theta) = -1$) with higher and higher probability.

In conclusion, we have studied the statistics of turbulent polymer solutions within the uniaxial model of viscoelastic flow. We have found numerical evidence for a coil-stretch transition at Wi^* above which polymers affect the small scales of the turbulent flow. For $Wi > Wi^*$, the energy flux is only partially removed by polymer elasticity at the Lumley scale and the turbulent cascade proceeds to smaller scales. As a consequence, small-scale statistics, such as acceleration, display features typical of Newtonian turbulence.

We remark that the above results have been obtained within the linear uniaxial model which is very attractive both analytically and numerically for its simplicity. Yet, it is worthy to remind that such model has some limitations, *e.g.* it does not allow to capture possible feedback effects in the coiled state. Therefore, it would be extremely interesting to compare our findings with the outcome of more realistic viscoelastic models.

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