Chapitre

1

Eulerian Turbulence

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The contribution of ANK to the foundation of a statistical theory of fully developed turbulence cannot be overestimated. His fundamental papers radically changed physicist's approach to the study of turbulence and are still source of inspiration for recent developments in the fields.
The intention of this paper is to give a up-to-date vision of our knowledge about fully developed turbulence with particular emphasis on the evolution of Kolmogorov pioneering ideas. We start from the discussion of what is now called "Kolmogorov 1941" theory of turbulence and his relation with the state of the art at that time. The second important contribution of Kolmogorov (the "refined similarity hypothesis" of 1962) is discussed in relation to the Landau objections to the original 1941 theory. The last part of the review is devoted to recent developments and gives a modern view of concepts, such as isotropy and universality of small scale fluctuations, already present in the original papers by ANK.

1. Richardson cascade

The equation of motion describing the spatio/temporal evolution of incompressible three dimensional flows are known since Navier (1923) :

$$\begin{aligned} &\partial_t \boldsymbol{v} + \boldsymbol{v} \cdot \boldsymbol{\partial v} = -\boldsymbol{\partial} P + \nu \Delta \boldsymbol{v} \\ &\boldsymbol{\partial} \cdot \boldsymbol{v} = 0 \end{aligned} \tag{1.1}$$

where with $\boldsymbol{v}(\boldsymbol{x},t), P(\boldsymbol{x},t), \nu$ we denote the local instantaneous velocity field, the pressure field and the molecular viscosity, respectively. Nevertheless, as noticed by Frisch [1] this is one of those interesting situation where the perfect knowledge of the deterministic equation of motion does not help very much in the understanding of the global and local behavior of the flow, either for short or long times. Very few rigorous results are known on the behavior of (1.1), especially in the *fully developed turbulent* regime, i.e. when the external forcing mechanism injects enough energy to produce a chaotic *unpredictable* flow evolution. Fully developed

turbulence is described by the Navier-Stokes equations (1.1) in the regime of high Reynolds numbers Re, i.e. in the limit when the ratio between the non-linear terms and the viscous, linear, contribution become large : $Re \sim \frac{v\partial v}{\nu\Delta u} \sim\to \infty$.
In this chapter we review the seminal contributions given by A. N. Kolmogorov on the phenomenological, theoretical and analytical understanding of turbulent flows starting from the pioneering works in the 1939-41 years [4, 5] and ending with the paper published in the occasion of the *Marseille Conference* in 1962 [2]. We also discuss the importance and fallout of A. N. Kolmogorov ideas on modern turbulence research, in both experiments and numerical simulations.
The basic brick of the entire work of Kolmogorov was the celebrated Richardson energy cascade as himself stated in the 1962 paper [2] :"[...] The hypothesis concerning the local structure of turbulence at high Reynolds number, developed in the years 1939-41 were based physically on the Richardson's idea of the existence in the turbulent flow of vortices on all possible scales $\eta < r < L$ between the external scale L and the internal scale η and of certain uniform mechanism of energy transfer the coarser-scaled vortices to the finer".
Richardson cascade is a pictorial and phenomenological way to explain the transfer of fluctuations from the largest scale in the system, L to the dissipative scale η where turbulent fluctuations are overwhelmed by the viscous term. Richardson proposed that the energy transfer from the pumping scale to the dissipative scales happens though a multi-step process, where eddies break up to smaller eddies, small-eddies break up to smaller and smaller eddies and so on to viscosity. The main assumption is the locality of interactions in Fourier space, i.e. there is not any important direct transfer of energy connecting very large scales with very small scales. Moreover, respecting the scale-invariance of NS equations, in the limit of vanishing viscosity, the mechanism is supposed to happen in a self-similar way, i.e. typical velocity, scales and times of eddies of size ℓ can be estimated on the basis of dimensional analysis without the introduction of any *outer* or *inner* length. Namely, by denoting with $\delta\boldsymbol{v}(\boldsymbol{x},\boldsymbol{\ell},t) = \boldsymbol{v}(\boldsymbol{x}+\boldsymbol{\ell}) - \boldsymbol{v}(\boldsymbol{x})$ the velocity increments over a scale ℓ, the typical local eddy-turn-over time, needed for a turbulent fluctuations of scale ℓ to change its energy content becomes : $t_\ell = \ell/\delta v(\ell)$. Richardson's scenario predicts that typical eddy-turn-over times of eddies of size ℓ become faster and faster by going to smaller and smaller scales : $t_{\ell'} < t_\ell$, if $\ell' < \ell$. In the cascade process, small scales velocity fluctuations may therefore expect to loose the memory of the large scale fluctuations. This is a qualitative explanation of the supposed –and experimentally observed– *universality* of small scales turbulent fluctuations. Where for *universality* we mean the independence of large-scales forcing mechanism used to maintain the flow. Universality of small scales fluctuations is also strictly linked with the so-called *return-to-isotropy*, i.e. to the assumption that small-scales turbulent statistics is dominated by the isotropic component. Both self-similar behavior and small-scales universality may hold only far enough from the boundaries, where, on the other hand, non-homogeneous effects and the presence of solid walls may strongly perturb the flow at all scales.
In the years 1939-41, A. N. Kolmogorov published a series of paper where the above phenomenological scenario reached a level of solidity such as, later on, people refer to them as *"the 1941 Kolmogorov theory"* or, briefly, *K41*. First, A. N. Kolmogorov listed the minimal set of hypotheses required to deduce in a systematic way the *self-similar* and *universal* character of small-scales fluctuations. Second, he determined quantitative information on the statistical properties of velocity fluctuations at small scales. Third, he derived the 4/5-law, the only known exact result describing the scaling properties of the energy flux at large Reynolds.
Another important contribution of A. N. Kolmogorov was to propose the most suitable set of correlation functions able to highlight small-scales fluctuations, the

so-called structure functions. A generic velocity structure function of order p, at scale ℓ, is a p-rank tensor defined in terms of the correlation function between p simultaneous velocity difference components, $\delta\boldsymbol{v}(\boldsymbol{x}, \boldsymbol{\ell}, t)$, at scales $\boldsymbol{\ell}$:

$$T^{\{\alpha\}}(\boldsymbol{\ell}) = \langle \delta v_{\alpha_1}(\boldsymbol{\ell}) \ldots \delta v_{\alpha_p}(\boldsymbol{\ell}) \rangle; \quad \textbf{(1.2)}$$

where with the shorthand notation, $\{\alpha\}$, we denote the set of p vectorial indexes, $\alpha_1, \ldots, \alpha_p$. In (1.2) we have dropped both dependencies on time, t, and on the spatial location, $\boldsymbol{x}$, because we assume to be in a stationary and homogeneous statistical ensemble. An simpler set of observable is made of *longitudinal structure functions*, i.e. the projections of the p-th rank tensor (1.2) on the unit separation vector, $\hat{\boldsymbol{\ell}}$:

$$S^{(p)}(\boldsymbol{\ell}) = \langle [\delta\boldsymbol{v}(\boldsymbol{\ell}) \cdot \hat{\boldsymbol{\ell}}]^p \rangle \quad \textbf{(1.3)}$$

which, in isotropic ensemble, are scalars functions which depend only on the modulus of the separation, $\ell = |\boldsymbol{\ell}|$.

This contribution is organized as follow. In Section 2 and subsection therein, we review the main bricks of the 1941 theory, focusing in particular on (i) the ideas that turbulent flows are characterized by a statistical recovery of symmetries at scales small-enough ; (ii) on the experimental evidences that there exist a *dissipative anomaly*, i.e. the energy dissipation tends to a non-vanishing constant in the limit of high Reynolds numbers ; (iii) on the importance of the 4/5 law and, finally, on the universality of small scales statistics. In Section 3 we review the important step forward made by A. N. Kolmogorov in 1962, thanks to the criticisms made by Landau and Obhukov on the 1941 theory. There, A. N. Kolmogorov further elaborated his 1941 theory to include also the possibility to describe *intermittency* and *anomalous scaling* of small scales statistics. In Section 4 we discuss the modern fallout of Kolmogorov contributions, including recent issues raised by experimental and numerical evidences in –apparent– contrast with some of the hypotheses at the basis of 1941 and 1962 theories.

2. Kolmogorov 1941 theory

Fully developed turbulence is a spatio-temporal disorder motion involving a wide range of scales. Large scales, of the order of the flow domain, are the most important from an engineering point of view as they contain most of the energy and dominate the transport of momentum, mass and heat. Small turbulent scales, including the so-called inertial scales and dissipative scales, are more interesting for a fundamental approach as they display properties which are universal with respect to the flow configuration and/or forcing mechanism. The fundamental contribution of A. N. Kolmogorov to the theory of turbulence is based on a statistical description of the fluctuations of these small scales which leads to very simple and universal predictions. His first work is contained in three short papers appeared in URSS in 1941 which constitute what is now called the K41 theory.

2.1. Symmetries for Navier–Stokes

At the basis of the concept of universality there is the idea that small scale turbulence, at sufficiently high Reynolds numbers, is statistically independent on the large scales and can thus locally recover homogeneity and isotropy. The concept of homogeneous and isotropic turbulence was already introduced by Taylor [6] for describing grid generated turbulence. The important step made by A. N. Kolmogorov in 1941 was to postulate that small scales are statistically isotropic, no matter how turbulence is generated. This hypothesis is based on intrinsic properties of the dynamics, i.e. the invariance of Navier-Stokes equations (1.1) under the following set of symmetries [1] :

- space translations : $\boldsymbol{x} \to \boldsymbol{x} + \boldsymbol{r}$
- space rotations : $(\boldsymbol{x}, \boldsymbol{v}) \to (A\boldsymbol{x}, A\boldsymbol{v})$ with $A \in SO(3)$
- scaling : $(t, \boldsymbol{x}, \boldsymbol{v}) \to (\lambda^{1-h}t, \lambda\boldsymbol{x}, \lambda^h\boldsymbol{v})$ for any h and $\lambda > 0$

The first two symmetries are consequence of space homogeneity and isotropy and are broken in presence of boundaries and or forcing mechanism. Concerning scaling symmetry, a classical example is the so called similarity principle of fluid mechanics which states that two flows with the same geometry and the same Reynolds number are similar. The similarity principle is at the basis of laboratory modeling of engineering and geophysical flows.
The idea at the basis of Kolmogorov's treatment of small scale turbulence is the hypothesis that in the limit of high Reynolds numbers and far from boundaries the symmetries of Navier-Stokes equation are restored for statistical quantities. To be more precise, let us consider the velocity increment $\delta\boldsymbol{v}(\boldsymbol{x}, \boldsymbol{\ell})$ over the scales $\ell \ll L$. Restoring of homogeneity in statistical sense demands $\delta\boldsymbol{v}(\boldsymbol{x}+\boldsymbol{r}, \boldsymbol{\ell}) \stackrel{law}{=} \delta\boldsymbol{v}(\boldsymbol{x}, \boldsymbol{\ell})$, where equality in law means that the PDF of $\delta\boldsymbol{v}(\boldsymbol{x}+\boldsymbol{r}, \boldsymbol{\ell})$ and $\delta\boldsymbol{v}(\boldsymbol{x}, \boldsymbol{\ell})$ are identical. Similarly, statistical isotropy, also used by Kolmogorov in his 1941 papers, requires $\delta\boldsymbol{v}(A\boldsymbol{x}, A\boldsymbol{\ell}) \stackrel{law}{=} \delta A\boldsymbol{v}(\boldsymbol{x}, \boldsymbol{\ell})$ where A is a rotation matrix. The issue of restoring of small scale isotropy will be discussed in details in Section 4.
In the limit of large Reynolds number, the second Kolmogorov similarity hypothesis [4] states that for separation in the inertial range of scales $\eta \ll \ell \ll L$ the PDF of $\delta\boldsymbol{v}(\boldsymbol{x}, \boldsymbol{\ell})$ becomes independent on viscosity ν. As a consequence in this limit and in this range of scales, scaling invariance is statistically recovered with a larger freedom in the values of the scaling exponent :

$$\delta\boldsymbol{v}(\boldsymbol{x}, \lambda\boldsymbol{\ell}) \stackrel{law}{=} \lambda^h \delta\boldsymbol{v}(\boldsymbol{x}, \boldsymbol{\ell}). \tag{1.4}$$

The values of the scaling exponent, h are now limited only by the requiring that the velocity fluctuations do not break incompressibility, $h \geqslant 0$ [1]. In this Section we discuss the self-similar $K41$ theory, i.e. the limitation to the case when the turbulent flow possesses a global scaling invariance with unique scaling exponent $h = 1/3$. In Sections 3.3 we relax the requirement of global scaling invariance and we consider a possible extension of the theory to the case of local scaling invariance.

2.2. Dissipative anomaly

Because we are interested in the statistical properties of small scale turbulence far from boundaries, we consider in the following the simple geometry without boundary of a periodic box of size L. This geometry is very convenient also for numerical simulations which can profit of the Fourier decomposition of the velocity field. In the inviscid limit $\nu = 0$ Navier-Stokes equation (1.1) conserves the kinetic energy

$$E = \langle \frac{1}{2} v^2 \rangle \equiv \frac{1}{V} \int_V v^2(\boldsymbol{x}, t) d^3x \tag{1.5}$$

where the brackets denote the average over the periodic box of volume $V = L^3$. Indeed, using (1.1) in (1.5) one obtains :

$$\frac{dE}{dt} = -\langle \frac{1}{2} \sum_{ij} \nu(\partial_i v_j + \partial_j v_i)^2 \rangle \tag{1.6}$$

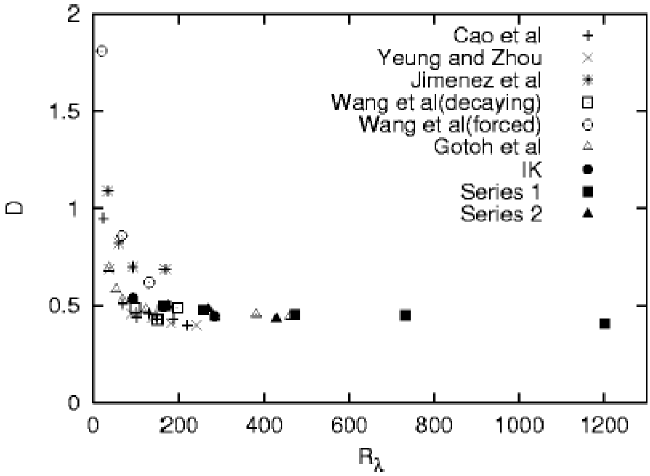

Figure 1.1. Dimensionless mean energy dissipation $D = \varepsilon L/U^3$ as a function of $R_\lambda = \sqrt{15 Re}$ from different direct numerical simulations (from [8]).

and thus the mean energy dissipation ε vanish if $\nu = 0$. Expression (1.6) introduces the most important experimental fact in the theory of fully developed turbulence : In the limit $Re \to \infty$ (which is equivalent to the limit $\nu \to 0$) the mean energy dissipation attains a finite limit ε [3, 7, 8]. This is actually true again far from boundaries, otherwise boundary layers give in general a Re dependence. Formally, we have

$$\lim_{\nu \to 0} \frac{dE}{dt} = -\varepsilon \neq \frac{dE}{dt}\|_{\nu=0} \tag{1.7}$$

which is the essence of the so-called *dissipative anomaly.* In Figure 1.1 we present a collection of numerical data on the Reynolds dependence of mean energy dissipation which supports the existence of a finite asymptotic for $Re \to \infty$.

The observation that kinetic energy is asymptotically dissipated at a constant rate has the important consequence that the flow develops small scales in order to compensate the $\nu \to 0$ limit in (1.6). This picture was already in the mind of Richardson about twenty years before Kolmogorov [9] when he developed a qualitative theory of turbulence based on the idea of a turbulent cascade in which the energy is transferred from the largest scales (where energy is injected by mechanical forcing) down to the smallest scales (where it is dissipated by molecular viscosity). The intermediate range of scales at which energy is simply transferred, the inertial range, can be expected to displays universal statistics, independent on the forcing mechanism and on the Reynolds number. Kolmogorov 1941 theory is the first attempt for a quantitative theory of inertial scale statistics.

2.3. The 4/5 law and self-similarity

In the first of his 1941 papers, A. N. Kolmogorov made the fundamental assumption that in the inertial range of scales the distribution function for $\delta \boldsymbol{v}(\boldsymbol{\ell})$ depends on $\boldsymbol{\ell}$ and ε only (and not on viscosity). As a consequence, dimensional analysis leads, assuming self-similarity, to the power-law behavior for the structure functions in the inertial range

$$S^{(p)}(\ell) = C_p \varepsilon^{p/3} \ell^{p/3} \tag{1.8}$$

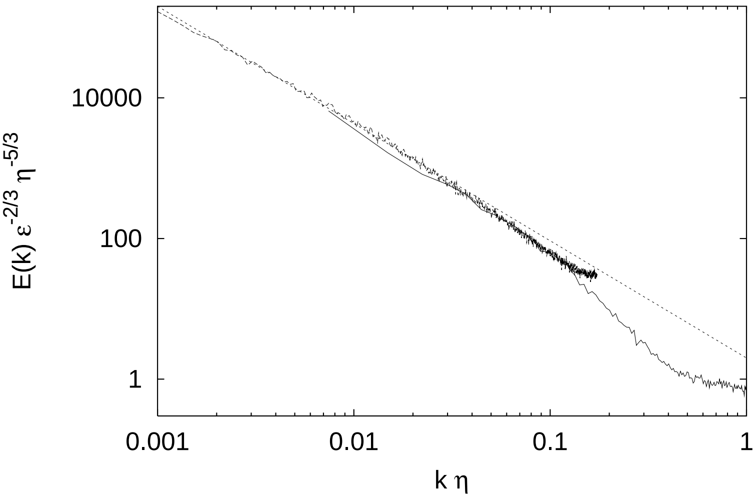

Figure 1.2. Velocity power spectra at $R_\lambda = 230$ (thin line) and $R_\lambda = 1840$ (tick line) normalized with Kolmogorov scale and velocity. Dashed line represent the Kolmogorov spectrum, $E(k) = Ck^{-5/3}$ with $C = 2$. From [11].

where the C_p are dimensionless, universal, constant. In particular he made the prediction for the second order structure function $S^{(2)}(\ell)$ or, which is the same [3], for the energy spectrum

$$E(k) = C\varepsilon^{2/3}k^{-5/3}. \tag{1.9}$$

Starting from the work of Grant et al. [10] in a tidal channel, there are now many experiments (both in field and in laboratory) and numerical simulations showing an energy spectrum following Kolmogorov prediction (1.9) with constant $C \simeq 2.0$ [3, 1]. We now know that spectrum (1.9) is not exact, as intermittency induces some deviations from the Kolmogorov exponent 5/3. Nevertheless, these effects are small and their existence have been definitively accepted only very recently (see the discussion in Section 4). For this reason, K41 theory is still important not only for historical reasons but also for many applications in modeling engineering and geophysical flows.
In Figure 1.2 we present an example of experimental energy spectrum obtained from a water jet experiment. Observe as the inertial range showing Kolmogorov scaling (1.9) increases with Reynolds number.

In his third 1941 paper, A. N. Kolmogorov is able to obtain, starting from Navier-Stokes equation, the exact expression for the third-order longitudinal structure function, called the "4/5 law" after the numerical prefactor :

$$S^{(3)}(\ell) = -\frac{4}{5}\varepsilon\ell. \tag{1.10}$$

This is still the only exact dynamical result on the behavior of velocity differences in the inertial range. The fact that the third moment of velocity differences does not vanish is a consequence of the directional transfer (from large to small scales) of energy on average. An important consequence, which will be discussed in details in connection with the intermittency issue, is that the PDF of velocity differences cannot be Gaussian. In the following we will give a dimensional argument for K41

predictions. The interested reader can find in [1] a complete derivation of the 4/5 law.
Let us consider the typical velocity fluctuation v_ℓ at scale ℓ, for example the rms of the velocity difference $\delta v(\ell)$. The eddy turnover time, i.e. the characteristic time of variation of v_ℓ, can be estimated dimensionally as $t_\ell \sim \ell/v_\ell$. This is also the time for the transfer of kinetic energy to smaller scales in the cascade. The flux of kinetic energy from scale ℓ to smaller scales is thus

$$\Pi_\ell \sim \frac{v_\ell^2}{t_\ell} \sim \frac{v_\ell^3}{\ell}. \tag{1.11}$$

Because in the inertial range the energy is neither injected nor dissipated, the energy flux (1.11) has to be independent on the scale ℓ and thus it must balance the mean energy dissipation ε. From (1.11) one obtains the prediction

$$v_\ell^3 \sim \varepsilon \ell, \tag{1.12}$$

which is the dimensional analogous of the four-fifths law (1.10). We observe that in this dimensional derivation (as in the complete derivation of (1.10) from Navier–Stokes equation) we have only made use of the hypothesis finite energy dissipation in the $Re \to \infty$ limit. Under the additional assumption of self similarity, the scaling exponent in (1.4) is forced to the value $h = 1/3$ by (1.12) and thus one obtains the K41 prediction for structure function exponents, $S^{(p)}(\ell) = C_p \ell^{\zeta(p)}$ with $\zeta(p) = p/3$ (1.8).
The inertial range has a viscous small scale cutoff at the Kolmogorov scale η. This scale can be dimensionally estimated as the scale at which the local Reynolds number $v_\ell \ell/\nu$ (which measures the relative importance of the inertial and viscous term in (1.1) is of order unit. Using Kolmogorov scaling one obtains

$$\eta \simeq L Re^{-3/4}. \tag{1.13}$$

Below the dissipative scale, the energy spectrum presents an exponential or more than exponential decay as a consequence of the smoothness of the velocity field. The extension of the inertial range, L/η thus increases as $Re^{3/4}$.
We conclude this Section by introducing the problem of intermittency. It is now accepted that K41 theory is not exact because higher order structure functions display unambiguous departure from the scaling exponents (1.8) :

$$S^{(p)}(\ell) = C_p (\frac{\ell}{L})^{\zeta(p)} \quad with \quad \zeta(p) \neq p/3. \tag{1.14}$$

In Figure 1.3 we report a famous collection of scaling exponents $\zeta(p)$ extracted from several experimental data [28] using the so-called ESS procedure [29]. Let us recall that the scaling exponents are not completely free as (1.11) requires $\zeta(3) = 1$. Under very general hypothesis, one can also demonstrate that ζ_p has to be a concave and nondecreasing function of p [1]. From Figure 1.3 it is evident that the $\zeta(p)$ exponents are firstly *universal* and secondarly *anomalous*, i.e. they are expressed by a non-linear function of p. This means that the pdf's of velocity differences $\delta \boldsymbol{v}(\ell)$ at different scales ℓ differ and the skewness of velocity differences increases without bound going to small scales (similarly, skewness based on longitudinal gradients increases as a function of Reynolds, instead of staying constant as predicted by the K41 theory).

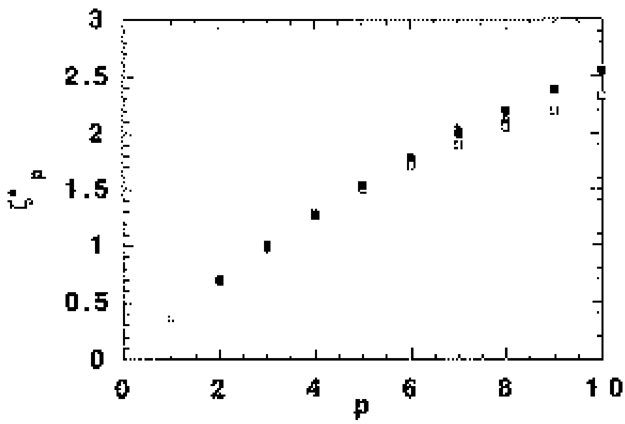

Figure 1.3. Velocity structure function exponents ζ_p for different experimental conditions. From Arneodo *et al.*, *Europhys. Lett.* **34**, 411 (1996).

3. Kolmogorov 1962

On September 1961, a Colloquium was organized by the CNRS in Marseille on Turbulence. Organizers asked A.N. Kolmogorov to give a talk on the status of the knowledge. He took this opportunity to stress on a contemporary contribution of Oboukhov, gave the new hypotheses which could justify it, and the consequences. Due to their importance, this contribution and that of Oboukhov were translated in English, and published by the Journal of Fluid Mechanics ("under the Editor responsibility") in two short papers [2]. They constitute the basis of the Kolmogorov-Oboukhov 62 (KO62) theory. At that time, there were not strong experimental motivations to call for an improvement of the 1941 theory. The main criticisms were based on a theoretical ground. The motivation invoked by Kolmogorov himself in his paper lays in a remark of L. Landau [2] : "But quite soon after [the K41 hypotheses] originated, Landau noticed that they did not take into account a circumstance which arises directly from the essentially accidental and random character of the mechanism of transfer of energy from the coarser vortices to the finer" Indeed, when looking at the various Landau's remarks, as reported for instance by Frisch [1], one can see that he emphasizes two points. First, that the constants, for instance the "Kolmogorov constant" C_2 in the K41 expression for the second order longitudinal structure function :

$$S^{(2)}(\ell) = C_2 \epsilon^{2/3} \ell^{2/3} \tag{1.15}$$

cannot be universal. As $\langle \epsilon^{2/3} \rangle$ differs from $\langle \epsilon \rangle^{2/3}$, it depends from the distribution of ϵ at large scales, close to the integral one, which cannot be universal. The first point is not even discussed by Kolmogorov. There will be no universality for constants. The second point is what emphasizes Kolmogorov, and it is directly connected to the issue of intermittency introduced in the previous section : vorticity, and thus probably dissipation too, is known through direct observations to be concentrated in tiny regions of the flow. This may lead to *anomalous* values for the scaling exponents, $\zeta(p)$, of velocity structure functions in the inertial range (1.14). To take into account this point, Kolmogorov emphasizes the role of the local dissipation. Namely, by denoting with

$$\epsilon_\ell(\mathbf{x}, t) = \frac{1}{4/3\pi\ell^3} \int_{|y|<\ell} d\boldsymbol{y}\, \epsilon(\boldsymbol{x} + \boldsymbol{y}, t) \tag{1.16}$$

the coarse grained energy dissipation on a ball of radius ℓ centered on $\mathbf{x}$, and with $\delta v(\ell)$ the velocity difference over a scale ℓ, Kolmogorov postulated that the adimensional ratio :

$$\frac{\delta v(\ell)}{\epsilon_\ell^{1/3}\ell^{1/3}} \tag{1.17}$$

has a probability distribution independent of the local Reynolds number $Re_\ell = \delta v(\ell)\ell/\nu$ in the limit $Re_\ell \to \infty$. This is the Refined Similarity Hypothesis (RSH). It links the scaling laws of velocity structure functions with the scaling properties of the energy dissipation :

$$S^{(p)}(\boldsymbol{\ell}) = \langle[\delta \boldsymbol{v}(\boldsymbol{\ell}) \cdot \hat{\boldsymbol{\ell}}]^p\rangle = C_p\langle\epsilon_\ell^{p/3}\rangle\ell^{p/3} \tag{1.18}$$

In a second group of hypotheses, Kolmogorov assumes that $\ln(\epsilon_\ell)$ has a Gaussian distribution, whose variances behaves as :

$$\sigma_\ell^2 = A + 9\mu\ln(L/\ell), \tag{1.19}$$

where experiments give $\mu \simeq 0.025$ [43] [44]. We shall first go to the consequences of these hypotheses, explicating what is intermittency and its physical interpretations. Then we shall go to the experimental verifications, and the discussions around the hypotheses.

3.1. Intermittency and anomalous scaling

From the log-Gaussian assumption it is easy to derive a parabolic shape for the behavior of the scaling exponents $\zeta(p)$. In particular, the behavior of the flatness factor $F(\ell)$, which characterizes the shape of the distribution of $\delta v(\ell)$ is given by :

$$F(\ell) = \frac{\langle\delta v^4(\ell)\rangle}{\langle\delta v^2(\ell)\rangle^2} \sim \ell^{\zeta(4)-2\zeta(2)} \sim \ell^{-4\mu}. \tag{1.20}$$

For a Gaussian distribution, $F(\ell) = 3$ independently of the scale. Larger values of $F(\ell)$ correspond to thicker wings compared to the center. Velocity differences much larger than their *rms* are more and more probable as ℓ becomes smaller and smaller. This evolution of the probability density function (PDF) of velocity differences is called *intermittency*. It is well observed [45], both along the scales in the inertial range ($\eta < \ell < L$) and *via* its consequences for the PDF of velocity gradients [46, 15]. Let us present now a simple phenomenological interpretations of this PDF evolution.

3.2. Multiplicative cascade

As suggested by Kolmogorov himself, the laws he proposed can follow from very simple arguments. Let us consider a set of reference scales $\ell_n = Lb^{-n}$ in the inertial range, with b, the inter-scale ratio, usually chosen equal to 2. The simplest relation one may define among the pdfs of velocity differences at scales ℓ_n and ℓ_{n+1} is given by a multiplicative convolution :

$$\delta v(\ell_{n+1}) = \alpha_{n+1,n}\delta v(\ell_n), \tag{1.21}$$

where with $\alpha_{n+1,n}$ we denote a suitable stochastic variable, whose logarithm $\phi = \ln(\alpha_{n+1,n})$ has the probability distribution $G_{n+1,n}[\phi]$, defining the statistical dependency between eddies of different size. Intermittent scaling laws, are then easily obtained through three assumptions :

- Scale invariance : $G_{n+1,n} = G_b$

– $\alpha_{n+1,n}$ and $\delta v(\ell_n)$ are uncorrelated

– $\alpha_{n+1,n}$ and $\alpha_{n+2,n+1}$ are uncorrelated.

Then one obtains for the scaling of longitudinal structure functions :

$$S^{(p)}(\ell) = C_p(\frac{\ell}{L})^{\zeta(p)} \quad \textbf{(1.22)}$$

with the scaling exponents uniquely fixed in terms of the basic distribution $G_b[\phi]$: $\zeta(p) = \log_b\langle\exp(p\phi)\rangle$. In (1.22) the constants C_p depend on the – non universal– velocity distribution at the integral scale, L. With the identification $\alpha_{n+1,n} = \epsilon_{\ell_{n+1}}^{1/3}\ell_{n+1}^{1/3}/\epsilon_{\ell_n}^{1/3}\ell_n^{1/3}$ the multiplicative model recovers the Refined Similarity Hypothesis (RSH) of Kolmogorov. The log-normal distribution proposed by Kolmogorov in the 1962 is only one of the possible choice for the basic distribution G_b. The above model ensures the growth of the variance of $\ln(\epsilon_\ell)$ in agreement with (1.19). The Central Limit Theorem is of no help as the evaluation of $\langle\epsilon_\ell^{p/3}\rangle$ involves parts of the distribution which are out of its range.
Notice that the presentation of the above multiplicative cascade model can be made free of the discretized inter-scale ratio b [45, 14]. Attempt has been made to characterize the multiplicative stochastic process in terms of *structures* in the flow [13]. Also, important developments have been achieved using deterministic dynamical models for the turbulent energy cascade (Shell Models), originating from the pioneering ideas of the Russian school [22, 1, 23, 24].
Through the Refined Similarity Hypothesis, the intermittency built by multiplicative models for velocity differences can be directly translated to the scaling properties of the coarse grained energy dissipation, $\epsilon(\ell)$. The Refined Kolmogorov Hypothesis can therefore be read as a statement about the multifractal properties of three dimensional measure defined by the local energy dissipation.

3.3. Multifractal

The above process yield to the concentration of the dissipation on small spots which well corresponds to the Landau remark [1]. This in turn suggests a geometrical, rather than dynamical, interpretation of intermittency : intense spots surrounded by calm regions. This has been generalized and formalized by Parisi and Frisch [48], in the multifractal picture of turbulence. The starting point is the invariance of Navier-Stokes equations under scaling transformation (1.4). If no characteristic scale appear in the inertial range, there should exist locally self-similar solutions where velocity differences scale with ℓ^h. In general, different exponents h are possible, but, again due to scale invariance, the subset of points where the exponent is h should be self similar, with fractal dimension $D(h)$.
Covering the flow, of size L, with balls of size ℓ gives $(L/\ell)^3$ balls. Those which contain a point where the exponent is h are of number $(L/\ell)^{D(h)}$. Thus the probability to find an exponent h at size ℓ is :

$$P(h) = (L/\ell)^{D(h)-3} \quad \textbf{(1.23)}$$

In the neighborhood of such a point, velocity differences at scale ℓ should be of order $\delta v(L)(\ell/L)^h$. Thus an estimate of $\langle\delta v^p(\ell)\rangle$ is given by :

$$\langle\delta v^p(\ell)\rangle \simeq \int \delta v^p(L)(\ell/L)^{ph}P(h)dh \simeq \delta v^p(L)\int(\ell/L)^{3+ph-D(h)}dh \quad \textbf{(1.24)}$$

Such an integral, in the limit $\ell/L \to 0$ can be estimated by the saddle point :

$$\langle \delta v^p(\ell) \rangle \sim (\ell/L)^{\zeta(p)} \text{ where } \zeta(p) = min_h(3 + ph - D(h)) \qquad \textbf{(1.25)}$$

The fractal dimension of singularities $D(h)$ is not completely free as (1.25) must be consistent with the 4/5 law (1.10), i.e. $\zeta(3) = 1$. This imposes in general the inequality $D(h) \leqslant 3h + 2$ and that there exist a value h for which the equality holds.

In spite of apparent differences, the present formalism is close to the previous one. In particular, it can be verified that the log-Normal hypothesis corresponds to a parabolic shape for $D(h)$:

$$D(h) = 3 - \frac{(h - h_o)^2}{2\mu} \qquad \textbf{(1.26)}$$

The original K41 theory is recovered assuming perfect selfsimilarity, i.e. a single scaling exponent h_0 on the whole domains, i.e. $D(h_0) = 3$. The above inequality on $D(h)$ then imposes $h_0 = 1/3$.

3.4. Tests of Kolmogorov Hypotheses

A large body of work in the past decades have been devoted to test the various hypotheses implicitely or explicitely proposed by Kolmogorov.

3.4.1. The Refined Similarity Hypothesis

An experimental validation of the RSH ask for large conditional statistics which were possible only recently. Even if attempts have been proposed earlier, a systematic study has been made in [49], where a one-dimensional record of turbulent velocity measured through a hot wire technique were used. By using Taylor Hypothesis, the estimation of the probability to have a coarse grained energy fluctuation conditioned to a given velocity increment, $P[\epsilon(\ell)|\delta v(\ell)]$ gave a Gaussian distributions at large ℓ and a bimodal distributions at small ℓ. This result was corrected by Gagne *et al.* [50], where a Gaussian statistics at all scales with a variance depending both on the scale and $\epsilon(\ell)$ was obtained, in agreement to Kolmogorov law. But for the test having a physical meaning, it should not give positive result with any random signal. Indeed, applying the same procedure to a temperature signal instead of velocity gave the same Gaussian conditionnal statistics, while it should not [51]. The uncontestable success of this test could then be due to a statistical effect and to the long range correlation of dissipation (see later).

3.4.2. The log-normal distribution

The log-normal hypothesis for ϵ_ℓ has been soon attacked on a general basis [52]. The quantity which plays the role of $\epsilon(\ell)$ is not easy to measure. But even within the general frame of multiplicative cascades, without refering to dissipation, theorems show that a pure Gaussian form for G_b yield incoherencies in the far tails of velocity differences distributions. Moreover, log-normal distribution violates a condition for the boundedness of the velocity field for large enough Reynolds [1]. However, these objections have not to be took into account as they concern incredibly small levels of probability. To take an example, the Fourier heat law allows (small) signals to propagate faster than the velocity of light, but nobody would discard it for this. Up to now, all serious attempts to evidence deviations from the log-normal predictions have failed.

3.4.3. The Fokker-Planck approach

Some recent approaches [58, 59], looking directly at multipliers statistics, could test their correlations. They experimentally verified the Markovian properties of the cascade (independence between successive steps). They obtained a Fokker-Planck equation for the velocity differences PDF $P_\ell[\delta v(\ell)]$:

$$\frac{\partial P_\ell}{\partial \ln(L/\ell)} = \frac{\partial (D_1 P_\ell)}{\partial \delta v(\ell)} + \frac{\partial^2 (D_2 P_\ell)}{\partial \delta v(\ell)^2} \tag{1.27}$$

where the functions $D_1[\delta v(\ell)]$ and $D_2[\delta v(\ell)]$ can be experimentally determined. A pure multiplicative cascade would give D_1 linear and D_2 purely quadratic. Deviations appear at large scale, but these functions seem to converge on this behaviour as ℓ goes down.

3.4.4. Scale invariance

An extension of scale invariance has been proposed in the following way [45, 55] : instead of using the scale separation, b, as a scaling factor, is it possbile to find scaling laws in terms of more general scaling funcitons ? Is it possible to adsorbe Finite-Reynolds effects with a suitable redefinition of scaling ? One possibility is to use the third order longitudinal structure function, $S^{(3)}(\ell)$ as reference function. This is because we know from the 4/5 law that it is exactly scaling invariant in the limit of infinite Reynolds. Than, we may look at the scaling of different orders of structure functions with respect to the third order. This is the Extended Self Similarity (ESS), discovered independently by Benzi and coworkers [29]. Confirmation of this extension of scaling was found also when looking at the flatness $\langle \delta v(\ell)^4\rangle / \langle \delta v(\ell)^2\rangle^2$ *versus* $\langle \delta v(\ell)^2\rangle$ instead of ℓ. However, these effects have another possible explanation [43] [56]. In the inertial range, typical velocity differences $\delta v(\ell)$ go like $\ell^{1/3}$ (K41 theory). In the dissipative range, where the local Reynolds number is smaller than 1, they behave as ℓ. Then the decrease of all the $\langle \delta v(\ell)^p\rangle$ is faster when in the dissipative range than in the inertial one. But large energy events contribute relatively more in $\langle \delta v(\ell)^4\rangle$ than in $\langle \delta v(\ell)^2\rangle$. $\langle \delta v(\ell)^4\rangle$ thus enters the dissipative range at smaller ℓ than $\langle \delta v(\ell)^2\rangle$ [60], and the flatness $\langle \delta v(\ell)^4\rangle / \langle \delta v(\ell)^2\rangle^2$ rapidly increases. For high Re, this growth is much faster than the decrease of $\langle \delta v(\ell)^2\rangle$ and the Self Similarity cannot be extended in this "Intermediate Dissipative Range" [60]. However, for moderate Re, the two behaviours are accidentally similar, which gives the illusion of ESS.
But ESS cannot improve the precision on ζ_p, as it fails to extrapolate to high Re [43, 56].

3.4.5. Two points correlation

It has been soon recognized [3] that Kolmogorov hypotheses implied a long range correlation between local dissipations, and thus velocity differences. It has been checked only recently, by J. Delour *et al.* [44]. They considered the "magnitude" (in the astronomical sense) $M(\boldsymbol{x}) = \ln|\delta v(\ell_c)|$ for an interval of length ℓ_c centered on $\boldsymbol{x}$, and the correlation of this magnitude between two points distant of $\boldsymbol{\ell}$:

$$\langle M(\boldsymbol{x}) M(\boldsymbol{x}+\boldsymbol{\ell})\rangle - \langle M(\boldsymbol{x})\rangle^2 \tag{1.28}$$

with $\ell > \ell_c$. After a rapid variation due to the short range correlation of the signs of $\delta v(\ell_c)$, they observe a slow decrease, with the correlation proportional to $\ln(L/\ell)$ This is in perfect agreement with a multiplicative cascade model. Using the notations above, let us assume that $\ell_c = \ell_n = b^{-n}L$ and $\ell = \ell_j = b^{-j}L$. Then, in the expression of $\delta v(\ell_c)$:

$$\delta v(\ell_c) = \alpha_{n,n-1} \cdots \alpha_{j,j-1} \cdots \alpha_{1,0} L \tag{1.29}$$

all multipliers $\alpha_{j,j-1}, \cdots, \alpha_{1,0}$ are common to the two velocity differences considered, which reduces the variance of the logarithm of their ratio. Starting from this observation, they could produce a "Multiplicative Random Walk", having this long range correlation, with the intermittency properties of real velocity records.

4. Kolmogorov's legacy on modern turbulence

As already stated, Kolmogorov theories are founded on two major statements. First, that for Reynolds high enough –and far from the boundaries– small-scales turbulent flows are dominated by isotropic fluctuations. Second, that small-scales fluctuations are *universal*, i.e. independent of the large scale set up. Both properties are strongly linked. Anisotropic fluctuations are injected in the flow only trough the large scale forcing (or by boundary effects). Therefore, any anisotropic fluctuations left at small scales must be seen as the legacy of the large-scale physics. Moreover, the total amount of small-scales anisotropy cannot be universal, being, case-by-case the direct effect of the particular forcing used to maintain that particular flow.

On the whole, experimental tests of Kolmogorov's theories ran into increasing difficulties when the data was analyzed with greater detail. The first systematic attempt to check the isotropic scaling (1.8) for high Reynolds number turbulence was done by Anselmet and coll. [17]. In [17] the authors performed, for the first time, a high statistical test of K41 theory by going beyond the usual two-point correlation. They measured also structure functions of higher order, reaching good evidences that *anomalous* deviations to the $p/3$ scaling exponents existed. At that time, and for many year later, the situations was very controversial both theoretically and experimentally, Many claims that the observed deviations were due to sub-leading finite-Reynolds effects appeared. One should not underestimate the difficulties of getting reliable estimates of the scaling exponents. First, one must expect finite Reynolds numbers corrections which may *strongly* reduce the inertial range extension where scaling laws are expected and/or introduce anisotropic corrections to the isotropic K41 predictions. Both effects are usually present in all experiments and numerical simulations. Nowadays, state-of-the-art experiments of turbulence in controlled geometry reach a maximum Reynolds numbers measured on the gradient scales, of $R_\lambda \sim 5000$, which can be pushed to $R_\lambda \sim 10000$ for atmospheric boundary flows (but highly anisotropic!). More complex is the situation of Direct Numerical Simulations (DNS) where the best resolution ever reached up to now is 4096^3 corresponding to a $R_\lambda \sim 800$. DNS allow a minimization of the anisotropic corrections, thanks to the possibility to implement periodic boundary conditions and fully isotropic forcing, something which is out of reach in any real experiments. However, also in DNS the discrete symmetries induced by the finite lattice-spacing do not allow for a perfect isotropic statistics. We thus either have high-Reynolds-numbers experiments which are strongly perturbed by anisotropic effects, or DNS isotropic flow at moderate Reynolds numbers. Therefore, one has to face the problem of how to disentangle isotropic from anisotropic fluctuations and how to extract information on the asymptotic scaling with a finite –often short– inertial-range extension. Only recently, after many experimental and numerical confirmations of the results of [17], the situation became clearer [29]. We may affirm now with some degrees of certitude that the isotropic scaling exponents are *anomalous*, the K41 prediction $\zeta(p) = p/3$ is wrong, except for $p = 3$ which is

fixed to be $\zeta(3) = 1$ by the exact 4/5 law. More recently, the possibility to show analytically the existence of anomalous scaling in turbulent advection [28], definitely eliminated those arguments supporting the *impossibility* to have a Reynolds independent anomalous scaling in any hydrodynamical system.
As already said, according to the Refined Similarity Hypothesis, anomalous scaling of isotropic structure functions is connected to the multifractal properties of the three dimensional measure defined in terms of the energy dissipation [1]. It should be noted however that RSH related inertial range scaling to scaling of dissipative quantities, and delicate issues connected to small distance expansions and fusion rules are being disregarded here [25, 26, 27]. At any rate, the RSH did not advance the calculation of the scaling exponents beyond crude phenomenology.

4.1. Universality of small-scales fluctuations

Universality of small-scale forced turbulence is at the forefront of both theoretical and experimental investigation of real turbulent flows [1]. The problem is to identify those statistical properties which are robust against changes of the large-scale physics, that is against changes in the boundary conditions and the forcing mechanisms. The brute force method to check universality quickly run in bad problems. In nature or in labs one finds an enormous variety of turbulent flows with different large-scale physics, as for example channel flows, convective flows, flows maintained by counter-rotating disks to list only some of them. The problem is that all this different experimental set-up suffer also of very different anisotropic corrections. Therefore it may be very difficult on the experimental data to clearly disentangle the isotropic component. A systematic study of small-scales universality is therefore limited to some degree of incertitude (see also later on the anisotropic fluctuations). Still, the beautiful connection and comparison of data coming from more than 10 different experiments with different large-scale set up and with different Reynolds numbers presented in [28] certainly supports the *universal* picture (see also figure 1.3).
More recently, a different proposal to test small-scales universality has been made [12]. The idea is to relate the small-scale universal properties of forced turbulent statistics to those of short-time decay for an ensemble of initial configurations. An immediate remark is that one cannot expect an universal behavior for all statistical observables as the very existence of anomalous scaling is the signature of the memory of the boundaries and/or the external forcing throughout all the scales (due to the appearance of the *outer* scale in the expression for structure functions). Indeed, the maximal expectation one may suggest is that only the scaling of small-scale fluctuations is universal, at least for forcing concentrated at large scales. The prefactors are not expected to be so. There is therefore no reason to expect that quantities such as the skewness, the kurtosis and in fact the whole PDF of velocity increments or gradients be universal.
This is the same behavior as for the passive transport of scalar and vector fields (see [34] and references therein). For those systems both the existence and the origin of the observed anomalous scaling laws have been understood and even calculated analytically for some instances in the special class of Kraichnan flows [33]. However, carrying over the analytical knowledge developed for linear hydrodynamical problems involve some nontrivial, yet missing, steps. For the Navier-Stokes dynamics, linear equations of motion appear at the functional level of the whole set of correlation functions, only. In a schematic form :

$$\partial_t C^{(p)} = \Gamma^{(p+1)} C^{(p+1)} + \nu D^{(p)} C^{(p)} + F^{(p)}, \tag{1.30}$$

where $\Gamma^{(p+1)}$ is the integro-differential linear operator coming from the inertial and pressure terms and $C^{(p+1)}$ is a shorthand notation for a generic $(p+1)$-point correlation. The molecular viscosity is denoted by ν and $D^{(p)}$ is the linear operator describing dissipative effects. Finally, $F^{(p)}$ is the correlation involving increments of the large-scale forcing $\boldsymbol{f}$ and of the velocity field. Balancing inertial and injection terms gives dimensional scaling, and anomalously scaling terms must therefore have a different source. A natural possibility is that a mechanism similar to the one identified in linear transport problems be at work in the Navier-Stokes case as well. The anomalous contributions to the correlation would then be associated to statistically stationary solutions of the unforced equations (1.30). The scaling exponents would *a fortiori* be independent of the forcing and thus universal. As for the prefactors, the anomalous scaling exponents are positive and thus the anomalous contributions grow at infinity. They should then be matched at the large scales with the contributions coming from the forcing to ensure that the resulting combination vanish at infinity, as required for correlation functions. To prove the previous points is still out of any analytical reach. Rather one may check over the most obvious catch : the Navier-Stokes equations being integro-differential, non-local contributions might directly couple inertial and injection scales and spoil the argument. In order to investigate the previous point in [12] a comparison between the behavior of *decaying turbulence* with two different ensemble of initial conditions was made. The first ensemble is made of initial conditions taken from a forced stationary run, the second ensemble is made of random initial conditions. The fact that only the decaying experiment with the initial conditions picked from the ensemble of forced stationary configuration does not decay for times up to the largest eddy turn over time can be seen as direct support to the statement that *forced* small-scales turbulence is *universal.*

4.2. Anisotropic Turbulence

Despite the background phenomenological and experimental framework of fully-developed turbulent flows seem well founded, there are still many problems which lack clear experimental validation and/or a theoretical understanding. Most of them have to do with anisotropic fluctuations.
For example, the central issue of K41 phenomenology is the assumption of *return-to-isotropy* for smaller and smaller scales. Recently, some detailed analysis of small-scale isotropy on experimental and numerical shear turbulence have been performed [38, 30, 42, 35]. The ideal experimental set-up to test such a problem is the case of an *homogeneous* shear flow. There, we have a shear pointing in one direction and spatially homogeneous, i.e. the large-scale mean-velocity has the perfect linear profile : $\boldsymbol{V} = (V_0 y, 0, 0)$. The shear is given by $\mathcal{S}_{ij} = \partial_i V_j = \delta_{iy}\delta_j x V_0$. It is easy to realize that we have an homogeneous but anisotropic flow, the ideal case to study the influence of anisotropies on small scales statistics. Small scales must be defined compared to the characteristic shear length, $L_{\mathcal{S}} = \epsilon^{1/3}/\mathcal{S}$, only for $r \ll L_{\mathcal{S}}$ we may expect that anisotropic fluctuations are sub-leading with respect to the isotropic ones. The case $r \gg L_{\mathcal{S}}$ is of some interest in all situations where the shear may become very intense, as very close to the walls in bounded flows. In the latter case a quite different physics from the K41 phenomenology must be expected [16]. Fortunately enough, it is not that difficult to design ad-hoc experiments or DNS possessing an almost perfect linear profile as the one of homogeneous shear [38, 37, 42]. A popular way to measure small-scales anisotropies is to focus on the Reynolds dependencies of isotropic and anisotropic observables built in terms of velocity gradients. For example, due to the symmetries of the mean flow, gradients

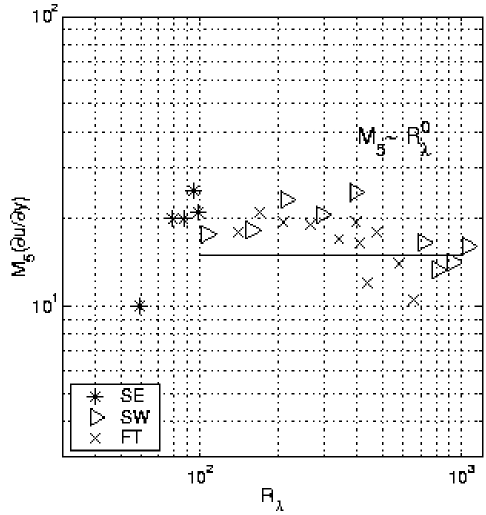

Figure 1.4. A collection of experimental and numerical results for the fifth order skewness $M^{(5)}(R_\lambda)$ as a function of Reynolds numbers taken from fig. 2 of J. Schumacher et al. *Phys. Fluids* **15** 84 (2003). Notation in the legend stands for SE reference [39] ; FT reference [41], SW reference [40].

of the stream-wise component in the shear direction, $\partial_y v_x$, may have a skewed distribution only due to the anisotropic fluctuations ; they should have a perfectly symmetric PDF in a perfectly isotropic flow. A natural way to measure the residual anisotropy at small scales as a function of the Reynolds numbers is to build mixed generalized Skewness based on gradients :

$$M^{(2p+1)}(R_\lambda) = \frac{\langle(\partial_y v_x)^{2p+1}\rangle}{\langle(\partial_y v_x)^2\rangle^{\frac{2n+1}{2}}}. \tag{1.31}$$

The above generalized Skewness should be exactly zero in an isotropic ensemble, because its numerator vanishes. Of course, at any finite Reynolds numbers one cannot expect that the large-scale anisotropy introduced by the shear is completely decayed on the gradient scale. Therefore, a direct measure of the rate of decaying of (1.31) as a function of Reynolds is a quantitative indication on the rate of decay of anisotropy at small scales, i.e. is a direct check of the assumption of *local isotropy* made by Kolmogorov for Reynolds large enough. Lumley set up a dimensional argument for anisotropic fluctuations predicting as a function of the Reynolds numbers based on the Taylor scale, R_λ :

$$M^{(2p+1)}(R_\lambda) \sim R_\lambda^{-\frac{1}{2}} \tag{1.32}$$

independently of the order of the moment, n. Surprisingly enough, both numerical [42] (at very low Reynolds numbers) and experimental tests (up to $R_\lambda \sim 1000$) showed a clear discrepancy from the dimensional prediction (1.32), see fig.(1.4).

For example in [37] the authors quote an almost constant behavior as a function of Reynolds for the fifth order, $M^{(5)}(R_\lambda) \sim O(1)$ and an *increasing* behavior for the seventh order $M^{(7)}(R_\lambda)$ although in the latter case some problems of statistical convergence may have spoiled the result. These results have cast severe doubts on the basis of the K41 and K62 theories.
In the last years a net advancements in the understanding of anisotropic turbulence

has been obtained thanks to systematic decomposition of velocity correlation functions in the basis of the rotational operator, the SO(3) group [18, 19, 20, 30, 31]. These works defined a clear theoretical, experimental and numerical basis from which starting to attack the *return-to-isotropy* discussed before. Moreover, the net improvement in the scaling quality measured after the SO(3) decomposition allowed for a better quantitative understanding of both isotropic and anisotropic turbulence. For scalar objects, as the longitudinal structure functions, the SO(3) decomposition reduces to the projection on the spherical harmonics :

$$S^{(p)}(\boldsymbol{\ell}) = \sum_{j=0}^{\infty} \sum_{m=-j}^{j} s_{jm}^{(p)}(\ell) Y_{jm}(\hat{\boldsymbol{\ell}}). \tag{1.33}$$

As customary, the indices (j, m) label the total angular momentum and its projection on a reference axis, respectively. Our interest here in concentrated on the behavior of the projection on the isotropic sector, $s_{j=0m=0}^{(p)}(\ell)$, or any of the anisotropic sectors, $s_{jm}^{(p)}(\ell)$ with $j > 0$. Theoretical and phenomenological reasonings suggest that each projections has its own scaling behavior, $s_{jm}^{(p)}(\ell) \sim \ell^{\zeta^j(p)}$, with the exponent depending on the j sector only. Now, the *recovery of isotropy* can be restated in terms of the values of the scaling exponents. The *return-to-isotropy* hypothesis made by Kolmogorov implies that the scaling exponents are organized in a hierarchical way :

$$\zeta^{j=0}(p) < \zeta^j(p) \tag{1.34}$$

for any order p. All numerical [31, 35], experimental [30, 36] and phenomenological [32] works on this subject confirm the existence of the hierarchy (1.34). Any turbulent flow should become more and more isotropic by going to smaller and smaller scales. The persistence of small scales anisotropy as measured by the generalized Skewness defined in (1.31) can be simply explained by noticing that there we are balancing isotropic and anisotropic effects of two different order of velocity correlation functions, the $(2p + 1)$th order in the numerator and the 2nd order in the denominator. It is easy to realize that thanks to the presence of intermittency also in the anisotropic sectors one may have a recovery of isotropy in the sense that the hierarchy (1.34) holds and still observe a persistence of anisotropies based on the gradients statistics as shown in figure (1.4) [35].

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