

Predictability in chaotic systems and turbulence

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Abstract

A method for characterizing the predictability of complex chaotic systems based on a generalization of the Lyapunov exponent is introduced. The method is illustrated on a toy system with two time scales and on a model of fully developed turbulence where universal features are found.

*... misi me per l'alto mare aperto
sol con un legno e con quella compagna
picciola da la qual non fui diserto.
Dante, Inferno, Canto XXVI*

1 INTRODUCTION

Our possibility to forecast is based on the assumption of a deterministic world, assumption which is summarized by the well known Laplace statement: the knowledge of the equations of motion and of the initial conditions is sufficient, in principle, to predict the future state of the world at any time. From a practical point of view, Laplacian determinism is applicable only to systems with regular behavior. In this case, small unavoidable uncertainties in the equations or in the initial conditions do not affect our possibility to make long time predictions within a given accuracy. The situation is completely different in the case of chaotic systems which, even being deterministic, are strongly unpredictable in practice because of their unstable dynamics. Sufficiently complex systems display chaos which turn out to be the rule in the physical world. A very popular example is the atmosphere circulation whose chaotic dynamics makes long times weather predictions impossible.

The topic of the present contribution is the investigation of the intrinsic predictability of many degree of freedom chaotic systems, using an hydrodynamic turbulence model as an example. Turbulence is characterized by a complex dynamics involving many spatial and temporal scales. Its ubiquity (for example in geophysical flows) and the existence of a well developed theoretical investigation make turbulence an ideal prototypical model for predictability study.

The sources of the errors in a forecasting are in general due both to the uncertainty in the initial conditions (first kind predictability problem) and to the imperfections of the model (second kind predictability). In this contribution we consider only the former case and we will assume to have a perfect model of the physical system. By definition, chaotic dynamical systems display sensitive dependence on initial conditions: two initially close trajectories will diverge exponentially in the phase space with a rate given by the leading Lyapunov exponent λ_{max} [1]. Because of the finite uncertainty δ in the initial conditions, we can forecast the future state of the system at a tolerance

level Δ only up to a maximum predictability time which is estimated, in terms of the Lyapunov exponent, as

$$T_p \simeq \frac{1}{\lambda_{max}} \ln \left(\frac{\Delta}{\delta} \right). \quad (1)$$

It is important to observe that definition (1) implies a weak dependence on the precision of the initial condition and on the tolerance; therefore according to this definition, the predictability time is an intrinsic quantity of the system which cannot be significantly improved by accepting larger tolerances.

This definition for the predictability time holds only for infinitesimal perturbations (and in non intermittent systems [2]); in the more physical situation of finite (and typically rather large) errors, the naive application of (1) leads to a series of paradoxes and subtle points which have been the object of studies in the last years [3, 4]. A familiar example of this kind of problems is given by weather forecasting: although the Lyapunov exponent for the atmosphere (as a whole) is presumably rather large (due to the small scale turbulence), the large scale behavior of the system can be forecast with good accuracy for several days [5, 6].

It should thus be clear that the knowledge of the (positive) leading Lyapunov exponent is not sufficient for a quantitative characterization of the predictability properties. The leading Lyapunov exponent is associated to the fastest, smallest scales in the system which rule the initial exponential growth of infinitesimal errors. When the uncertainty is sufficiently large, the error in the small scales is saturated (i.e. is of the same size of the variable fluctuations) and do not play a role any more in the error growth law. Large errors will grow with the characteristic time of the large scales which is, in general, independent on the Lyapunov exponent. It is thus natural to introduce a generalization of the Lyapunov exponent to finite errors, from which one can compute a more realistic predictability time. The Finite Size Lyapunov Exponent is introduced to this scope in section 2 with an application to a simple toy model. Section 3 is devoted to illustrate the application of the FSLE to the more complex situation of a model of turbulent flows. Section 4 contains some conclusions.

2 THE FINITE SIZE LYAPUNOV EXPONENT

The standard algorithm for the computation of the Lyapunov exponent [7] evaluates the average exponential separation of two close trajectories in the phase space obtained by integrating two couples of the system with slightly different initial conditions. Periodic rescaling of trajectory separation is needed in order to avoid nonlinear effects. The Finite Size Lyapunov Exponent (FSLE) is introduced by a generalization of this algorithm which relaxes the request of small separations. In this way one is able to compute the average separation rate at finite scales.

For computing the FSLE one has to integrate two trajectories $x(t)$, $x'(t)$ starting at small initial separation $\delta_{min} = |x'(0) - x(0)|$. Defined a set of thresholds $\delta_n = r^n \delta_0$ one computes the "doubling times" $T_r(\delta_n)$ that the error takes to grow from the threshold δ_n up to the next one δ_{n+1} . Given many realizations of this experiment (ensemble average) the FSLE is defined, for any error level δ , as

$$\lambda(\delta) = \frac{1}{\langle T_r(\delta) \rangle} \ln r \quad (2)$$

It is easy to show that definition (2) recovers the leading Lyapunov exponent λ_{max} in the infinitesimal limit $\delta \rightarrow 0$ [8]. In the opposite limit of large δ , the FSLE goes to zero indicating error saturation and complete decorrelation of the two trajectories. The initial error must be $\delta_{min} \ll \delta_0$ in order to allow the initial perturbation to align along the most unstable direction.

Definition (1) of the predictability time can now be generalized in terms of the FSLE. The average time that an initial error of size δ takes to grow up to the tolerance Δ is

$$T_p = \int_{\delta}^{\Delta} \frac{d \ln \delta'}{\lambda(\delta')} \quad (3)$$

Observe that this definition reduces to (1) in the case of constant λ . From general considerations, one expects that $\lambda(\delta)$ is a decreasing function of δ and thus (4) gives longer predictability time than (1). We want to stress the fact that the computation of the FSLE is numerically no more expensive than the computation of the leading Lyapunov exponent. The FSLE tool seems promising also for the analysis of experimental data [9].

We now illustrate the application of the FSLE to a simple system which presents two characteristic times. The system is not intended to be a realistic model of any physical situation; it should be rather considered as a toy model of a situation presenting two well separated time scales. The example is obtained by coupling two Lorenz models [10], the first ($x_i^{(s)}$) representing the slow dynamics and the second ($x_i^{(f)}$) the fast dynamics

$$\left\{ \begin{array}{l} \frac{dx_1^{(s)}}{dt} = \sigma(x_2^{(s)} - x_1^{(s)}) \\ \frac{dx_2^{(s)}}{dt} = (-x_1^{(s)}x_3^{(s)} + r_s x_1^{(s)} - x_2^{(s)}) - \epsilon_s x_1^{(f)} x_2^{(f)} \\ \frac{dx_3^{(s)}}{dt} = x_1^{(s)} x_2^{(s)} - b x_3^{(s)} \\ \frac{dx_1^{(f)}}{dt} = c\sigma(x_2^{(f)} - x_1^{(f)}) \\ \frac{dx_2^{(f)}}{dt} = c(-x_1^{(f)}x_3^{(f)} + r_f x_1^{(f)} - x_2^{(f)}) + \epsilon_f x_1^{(f)} x_2^{(s)} \\ \frac{dx_3^{(f)}}{dt} = c(x_1^{(f)} x_2^{(f)} - b x_3^{(f)}) \end{array} \right. \quad (4)$$

The particular form of the coupling is not important for the following discussion. With the present choice the trajectory is constrained within a bounded region of the phase space [11]. We fix the parameters at the standard values $\sigma = 10$, $b = 8/3$ and $c = 10$, the latter giving the relative time scale between the fast and slow dynamics. The two Rayleigh numbers are taken different, $r_s = 28$ and $r_f = 45$ for generality. With the present choice, both the uncoupled systems (i.e. $\epsilon_s = \epsilon_f = 0$) display chaotic dynamics with Lyapunov exponents $\lambda^{(f)} \simeq 12.17$ and $\lambda^{(s)} \simeq 0.905$ respectively.

By switching on the couplings ϵ_s and ϵ_f we obtain a single dynamical system whose maximal Lyapunov exponent λ_{max} is close to the Lyapunov exponent of the faster decoupled system ($\lambda^{(f)}$). In the following we will use $\epsilon_f = 10$ and $\epsilon_s = 10^{-2}$. With this choice for the couplings, the leading global Lyapunov exponent is found to be $\lambda_{max} \simeq 11.5$ which is indeed close to $\lambda^{(f)}$.

With regard to predictability, one expects reasonably that for small coupling ϵ_s the slow component of the system \mathbf{x}_s remains predictable up to its own characteristic time. On the other hand, for any coupling $\epsilon \neq 0$ we obtain a single dynamical system in which the errors grow with the leading Lyapunov exponent $\lambda_{max} \simeq \lambda^{(f)}$. This apparent paradox shows that the Lyapunov exponent is unable to characterize the predictability time in general.

To clarify the situation, Figure 1 shows the average error growth for both the slow and fast variables. We observe that since in the beginning both the errors are very small, their growth rate is given by the leading Lyapunov exponent λ_{max} . For larger times ($t > 2$), the fast component of the error, δx_f , has reached the saturation, the trajectories separation evolves according to the full non linear equations of motion and the growth rate for the slow component is strongly reduced. From Figure 1 one observes that the slow component error δx_s is still well below the saturation value, and grows with a rate close to its characteristic inverse time $\lambda^{(s)}$.

The application of the FSLE method to the slow component of the error, δx_s , is shown in Figure 2. As expected, for very small δ , the FSLE recovers the leading Lyapunov exponent λ_{max} , indicating that in small error predictability the fast component has indeed a dominant role. As soon as the error grows above the coupling ϵ_s , $\lambda(\delta)$ drops to a value close to $\lambda^{(s)}$ and the characteristic time of small scale dynamics is no more relevant.

In Figure 3 we plot the slow component predictability time (3) for fixed initial error $\delta x_s = 10^{-6}$ as a function of the tolerance Δ . We observe a strong enhancement of T_p as soon as the accepted tolerance is larger than the coupling ϵ_s . Observe that the naive application of (1) would heavily underestimate the predictability time for large tolerances (dashed line).

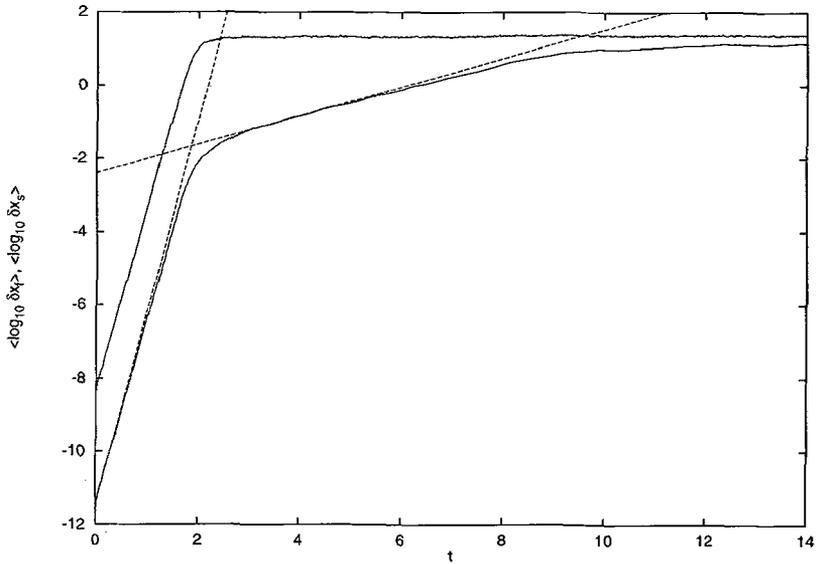


Figure 1: Typical error growth for the fast component δx_f (upper curve) and for the slow component δx_s in the coupled Lorenz models with $\delta x_f(0) = 10^{-8}$ and $\delta x_s(0) = 10^{-12}$, averaged over 500 samples. The dashed lines show the exponential growths with exponents $\lambda^{(f)}$ and $\lambda^{(s)}$.

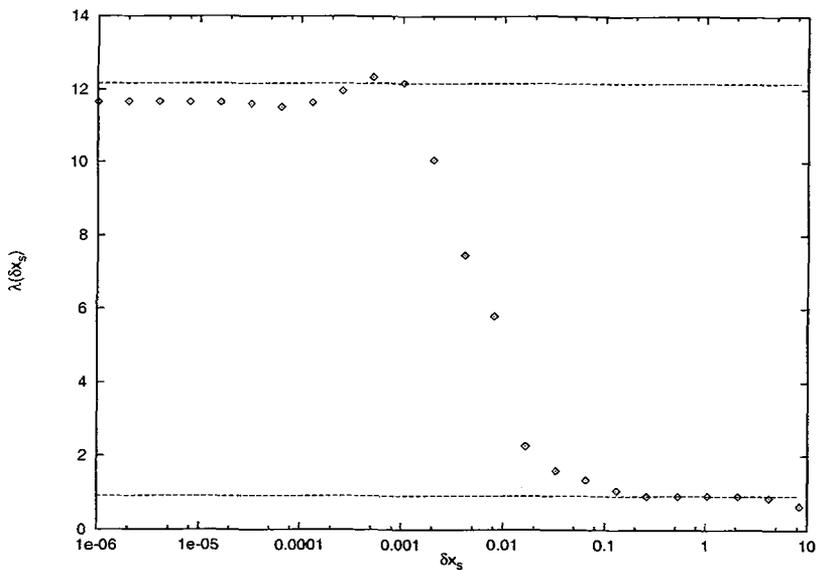


Figure 2: FSLE for the two coupled Lorenz models computed from the slow variable error. The initial error is $\delta_0 = 10^{-6}$ and the average is over 500 realizations. The two horizontal lines represent the uncoupled Lyapunov exponents $\lambda^{(f)}$ and $\lambda^{(s)}$.

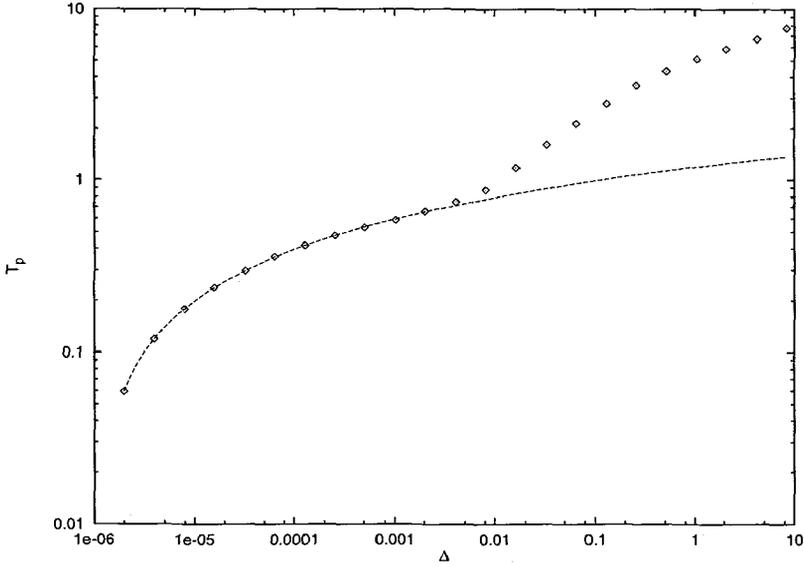


Figure 3: Predictability time for the slow component of the two coupled Lorenz models as a function of the tolerance Δ . The initial error is fixed at $\delta = 10^{-6}$. The dashed line represent the Lyapunov estimation $T_p \sim \lambda_{max}^{-1} \ln(\Delta/\delta)$.

3 FINITE SIZE PREDICTABILITY IN TURBULENCE

We now consider fully developed turbulence as a well known example of system with many characteristic scales. Because of the ubiquity of turbulence in nature, the example is also of physical relevance. Following the original picture of Kolmogorov, turbulence is characterized by an wide range of locally interacting scales (inertial range) in which the energy is simply transfer from large to small scale. The energy cascade is maintained stationary by an energy source a large scales (forcing) and an energy sink at small scales (viscous dissipation). The typical transfer time at scale ℓ (eddy turnover time) is dimensionally given by

$$\tau_\ell \simeq \frac{\ell}{u_\ell} \quad (5)$$

where $u_\ell = |\mathbf{u}(\mathbf{x}') - \mathbf{u}(\mathbf{x})|$ is the turbulent velocity difference at scale $\ell = |\mathbf{x}' - \mathbf{x}|$. In stationary conditions the average energy transfer rate ϵ in the inertial range must be scale independent and thus

$$\left\langle \frac{u_\ell^2}{\tau_\ell} \right\rangle \simeq \frac{\langle u_\ell^3 \rangle}{\ell} \simeq \epsilon \quad (6)$$

which leads to the well known Kolmogorov scaling

$$\langle u_\ell^p \rangle \simeq (\epsilon \ell)^{p/3} \quad (7)$$

As a consequence of (7), the eddy turnover time in the inertial range scales like $\tau_\ell \simeq \epsilon^{-1/3} \ell^{2/3}$. The inertial transfer of energy terminates at the scale $\ell \simeq \eta$ at which the dissipation timescale becomes smaller than the transfer timescale. Dimensional considerations lead to the Kolmogorov scale $\eta = L Re^{-3/4}$ where Re is the integral Reynolds number expressed in term of the large scale velocity U at scale L as $Re = LU/\nu$. The leading Lyapunov exponent in a turbulent flow can be estimated proportional to the inverse smallest characteristic time in the cascade [12, 2], i.e.

$$\lambda_{max} \sim 1/\tau_\eta \sim Re^{1/2}. \quad (8)$$

Because the Lyapunov exponent grows with the Reynolds number, the small scale predictability time (1) goes to zero in the limit of fully developed turbulence ($Re \rightarrow \infty$).

Let us now consider the growth of finite errors in a turbulent velocity field. We assume to have two realizations of the turbulent flow $\mathbf{u}(\mathbf{x}, t)$ and $\mathbf{u}'(\mathbf{x}, t)$ at a distance $\delta(t)$. Following the phenomenological ideas of Lorenz [5], the growing rate for δ can be identified with the inverse eddy turnover time τ_ℓ at the scale ℓ at which $u_\ell \sim \delta$. Indeed, at smaller scales ℓ' for which $u_{\ell'} \ll \delta$ we can assume that the error is already saturated, as in the example of section 2; larger scales have longer characteristic time and thus are subleading in the error growth rate. Expressing the eddy turnover time (5) in terms of the velocity difference $u_\ell \simeq \delta$, we obtain through (7) the prediction for the FSLE in turbulence [8]

$$\lambda(\delta) \simeq \epsilon \delta^{-2} \quad (9)$$

which is valid within the inertial range $u_\eta < \delta < U$. For $\delta < u_\eta$ (dissipative range) the FSLE is expected to recover the leading Lyapunov exponent (8).

It is well known [13, 14] that fully developed turbulence is not completely self similar and the scaling exponents for statistical quantities deviate from the dimensional estimations recalled above. An easy way to introduce intermittency in the energy cascade is to consider the moments of the energy dissipation ϵ_ℓ averaged over a domain of scale ℓ . The hypothesis of constant energy cascade in the inertial range implies that $\langle \epsilon_\ell \rangle = \epsilon = \text{const}$ but the other moments display a ℓ -dependence as a consequence of a non uniform dissipation in the physical space. As a consequence, a generic statistical quantity which depends dimensionally on the energy dissipation as ϵ^α displays anomalous scaling unless $\alpha = 1$ (as in the case of third-order structure functions, $p = 3$ in (7)). Because in (9) there appears the average energy dissipation ϵ , we expect that the scaling exponent for $\lambda(\delta)$ is not affected by intermittency corrections and is thus a new universal exponent in the statistical description of energy cascade. The effect of intermittency is to modulate the crossover between the scaling regime (9) and the constant regime (8) as a consequence of the fluctuations of the dissipative scale (see [8] for a detailed discussion).

Introducing now the scaling law (9) into (3) we obtain, as expected, that the predictability time for large errors is strongly dependent on the tolerance, $T_p(\Delta) \simeq \Delta^2$, which has to be compared with (1).

Prediction (9) cannot be checked in DNS because of the limited inertial range achievable with nowadays computers. An alternative is to consider simplified dynamical models of turbulence, referred to as shell models, with much less degrees of freedom. Without entering in the details (see [14] for a recent overview) in shell models the velocity fluctuations over a scale ℓ_n are collectively represented by a single complex variable u_n ($n = 1, 2, \dots, N$). Because one is interested in power-law scaling, the scales are spaced geometrically as $\ell_n = L2^{-n}$; this allows to reach huge Reynolds numbers with a relatively few variables. In the following we will consider the most popular shell model, called GOY model, which displays a Kolmogorov-like energy cascade with chaotic dynamics which leads to corrections in the scaling exponents very close to those observed in fluid experiments. To compute the FSLE in shell model turbulence, we integrate two realizations of the same system $u_n(t)$ and $u'_n(t)$ starting at very close initial conditions. For each experiment we compute the doubling times $T_r(\delta)$ for the error $\delta^2 = \sum_n |u'_n - u_n|^2$ until it reaches the largest threshold δ_{max} . The average over many error-doubling experiments gives the FSLE according to (2).

In Figure 4 it is shown the FSLE computed for shell models at different Reynolds numbers. At small errors δ $\lambda(\delta)$ displays a plateau at λ_{max} . The value of λ_{max} increases with the Reynolds number according to (8). For larger errors, the inertial range scaling (9) is evident. Observe that in this range (large errors) the FSLE is independent on the Reynolds number: this fact explains why is possible to perform finite-time forecasting even in the limit $Re \rightarrow \infty$.

To demonstrate more quantitatively the scaling of the FSLE with the Reynolds number, in Figure 5 we plot the rescaled $\lambda/\lambda_{max} \sim \lambda/Re^{1/2}$ as a function of the rescaled error $\delta/u_\eta \sim \delta/Re^{-1/4}$. We

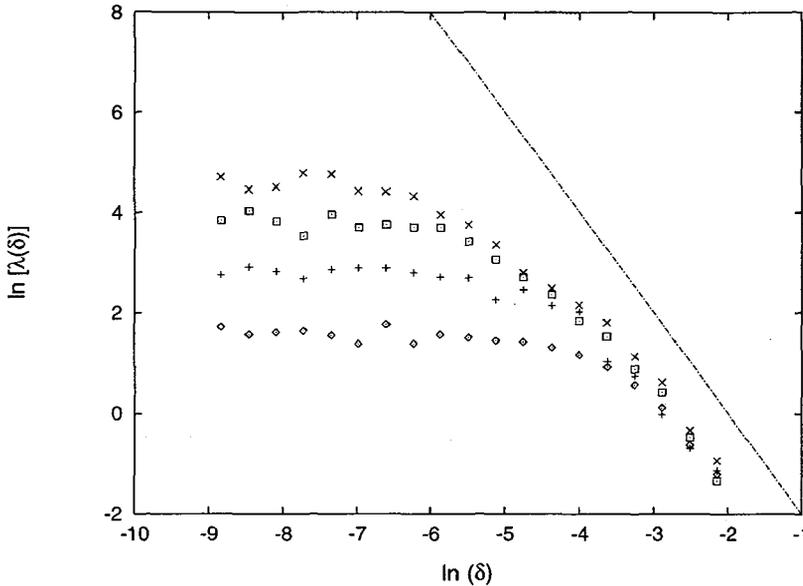


Figure 4: FSLE for the Shell Model at different Reynolds numbers: $Re = 10^8, 10^9, 10^{10}, 10^{11}$ from bottom to top. The initial error is $\delta = 10^{-6}$ and the number of experiments is 400. The dashed line, representing the theoretical prediction, has slope -2 .

see that the collapse is rather good. A better collapse can be obtained by taking into account intermittency corrections for λ_{max} and u_η [8].

4 CONCLUSIONS

We have shown that in systems with possess different characteristic time scales, the predictability time can be an independent quantity of the leading Lyapunov exponent. The latter is associated to the faster characteristic time and dominates the exponential growth of infinitesimal errors. Finite errors will evolve in general with large scale characteristic time which thus rules large scale predictability.

We have discussed a generalization of the Lyapunov exponent which allows one to compute the average exponential error growth at a given error size δ . The Finite Size Lyapunov Exponent is expected to converge to the leading Lyapunov exponent for very small errors. For larger errors, $\lambda(\delta)$ is decreasing with δ and thus the FSLE analysis predicts an enhancement of the predictability time at large tolerances.

The method have been illustrated in a toy model with two timescales and in a Shell Model of turbulence where we have found an universal scaling law for the error growth rate.

Acknowledgments

The characterization of predictability is one of the fields in which Giovanni gave many substantial contributions in the last years. I started working on this subject also thanks to the enthusiastic attitude of Giovanni in all the manifestation of life, including science. I hope that this short contribution will help to remember who was, more than a collaborator, a good friend.

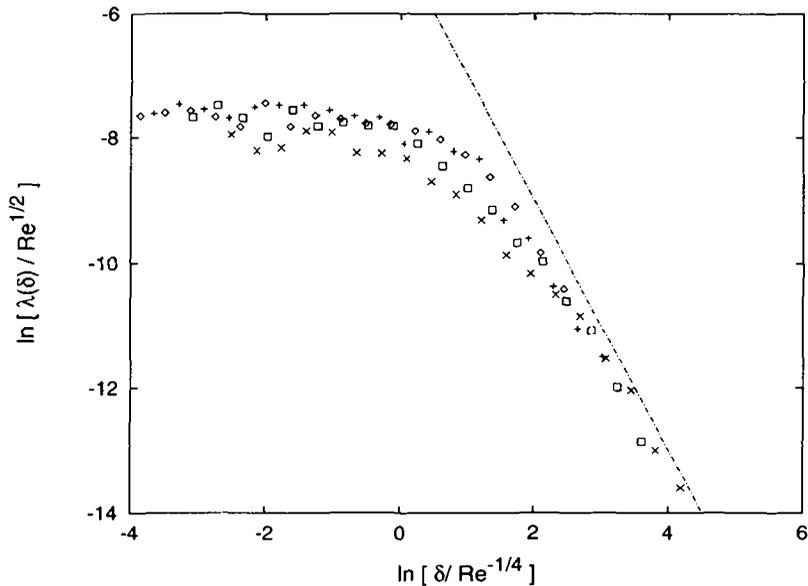


Figure 5: The rescaled FSLE $\lambda/Re^{1/2}$ versus the rescaled error size $\delta/Re^{-1/4}$ at the different Reynolds numbers as in the previous figure. The dashed line has slope -2 .

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