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Transient anomalous dispersion in random walkers

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Abstract

A simple model of dispersive tracers which display a transient anomalous regime is presented. It is based on an ensemble of random walkers belonging to two independent populations characterized by different Lagrangian decorrelation times. Apart from short-time ballistic and long-time diffusive behavior, the dispersion shows anomalous scaling at intermediate times over a wide range of variability for the free parameters of the model. © 1997 Elsevier Science B.V.

Full comprehensive of the general properties of particle dispersion in turbulent flows is of great practical importance for its geophysical and engineering implications. Single particle Lagrangian statistics is usually described by the square absolute dispersion

$$D_2(t) = \langle |x(t) - x(0)|^2 \rangle, \tag{1}$$

where $x(t)$ is a d-dimensional vector representing the position of the particle and the average is taken over many independent dispersion experiments with different initial positions and/or times. Particles are advected by the Lagrangian velocity according to the differential equation $\dot{x}(t) = u(t)$.

The asymptotic behavior of (1) follows immediately by the assumption that the velocity autocorrelation $\langle u(t)u(t+\tau)\rangle$ decays to zero in a finite time T. For very short time $t \ll T$ the particle is approximately advected by constant velocity, so $x(t) \sim x(0) + u(0)t + O(t^2)$ and thus one expects for the dispersion the ballistic law

$$D_2(t) = 2Et^2, \tag{2}$$

where E is the average kinetic energy of the flow. On the other hand, for $t \gg T$ the particle has experienced several independent and incoherent position increments and one expects the diffusive law to hold [1],

$$D_2(t) = 4ETt. \tag{3}$$

Recently has shown in various contexts the emergence of a dispersion regime which is neither ballistic nor diffusive. In the case of a power-law regime, $D_2(t) \sim t^\alpha$ with non-integer exponent α, the dispersion is called anomalous.

Many simple models of flows which lead to anomalous dispersion laws have been recently proposed (see Refs. [2,3] for a review). The basic mechanism leading to anomalous dispersion in these models is the existence of long-range correlations in the random displacements of the walker. This leads to the breakdown of the central limit theorem and thus to a deviation from the general result (3) [4].

Since the anomalous dispersion is a symptom of surviving correlations in the velocity field, its presence

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has been often interpreted as an indication of complex structures in the velocity field. For example, anomalous regimes have been found in numerical simulation of two-dimensional turbulence [5] where it has been linked to the presence of coherent vortices which can trap passive particles for very long times [6,7]. Anomalous dispersion laws have also been found in experimental data of surface buoys in the ocean [8]; the anomalous regime here holds at intermediate times while at longer times diffusive behavior is recovered. Although anomalous scaling in turbulent dispersion looks quite common, one cannot identify a unique scaling exponent α; this suggests that the generating physical mechanism should be non-universal.

In this Letter we investigate the average absolute dispersion for an ensemble of independent random walkers. Each walker is characterized by a decorrelation time T at which the transition from the ballistic regime (2) and the diffusive regime (3) takes place. For a very broad distribution of decorrelation times we recover, as expected, asymptotic anomalous dispersion.

We furthermore show that the presence of two populations with well separated decorrelation times T_1 and T_2 (bimodal distribution) is sufficient for recovering anomalous scaling at intermediate times. The standard asymptotic regime (3) still holds for very long times $t \gg T_2$ but in the intermediate regime $T_1 < t < T_2$ the average over fast walkers (which are already in the diffusive regime) and slow walkers (still in the ballistic regime) leads to dispersion behavior which can be approximatively described by a scaling law t^α with a single exponent $1 < \alpha < 2$.

We think that the proposed mechanism of generation of an intermediate anomalous regime by a superposition of different contributions at a fixed time is very general and can have physical relevance.

Consider an ensemble of one-dimensional random walkers (passive tracers) performing steps of length ± 1. Each walker is characterized by a single parameter ϵ, which is the probability to make a step in the opposite direction of the previous one. It is tightly related to the Lagrangian correlation time for the walker, since the average number of steps in the same direction is $n_c = 1/\epsilon$.

Introducing the (discrete) velocity $u_n = \pm 1$ at time step n, we consider the Markovian stochastic process

$$\begin{aligned} u_{n+1} &= u_n \qquad \text{with probability } 1-\epsilon, \\ &= -u_n \qquad \text{with probability } \epsilon. \end{aligned} \tag{4}$$

The tracer position x_n is governed by the first-order difference equation

$$x_{n+1} = x_n + u_n \tag{5}$$

and we will always assume for the initial position $x_0 = 0$.

To compute the average quantities one defines a 2×2 matrix P with the elements

$$\langle u|P|u'\rangle = [\tfrac{1}{2} + (\tfrac{1}{2} - \epsilon)uu']e^{hu}, \tag{6}$$

which for $h = 0$ gives the probability (4) of the transition from u to u'. Averages can be computed by means of the function

$$\begin{aligned} Z(n) &= \sum_{u_0}\sum_{u_1}\dots\sum_{u_n}\langle u_0|P|u_1\rangle\langle u_1|P|u_2\rangle\dots \\ &\quad\times \langle u_{n-1}|P|u_n\rangle = \sum_{u_0}\sum_{u_n}\langle u_0|P|u_n\rangle \\ &= \sum P^n, \end{aligned} \tag{7}$$

where the last expression indicates the sum of the four elements of matrix P^n. $Z(n)$ still contains the auxiliary variable h from which one obtains

$$D_p(n,\epsilon) = \langle (x_n)^p \rangle = \frac{1}{2}\left(\frac{\partial^p Z(n)}{\partial h^p}\right)_{h=0}, \tag{8}$$

where the factor $1/2$ comes from the normalization $Z(n)_{h=0} = 2$.

It is interesting to observe that expression (7) is exactly the partition function for a one-dimensional Ising model of n spins with ferromagnetic coupling $J = \frac{1}{2}\ln[(1-\epsilon)/\epsilon]$ [9]. The external magnetic field h is set to zero in the absence of an underlying drift. In this case all the odd moments of (8) are identically zero and we have for the absolute square dispersion the exact expression

$$D_2(n,\epsilon) = \frac{1-\epsilon}{\epsilon}n + \frac{1-2\epsilon}{2\epsilon^2}[(1-2\epsilon)^n - 1]. \tag{9}$$

The asymptotic regimes (ballistic and Brownian) are recovered from (9) in the two limits

$$D_2(n,\epsilon) \sim n^2, \qquad \text{for } n\epsilon \ll 1,$$

$$D_2(n,\epsilon) \sim \frac{1-\epsilon}{\epsilon} n, \quad \text{for } n\epsilon \gg 1. \qquad (10)$$

with the crossover between the two regimes at $n\epsilon \sim 1$, i.e. at the average inversion time $n_c = 1/\epsilon$. The decorrelation time N can be computed in a similar way and turns out to be

$$N = \left\langle \sum_n u_0 u_n \right\rangle = \frac{1-2\epsilon}{2\epsilon}. \qquad (11)$$

The continuous time limit, which is more suitable in view of the following considerations, is obtained by considering steps of width δt with a velocity inversion rate $\omega = \epsilon/\delta t$; letting ϵ, $\delta t \to 0$ keeping ω finite, introducing $t = n\delta t$ and considering steps of size $u_n \delta t$, the dispersion (9) reads [10]

$$D_2(t,\omega) = \frac{t}{\omega} + \frac{e^{-2\omega t} - 1}{2\omega^2} \qquad (12)$$

from which one recovers the ballistic regime (2) for $t \ll T$ and the standard diffusive law (3) for $t \gg T$ where the decorrelation time $T = N\delta t = 1/2\omega$ and $E = \frac{1}{2}\langle u^2 \rangle = 1/2$.

Now, take an ensemble of random walkers with a stationary distribution of inversion rates, described by a p.d.f. $p(\omega)$; for non-interacting walkers, the average absolute dispersion is simply given by

$$D_2(t) = \int_{\Omega_1}^{\Omega_2} d\omega p(\omega) D_2(t,\omega) \qquad (13)$$

with regularizing cutoff Ω_1 and Ω_2.

Anomalous dispersion laws are characterized by $D_2(\lambda t) = \lambda^\alpha D_2(t)$. To fulfill this scaling relation for the dispersion in the form of (13) one is forced to choose a power law distribution $p(\omega) \sim \omega^{1-\alpha}$ for inversion rates. By this choice, we obtain, for $1/\Omega_2 \ll t \ll 1/\Omega_1$,

$$D_2(t) \simeq t^\alpha. \qquad (14)$$

Thus we have recovered an anomalous dispersion law for a set of independent random walkers with a broad distribution (power-law) of inversion rates. This result is well known and applies to a wide class of physical situations.

Actually, we surprisingly found that intermediate anomalous regimes are quite common among different simple inversion rate distributions. The simplest distribution in this class is a bimodal distribution for two populations with two well separated decorrelation times T_1 and T_2. As shown below, in the long crossover $T_1 < t < T_2$ the average dispersion displays a pseudo-power-law behavior with a non-integer exponent. This is a pseudo-anomalous regime since no exact relation like (14) holds and the standard diffusive regime is asymptotically (for $t \gg T$) recovered.

Let us thus consider an inversion rate distribution of the form

$$p(\omega) = \lambda\delta(\omega - \omega_1) + (1-\lambda)\delta(\omega - \omega_2) \qquad (15)$$

with $\omega_1 > \omega_2$. This distribution corresponds to two independent components in the random walker population: one with shorter decorrelation time $T_1 = 1/2\omega_1$ and relative weight λ, the other with longer $T_2 = 1/2\omega_2$ and weight $1-\lambda$. Such a bimodal distribution can be assumed as a first approximation in situations where one can recognize two different characteristic timescales in the system. One interesting application is the dispersion of passive particles in two-dimensional turbulence in the presence of vortices [5,11]. Particles trapped within a vortex will follow the relatively regular vortex motion with long decorrelation times while particles in the turbulent fluid between vortices will experience a much faster decorrelating motion. A somewhat similar situation is the dispersion of vortices in a point vortex system where anomalous dispersion has also been found [12]. In this case the two populations can be associated to pairs of close vortices with like circulations (fast decorrelation) and opposite ones (slow decorrelation).

In Fig. 1 we plot the absolute dispersion (1) obtained by means of (13) and (15) with $\omega_1 = 0.2$, $\omega_2 = 0.008$ and $\lambda = 0.65$. At very short ($t < T_1$) and long ($t > T_2$) times one recognizes the asymptotic behavior (2) and (3). For intermediate times $T_1 < t < T_2$ a pseudo-scaling-law behavior $D_2(t) \sim t^\alpha$ ($\alpha \simeq 1.67$) is evident. Actually, the anomalous scaling lasts for less than a decade, as displayed by the instantaneous logarithmic slope in Fig. 2.

Extensive investigations of the absolute dispersion with a distribution of the form (15) demonstrate that anomalous scaling at intermediate times is quite common, since it is found for several values of the parameters ω_1, ω_2, λ. As an example, we plot in Fig. 3 the intermediate scaling exponent α as a function of the

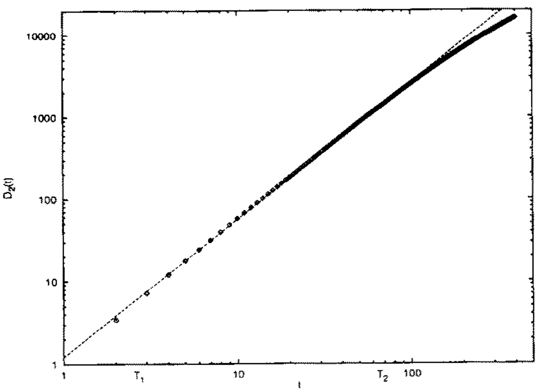

Fig. 1. Absolute dispersion $D_2(t)$ for a two component population in log-log plot. The decorrelation times are $T_1 = 2.5$ and $T_2 = 62.5$ and the relative weight is $\lambda = 0.65$. The dashed line represents the power law fit $D_2(t) \sim t^{1.67}$.

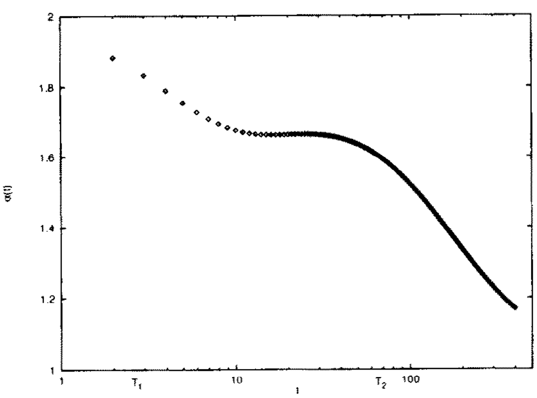

Fig. 2. Instantaneous logarithm slope $\alpha(t)$ of Fig. 1. Anomalous scaling is apparent from the constancy of the scaling exponent at intermediate times.

weight parameter λ for fixed $\omega_1 = 0.2$ and $\omega_2 = 0.008$. The value of α is numerically obtained by a log-log fit of the absolute dispersion over times $T_1 \ll t \ll T_2$. For $\lambda < 0.5$ we do not observe a clear intermediate scaling. Fig. 3 shows that there is no need of fine tuning of the parameters in order to obtain an anomalous scaling exponent within the range $1 < \alpha < 2$.

In conclusion, we have obtained an exact expression for the square absolute dispersion of a one-dimensional random walker with a finite decorrelation time. We have demonstrated that a population of independent random walkers with a power law distribution of decorrelation times gives rise to anomalous dispersion laws. We have then investigated the simple situation of a bimodal distribution and found that in this case too the absolute dispersion shows a pseudo-power-law behavior at intermediate times, strongly reminiscent of what is observed in many experimental data. We have shown that in this case the intermediate scaling exponent can assume any value within the interval $1 < \alpha < 2$ depending on the weight parameter λ.

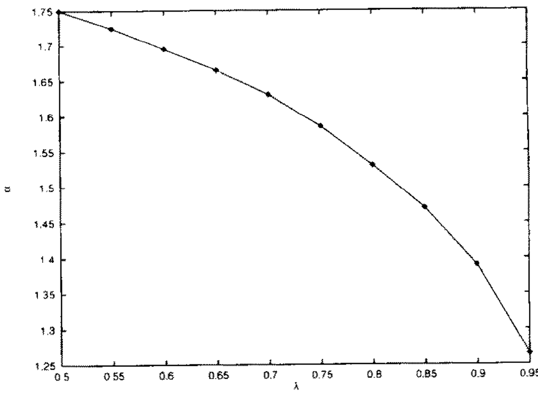

Fig. 3. Intermediate scaling exponent α for $T_1 = 2.5$, $T_2 = 62.5$ at different values of λ. The value of α is determined by a fit of Fig. 2 in the interval $10 < t < 50$.

It would be interesting to check whether real flows displaying anomalous dispersion regimes at intermediate times do share with our model the property of having Lagrangian decorrelation times with a bimodal distribution. If so, this could be argued to be the mechanism for anomalous dispersion.

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