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A new assessment of the second-order moment of Lagrangian velocity increments in turbulence

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The behaviour of the second-order Lagrangian structure functions on state-of-the-art numerical data both in two and three dimensions is studied. On the basis of a phenomenological connection between Eulerian space-fluctuations and the Lagrangian time-fluctuations, it is possible to rephrase the Kolmogorov 4/5-law into a relation predicting the linear (in time) scaling for the second-order Lagrangian structure function. When such a function is directly observed on current experimental or numerical data, it does not clearly display a scaling regime. A parameterisation of the Lagrangian structure functions based on Batchelor model is introduced and tested on data for $3d$ turbulence, and for $2d$ turbulence in the inverse cascade regime. Such parameterisation supports the idea, previously suggested, that both Eulerian and Lagrangian data are consistent with a linear scaling plus finite-Reynolds number effects affecting the small- and large timescales. When large-time saturation effects are properly accounted for, compensated plots show a detectable plateau already at the available Reynolds number. Furthermore, this parameterisation allows us to make quantitative predictions on the Reynolds number value for which Lagrangian structure functions are expected to display a scaling region. Finally, we show that this is also sufficient to predict the anomalous dependency of the normalised root mean squared acceleration as a function of the Reynolds number, without fitting parameters.

Keywords: isotropic turbulence; homogeneous turbulence; direct numerical simulation; two-dimensional turbulence

1. Introduction

The knowledge of the statistical properties of turbulence, and in particular its non-Gaussian statistics, is a key open problem in classical physics with important consequences for applications [1]. The description of a fluid flow can be equally done in the Eulerian frame, where the velocity field at any position and time is known, $\boldsymbol{u}(\boldsymbol{x}, t)$, or in the Lagrangian frame where the evolution of fluid tracers, $\boldsymbol{x}(t)$, is followed in time, $\boldsymbol{v}(t) = \boldsymbol{u}(\boldsymbol{x}(t), t)$ and $\boldsymbol{v}(t) = \dot{\boldsymbol{x}}(t)$. Although the two descriptions are mathematically equivalent, the second bears premises to better shed light into the dynamics of (small) particles dispersed and transported by turbulent flows [2,3].

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One of very few exact results known for three-dimensional homogeneous and isotropic turbulence is the Kolmogorov 4/5-law for inertial range of scales; for d-dimensional flows with $d = 2, 3$, it reads as

$$S_3(r) = \langle [(\boldsymbol{u}(\boldsymbol{x}+\boldsymbol{r}) - \boldsymbol{u}(\boldsymbol{x})) \cdot \hat{\boldsymbol{r}}]^3 \rangle = -\frac{12}{d(d+2)}\, \varepsilon r, \tag{1}$$

where longitudinal velocity increments are considered.

This relation connects velocity differences at scale r with the presence of a non-vanishing energy flux, ε. In the $3d$ direct cascade, the energy flux remains constant and positive at increasing the Reynolds number, giving rise to the dissipative anomaly of turbulence [1]. The translation of Equation (1) to the Lagrangian domain has been suggested long time back [4,5], but it only relies on phenomenological bases. It connects Eulerian fluctuations at separation r, $\delta_r u = u(x+r) - u(x)$, with Lagrangian temporal velocity difference over a time interval τ, $\delta_\tau v = v(t+\tau) - v(t)$, where space and time are connected through the local eddy turnover time:

$$\delta_\tau v \sim \delta_r u, \qquad \tau \sim r/\delta_r u. \tag{2}$$

Here due to the *dimensional* and phenomenological nature of the relation, all geometrical and vectorial properties are neglected. Moreover, it is important to stress that the symbol $\sim$ in Equation (2) is meant as *scale-as* in a pure *statistical* sense and not as a deterministic constraint holding point-by-point, as sometimes suggested [6]. It results that the phenomenological equivalent of the exact law (1) in the Lagrangian domain reads

$$S_2(\tau) \equiv \langle (\delta_\tau v)^2 \rangle \sim \varepsilon \tau, \tag{3}$$

where the prefactor cannot be exactly controlled. Another important difference with respect to (1) is that the sign of the right-hand side is also fixed, implying that (3) cannot be exact in principle because of the energy flux differently sign-defined in $2d$ and in $3d$ turbulence.

This relation is intimately connected with the picture of the Richardson cascade, built in terms of a superposition of eddies at different scales and with different characteristic times (eddy turn over times). The idea is to imagine that Lagrangian fluctuations, $\delta_\tau v$, at a given timescale, τ, are dominated by those Eulerian eddies, $\delta_r u$, which have a typical decorrelation time (2) of the order of the time lag, τ. Indeed eddies at smaller scales are much less intense, i.e. if $r' \ll r$, then $\delta_{r'} u \ll \delta_r u$, while eddies at larger scales do not contribute to Lagrangian fluctuations being almost frozen on the time lag, τ. The *bridge* relation (2) must be considered the zero-*th* order approximation connecting Lagrangian and Eulerian domains. It cannot be exact and it cannot be applied straightforwardly to all hydrodynamical systems, being strongly based on the hypotheses of locality of the energy transfer process and on the existence of a unique typical eddy turn over time at each scale. Therefore, it is not expected that it can straightforwardly explain Lagrangian–Eulerian correlations in conducting flows, as investigated in [7].

In considering the application of the bridge relation for Lagrangian scaling in $2d$ and $3d$ hydrodynamical turbulence, the situation is not at all yet clear. On the one hand, it has been successfully used to predict the probability density function of accelerations and the relative scaling between Lagrangian structure functions [8–11]. On the other hand, when looking at direct scaling versus the time lag, inconclusive results have been obtained [12–14]. As a consequence, different scaling behaviours have been proposed to overcome doubts raised due to the consistently poorer quality of the validation from both numerical and experimental tests [15,16], when compared to the Eulerian counterpart. Moreover, by means of a stochastic model, it has been argued that the observed reduced scale separation in the Lagrangian frame is the main reason for the departure from Kolmogorov scaling in data [17], and that the inertial sub-range linear scaling is eventually reached only at Reynolds numbers beyond $Re_\lambda = 30{,}000$ [18].

Note that acceleration probability and relative scaling of Lagrangian structures functions are a probe of intermittent fluctuations over time lags τ, which can be assessed independently of the scaling of the second-order moment $S_2(\tau)$. However, this deserves a particular interest since it is a key ingredient of Lagrangian stochastic models for turbulent diffusion and dispersion [3,19–21].

In this manuscript, we specifically address the issue of poor inertial range scaling of $S_2(\tau)$, basing our analysis on currently available numerical data. Our analysis points in the direction of an enhanced sensitivity to finite-*Re* corrections in the Lagrangian framework with respect to the Eulerian one. We show that a simple modelling of finite-Reynolds number effect, affecting the small and the large scales, can be enough to interpret present data on the basis of the *bridge* relation (2). This result confirms two things. First, that the dimensional *Kolmogorov-like* argument of (2), even if not supported by any exact theoretical statement, represents a very good first start to guess the statistical connections between Eulerian and Lagrangian statistics. Second, that any possible *new physics* beyond the relation (2) is to be compared with more refined data at higher Reynolds numbers.

We remark that on the basis of the refined similarity approach [22,23], no intermittency correction of the second-order structure function is expected. Alternatively, in [15], a small modification of the linear scaling for the second-order structure function has been proposed on the basis of the observed behaviour of the acceleration spectrum. While further data are needed to definitely discriminate between anomalous scaling or finite-Reynolds number effects in the second-order moment, we point out that the simple parameterisation here proposed gives very good results without invoking any intermittent correction. Finally, we stress that recently, by using Hilbert–Huang transform, further evidences for a linear scaling of second-order Lagrangian moment have been presented [24].

2. Batchelor parameterisation for the Lagrangian second-order structure function

We consider the second-order moment of velocity increments measured along tracer trajectories in statistically stationary, isotropic and homogeneous (HIT) 3*d* dimensional turbulence:

$$S_2(\tau) \equiv \langle [v_i(t+\tau) - v_i(t)]^2 \rangle, \tag{4}$$

where $v_i(t)$ is one component of the turbulent Lagrangian velocity field. As mentioned, the Kolmogorov scaling for the Eulerian velocity increments once translated into the time domain via the *bridge* relation gives – for any velocity component – the linear prediction $S_2(\tau) = C_0\,\varepsilon\tau$, where C_0 is a dimensionless constant of order unity. Observations [25] suggest that in 3*d* HIT, $C_0 \in [6\text{–}7]$; however, since even at the largest Reynolds number achieved, both experimental and numerical data do not show a well-developed scaling range in $S_2(\tau)$, the value of C_0 measured displays a weak yet detectable *Re* dependence [14–16,18,25,26].

The point that we address here is to understand if this poor scaling reflects a real deviation from the linear scaling of Lagrangian turbulence, or if it is just the result of finite-Reynolds numbers effects, coming from both ultraviolet and infrared cut-offs. In the latter case, one could expect that future DNS and experiments might be able to directly display scaling properties also in the Lagrangian domain, including intermittency. In fact, at the moment current practice is analysing intermittency in the Lagrangian domain only by using Extended Self-Similarity approach [10,11], hence bypassing the need for well defined power-law behaviour in the inertial range.

In order to understand the above issue, it is mandatory to have a control on the effects of viscous and integral scales on the *supposed* inertial range. Due to the lack of control on the analytical side, one possible way is to resort to phenomenological models [10,17,27], trying to reproduce the behaviour of the velocity increments over the entire range of scale/frequency. In particular, according to studies over the last few decades [10,11,27–29], a parameterisation proposed by Batchelor became quite popular because of its simplicity and capability to include non-trivial viscous-effects (such as the intermittency of velocity gradients and acceleration) [10,27], as well as the saturation effects observed at the large scales [11,28,29].

In the following, we test the possibility to get a suitable *Batchelor*-like parameterisation able to capture the poor scaling behaviour observed on the data. The anticipated success of this goal implies two facts. First, it shows that the absence of a genuine scaling observed at moderate Reynolds numbers *is not in contradiction* with the possibility to have scaling at higher Reynolds numbers. Second, it gives a first hint on how far in Reynolds number one needs to go before expecting an observable scaling behaviour. Of course, the Batchelor parameterisation is not based on any analytical result and finds a justification only on its ability to reproduce data. Other parameterisations are very much possible as well, and whether the Batchelor one will agree or not with data at higher Reynolds numbers is an open question for the future.

On a dimensional ground, a parameterisation for the time behaviour of $S_2(\tau)$ has to reproduce the three following regimes:

$$\begin{cases} S_2(\tau) \sim \epsilon\, \tau^2/\tau_\eta & \tau \ll \tau_\eta, \\ S_2(\tau) \sim \epsilon\, \tau (\tau/T_L)^{z_2-1} & \tau_\eta \ll \tau \ll T_L, \\ S_2(\tau) \sim \epsilon\, T_L & \tau \gg T_L, \end{cases} \tag{5}$$

where τ_η is the Kolmogorov timescale and T_L is the large-scale Lagrangian eddy turnover time. If we assume a Kolmogorov scaling in the temporal inertial range then $z_2 = 1$, otherwise it can be kept as a free parameter (see also Sec. 3). We recall that by dimensional arguments we have $T_L/\tau_\eta \propto Re_\lambda$. A functional form which interpolates between the above behaviours is simply obtained as [27–29]

$$S_2(\tau) = C_0\, \epsilon\, \frac{\tau^2}{\left(c_1\, \tau_\eta^2 + \tau^2\right)^{\frac{(2-z_2)}{2}}} \left(1 + c_3 \tau/T_L\right)^{-z_2}, \tag{6}$$

where c_1 and c_3 are order one dimensionless constants.

In Figure 1, we show the results for the linearly compensated second-order moment, when we take $T_L/\tau_\eta = 0.1 Re_\lambda$ [25]. It turns out that the effect of finite-Reynolds number induced by the large-scale saturation are big, since a plateau develops only for very large Reynolds numbers currently unreachable. In the inset, we zoom in the scaling region: starting for Reynolds number $Re_\lambda = 5000$, a scaling shows up.

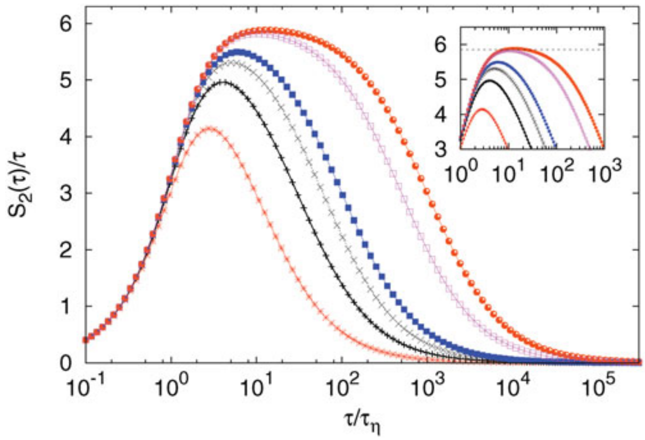

Figure 1. The linearly compensated second-order Lagrangian structure function as obtained with the Batchelor parameterisation (6), for different Re_λ. Starting from bottom curve, they refer to structure functions at the following values of Taylor-scale based Reynolds numbers $Re = 100; 300; 600; 1000; 5000$ and $Re_\lambda = 10{,}000$. The inertial range scaling exponent is fixed to $z_2 = 1$. Inset: a zoom in the scaling region to highlight a plateau starting to develop already at $Re_\lambda = 5000$.

One can of course play with the parameterisation in order to modify the transitions from viscous to inertial, and from inertial to integral ranges. In particular, by changing the functional form of the denominator in Equation (6) and of the saturation factor, these transitions can be made sharper or smoother [10].

We also note that in order to be consistent with an exponential decay for the velocity correlation function, one can possibly slightly refine the functional form of the saturation factor for large times (see below). It is thus probable that the observed absence of a clear and well-developed plateau in numerical and experimental data is just a finite-Reynolds number effect that, as we mentioned, are more pronounced in the Lagrangian statistics than in the Eulerian case (dimensionally, $L/\eta \propto Re_\lambda^{3/2}$ and $T_L/\tau_\eta \propto Re^{1/2}$).

In Figure 2, we present an analysis of DNS data of $3d$ HIT at $Re_\lambda = 180, 280, 400, 600$ (see [11,30,31]). In particular, we compare the linearly compensated second-order Lagrangian structure functions at the four Reynolds numbers (left panel), with curves obtained according to Equation (6). As one can see the fit is very good. Moreover, in the same figure we also show, to guide the eyes, the result of the Batchelor parameterisation for a much higher Reynolds number ($Re_\lambda = 5000$).

It is well known that time correlations along tracer trajectories decay very slowly. Hence, when considering the second-order Lagrangian structure function, there is the issue of the long time decaying of the velocity correlation functions. Here, we compare the power-law saturation factor $\propto (1 + \tau/T_L)^{-z_2}$ appearing in Equation (6), with an exponential saturation factor ruling the large times behaviour. We used the following interpolation:

$$S_2^*(\tau) = C_0\, \epsilon\, T_L \frac{\tau}{\left(c_1 \tau_\eta^2 + \tau^2\right)^{1/2}} (1 - \exp(-c_3 \tau / T_L)), \tag{7}$$

where in comparison to expression (6), we have fixed the exponent $z_2 = 1$ and C_0, c_1 and c_3 are free parameters. In the right panel of Figure 2, we compare the results of the

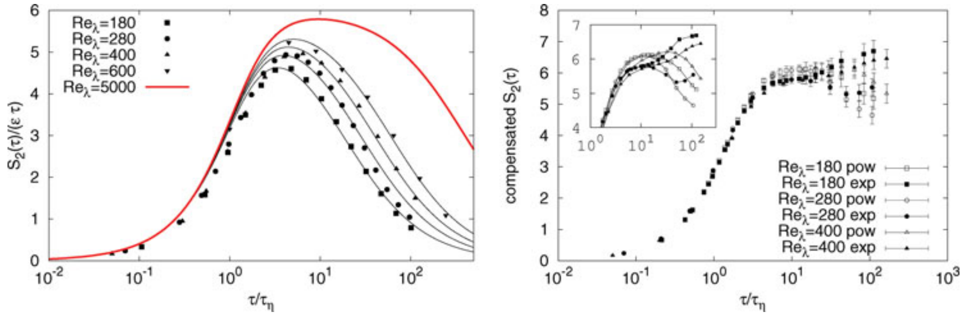

Figure 2. (Left panel) DNS of 3d HIT at $Re_\lambda \sim 180, 280, 400, 600$ [11,30,31]. $S_2(\tau)$ compensated with $\varepsilon\tau$ versus the Batchelor fit (solid lines) with a power-law large-scale saturation term. (Right panel) Same DNS data of HIT at three different Reynolds numbers, compensated such as to highlight inertial range behaviour according to the two Batchelor parameterisations (with large times exponential or power-law behaviour, see Equation (8)). Fitting parameters are: $c_1 = 2.2$ in the power-law and $c_1 = 2.5$ in the exponential form; $c_3 = 1.0$ in the power-law expression, while $c_3 = 1.5$ in the exponential expression; in all cases $C_0 = 6$. Error bars are estimated from the anisotropy of velocity components at large scales. (Inset of right panel) same curves as in the body, to highlight large scale behaviour. Error bars have been omitted for clarity.

two different functional forms for large timescales. In order to do it properly, we plot the second-order structure function compensated with its whole inertial and integral timescale regime, that is,

$$\frac{S_2(\tau)}{(\tau\,(1 + c_3\tau/T_L)^{-1})}; \qquad \frac{S_2^*(\tau)}{(\tau\,(1 - \exp(-c_3\tau/T_L)))}, \tag{8}$$

though the power-law and exponential forms are very close. Here clearly it is important to consider that at large scales we expect to have quite *large* statistical and systematic error bars, due to anisotropy and/or finite-size effects (see error bars in the right panel of Figure 2). Moreover, large scale fluctuations are not expected to be universal. It is interesting to note that once the large-scale contamination is removed, compensated data start to show a well-defined plateau already at moderate Reynolds numbers, independently of the functional form for the large-scale behaviour.

Scaling relations have to be consistent with kinematic constraints of – statistically stationary and isotropic – turbulence. One of such is that the time integral of the acceleration autocorrelation function is zero [32,33] ,

$$\int_0^\infty R_a(\tau)d\tau = \int_0^\infty \langle a_i(\tau)a_i(0)\rangle d\tau = 0. \tag{9}$$

Hence, the acceleration autocorrelation, which is positive at small time lags, should then be negative to match the kinematic constraint.

Provided a linear leading scaling is prescribed in the Lagrangian second-order structure function, the acceleration autocorrelation function is further constrained to be zero in the inertial range of scales [33]. In Figure 3, we plot the behaviour of the acceleration covariance obtained from the Batchelor parameterisation of the Lagrangian second-order structure function, which shows consistency vanishing behaviour in the scaling range. These findings are valid as possible working hypothesis, until as suggested in [33] and [15], a

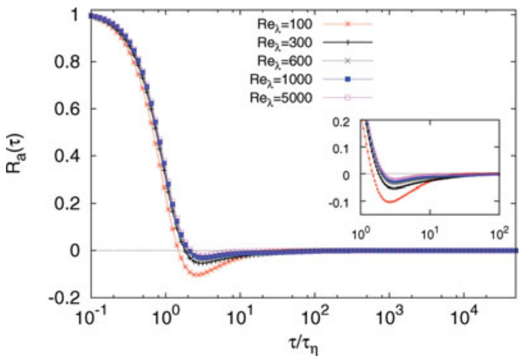

Figure 3. The temporal behaviour of acceleration autocorrelation function $R_a(\tau) = \langle a(\tau)a(0)\rangle$ derived from the Batchelor parameterisation of the Lagrangian second-order structure function, with exponential large-scale saturation, see Equation (7). Parameters are the same used for Figure 2. In the inset, a zoom in the negative region of the autocorrelation function.

precise form of the acceleration autocorrelation is known both in the dissipative and inertial sub-ranges for finite-Reynolds number $3d$ turbulence.

Hence, at least for $3d$ turbulence, we summarise these indications as follows: (i) the absence of a plateau can be related to the presence of strong large-scale and small-scale effects, competing with the inertial range behaviour; (ii) as it appears from the left and the right panels of Figure 2, the Lagrangian inertial range does not coincide with the plateau region, where the second-order structure function linearly compensated shows a peak, since the large-scale contamination is still present.

3. Intermittency corrections

It is well known that Lagrangian statistics in $3d$ is affected by intermittent corrections. In particular, acceleration statistics does not obey dimensional scaling: the normalised acceleration rms, $\langle a^2\rangle\tau_\eta/\varepsilon$ is observed to have a *weak* anomalous dependency on Re_λ:

$$a^2_{rms} = \langle a^2\rangle \sim a_0 \frac{\varepsilon}{\tau_\eta} Re_\lambda^\gamma, \qquad (10)$$

with $\gamma \sim 0.2$ (see also Figure 4). Similarly, the probability density function of the normalised acceleration $P(a/a_{\rm rms})$ possesses strong non-Gaussian and Reynolds-dependent tails [8]. It is remarkable that such intermittent corrections can be explained by invoking again the *bridge* relation previously discussed: so doing, it is possible to predict Lagrangian intermittent properties once the Eulerian ones are given, and viceversa [5,8,10,11,30]. In Figure (4), we report the compilation of data sets at different Reynolds numbers for the root-mean-square acceleration (10). On these data three curves are superposed: (1) a phenomenological fit proposed in [34], and the predictions obtained by using the bridge relation (2) with two different multi-fractal estimates of the Eulerian statistics, based on the longitudinal and on the transverse spatial increments [11]. The numerical data fall well within the two multi-fractal predictions, confirming the ability of the bridge relation to reproduce Lagrangian properties *without* any additional free parameter. We notice that the bridge relation still predicts that (3) holds true, i.e. intermittency is absent for third-order

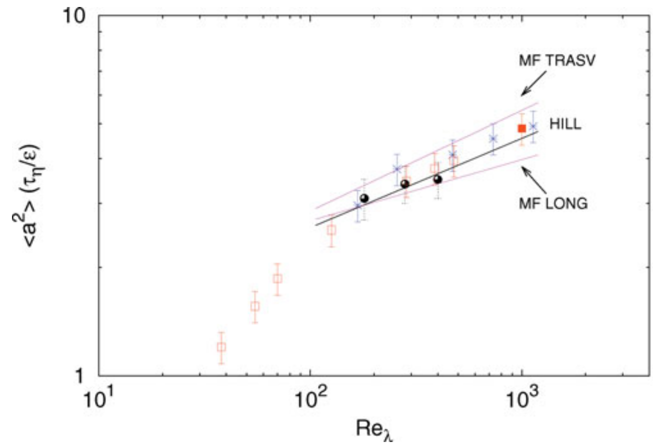

Figure 4. Collection of different numerical data of the scaling of normalised root-mean-square acceleration as a function of the Taylor-scale based Reynolds number Re_λ. Two lines correspond to the multi-fractal prediction using the bridge relation for transverse increments (MF TRASV) leading to $\gamma = 0.17$, or the bridge relation for longitudinal increments (MF LONG) leading to $\gamma = 0.28$ (see [10] for details). These lines can be shifted up or down arbitrarily, being the multi-fractal prediction valid scaling-wise and not for the prefactors. A third line is a fit proposed by Hill in [34], as a superposition of two power laws of exponents $\gamma_1 = 0.25$ and $\gamma_2 = 0.11$. Data are taken from Refs. [11,30,31,35–37]. Error bars are estimated considering a typical 10% uncertainty in the energy dissipation rate.

quantities in the Eulerian domain and *hence* for second-order quantities in the Lagrangian one.

An alternative approach can be followed by assuming *independent* anomalous scaling properties for Lagrangian and Eulerian domains, i.e. without using the bridge relation. In this case, $S_2(\tau)$ is not constrained to scale linearly and one could assume a pure inertial range intermittent correction as in Equation (5) with $z_2 = 1 - \gamma$,

$$S_2(\tau) \sim \varepsilon\tau \left(\frac{\tau}{T_L}\right)^{-\gamma} \quad \tau \gg \tau_\eta, \tag{11}$$

where γ is no longer linked to any Eulerian properties; moreover, τ_η must fluctuate independently of the Eulerian fields, too. This is another route to explain the anomalous scaling of the acceleration variance as a function of Re_λ, which has been investigated in [15] and which is not in contradiction with any exact scaling law in Lagrangian turbulence. In [15], the intermittent correction γ was obtained from a fit of the scaling of (10).

In Figure 5, we apply the intermittent compensation $\tau^{1-\gamma}$ to the DNS data shown in previous sections and observe that the plateau is slightly increased, but it is still very narrow. *Finite-Reynolds number* effects are overwhelming.

The question whether $S_2(\tau)$ scales linearly, or with an intermittent correction, or does not scale at all, needs data at higher Reynolds numbers to further support present ideas.

4. Inverse cascade in 2*d* turbulence

In this section, we present results on Lagrangian structure functions measured in the inverse cascade regime of 2*d* homogeneous and isotropic turbulence. Again, the general question

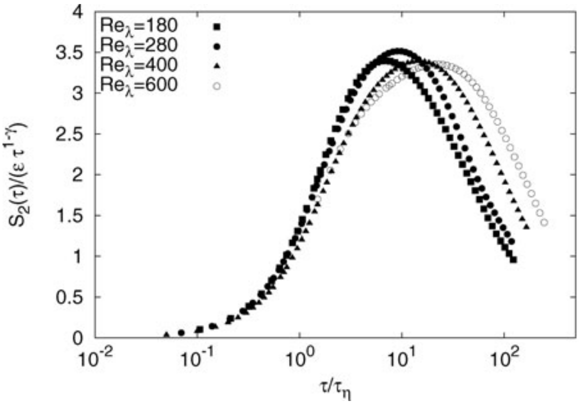

Figure 5. The second-order Lagrangian structure function compensated as $S_2(\tau)/(\epsilon\tau^{1-\gamma})$, with $\gamma = 0.22$. The anomalous correction γ is extracted from acceleration data shown in Figure (4).

we want to address is whether Lagrangian statistics are compatible with Eulerian statistics, i.e. if a suitable transformation from space to time is able to reproduce Lagrangian statistics given the Eulerian one. We remind ourselves that, in spite of the fact that the inverse cascade is statistically *simpler* than the direct cascade in $3d$ (since in $2d$ inverse cascade, the Eulerian statistics displays Kolmogorov scaling without intermittency corrections [38]), a recent work [39] claims that Lagrangian statistics do not reflect this simplicity and cannot be related to Eulerian statistics.

In the following, we consider Eulerian and Lagrangian structure functions obtained from numerical simulations of $2d$ Navier–Stokes equations for the vorticity $\omega = \nabla_x u_y - \nabla_y u_x$:

$$\frac{\partial \omega}{\partial t} + (\mathbf{u} \cdot \nabla)\omega = \nu \nabla^2 \omega - \alpha\,\omega + f_\omega, \tag{12}$$

in the inverse cascade regime at resolutions 2048^2. The forcing f_ω is active on a range of wavenumbers around $k_f \simeq 256$, is δ-correlated in time and injects energy at a fixed rate ε_I. About one half of the injected energy flows to large scale generating the inverse cascade with a flux ε. The $\alpha\,\omega$ friction term is necessary to reach a stationary state, and defines the large-scale eddy turnover time $T_L \simeq 1/\alpha$. Different runs correspond to different values of α and, therefore, to different extension of the inertial range of scales. The smallest characteristic time, the *Kolmogorov time* τ_η, is given in the inverse cascade by the time at the forcing scale $l_f \sim 1/k_f$, that is kept fixed. The extension of the inertial range in the time domain is thus growing as $T_L \propto 1/\alpha$.

To compare Eulerian and Lagrangian structure functions, a simple model, motivated by the cascade model for turbulence, can be introduced. We represent turbulent Eulerian velocity fluctuations $\delta_r u$ as the superposition of the contributions from different eddies in the cascade [40]: $\delta_r u = \sum_n u_n f(r/r_n)$, where u_n is the typical fluctuation at the scale r_n. The decorrelation function $f(x)$ is such that $f(x) \sim x$ as $x \ll 1$ and $f(x) \sim 1$ for $x \gg 1$: here, we choose the simple function $f(x) = 1 - exp(-x)$.

Within this framework, it is natural to represent the corresponding Lagrangian velocity fluctuation as $\delta_\tau v = \sum_n v_n f(\tau/\tau_n)$, where $\tau_n \sim r_n^{2/3}$ is the correlation time of the eddy at scale r_n. A minimal realisation of this model requires the presence of two scales that govern the crossover from dissipative to inertial scales, η, and from inertial to integral scales, L.

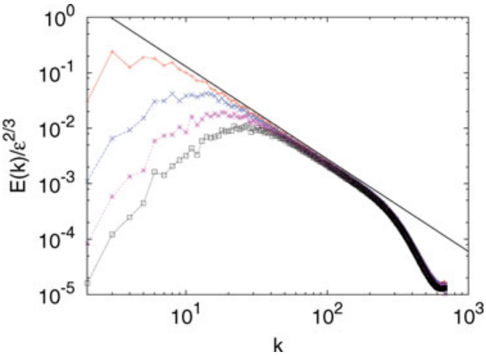

Figure 6. Kinetic energy spectra from $2d$ direct numerical simulations at resolution $N^2 = 2048^2$, with $\alpha = 0.02$ (red +), $\alpha = 0.04$ (blues $\times$), $\alpha = 0.06$ (pink $*$) and $\alpha = 0.08$ (black $\square$). The line represents Kolmogorov spectrum $E(k) = C\varepsilon^{2/3}k^{-5/3}$ with $C = 6$.

We can, therefore, write, introducing explicitly the scaling behaviour in the inertial range, the following relation:

$$\delta_r u = U\, f\left(\frac{r}{L}\right) + U\left[1 - f\left(\frac{r}{L}\right)\right] f\left(\frac{r}{\eta}\right)\left(\frac{r+\eta}{L}\right)^{1/3}, \qquad (13)$$

which, for Lagrangian increments, translates into

$$\delta v(\tau) = U_L f\left(\frac{\tau}{T_L}\right) + U_L\left[1 - f\left(\frac{\tau}{T_L}\right)\right] f\left(\frac{\tau}{\tau_\eta}\right)\left(\frac{\tau+\tau_\eta}{T_L}\right)^{1/2}. \qquad (14)$$

U and U_L are the root-mean-square velocities in the Eulerian and Lagrangian domain, respectively. In Figure 6, it is shown that in the stationary state, we observe an inverse cascade with a Kolmogorov spectrum that extends from the forcing wavenumber $k_f = 256$ to the friction wavenumber $k_\alpha \simeq \varepsilon^{-1/2}\alpha^{3/2}$ [38,41].

In Figure 7, we show the Eulerian second-order structure functions $S_2(r) = \langle(\delta_r u)^2\rangle$, compensated with dimensional scaling $(\varepsilon r)^{2/3}$, for different values of α. An important remark is that, in spite of the clear power-law scaling in the spectra, we do not observe any inertial range scaling for the Eulerian structure functions, even for the most resolved simulation. Nonetheless, the simple two-scales model, for the Eulerian statistics (13) and for the Lagrangian one (14), is able to reproduce quite accurately the crossovers from dissipative and to integral scales, with parameters $(\eta/L, \tau_\eta/T_L, U, U_L)$ which change according to dimensional predictions (cfr. caption of Figures 7 and 8). Lagrangian structure functions $S_2(\tau)$ linearly compensated with $\varepsilon\tau$ are shown in Figure 8, together with the prediction obtained from model (14). The model fits well the data, at least at small and intermediate times and with parameters that change with the extension of the inertial range.

We remark that the model parameterisation and the Batchelor model or any other model (see [17,42]) are all constructed on the hypothesis of a linear scaling in the inertial subrange. The point we want to make here is that, within the approximation model, the quality of data fit is comparable for Eulerian and Lagrangian statistics.

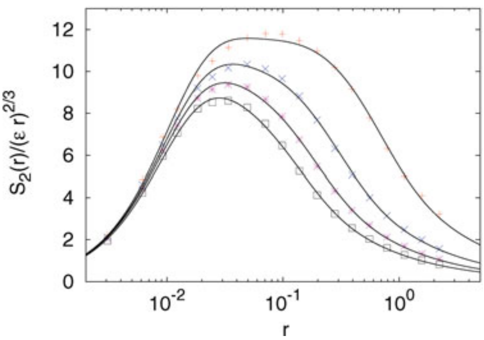

Figure 7. Eulerian second-order structure function $S_2(r)$ in the inverse energy cascade, compensated with Kolmogorov scaling $(\varepsilon r)^{2/3}$. Colours and symbols as in Figure 6. Lines represent the fit with $(\delta_r u)^2$ as in Equation (13), which gives the ratio $L/\eta = 12$ ($\alpha = 0.08$), $L/\eta = 16$ ($\alpha = 0.06$), $L/\eta = 25$ ($\alpha = 0.04$) and $L/\eta = 54$ ($\alpha = 0.02$).

More sophisticated multi-scale models can be envisaged, e.g. based on the superposition of a hierarchy of characteristic scales and times, at the price of a complex form of the parameterisation.

One interesting result discussed in [39] is that Lagrangian statistics in two dimensional turbulence are not Gaussian, even if Eulerian statistics are very close to Gaussian in the inverse cascade. Our simulations confirm this result but suggest that this is a delicate point as the statistics may depend on the observable. Figure 9 shows the excess kurtosis for Lagrangian structure function $\lambda(\tau) = \left[S_4(\tau)/S_2(\tau)^2 - 3\right]$ and for Eulerian structure function $\lambda(r) = \left[S_4(r)/S_2(r)^2 - 3\right]$, measured for both the x-component velocity increments and for the longitudinal velocity increments. The kurtosis of longitudinal velocity increments is constant and close to Gaussian value at all scales, but this is not the case for Eulerian increments of a single component of the velocity. We do not have a simple explanation for this observation, but we think that this is a possible origin of the deviation from Gaussianity observed in Lagrangian statistics. Indeed, Figure 9 also shows that the Lagrangian excess kurtosis $\lambda(\tau)$ is very close to $\lambda(r)$, when time is rescaled using the

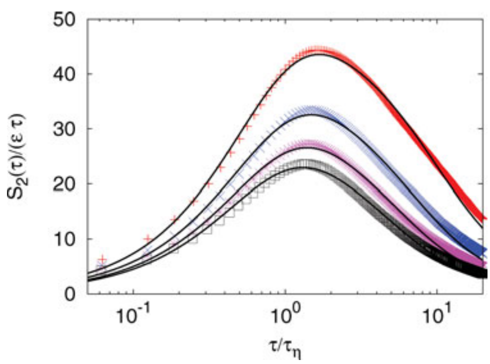

Figure 8. Second-order Lagrangian structure function $S_2(\tau)$ in the inverse energy cascade, compensated with $\varepsilon\tau$. Colours and symbols as in Figure 6. Lines represent the fit with $(\delta_\tau v)^2$ as in Equation (14), which gives the ratio of times $T_L/\tau_\eta = 7.8$ ($\alpha = 0.08$), $T_L/\tau_\eta = 8.9$ ($\alpha = 0.06$), $T_L/\tau_\eta = 11.2$ ($\alpha = 0.04$) and $T_L/\tau_\eta = 16.1$ ($\alpha = 0.02$).

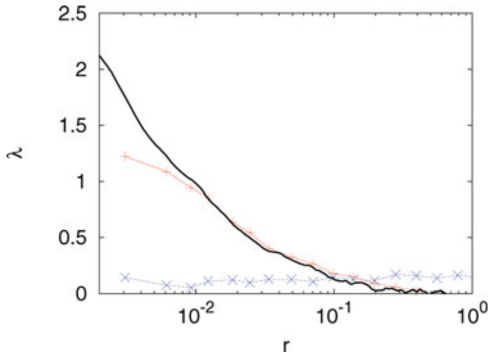

Figure 9. Excess kurtosis $\lambda(r) = S_4/S_2^2 - 3$ for Eulerian longitudinal structure function ($\times$ blue); for the Eulerian x-component structure function (+ red) averaged over the increment vector $\mathbf{r}$ taken in all directions; for the Lagrangian structure function (black line) with time rescaled as $r = 0.035\tau^{3/2}$. Data refer to the run with $\alpha = 0.02$.

bridge relation $\tau_r = cr^{2/3}$. Of course, this rescaling can work only in the inertial range and, therefore, we observe deviations at small separation r.

5. Conclusions

The Lagrangian and Eulerian description of the velocity field of a fluid are of course correlated and it should be possible to rephrase some statistical properties of Eulerian turbulence in terms of Lagrangian counterparts and viceversa. The question is 'how much' and 'what' one can bridge by using relation (2).

The simplest phenomenological description connects velocity fluctuations in time to velocity fluctuation in space, $\delta_r u \sim \delta_\tau v$, where the time-lag, τ, and space separation, r, are connected by the relation $\tau \sim r/\delta_r u$. From such a connection, one can obtain the prediction that $S_2(\tau) \sim \varepsilon\tau$, independently of the Eulerian intermittency, which is the Lagrangian rephrasing of the Kolmogorov 4/5-law.

As we discussed, both the linear scaling relation of $S_2(\tau)$ and the Eulerian vs. Lagrangian mapping could be objected to. Reasons for questioning their validity are: (i) the fact that such a relation, contrarily to the 4/5-law, is not rigorously derived; (ii) the fact that the scaling of the $S_2(\tau)$ appears to be of poorer quality than its Eulerian counterpart.

In the present manuscript, we have addressed the issue of the consistency of present state-of-the-art numerical data with the linear dimensional scaling for the $S_2(\tau)$, both in $3d$ and $2d$ turbulence. More specifically we have tried to shed further light on the question whether or not the present data are consistent with the linear scaling for the $S_2(\tau)$ plus finite-Reynolds number effects. Eulerian and Lagrangian data, both for $3d$ and $2d$ turbulence, appear to agree equally well with a Batchelor-like parameterisation, which takes into account dissipative and integral effects in a phenomenological way.

This indicates that present $3d$ and $2d$ Lagrangian data are *not inconsistent with* the relation (2), once finite-Reynolds number effects are kept into account. Furthermore, the use of the Batchelor parameterisation in $3d$ turbulence allows to make prediction on the values of Reynolds number for which a given window of direct scaling is expected to appear in the second-order moment.

Alternatively, one might not follow the Ockham's suggestion *'Entia non sunt multiplicanda praeter necessitatem'* [43] and invoke a genuine – i.e. not Reynolds number dependent – departure of $S_2(\tau)$ from the linear scaling predicted by (2). For instance, in [15] it was investigated the possibility that anomalous scaling develops already for $S_2(\tau)$ and it was showed that also this option *is not inconsistent with the* data.

More investigation will be needed to understand whether the simple description (2), plus Reynolds number effects, is all we need – as far as scaling properties are concerned – or if anomalous scaling as suggested in [15] is correct.

Here, we also showed that (2) is able to predict the Reynolds number dependency of the normalised root mean squared acceleration without the need to introduce any free parameter, if multi-fractal fluctuations in the Eulerian statistics are considered.

Finally, let us comment that in $3d$ turbulence, different scaling exponents for transverse and longitudinal spatial increments are observed [11,44], something not fully understood. Along a Lagrangian trajectory, both longitudinal and transverse Eulerian fluctuations are naturally mixed and entangled, introducing some uncertainties in the *bridge* relation as discussed here. In the $2d$ inverse cascade regime, Eulerian longitudinal increments do not show any deviation from Gaussianity, while the excess kurtosis measured on a mixed longitudinal and transverse Eulerian increments is different from zero. The Lagrangian equivalent of the latter Eulerian measurement is also non-Gaussian and in agreement with the *bridge* relation. Therefore, there are still many open points that must be further clarified.

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