The quantum structure of EFT

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constructing SMEFT

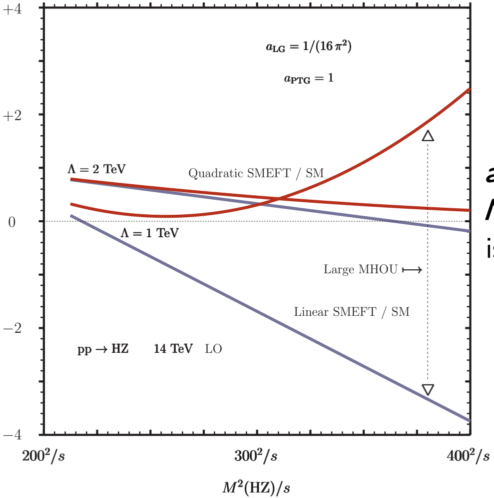
- Experiments occur at finite energy and measure $\mathbf{S^{eff}(\Lambda)}$
- Whatever QFT should give low energy $\mathbf{S^{eff}(\Lambda)}$, $\forall \Lambda < \infty$
- There is no fundamental scale above which $\mathbf{S^{eff}(\Lambda)}$ is not defined (K. Costello, Renormalization and EFT, AMS)
- $\mathbf{S^{eff}(\Lambda)}$ loses its predictive power if a process at $\boldsymbol{E = \Lambda}$ requires $\boldsymbol{\infty}$ renormalized parameters (J. Preskill, CALT-68-1493)

No phenomenology here, not even a discussion on the *validity* range of the SMEFT, i.e.

- SMEFT (linear representation) vs. HEFT (non-linear representation),
- linear vs. quadratic implementation of $\mathrm{dim} = 6$ operators, i.e. positivity,
- unitarity bounds
- analyticity, etc.

Waiting for the next SM or the next EFT, is SMEFT a consistent QFT from the point of view of removing UV poles (beware, an EFT is not stricty renormalizable)?

Note a/Λ^2 degeneracy



a fixed
Λ small
is large MHOU

EFT is just the effective theory only designed for perturbative calculations of the S-matrix elements. Green functions may contain divergent contributions: we are only interested in Dyson-perturbation- scheme of calculating the S matrix.

Though the effective theory is renormalizable by construction it presents no interest until the problem of redundant operators is solved.

Thus it is necessary to point out an infinite number of renormalization prescriptions that allow one to fix the finite parts of counterterms. If this is done arbitrarily, the theory loses its predictive power. Unfortunately, we do not have an infinite number of corresponding physical principles needed to avoid the problem.

Therefore, one must either indicate new (sufficiently powerful) principles or radically reduce the number of free parameters in the theory. Anyway, we need to know the whole list of free parameters which S-matrix depends upon.

Technically speaking, we need to make all Green functions UV finite, including $\dim = 6$ operators and, at least, $\dim = 8$ operators. This means computing the UV divergent part of a huge number of diagrams. The problem is already clear for the $\dim = 4$ part of the Lagrangian. To give an example, think about one-loop CTs for QGR. Is there a simple path?

Yes, it was *invented* by G. 't Hooft and it is based on the so-called background-field-method (BFM).

For the SMEFT the problem is even more severe, since the number of CTs grows with the order in PT. Any calculation will produce a huge number of terms, most of them being related by EoMs or differing by a total divergence. Consider the following Lagrangian

Consider the following Lagrangian

$$\begin{aligned}\mathscr{L}(\mathrm{A}+\phi) &= \mathscr{L}(\mathrm{A})+\phi_i\,\mathscr{L}_i'(\mathrm{A})+\frac{1}{2}\,\partial_\mu\,\phi_i\,\mathrm{W}_{ij}^{\mu\nu}(\mathrm{A})\,\partial_\nu\,\phi_j \\ &+ \phi_i\,\mathrm{N}_{ij}^{\mu}\,\partial_\mu\,\phi_j+\frac{1}{2}\,\phi_i\,\mathrm{M}_{ij}(\mathrm{A})\,\phi_j+\mathscr{O}(\phi^3)+\text{total derivative}\end{aligned}$$

Classical equations of motion are $\mathscr{L}_i'(\mathrm{A})=0$. All one loop diagrams are generated by $\mathscr{L}(\phi)$, the part quadratic in ϕ.

Assume

$$
\begin{aligned}
\mathrm{W}_{ij}^{\mu\nu} &= -\delta^{\mu\nu}\,\delta_{ij} \\
\mathscr{L}(\phi) &= -\frac{1}{2}\left(\partial_\mu\,\phi\right)^2 + \phi\,\mathrm{N}^\mu\,\partial_\mu\,\phi + \frac{1}{2}\,\phi\,\mathrm{M}\,\phi
\end{aligned}
$$

The Counter- Lagrangian is

$$
\begin{aligned}
\Delta\mathscr{L} &= \frac{1}{8\,\pi^2\,(d-4)}\Big[a_0\,M^2 + a_1\,\left(\partial_\mu\,\mathrm{N}_\nu\right)^2 + a_2\,\left(\partial_\mu\,\mathrm{N}_\mu\right)^2 + a_3\,\mathrm{M}\,\mathrm{N}^2 \\
&+ a_4\,\mathrm{N}_\mu\,\mathrm{N}_\nu\,\partial_\mu\,\mathrm{N}_\nu + a_5\,\left(\mathrm{N}^2\right)^2 + a_6\,\left(\mathrm{N}_\mu\,\mathrm{N}_\nu\right)^2\Big]
\end{aligned}
$$

However

$$\mathscr{L}(\phi) \quad = \quad -\frac{1}{2}\left(\partial_\mu \phi + \mathrm{N}_\mu \phi\right)^2 + \frac{1}{2}\phi X \phi$$

$$X \quad = \quad \mathrm{M} - \mathrm{N}^\mu \mathrm{N}_\mu$$

is invariant under

$$\begin{aligned} \phi' &= \phi + \Lambda \phi \\ \mathrm{N}'_\mu &= \mathrm{N}_\mu - \partial_\mu \Lambda + \left[\Lambda, \mathrm{N}_\mu\right] \\ X' &= X + \left[\Lambda, X\right] \end{aligned}$$

Therefore $\Delta\mathscr{L}$ also will be invariant

$$\begin{aligned} \Delta\mathscr{L} &= \frac{1}{\varepsilon}\,\mathrm{Tr}\left(a X^2 + b\, Y^{\mu\nu}\, Y_{\mu\nu}\right) \\ Y_{\mu\nu} &= \partial_\mu \mathrm{N}_\nu - \partial_\nu \mathrm{N}_\mu + \mathrm{N}_\mu \mathrm{N}_\nu - \mathrm{N}_\nu \mathrm{N}_\mu \end{aligned}$$

To understand the EFT problem we consider a toy model, one complex scalar field, but will include both $\mathrm{dim} = 6$ and $\mathrm{dim} = 8$ operators. The Lagrangian looks as follows:

$$\begin{aligned}
\mathscr{L} &= \mathscr{L}_4 + \frac{g_6}{M^2}\,\mathscr{L}_6 + \frac{g_6^2}{M^4}\,\mathscr{L}_8 \\
\mathscr{L}_4 &= -\partial_\mu \phi^* \,\partial_\mu \phi - m^2 \phi^* \phi - \frac{1}{2}\lambda\,(\phi^* \phi)^2 \\
\mathscr{L}_6 &= g^2 \Big[a_1 \mid \phi \mid^2 \Box \mid \phi \mid^2 + a_2 \left(\phi^* \partial_\mu \phi\right)^* \phi^* \partial_\mu \phi \\
&+ g^2 a_3 \left(\mid \phi \mid^2 - \frac{1}{2}\mathrm{v}^2 \right)^3 \Big]
\end{aligned}$$

$$
\begin{aligned}
\mathscr{L}_8 \quad &= \quad g^6\, a_4 \left(|\,\phi\,|^2 - \frac{1}{2}\,\mathrm{v}^2 \right)^4 + g^4\, a_5 \left(|\,\phi\,|^2 - \frac{1}{2}\,\mathrm{v}^2 \right)^2 \mathrm{B} \\
&+ \quad g^4\, a_6\, |\,\phi\,|^2\, \mathrm{V}^*_\mu\, \mathrm{V}_\mu + g^2\, a_7\, \mathrm{B}^2 \\
&+ \quad \frac{1}{2}\, g^2\, a_8 \left(\mathrm{V}^*_\mu\, \mathrm{U}_\mu + \mathrm{V}_\mu\, \mathrm{U}^*_\mu \right) + a_9\, \mathrm{S}\, \mathrm{B} \\
&+ \quad a_{10}\, \mathrm{T}_{\mu\nu}\, \mathrm{T}_{\nu\mu} + a_{11}\, \mathrm{Z}^*_{\mu\nu}\, \mathrm{Z}_{\mu\nu} + a_{12} \left(\mathrm{Re}\, \mathrm{Z}_{\mu\nu} \right) \left(\mathrm{Re}\, \mathrm{T}_{\mu\nu} \right)
\end{aligned}
$$

$$
\begin{aligned}
\mathrm{B} \quad &= \quad 2\,\partial_\mu\, \phi^*\, \partial_\mu\, \phi + \phi^*\, \Box\, \phi + \phi\, \Box\, \phi^* \\
\mathrm{V}_\mu \quad &= \quad \phi^*\, \partial_\mu\, \phi \\
\mathrm{U}_\mu \quad &= \quad (\Box\, \phi^*)\, \partial_\mu\, \phi + 2\,\partial_\nu\, \phi^* \partial_\mu\, \partial_\nu\, \phi + \phi^*\, \Box\, \partial_\mu\, \phi \\
\mathrm{S} \quad &= \quad \partial_\mu\, \phi^*\, \partial_\mu\, \phi \\
\mathrm{T}_{\mu\nu} \quad &= \quad \partial_\mu\, \phi^*\, \partial_\nu\, \phi \\
\mathrm{Z}_{\mu\nu} \quad &= \quad \phi^*\, \partial_\mu\, \partial_\nu\, \phi
\end{aligned}
$$

Introduce

$$
\begin{aligned}
\phi &= \frac{1}{\sqrt{2}}\,(h + \mathrm{v} + i\,\psi) \\
m^2 &= \beta_h - \frac{1}{2}\,\mathrm{v}^2 \\
\mathrm{v} &= \frac{M}{g}
\end{aligned}
$$

where β_h fixes tadpoles

However, the Lagrangian is not canonically normalized (LSZ etc.), therefore we have to transform the fields, which is allowed by the Equivalence Theorem (ET)

$$
\begin{aligned}
h &= \left(1+g_6\,x_h^{(1)}+g_6^2\,x_h^{(2)}\right)h'+\frac{g_6^2}{M^2}\,X_h\,\Box\,h' \\
\psi &= \left(1+g_6\,x_\psi^{(1)}+g_6^2\,x_\psi^{(2)}\right)\psi'+\frac{g_6^2}{M^2}\,X_\psi\,\Box\,\psi'
\end{aligned}
$$

$$
M = \left(1+g_6\,x_M^{(1)}+g_6^2\,x_M^{(2)}\right)M'
$$

$$
\begin{aligned}
x_M^{(1)} &= -x_h^{(1)} \quad x_h^{(1)}=\frac{1}{4}\,a_2-a_1 \quad x_\psi^{(1)}=\frac{1}{4}\,a_2 \\
X_\psi &= \frac{1}{4}\,(a_8-a_1 1) \quad x_M^{(2)}=-x_h^{(2)}+a_1^2-\frac{1}{2}\,a_1\,a_2+\frac{1}{16}\,a_2^2 \\
X_h &= -a_7+\frac{1}{4}\,a_8-\frac{1}{4}\,a_{11} \quad x_\psi^{(2)}=\frac{3}{32}\,a_2^2+\frac{1}{8}\,a_6 \\
x_h^{(2)} &= \frac{3}{2}\,a_1^2-\frac{3}{4}\,a_1\,a_2+\frac{3}{32}\,a_2^2+\frac{1}{8}\,a_6-a_7+\frac{1}{4}\,a_8-\frac{1}{4}\,a_{11}
\end{aligned}
$$

BFM obtains by expanding, $h \to \mathrm{H} + h, \psi \to \Psi + \psi$. The resulting Lagrangian is still invariant under the following (shifted) transformations

$$\mathrm{H} \to \mathrm{H} + \Lambda \Delta \mathrm{H} \qquad\qquad \Psi \to \Psi + \Lambda \Delta \Psi$$

$$\begin{aligned}
\Delta \mathrm{H} &= -\Psi - g_6\, a_1\, \Psi \\
&+ g_6^2 \left[\left(\frac{1}{2}\, a_1^2 - \frac{1}{2}\, a_1\, a_2 - a_7 + \frac{1}{4}\, a_8 - \frac{1}{4}\, a_{11} \right) \Psi - \frac{a_7}{M^2} \Box \Psi \right] \\
\Delta \Psi &= \mathrm{H} + \frac{M}{g} + g_6 \left[\frac{1}{2}\, (2\, a_1 - a_2)\, \frac{M}{g} - a_1\, \mathrm{H} \right] \\
&+ g_6^2 \left[\frac{1}{4} \left(6\, a_1^2 - 2\, a_1\, a_2 - 4\, a_7 + a_8 - a_{11} \right) \mathrm{H} \right. \\
&- \left. \frac{1}{4} \left(2\, a_1^2 + a_6 - 4\, a_7 + a_8 - a_{11} \right) \frac{M}{g} - \frac{a_7}{M^2} \Box \mathrm{H} \right]
\end{aligned}$$

Remarks:

- Once we have the Lagrangian written in the language of the BFM we can use EoMs and the relevant message is that EoMs must only be used for classical fields
- Next, the integration over the quantum fields is Gaussian and can be performed according to a well-known algorithm
- It is worth noting that the problem of finding UV poles is reduced to the one of computing tadpoles
- Still we will have plenty of terms that are equivalent when using EoMs and neglecting total derivatives. We need an algorithm for that
- Only at this point we can determine field/parameter CTs and the mixing among Wilson coefficients. The presence of $\dim = 8$ operators is not trivial

Redundancy algorithm

$$
\begin{aligned}
\mathsf{TD}_1 &= \partial_\mu\,\partial_\mu\,\partial_\nu\,\partial_\nu\,\mathrm{H}^2 = 0 \\
\ldots & \qquad \ldots \\
\mathsf{TD}_{18} &= \partial_\mu\,\left(\partial_\mu\,\partial_\nu\,\Psi\right)\,\partial_\nu\,\Psi = 0
\end{aligned}
$$

Apply EoMs and introduce scalars

$$
\begin{aligned}
B_1 &= \left(\partial_\mu\,\partial_\nu\,\mathrm{H}\right)^2 \\
B_2 &= \left(\partial_\mu\,\partial_\nu\,\Psi\right)^2 \\
B_3 &= \left(\partial_\mu\,\partial_\nu\,\mathrm{H}\right)\,\left(\partial_\mu\,\partial_\nu\,\Psi\right) \\
B_4 &= \left(\partial_\mu\,\mathrm{H}\right)\,\left(\partial_\mu\,\Psi\right)
\end{aligned}
$$

Solve and obtain

$$
\begin{aligned}
B_1 &= -M^2\,\mathrm{K}_h + \mathscr{O}(\text{fields}^3) + \mathscr{O}(\text{fields}^4) \\
&\text{etc.}
\end{aligned}
$$

$$
\mathrm{K}_h = \partial_\mu \mathrm{H}\, \partial_\mu \mathrm{H}
$$

Move to 3 fields, e.g.

$$
\begin{aligned}
\mathrm{TD}_1 &= \partial_\mu\, \partial_\mu\, \partial_\nu\, \partial_\nu\, \mathrm{H}^3 = 0 \\
&\cdots \\
\mathrm{TD}_{46} &= \partial_\mu \left(\partial_\mu \Psi\right) \mathrm{H}^2 = 0
\end{aligned}
$$

Apply EoMs, introduce scalars

$$
\begin{aligned}
C_1 &= \mathrm{H}\left(\partial_\mu \partial_\nu \mathrm{H}\right)^2 \\
&\dots \\
C_{20} &= \Psi \partial_\mu \Psi \partial_\mu \Psi
\end{aligned}
$$

Solve the equations, e.g.

$$
C_1 = \frac{1}{4} M^4 \mathrm{H}^3 + \text{fields}^4 + \cdots + \text{fields}^7
$$

until all redundant expressions are eliminated.

The Lagrangian has the following structure

$$
\begin{aligned}
\mathscr{L} \quad = \quad & \sum_{n=0,2} g_6^n\, \mathrm{M}_{0n}(1,1)\, h^2 + \sum_{n=0,2} g_6^n\, \mathrm{M}_{0n}(2,2)\, \psi^2 \\
+ \quad & \sum_{n=0,2} g_6^n\, \mathrm{M}_{0n}(1,2)\, h\, \psi + \sum_{n=1,2} g_6^n\, \mathrm{M}^{\mu}_{1n}(1,2)\, h\, \partial_\mu\, \psi \\
\ldots & \\
+ \quad & g_6^2\, \mathrm{M}_4(1,2)\, h\, \Box\, \Box\, \psi
\end{aligned}
$$

Gaussian integration amounts to compute the trace of a logarithm where, in the M-matrices one replaces

$$
\partial_\mu \quad \to \quad \partial_\mu + i\, q_\mu
$$

integration over q follows

Remark: we are looking for UV divergent parts, therefore the expansion stops after a finite number of terms. Typical integrals (tadpoles) are:

$$\int d^d q \frac{1}{(q^2+M^2)^j} \quad = \quad \mathrm{F}(j\,;M)$$
$$\cdots$$

Only $j = 1,2$ give UV poles, etc.

$$\begin{aligned}\ln \mathrm{M}(x, \partial_x)\, \delta(x-y) &= \ln \mathrm{M}(x, \partial_x) \int \frac{d^4 q}{(2\pi)^4} \exp\{i\, q \cdot (x-y)\} \\ &= \int \frac{d^4 q}{(2\pi)^4} \exp\{i\, q \cdot (x-y\} \ln \mathrm{M}(x, \partial_x + i\, q)\end{aligned}$$

we have to compute

$$\exp\{-\frac{1}{2} \mathrm{Tr} \ln \mathrm{M}\, \delta(x-y)\}$$

and we perform and expansion of the trace in inverse powers of propagators. Since we are looking for UV poles, the expansion will stop somewhere (up to M^6 is required when $\dim = 8$ operators are inserted).

It is worth noting that Tr ln M requires a symmetric matrix M and that particular care is due in avoiding spurious IR divergences (ψ is massless)

$$\ln(\mathrm{AB}) \quad = \quad \ln \mathrm{A} + \ln \mathrm{B} \quad \text{if} \quad \left[\mathrm{A}\,,\,\mathrm{B}\right] = 0$$

$$\mathrm{Tr}\,\ln(\mathrm{AB}) \quad = \quad \mathrm{Tr}\,\ln \mathrm{A} + \mathrm{Tr}\,\ln \mathrm{B} \quad \text{if} \quad \mathrm{AB} \quad \text{positive-definite}$$

Solution: write $\mathrm{M} = \lambda\,(\mathrm{I} + \mathrm{K})$ and use

$$\ln\left[\lambda\,(\mathrm{I} + \mathrm{K})\right] \quad = \quad \ln(\lambda\,\mathrm{I}) + \ln(\mathrm{I} + \mathrm{K}) = (\ln\lambda)\,\mathrm{I} + \mathrm{K} - \frac{1}{2}\mathrm{K}^2 + \ldots$$

Only at this point and after eliminating all redundant terms, we substitute

$$\begin{aligned} \mathrm{H} &= \mathrm{Z}_h \, \mathrm{H} \qquad \Psi = \mathrm{Z}_\psi \, \Psi \\ M &= \mathrm{Z}_M \, M \qquad g = \mathrm{Z}_g \, g \end{aligned}$$

$$\begin{aligned} \mathrm{Z}_h &= 1 + g^2 \left(\delta \mathrm{Z}_h^{(4)} + g_6 \, \delta \, \mathrm{Z}_h^{(6)} + g_6^2 \, \delta \mathrm{Z}_h^{(8)} \right) \frac{1}{\varepsilon} \\ &\cdots \\ \mathrm{Z}_g &= 1 + g^2 \left(\delta \mathrm{Z}_g^{(4)} + g_6 \, \delta \, \mathrm{Z}_g^{(6)} + g_6^2 \, \delta \mathrm{Z}_g^{(8)} \right) \frac{1}{\varepsilon} \end{aligned}$$

$$
\begin{aligned}
\delta Z_g^{(6)} &= \sum_{n=1,3} \delta Z_g^{(6)}(n)\, a_n^{\mathrm{r}} \\
\delta Z_g^{(8)} &= \delta Z_g^{(8)}(1,1)\, a_1^{\mathrm{r}}\, a_1^{\mathrm{r}} + \delta Z_g^{(8)}(2,2)\, a_2^{\mathrm{r}}\, a_2^{\mathrm{r}} + \delta Z_g^{(8)}(1,2)\, a_1^{\mathrm{r}}\, a_2^{\mathrm{r}} \\
&+ \delta Z_g^{(8)}(3,3)\, a_3^{\mathrm{r}}\, a_3^{\mathrm{r}} + \delta Z_g^{(8)}(1,3)\, a_1^{\mathrm{r}}\, a_3^{\mathrm{r}} + \delta Z_g^{(8)}(2,3)\, a_2^{\mathrm{r}}\, a_3^{\mathrm{r}} \\
&+ \sum_{j=4,12} \delta Z_g^{(8)}(j)\, a_j^{\mathrm{r}}
\end{aligned}
$$

$$
\begin{aligned}
a_i &= \sum_{j=1,3} \mathscr{Z}_{ij}\, a_j^{\mathrm{r}} + \sum_{j=4,12} \mathscr{Z}_{ij}^{(6)}\, a_j^{\mathrm{r}} \\
&+ \sum_{j=1,3} \sum_{l=j,3} \mathscr{Z}_{ijl}^{(6)}\, a_j^{\mathrm{r}}\, a_l^{\mathrm{r}} \quad i = 1,2,3
\end{aligned}
$$

$$
a_i = \sum_{j=4,12} \mathscr{Z}_{ij}\, a_j^{\mathrm{r}} + \sum_{j=1,3} \sum_{l=j,3} \mathscr{Z}_{ijl}\, a_j^{\mathrm{r}}\, a_l^{\mathrm{r}}
$$

$$
\begin{aligned}
\mathscr{Z}_{ij} &= \delta_{ij} + g^2\, \frac{1}{\varepsilon}\, \mathrm{C}_{ij} \\
\mathscr{Z}_{ijl} &= g^2\, \frac{1}{\varepsilon}\, \mathrm{C}_{ijl}^{(6)} \\
\mathscr{Z}_{ij}^{(6)} &= g^2\, g_6\, \frac{1}{\varepsilon}\, \mathrm{C}_{ij}^{(6)} \\
\mathscr{Z}_{ijl}^{(6)} &= g^2\, g_6\, \frac{1}{\varepsilon}\, \mathrm{C}_{ijl}^{(6)}
\end{aligned}
$$

Finally:

$\mathrm{dim} = 4$

$$
\begin{aligned}
\delta \mathrm{Z}_h^{(4)} &= \delta \mathrm{Z}_\psi^{(4)} = 0 \\
\delta \mathrm{Z}_M^{(4)} &= \delta \mathrm{Z}_g^{(4)} = -10
\end{aligned}
$$

$\mathrm{dim} = 6$

$$
\begin{aligned}
\delta \mathrm{Z}_h^{(6)}(i) &= 0 \\
\delta \mathrm{Z}_\psi^{(6)}(1) &= 4 \quad \delta \mathrm{Z}_\psi^{(6)}(2) = 2 \quad \delta \mathrm{Z}_\psi^{(6)}(3) = 0 \\
\delta \mathrm{Z}_M^{(6)}(1) &= 108 \quad \delta \mathrm{Z}_M^{(6)}(2) = -26 \quad \delta \mathrm{Z}_M^{(6)}(3) = -66 \\
\delta \mathrm{Z}_g^{(6)}(1) &= 112 \quad \delta \mathrm{Z}_g^{(6)}(2) = -24 \quad \delta \mathrm{Z}_g^{(6)}(3) = 66 \\
\mathrm{C}_{11} &= -4 \quad \mathrm{C}_{12} = -2 \quad \mathrm{C}_{13} = 0 \\
\mathrm{C}_{21} &= -24 \quad \mathrm{C}_{22} = -12 \quad \mathrm{C}_{23} = 0 \\
\mathrm{C}_{31} &= -104 \quad \mathrm{C}_{32} = 24 \quad \mathrm{C}_{33} = -76
\end{aligned}
$$

A longer solution for $\mathrm{dim} = 8$

Thank you for your attention