Unstable Particles in Perturbation Theory Fermion-Loop Scheme and its Applications

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Abstract

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Contents

1 Definition of Vector-Boson Self-Energies

For the $\delta_{\mu\nu}$ part of the vector–vector transitions all the results can be cast in the following form:

$$
\begin{aligned}
S_{\gamma\gamma} &= (2\pi)^4\, i\, \frac{g^2 s_\theta^2}{16\,\pi^2}\, \Pi_{\gamma\gamma}(p^2)\, p^2, \quad S_{ZZ} = (2\pi)^4\, i\, \frac{g^2}{16\pi^2 c_\theta^2}\, \Sigma_{ZZ}(p^2), \\
S_{Z\gamma} &= (2\pi)^4\, i\, \frac{g^2 s_\theta}{16\pi^2 c_\theta}\, \Sigma_{ZA}(p^2), \\
S_{WW} &= (2\pi)^4\, i\, \frac{g^2}{16\pi^2}\, \Sigma_{WW}(p^2), \qquad S_{HH} = (2\pi)^4\, i\, \frac{g^2}{16\pi^2}\, \Sigma_{HH}(p^2), \qquad (1)
\end{aligned}
$$

where we have transformed from the (A, Z) basis to the $(3, Q)$ basis, defined by:

$$
\begin{aligned}
\Sigma_{ZZ}(p^2) &= \Sigma_{33}(p^2) - 2 s_\theta^2 \Sigma_{3Q}(p^2) + s_\theta^4 \Pi_{\gamma\gamma}(p^2)\, p^2, \\
\Sigma_{ZA}(p^2) &= \Sigma_{3Q}(p^2) - s_\theta^2 \Pi_{\gamma\gamma}(p^2)\, p^2. \qquad (2)
\end{aligned}
$$

Within the SM and in the $\xi = 1$ gauge we also have the following identities:

$$
\begin{aligned}
\Pi_{\gamma\gamma}(p^2) &= \frac{2}{3} - 12 B_{21}\left(p^2; M, M\right) + 7 B_0\left(p^2; M, M\right) + 4 \sum_f c_f Q_f^2 B_f\left(p^2; m_f, m_f\right), \\
\Sigma_{3Q}(p^2) &= p^2 \left[\frac{2}{3} - 10 B_{21}\left(p^2; M, M\right) + \frac{13}{2} B_0\left(p^2; M, M\right)\right] \\
&\quad - 2 M^2 B_0\left(p^2; M, M\right) + p^2 \sum_f c_f |Q_f| B_f\left(p^2; m_f, m_f\right). \qquad (3)
\end{aligned}
$$

Note that $\Sigma_{3Q}(0)$ is not zero and so it will be the Z–γ transition that is most important for the electric charge renormalization. Next we split the self-energies into a p^2-proportional part and the rest:

$$
\Sigma_{33}(p^2) = \Pi_{33}(p^2)\, p^2 + \Sigma_{33}^r(p^2), \quad \Sigma_{WW}(p^2) = \Pi_{WW}(p^2)\, p^2 + \Sigma_{WW}^r(p^2). \quad (4)
$$

The various components are given by:

$$
\begin{aligned}
\Pi_{33}(p^2) &= \frac{2}{3} - 9 B_{21}\left(p^2; M, M\right) + \frac{25}{4} B_0\left(p^2; M, M\right) - B_{21}\left(p^2; M_0, M_H\right) \qquad (5) \\
&\quad - B_1\left(p^2; M_0, M_H\right) - \frac{1}{4} B_0\left(p^2; M_0, M_H\right) + \frac{1}{2} \sum_f c_f B_f\left(p^2; m_f, m_f\right), \\
\Sigma_{33}^r(p^2) &= -2 M^2 B_0\left(p^2; M, M\right) + \frac{1}{2}\left(M_0^2 - M_H^2\right) B_1\left(p^2; M_0, M_H\right) \qquad (6)
\end{aligned}
$$

$$
\begin{aligned}
& +\frac{1}{4}\left(5M_{0}^{2}-M_{H}^{2}\right) B_0\left(p^2;M_0,M_H\right)-\frac{1}{2}\sum_f c_f m_f^2 B_0\left(p^2;m_f,m_f\right), \\
\Pi_{WW}(p^2) \;=\; & \frac{2}{3}-9B_{21}\left(p^2;M_0,M\right)-9B_1\left(p^2;M_0,M\right)+\frac{7}{4}B_0\left(p^2;M_0,M\right) \\
& -B_{21}\left(p^2;M,M_H\right)-B_1\left(p^2;M,M_H\right)-\frac{1}{4}B_0\left(p^2;M,M_H\right) \\
& +s_\theta^2[8B_{21}\left(p^2;M_0,M\right)-2B_0\left(p^2;M_0,M\right)+8B_1\left(p^2;M_0,M\right) \\
& -8B_{21}\left(p^2;0,M\right)-8B_1\left(p^2;0,M\right)+2B_0\left(p^2;0,M\right)] \\
& +\sum_{f=d} c_f B_f\left(p^2;m_{f'},m_f\right), \qquad (7) \\
\Sigma_{WW}^r(p^2) \;=\; & \frac{9}{2}\left(M_0^2-M^2\right)B_1\left(p^2;M_0,M\right)+\frac{1}{4}\left(13M_0^2-21M^2\right)B_0\left(p^2;M_0,M\right) \\
& +\frac{1}{2}\left(M^2-M_H^2\right)B_1\left(p^2;M,M_H\right)+\frac{1}{4}\left(5M^2-M_H^2\right)B_0\left(p^2;M,M_H\right) \\
& +s_\theta^2\{2\left(M^2-M_0^2\right)[2B_1\left(p^2;M_0,M\right)+B_0\left(p^2;M_0,M\right)] \qquad (8) \\
& -2M^2[2B_1\left(p^2;0,M\right)+B_0\left(p^2;0,M\right)]\}+\sum_f c_f m_f^2 B_1\left(p^2;m_{f'},m_f\right).
\end{aligned}
$$

It is worth presenting here the fermionic component of the Higgs boson self-energy,

$$
\Sigma_{HH}(p^2)=\sum_f c_f\frac{m_f^2}{M_W^2}\Big[A_0\left(m_f\right)-\frac{p^2+4m_f^2}{2}B_0\left(p^2;m_f,m_f\right)\Big]. \qquad (9)
$$

Note that the fermionic component of the Higgs–vector boson transition vanishes identically as it should, since it is proportional to

$$
[B_0\left(p^2;m_f,m_f\right)+2B_1\left(p^2;m_f,m_f\right)]p_\mu. \qquad (10)
$$

2 Fermion wave-function renormalization

For fermion fields the wave-function renormalization starts from considering the two-point Green function (see Fig. 1).

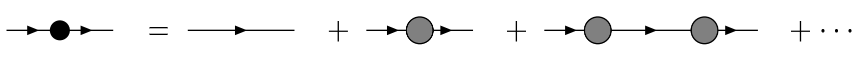


Figure 1: The re-summed fermion propagator.

We have

$$
G(p) \;=\; \frac{1}{(2\pi)^4 i}\left[i\not{p}+m_0-\frac{\Sigma(i\not{p})}{(2\pi)^4 i}\right]^{-1},
$$

$$\Sigma(i\not{p}) \;=\; (2\pi)^4\, i\, [a_1 + a_2\,\gamma_5 + (a_3 - a_4\,\gamma_5)\, i\not{p}]\,. \tag{11}$$

In the SM we always have $a_2 = 0$. This can be seen by considering the explicit expression for Σ. The case $a_2 \neq 0$ would require starting again from the Yukawa coupling in the original Lagrangian and rewriting the mass term as

$$\frac{1}{2}\,\overline{\psi}\,[\kappa_+ + \kappa_- + (\kappa_+ - \kappa_-)\,\gamma_5]\psi, \tag{12}$$

but $\kappa_+ \neq \kappa_-$ violates the hermiticity of the Lagrangian. The Green function must have a pole at the physical fermion mass, m.

Therefore, we expand the form factors around $i\not{p} = -m$, obtaining footnotethe original argument is due to M. Veltman, unpublished

$$a_i(p) = a_i(m) + 2\,m\;(i\not{p} + m)\;a_i'(m) + \mathcal{O}\left((i\not{p} + m)^2\right), \quad i = 1, 3. \tag{13}$$

After the expansion we can rewrite the propagator G as

$$\begin{aligned} G\,(p) \;&=\; \frac{1}{(2\pi)^4\, i}\,\left[(A + B\,\gamma_5)\;i\not{p} + a + \mathcal{O}\left((i\not{p} + m)^2\right)\right]^{-1}_{,} \\ A \;&=\; 1 - a_3(m) - 2\,m[a_1'(m) - m\,a_3'(m)], \qquad B = a_4(p)\,, \\ a \;&=\; m_0 - a_1(m) - 2\,m^2\,[a_1'(m) - m\,a_3'(m)], \end{aligned} \tag{14}$$

where $a' = da/dp^2$. Note that a_4 is not expanded since it has nothing to do with mass renormalization at this order in g. This fact is most easily seen by considering that a_i is $\mathcal{O}\left(g^2\right)$ and that

$$G\,(p) = \frac{1}{(2\pi)^4\, i}\,\frac{-\,(1 - a_3 + a_4\,\gamma_5)\;i\not{p} - m_0 + a_1}{(1 - 2\,a_3)\;p^2 + m_0^2 - 2\,m_0\,a_1}, \tag{15}$$

showing a pole at

$$p^2 = -m_0^2 + 2\,m_0\,a_1(p) - 2\,m_0^2\,a_3(p^2)\,. \tag{16}$$

Therefore, $a_4(p)$ will enter the fermion wave-function, not its mass.

The idea is to define the factors Z_{V}, Z_{A} such that

$$G\,(p) = \frac{1}{(2\pi)^4\, i}\,[(Z_{V} + Z_{A}\,\gamma_5)\;(i\not{p} + m)\;(Z_{V} - Z_{A}\,\gamma_5)]^{-1}_{.} \tag{17}$$

The Z-factors are expanded in perturbation theory,

$$Z_{V} = 1 - W_{V}, \qquad Z_{A} = W_{A}\,, \tag{18}$$

where W starts at $\mathcal{O}\left(g^2\right)$. By matching Eq.(14) with Eq.(17) and by requiring that G has indeed a pole at $p^2 = -m^2$ we derive the following conditions:

$$A = 1 - 2\,W_{\scriptscriptstyle V}, \qquad B = 2\,W_{\scriptscriptstyle A}\,, \qquad a = m\,\left(1 - 2\,W_{\scriptscriptstyle V}\right). \tag{19}$$

First, we impose mass renormalization, thus

$$m_0 = m + a_1(m) - m\,a_3(m), \tag{20}$$

which is allowed by the absence of any pseudo-scalar component of the one-loop corrected fermion propagator.

In summary, the solution will become

$$\begin{aligned} W_{\scriptscriptstyle V} &= \frac{1}{2}\left[a_3(m) + 2\,m a_1'(m) - 2\,m^2 a_3'(m)\right] + \mathcal{O}\left(p^2 + m^2\right), \\ W_{\scriptscriptstyle A} &= \frac{1}{2}\,a_4(m) + \mathcal{O}\left(p^2 + m^2\right), \end{aligned} \tag{21}$$

where the $\mathcal{O}\left(p^2 + m^2\right)$ terms will cancel in going to the $\mathcal{S}$-matrix. In this way the Green function G becomes, after inversion

$$\begin{aligned} G\left(p\right) &= \frac{1}{\left(2\pi\right)^4 i}\,\left(Z_{\scriptscriptstyle V} - Z_{\scriptscriptstyle A}\,\gamma_5\right)^{-1}\,\frac{-i\not{p} + m}{p^2 + m^2 + \mathcal{O}\left(\left(p^2 + m^2\right)^2\right)} \\ &\times\ \left(Z_{\scriptscriptstyle V} + Z_{\scriptscriptstyle A}\,\gamma_5\right)^{-1}. \end{aligned} \tag{22}$$

The properly normalized incoming particle wave-function will, therefore, be

$$J(p) = \frac{\sqrt{2\,E}}{2\,m}\,\left(Z_{\scriptscriptstyle V} + Z_{\scriptscriptstyle A}\,\gamma_5\right)\,u\left(p\right). \tag{23}$$

Now, if we consider arbitrary $\mathcal{S}$-matrix elements the net effect will be to replace

$$\begin{aligned} u\left(p\right) &\to \left(Z_{\scriptscriptstyle V} - Z_{\scriptscriptstyle A}\,\gamma_5\right)^{-1}\,u\left(p\right) = \left(1 + W_{\scriptscriptstyle V} + W_{\scriptscriptstyle A}\,\gamma_5\right)\,u\left(p\right), \\ \bar{u}\left(p\right) &\to \bar{u}\left(p\right)\left(Z_{\scriptscriptstyle V} + Z_{\scriptscriptstyle A}\,\gamma_5\right)^{-1} = \bar{u}\left(p\right)\,\left(1 + W_{\scriptscriptstyle V} - W_{\scriptscriptstyle A}\,\gamma_5\right), \end{aligned} \tag{24}$$

and similarly for v-spinors. In this way we have extended the QED result to include the axial part $\gamma_\mu\gamma_5$. Obviously, in QED we have $a_4 = 0$ and, moreover, in the non-QED part of the SM, whenever we can ignore fermion masses, the result simplifies into

$$W_{\scriptscriptstyle V} = \frac{1}{2}\,a_3\,, \qquad W_{\scriptscriptstyle A} = \frac{1}{2}\,a_4\,. \tag{25}$$

In this way we easily obtain wave-function factors for all the external fermion lines and for any $\mathcal{S}$-matrix elements, starting from the corresponding fermion self-energy.

3 Vector boson wave-function renormalization

The $\mathcal{S}$-matrix is obtained from transitions between physical sources that are put on the mass shell and therefore external line factors must be included. Consider an irreducible vector boson self-energy diagram which is given in Fig. 2 and may be expressed as

$$A(p^2)\delta_{\mu\nu} + B(p^2)p_\mu p_\nu\,, \tag{26}$$

where A and B are functions of p^2.

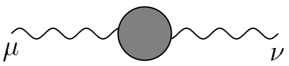


Figure 2: Irreducible vector boson self-energy diagram.

The bare propagator in the $\xi = 1$ gauge will be of the form

$$\Delta_{\mu\nu} = \frac{\delta_{\mu\nu}}{p^2 + M^2 - i\varepsilon}. \tag{27}$$

Repeated insertion of self-energy diagrams leads to the complete propagator:

$$\overline{\Delta}_{\mu\nu} = \frac{\delta_{\mu\nu}}{p^2 + M^2 - A} + \frac{Bp_\mu p_\nu}{(p^2 + M^2 - A)\,(p^2 + M^2 - A - Bp^2)}. \tag{28}$$

In our conventions a black dot on the vector boson line always indicates the re-summed propagators (Fig. 3). Next we consider a simple source–source transition with factors Z in the external lines. These factors are in general matrices and are chosen in order to obtain a residue 1 when taking the mass-shell limit. If we take $Z = Z_1\delta_{\mu\nu} + Z_2 p_\mu p_\nu$ and only consider physical sources, then Z_2 and B do not contribute.

The reason why Z_2 and B do not contribute may be seen as follows. Let us consider the (W, ϕ) sector of the Lagrangian for simplicity,

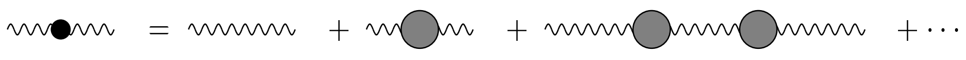


Figure 3: The re-summed vector boson propagator.

adding a source term to it:

$$\mathcal{L}_{\rm s} = J^w_\mu W_\mu + J^\phi \phi . \tag{29}$$

Then taking into account that a set of sources is physical if the product of the source and the field-independent part in the transformation law of the fields is zero, we arrive at the condition

$$\partial_\mu \, J^w_\mu - M \, J^\phi = 0 . \tag{30}$$

Sources that satisfy $\partial_\mu \, J^w_\mu = 0$ and , $J^\phi = 0$ separately are physical. This indeed is what we are used to for spin-1 particles. However, the general solution of Eq.(30) would not restrict J^w_μ and $M J^\phi = \partial_\mu J^w_\mu$. The source term for the Lagrangian takes the form

$$J^w_\mu \left(W_\mu - \frac{1}{M} \partial_\mu \phi \right) . \tag{31}$$

In going to the $\mathcal{S}$-matrix the external lines obtained from the longitudinal part of J^W_μ vanish. A longitudinal source would be coupled to the field combination

$$X = \partial_\mu W_\mu - \frac{1}{M} \partial^2 \phi . \tag{32}$$

According to the equations of motion and to the fact that

$$\mathcal{C} = - \frac{1}{\xi} \partial_\mu W_\mu + \xi M \phi , \tag{33}$$

is a free field thanks to the Ward identities of the theory, the associated propagator for the X-combination is simply a constant and has no pole. Therefore, it will not contribute to the $\mathcal{S}$-matrix. The source J^w_μ will always emit (absorb) the field combination of Eq.(31) and the Ward identities guarantee that adding a term

$$- \frac{1}{\xi M^2} \mathcal{C} \, \partial_\mu J^w_\mu \tag{34}$$

to the Lagrangian will not affect the $\mathcal{S}$-matrix between physical states, including vector bosons. By adding the contributions from Eq.(31) and Eq.(34) we obtain the following vertex:

$$V_\nu = J^w_\mu \left(\delta_{\mu\nu} + \frac{1}{\xi^2 M^2} p_\mu p_\nu\right), \tag{35}$$

where the ϕ-line has been replaced by a W-line by means of the Ward identity, $\mathcal{C}$ = free field. The W-line emerging from the source will be coupled to a W-propagator:

$$\begin{aligned} V_\nu \Delta_{\nu\alpha} &= J^w_\mu \left(\delta_{\mu\nu} + \frac{1}{\xi^2 M^2}\, p_\mu p_\nu\right) \frac{1}{p^2 + M^2} \\ &\times \left[\delta_{\nu\alpha} + \left(\xi^2 - 1\right) \frac{p_\nu p_\alpha}{p^2 + \xi^2 M^2}\right] \\ &= J^w_\mu \frac{1}{p^2 + M^2} \left(\delta_{\mu\alpha} + \frac{p_\mu p_\alpha}{M^2}\right). \end{aligned} \tag{36}$$

We now assume a longitudinal W-source, $J^w_\mu = p_\mu\, J^w_L$, and derive

$$V_\nu \Delta_{\nu\alpha} = J^w_L p_\mu \frac{1}{p^2 + M^2} \left(\delta_{\mu\alpha} + \frac{p_\mu p_\alpha}{M^2}\right) = J^w_L \frac{p_\alpha}{M^2}. \tag{37}$$

The pole of the propagator is at some value $p^2 = s_p$ with $s_p = -M^2 + A(s_p)$ and we may expand

$$A(p^2) = A_0 + \left(p^2 - s_p\right) A_1 + \left(p^2 - s_p\right)^2 A_{\text{rest}}(p^2), \tag{38}$$

with $A_0 = A(s_p)$. In going to the $\mathcal{S}$-matrix we multiply by $p^2 + M^2_p$, with $M^2_p = M^2 - \operatorname{Re} A_0$ and take the mass-shell limit; therefore, J^w_L cannot emit or absorb a W boson. Having completed the proof of the statement we may return to Eq.(28) where we see that near the pole the diagram behaves as

$$\frac{Z_1^2}{(1 - A_1)\,(p^2 - s_p)}, \tag{39}$$

showing that the external line factor for a physical vector boson is $(1 - A_1)^{1/2}$.

The correct treatment for the $\mathcal{S}$-matrix elements is shown in Fig. 4 where, according to our notations, an external source is denoted by the symbol $\times$.

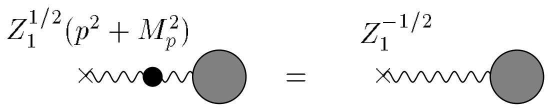


Figure 4: Wave-function factors for the $\mathcal{S}$-matrix.

Particular care has to be devoted to the proper treatment of Green functions with respect to the issue of gauge invariance. A complete derivation is quite involved and here we will only consider the bare propagators in the R_ξ gauge. As usual we have

$$\Delta_{Z\mu\nu} = \frac{1}{p^2 + M_{_0}^2}\left[\delta_{\mu\nu} + \left(\xi^2 - 1\right)\frac{p_\mu p_\nu}{p^2 + \xi^2 M_{_0}^2}\right],$$
$$\Delta_{\phi^0} = \frac{1}{p^2 + \xi^2 M_{_0}^2}. \tag{40}$$

Whenever we go from a Green function to the corresponding $\mathcal{S}$-matrix element the unphysical pole $p^2 = -\xi^2 M_{_0}^2$, as well as any ξ-dependence, must disappear.

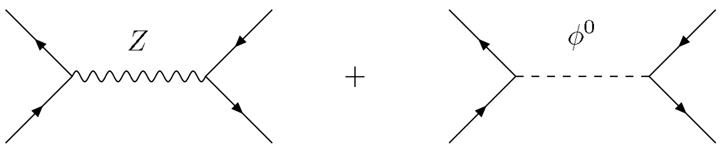


Figure 5: Diagrams for $f^+ f^- \to h^+ h^-$ annihilation.

Let us consider a simple $f^+ f^- \to h^+ h^-$ annihilation, depicted in Fig. 5, where f and h are arbitrary fermions. The Z-mediated diagram will read

$$D_{_Z} = (2\pi)^4\, i\left(-\frac{g^2}{4\, c_\theta^2}\right)\gamma_\mu\left(v_f + a_f\gamma_5\right)\otimes\gamma_\nu\left(v_h + a_h\gamma_5\right)\,\Delta_{Z\mu\nu}, \tag{41}$$

where, as usual, we have introduced v_f and a_f and where, moreover, we used the following abbreviation:

$$\gamma_\mu\left(v_1 + a_1\gamma_5\right)\otimes\gamma_\nu\left(v_2 + a_2\gamma_5\right)$$
$$= \bar{u}\left(p_+\right)\gamma_\mu\left(v_1 + a_1\gamma_5\right)v\left(p_-\right)\bar{v}\left(q_-\right)\gamma_\nu\left(v_2 + a_2\gamma_5\right)u\left(q_+\right), \tag{42}$$

with $P = p_+ + p_- = q_+ + q_-$. The ϕ^0-mediated diagram will be

$$D_{\phi^0} = (2\pi)^4\, i\left(-g^2\right) I_f^{(3)} I_h^{(3)}\,\gamma_5\otimes\gamma_5\frac{m_f m_h}{M^2}\Delta_{\phi^0}. \tag{43}$$

By using the Dirac equation we find

$$\begin{aligned}
\bar{u}\,(p_+)\,(\not{p}_+ + \not{p}_-)\,(v_1 + a_1\gamma_5)\,v\,(p_-) &= 2ia_1 m_f \bar{u}\,(p_+)\,\gamma_5 v\,(p_-)\,, \\
\bar{v}\,(q_-)\,(\not{q}_+ + \not{q}_-)\,(v_2 + a_2\gamma_5)\,u\,(q_+) &= -2ia_2 m_h \bar{v}\,(q_-)\,\gamma_5 u\,(q_+)\,. \quad (44)
\end{aligned}$$

At this point we collect the $p_\mu p_\nu$ part of the Z exchange diagram and the ϕ^0 exchange diagram to obtain

$$\begin{aligned}
&-\frac{g^2}{c_\theta^2} I_f^{(3)} I_h^{(3)} m_f m_h \frac{\xi^2 - 1}{\left(p^2 + M_{_0}^2\right)\left(p^2 + \xi^2 M_{_0}^2\right)} \gamma_5 \otimes \gamma_5 \\
&+ g^2 I_f^{(3)} I_h^{(3)} \frac{m_f m_h}{M^2} \frac{1}{p^2 + \xi^2 M_{_0}^2} \gamma_5 \otimes \gamma_5\,. \qquad (45)
\end{aligned}$$

After some simple algebra we derive

$$-g^2 I_f^{(3)} I_h^{(3)} \frac{m_f m_h}{M^2} \frac{1}{p^2 + M_{_0}^2} \gamma_5 \otimes \gamma_5\,, \qquad (46)$$

where we have used the relation $c_\theta^2 M_{_0}^2 = M^2$. In this way it is shown that only the physical pole, $p^2 = -M_{_0}^2$, will appear in the $\mathcal{S}$-matrix element. The same property must hold also when we include radiative corrections, moving from the bare propagators to the complete ones. However, ξ-independence requires more: self-energy diagrams alone are not gauge-invariant.

4 Renormalization with unstable particles

The QED radiative corrections become particularly transparent when we express their universal part in terms of the running of α, which will properly take into account the large (leading) logarithms arising in multi-loop diagrams. The effective coupling constant $\alpha(p^2)$ in QED is nothing but the effect of a re-summation of irreducible self-energy diagrams into the complete photon propagator.

As soon as we move from QED to the full SM the problem of a gauge-invariant re-summation procedure arises, since the bosonic part of the self-energy diagrams is not gauge-invariant by itself.

To give an example we consider the photon self-energy in the R_ξ gauge. We can split the Π_0 term into a fermionic part, a bosonic one in the $\xi = 1$ gauge and the remaining, ξ-dependent terms

$$
\begin{aligned}
\Pi_{\mu\nu} &= \Pi_0 \delta_{\mu\nu} + \Pi_1 p_\mu p_\nu\,, \quad \Pi_0 = \Pi_0^{\rm fer} + \Pi_0^{\rm bos}\,, \\
\Pi_0^{\rm bos} &= g^2 s_\theta^2 p^2 \left(\Pi_0^{\rm bos,1} + \Pi_0^{\rm bos,\xi}\right).
\end{aligned}
\tag{47}
$$

From known results and with $w = p^2/M^2$, we obtain

$$
\begin{aligned}
\Pi_0^{\rm bos,1} &= -\frac{4-3w}{w} B_0\left(p^2; M, M\right) - \frac{4}{w}[\frac{A_0\left(M\right)}{M^2} + 1]\Big\}, \\
\Pi_0^{\rm bos,\xi} &= -\frac{1}{3}\left(1 + \frac{1}{4}\,w\right) w[B_0\left(p^2; \xi M, \xi M\right) - B_0\left(p^2; M, M\right)] \\
&\quad - \left(\frac{8}{3} + w - \frac{1}{6}\,w^2\right)[B_0\left(p^2; M, \xi M\right) - B_0\left(p^2; M, M\right)] \\
&\quad + \frac{1}{3}\Big[\xi^2 - 1 + \frac{1}{2}\left(\xi^2 + 9\right)\,w\Big]\frac{A_0\left(\xi M\right) - A_0\left(M\right)}{p^2} \\
&\quad + \frac{\xi^2 - 1}{3}[2 - \,w B_0\left(p^2; \xi M, \xi M\right)] \\
&\quad + \frac{\xi^2 - 1}{6\,w}[2\left(\xi^2 - 1\right) + \left(\xi^2 + 3\right)\,w + 2\,w^2] B_0\left(p^2; M, \xi M\right).
\end{aligned}
\tag{48}
$$

Note that the ξ-dependent part is not ultraviolet finite, since

$$
\Pi_0^{\rm bos} = g^2 s_\theta^2 p^2\,\left(4 - \xi^2\right)\frac{1}{\bar{\varepsilon}} + \mathcal{O}\left(1\right).
\tag{49}
$$

A rather important object in all renormalization procedures will be $\Pi_{\gamma\gamma}(0)$, defined by

$$
\Pi_0^{\rm bos}|_{p^2 \to 0} = g^2 s_\theta^2 p^2 \Pi_{\gamma\gamma}(0).
\tag{50}
$$

Using the expansions, we obtain a non-zero result:

$$
\Pi_{\gamma\gamma}^{\rm bos}(0) = \left(4 - \xi^2\right)\,\Delta\left(M\right) - \frac{11}{6} - \frac{1}{2}\xi^2 + \left(1 + \frac{3}{\xi^2 - 1}\right)\xi^2 \ln \xi^2.
\tag{51}
$$

4.1 Fermion-loop approximation: massless fermions

Before going on, we recall the expression for the $\delta_{\mu\nu}$ part of the vector–vector transitions. To go further with the derivation, we make another drastic approximation; namely, we put to zero all fermions masses. Under this assumption we will introduce the following quantities:

$$
\begin{aligned}
B_n \equiv B_n\left(p^2;0,0\right) &= 2\,B_{21}\left(p^2;0,0\right) - B_0\left(p^2;0,0\right), \\
B_c \equiv B_c\left(p^2;0,0\right) &= B_{21}\left(p^2;0,0\right) + B_0\left(p^2;0,0\right) = \frac{1}{2}\,B_n\,.
\end{aligned}
\tag{52}
$$

Furthermore, we will use the fact that a p^2 can be factorized:

$$
\Sigma_{3Q}(p^2) = \Pi_{3Q}(p^2)\,p^2, \qquad \Sigma_{33}(p^2) = \Pi_{33}(p^2)\,p^2. \tag{53}
$$

As a result, all the vector boson–vector boson transitions simplify drastically:

$$
\begin{aligned}
\Pi_{\gamma\gamma}(s) &= \frac{32}{3}\sum_g B_n\,, \quad \Pi_{3Q}(s) = 4\sum_g B_n = \frac{3}{8}\,\Pi_{\gamma\gamma}(s)\,, \\
\Sigma_{33}(s) &= -4s\sum_g B_n = -s\Pi_{3Q}(s)\,, \\
\Sigma_{WW}(s) &= -8s\sum_g B_c = -4s\sum_g B_n = -s\Pi_{33}(s)\,,
\end{aligned}
\tag{54}
$$

where $p^2 = -s$ and where the sum is over the fermion generations.

4.2 Fermion-loop approximation: finite top mass

The resulting expressions retain some simplicity even if we do not ignore the top quark mass. Self-energies may still be written in compact form:

$$
\begin{aligned}
\Pi_{\gamma\gamma}(s) &= \frac{32}{3}\sum_g B_n + \frac{16}{3}\,\overline{B}_n\,, \quad \Pi_{3Q}(s) = 4\sum_g B_n + 2\,\overline{B}_n\,, \\
\Sigma_{33}(s) &= -s\,\Pi_{3Q}(s) + \frac{1}{2}s\,\overline{B}_n - \frac{3}{2}\,m_t^2 B_0\left(-s;m_t,m_t\right), \\
\Sigma_{WW}(s) &= -s\left[4\sum_g B_n + 6\,\overline{B}_c\right] - 3\,m_t^2\,B_{mc}\,,
\end{aligned}
\tag{55}
$$

where we have introduced new auxiliary functions, defined by

$$
\begin{aligned}
\overline{B}_n &= B_n\left(-s;m_t,m_t\right) - B_n\left(-s;0,0\right), \\
\overline{B}_c &= B_c\left(-s;m_t,0\right) - B_c\left(-s;0,0\right), \\
B_{mc} &= B_1\left(-s;m_t,0\right) + B_0\left(-s;m_t,0\right).
\end{aligned}
\tag{56}
$$

A useful way of presenting these results will be to split the universal part, proportional to $\Pi_{3Q}(s)$, from the remainder:

$$\Sigma_{33}(s) = -s\,\Pi_{3Q}(s) + f_Z(s), \quad \Sigma_{WW}(s) = -s\,\Pi_{3Q}(s) + f_W(s), \tag{57}$$

where the two f-functions are expressible as

$$\begin{aligned} f_W(s) &= -2\,s\,[3B_{21}(-s;m_t,0) - 2B_{21}(-s;m_t,m_t) - B_{21}(-s;0,0) \\ &+3B_1(-s;m_t,0) - 3B_1(-s;0,0) + B_0(-s;m_t,m_t) - B_0(-s;0,0)], \\ &-3\,m_t^2\,[B_1(-s;m_t,0) + B_0(-s;m_t,0)], \\ f_Z(s) &= \frac{1}{2}\,s\,[2B_{21}(-s;m_t,m_t) - 2B_{21}(-s;0,0) \\ &-B_0(-s;m_t,m_t) + B_0(-s;0,0)] - \frac{3}{2}\,m_t^2\,B_0(-s;m_t,m_t)\,. \end{aligned} \tag{58}$$

4.3 Running couplings

We now consider three parameters, the e.m. coupling constant e, the $SU(2)$ coupling constant g and the sine of the weak mixing angle s_θ. At the tree level they are not independent, but rather they satisfy the relation $g^2 s_\theta^2 = e^2$. The running of the e.m. coupling constant is easily derived and gives

$$\frac{1}{e^2(s)} = \frac{1}{g^2 s_\theta^2} - \frac{1}{16\,\pi^2}\,\Pi_{\gamma\gamma}(s)\,. \tag{59}$$

However, we have a quite natural scale to use since at $s = 0$ we have the fine structure constant at our disposal. Therefore, the running of $e^2(s)$ is completely specified in terms of α by

$$\frac{1}{e^2(s)} = \frac{1}{4\,\pi\alpha}\left[1 - \frac{\alpha}{4\,\pi}\,\Pi(s)\right], \qquad \text{with} \quad \Pi(s) = \Pi_{\gamma\gamma}(s) - \Pi_{\gamma\gamma}(0). \tag{60}$$

For the running of g^2 we derive a similar equation:

$$\frac{1}{g^2(s)} = \frac{1}{g^2} - \frac{1}{16\,\pi^2}\,\Pi_{3Q}(s)\,. \tag{61}$$

The running of the third parameter, $s_\theta^2(s)$, is now fixed by

$$s_\theta^2(s) = \frac{e^2(s)}{g^2(s)}\,. \tag{62}$$

In order to understand the role played by these running coupling constants let us consider the inverse Z-propagator where the re-summed self-energy part will be denoted by Σ_Z. The latter is given by

$$\Sigma_Z(s) = \Sigma_{33}(s) - 2\, s_\theta^2 \Sigma_{3Q}(s) - s_\theta^4\, \Pi_{\gamma\gamma}(s)\, s - \frac{s}{16\,\pi^2}\, \frac{\Pi_{ZA}^2(s)}{\dfrac{1}{g^2 s_\theta^2} - \dfrac{1}{16\pi^2}\Pi_{\gamma\gamma}(s)}. \qquad (63)$$

Note that in the re-summation we have included the higher-order terms which take into account the Z–γ transition. This effect is shown in Fig. 6 where $i, j, k = \gamma, Z$.

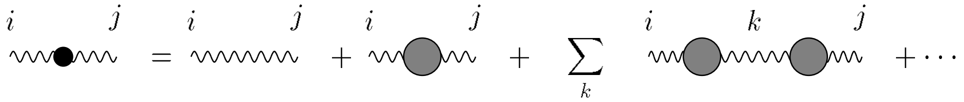


Figure 6: The coupled neutral vector boson propagators.

This expression can be simplified by virtue of the fermion-loop approximation. Indeed, we can write

$$\begin{aligned}
\frac{1}{s}\,\Sigma_Z(s) &= -\Pi_{3Q}(s) + s_\theta^2\,[2\,\Pi_{3Q}(s) - s_\theta^2\,\Pi_{\gamma\gamma}(s)] \\
&- \frac{1}{16\,\pi^2}\, \frac{[\Pi_{3Q}(s) - s_\theta^2 \Pi_{\gamma\gamma}(s)]^2}{\dfrac{1}{g^2 s_\theta^2} - \dfrac{1}{16\pi^2}\Pi_{\gamma\gamma}(s)} \\
&= -\Pi_{3Q}(s) + \frac{1}{g^2}\, \frac{2\,\Pi_{3Q}(s) - s_\theta^2 \Pi_{\gamma\gamma}(s) - \dfrac{g^2}{16\,\pi^2}\,\Pi_{3Q}^2(s)}{\dfrac{1}{g^2 s_\theta^2} - \dfrac{1}{16\pi^2}\Pi_{\gamma\gamma}(s)}. \qquad (64)
\end{aligned}$$

After introducing the running couplings we arrive at the following result:

$$\frac{1}{s}\,\Sigma_Z(s) = -\Pi_{3Q}(s) + e^2(s)\, \left[\frac{\Pi_{3Q}(s)}{g^2(s)} + \frac{\Pi_{3Q}(s) - s_\theta^2 \Pi_{\gamma\gamma}(s)}{g^2}\right]. \qquad (65)$$

If we now express the various self-energies in terms of the running couplings,

$$\frac{s_\theta^2 \Pi_{\gamma\gamma}(s)}{16\,\pi^2} = \frac{1}{g^2} - \frac{s_\theta^2}{e^2(s)}, \qquad \frac{\Pi_{3Q}(s)}{16\,\pi^2} = \frac{1}{g^2} - \frac{1}{g^2(s)}, \qquad (66)$$

then we obtain a very simple relation for the propagation function $\Sigma_Z(s)$. Indeed, we have

$$\frac{1}{s}\,\Sigma_z(s) = 16\,\pi^2\left[\frac{c_\theta^2(s)}{g^2(s)} - \frac{c_\theta^2}{g^2}\right], \qquad c_\theta^2(s) = 1 - s_\theta^2(s). \tag{67}$$

When the top quark mass is not ignored the above equation modifies to

$$\frac{1}{s}\,\Sigma_z(s) = 16\,\pi^2\left[\frac{c_\theta^2(s)}{g^2(s)} - \frac{c_\theta^2}{g^2} + \frac{f_z(s)}{16\,\pi^2 s}\right]. \tag{68}$$

Let us turn to the W-propagator, which becomes

$$\Delta_W = \frac{1}{(2\pi)^4\,i}\,\frac{1}{s - M^2 + \dfrac{g^2}{16\pi^2}\Sigma_{WW}(s)}. \tag{69}$$

In the fermion-loop approximation the corrections to μ-decay are concentrated only in the W self-energy and, therefore, we can write an equation, expressing bare quantities in terms of experimental data points.

The Fermi coupling constant in the fermion-loop approximation is expressed as

$$\frac{M^2}{g^2} = \frac{\sqrt{2}}{8\,G_F} + \frac{1}{16\,\pi^2}\,f_W(0). \tag{70}$$

The definition of the mass for an unstable vector boson requires some additional work. First, we will define the so-called line-shape masses.

The line-shape mass for the W boson is defined as the zero of the real part of the inverse W-propagator, i.e.

$$\begin{aligned} M_W^2 &= M^2 - \frac{g^2}{16\,\pi^2}\,\mathrm{Re}\,\Sigma_{WW}\left(M_W^2\right), \\ M_W^2\,\left[\frac{1}{g^2} - \frac{1}{16\,\pi^2}\,\mathrm{Re}\,\Pi_{3Q}(M_W^2)\right] &= \frac{\sqrt{2}}{8\,G_F} + \frac{1}{16\,\pi^2}\left[f_W(0) - \mathrm{Re}\,f_W(M_W^2)\right] \end{aligned} \tag{71}$$

The above definition allows us to fully determine the running of the $SU(2)$ coupling constant in terms of G_F and of M_W. Indeed, we obtain

$$\frac{1}{g^2(M_W^2)} = \frac{1}{g^2} - \frac{1}{16\,\pi^2}\,\mathrm{Re}\,\Pi_{3Q}(M_W^2), \tag{72}$$

$$
\begin{aligned}
\frac{1}{g^2(M_{_W}^2)} &= \frac{\sqrt{2}}{8\,G_{_F}M_{_W}^2} + \frac{1}{16\,\pi^2 M_{_W}^2}\,[f_{_W}(0) - \mathrm{Re}\,f_{_W}(M_{_W}^2)]\,,\\
\frac{8\,G_{_F}M_{_W}^2}{\sqrt{2}g^2(s)} &= 1 + \frac{G_{_F}M_{_W}^2}{2\sqrt{2}\,\pi^2}\,\mathrm{Re}\,\Big\{\frac{f_{_W}(0) - f_{_W}(M_{_W}^2)}{M_{_W}^2} + [\Pi_{_{3Q}}(M_{_W}^2) - \Pi_{_{3Q}}(s)]\Big\}.
\end{aligned}
$$

Equipped with these results we can now move on to the definition of the mass for an unstable vector boson. *The complex mass* of a resonance is defined to be the location of the complex pole in the corresponding complete propagator; thus, for the W boson it is the quantity $p_{_W}$ defined below:

$$
\frac{p_{_W}}{g^2} - \frac{M^2}{g^2} + \frac{1}{16\,\pi^2}\,[f_{_W}(p_{_W}) - p_{_W}\,\Pi_{_{3Q}}(p_{_W})] = 0\,. \tag{73}
$$

Despite the presence of bare quantities in the previous equation, we can easily see that the definition of the complex pole is actually free from ultraviolet singularities. Indeed, one can show that $p_{_W}$ is a solution of the equation

$$
\frac{p_{_W}}{g^2(p_{_W})} + \frac{1}{16\,\pi^2}\,[f_{_W}(p_{_W}) - f_{_W}(0)] = \frac{\sqrt{2}}{8\,G_{_F}}. \tag{74}
$$

We note the following: when evaluating functions with argument $p_{_W}$, some care has to be taken, because the complex pole does not lie on the usual physical (first) Riemann sheet. Its location is determined by the fact that it should smoothly approach the value for a stable gauge boson when the coupling of the theory tends to zero. On the basis of similar arguments we define the complex mass for the Z boson.

The complex mass for the Z is $p_{_Z}$ defined implicitly by the following equation:

$$
p_{_Z}\,\frac{c_\theta^2(p_{_Z})}{g^2(p_{_Z})} + \frac{1}{16\,\pi^2}\,[f_{_Z}(p_{_Z}) - f_{_W}(0)] = \frac{\sqrt{2}}{8\,G_{_F}}. \tag{75}
$$

The fermion-loop approximation allows us to carry out the complex-mass scheme renormalization by introducing complex pole positions and running couplings. One of the nice aspects of this renormalization scheme is that it is not necessary to fix the input parameters in order to carry out the renormalization. The running couplings can be adjusted to input parameters after renormalization, even if we have adopted the opposite strategy: first, input parameters are introduced and then we renormalize the gauge boson masses at their (gauge-invariant) complex poles. In this way things are more transparent since knowledge of the bare parameters allows us to compute all running (renormalized) couplings and the renormalized complex pole positions $p_{_W}$ and $p_{_Z}$.

Having performed the renormalization we can write down the vector boson propagators. First the W-propagator:

$$
\begin{aligned}
-g^2\,\Delta_W &= \left[\frac{1}{g^2}\left(s-M^2\right)+\frac{1}{16\,\pi^2}\,\Sigma_{WW}(s)\right]^{-1} = \frac{1}{1+\Delta\rho_W(s)}\,\frac{g^2(s)}{s-p_{_W}}, \\
\Delta\rho_W(s) &= \frac{g^2(s)}{16\,\pi^2}\,\frac{f_{_W}(s)-f_{_W}(p_{_W})+p_{_W}\left[\Pi_{_{3Q}}(p_{_W})-\Pi_{_{3Q}}(s)\right]}{s-p_{_W}}.
\end{aligned} \tag{76}
$$

The W-propagator is, therefore, fully specified by the position of the complex pole $p_{_W}$, by the running of $g^2(s)$ and by a running Veltman ρ-parameter, which we have denoted by $\Delta\rho_{_W}(s)$. Similarly, for the Z-propagator we find

$$
\begin{aligned}
-\frac{g^2}{c_\theta^2}\,\Delta_Z &= \left[\frac{c_\theta^2}{g^2}\left(s-M_{_0}^2\right)+\frac{1}{16\,\pi^2}\,\Sigma_Z(s)\right]^{-1} \\
&= \frac{1}{1+\Delta\rho_Z(s)}\,\frac{g^2(s)}{c_\theta^2(s)}\,\frac{1}{s-p_{_Z}}, \\
(s-p_{_Z})\,\Delta\rho_Z(s) &= 1-\frac{g^2(s)}{g^2(p_{_Z})}\frac{c_\theta^2(p_Z)}{c_\theta^2(s)}+\frac{1}{16\,\pi^2}\,\frac{g^2(s)}{c_\theta^2(s)}\left[f_{_Z}(s)-f_{_Z}(p_{_Z})\right].
\end{aligned} \tag{77}
$$

In the previous equation a new $\Delta\rho$ factor has been introduced. Having specified the (line-shape) W mass among the data points in the input parameter set we have adjusted the running of g according to Eq.(73). This is not a unique procedure, because the input parameter set is not unique. We could have made another choice, for instance the line-shape Z mass:

$$
\frac{c_\theta^2}{g^2}M_{_Z}^2-\frac{M^2}{g^2}+\frac{1}{16\,\pi^2}\,\mathrm{Re}\,\Sigma_Z(M_{_Z}^2)=0\,. \tag{78}
$$

The same equation can be rewritten as follows:

$$
\mathrm{Re}\,\frac{c_\theta^2(M_Z^2)}{g^2(M_{_Z}^2)} = \frac{\sqrt{2}}{8\,G_{_F}M_{_Z}^2}+\frac{1}{16\,\pi^2M_{_Z}^2}\left[f_{_W}(0)-\mathrm{Re}\,f_{_Z}(M_{_Z}^2)\right]
$$

$$= \operatorname{Re}\frac{1 - s_\theta^2 M_Z^2}{g^2(M_Z^2)} = \operatorname{Re}\left[\frac{1}{g^2(M_Z^2)} - 4\,\pi\,\frac{\alpha(M_Z^2)}{g^4(M_Z^2)}\right]. \tag{79}$$

To go further, let us introduce both the real and imaginary parts of $g^2(M_Z^2)$. We write

$$\frac{1}{g^2(M_Z^2)} = \frac{1}{\mathcal{G}_r} + i\frac{1}{\mathcal{G}_i}. \tag{80}$$

In this way the running of g^2 is fixed according to the following set of equations:

$$X = \frac{\sqrt{2}}{8\,G_F M_Z^2} + \frac{1}{16\,\pi^2 M_Z^2}\,[f_W(0) - \operatorname{Re} f_Z(M_Z^2)], \tag{81}$$

where the quantity X is defined by

$$X = -4\,\pi \operatorname{Re}\alpha(M_Z^2)\frac{1}{\mathcal{G}_r^2} + \left[1 + 8\,\pi \operatorname{Im}\alpha(M_Z^2)\frac{1}{\mathcal{G}_i}\right]\frac{1}{\mathcal{G}_r} + 4\,\pi \operatorname{Re}\alpha(M_Z^2)\frac{1}{\mathcal{G}_i^2}. \tag{82}$$

The imaginary part of the inverse running coupling at M_Z^2 and the coupling at an arbitrary scale are

$$\begin{aligned}
\frac{1}{\mathcal{G}_i} &= -\frac{1}{16\,\pi^2}\operatorname{Im}\Pi_{3Q}(M_Z^2), \\
g^2(s) &= \frac{\mathcal{G}_r}{1 + \dfrac{\mathcal{G}_r}{16\,\pi^2}[\operatorname{Re}\Pi_{3Q}(M_Z^2) - \Pi_{3Q}(s)]}.
\end{aligned} \tag{83}$$

Consider another relation,

$$\frac{1}{\alpha(s)} = \frac{1}{\alpha} - \frac{1}{4\,\pi^2}\,[\Pi_{\gamma\gamma}(s) - \Pi_{\gamma\gamma}(0)], \tag{84}$$

and define

$$\begin{aligned}
x_\alpha &= \operatorname{Re}\frac{1}{\alpha(M_Z^2)} = \frac{1}{\alpha} - \frac{1}{4\,\pi^2}\operatorname{Re}\Pi_{\gamma\gamma}(M_Z^2), \\
y_\alpha &= \operatorname{Im}\frac{1}{\alpha(M_Z^2)} = -\frac{1}{4\,\pi^2}\operatorname{Im}\Pi_{\gamma\gamma}(M_Z^2),
\end{aligned} \tag{85}$$

which gives

$$\operatorname{Re}\alpha(M_Z^2) = \frac{x_\alpha}{x_\alpha^2 + y_\alpha^2}, \qquad \operatorname{Im}\alpha(M_Z^2) = -\frac{y_\alpha}{x_\alpha^2 + y_\alpha^2}. \tag{86}$$

If we want to summarize the procedure, then we could say that starting from the bare re-summed propagators, the renormalization can be performed by first introducing renormalized masses and then by subsequently rewriting all bare couplings in terms of the running ones. It is easy to show that the running couplings, deduced from the input in the above way, are finite. From this the finiteness of the propagator functions and vertex functions follows. This also holds for the masses of the weak vector bosons.

The so-obtained renormalized—re-summed—gauge boson propagators automatically combine with the fermion–gauge boson vertices to form dressed Born matrix elements, involving running couplings and propagator functions.

4.4 Transition to the line-shape framework

It should be noted, however, that the real part of the complex pole differs only by terms of order Γ^2/M^2 from the gauge boson mass in the more familiar line-shape framework, which is defined as the zero in the real part of the Z-boson self-energy. Let us consider an arbitrary resonance V. We consider the unrenormalized V self-energy, $\hat{\Sigma}_{VV}$, which contains possible tadpole contributions. If $\hat{\mu}_V$ is the bare V-boson mass squared, then the complex pole p_V is defined by

$$p_V - \hat{\mu}_V + \hat{\Sigma}_{VV}(p_V; \hat{\mu}_V) = 0. \tag{87}$$

The complex pole is a property of the $\mathcal{S}$-matrix, and, therefore, gauge-invariant. Note that in the full theory one should actually write $\hat{\Sigma}_{VV}(s; \mu)$, where μ denotes the squared mass to be used in the propagators for the internal V-lines. In what follows the second argument of $\hat{\Sigma}_{VV}(s; \mu)$ is omitted, since we always understand that bare masses and couplings are inserted in the loop calculations. Alternatively, we could of course insert renormalized quantities inside the loops, but then additional counter-term contributions would have to be taken into account, which become relevant at the two-loop level.

For clarity, we briefly develop the formalism in the relations for the V-boson mass and width, and turn to the special case of fermionic corrections afterwards. The complex pole is rewritten in terms of real quantities $\overline{m}_V$ and $\overline{\Gamma}_V$ as

$$p_V = \overline{m}_V^2 - i\overline{\Gamma}_V\overline{m}_V\,, \tag{88}$$

so that Eq.(87) yields

$$\begin{aligned} \overline{m}_V^2 - i\overline{\Gamma}_V\overline{m}_V &= \hat{\mu}_V - \hat{\Sigma}_{VV}\left(\overline{m}_V^2 - i\overline{\Gamma}_V\overline{m}_V\right) \\ b &= \hat{\mu}_V - \hat{\Sigma}_{VV}(\overline{m}_V^2) + i\overline{\Gamma}_V\overline{m}_V\hat{\Sigma}'_{VV}(\overline{m}_V^2) \\ &+ \frac{1}{2}\overline{\Gamma}_V^2\overline{m}_V^2\hat{\Sigma}''_{VV}(\overline{m}_V^2) + \ldots . \end{aligned} \tag{89}$$

Taking the real and imaginary parts of Eq.(90), we obtain relations for the mass,

$$\overline{m}_V^2 = \hat{\mu}_V - \mathrm{Re}\hat{\Sigma}_{VV}(\overline{m}_V^2) - \overline{\Gamma}_V\overline{m}_V\mathrm{Im}\hat{\Sigma}'_{VV}(\overline{m}_V^2) + \ldots\ , \tag{90}$$

and the width,

$$\overline{\Gamma}_V\overline{m}_V = \mathrm{Im}\hat{\Sigma}_{VV}(\overline{m}_V^2) - \overline{\Gamma}_V\overline{m}_V\mathrm{Re}\hat{\Sigma}'_{VV}(\overline{m}_V^2) - \frac{1}{2}\overline{\Gamma}_V^2\overline{m}_V^2\mathrm{Im}\hat{\Sigma}''_{VV}(\overline{m}_V^2) + \ldots . \tag{91}$$

The width $\overline{\Gamma}_V$ can be eliminated from the right-hand sides of Eqs.(90) and (91) by iteration:

$$\overline{m}_V^2 = \hat{\mu}_V - \mathrm{Re}\hat{\Sigma}_{VV}(\overline{m}_V^2) - \left[\mathrm{Im}\hat{\Sigma}_{VV}(\overline{m}_V^2)\right]\left[\mathrm{Im}\hat{\Sigma}'_{VV}(\overline{m}_V^2)\right] + \ldots\ , \tag{92}$$

$$\begin{aligned} \overline{\Gamma}_V m_V &= \mathrm{Im}\hat{\Sigma}_{VV}(\overline{m}_V^2)\Big\{1 - \mathrm{Re}\hat{\Sigma}'_{VV}(\overline{m}_V^2) + \left[\mathrm{Re}\hat{\Sigma}'_{VV}(\overline{m}_V^2)\right]^2 \\ &-\frac{1}{2}\left[\mathrm{Im}\hat{\Sigma}_{VV}(\overline{m}_V^2)\right]\left[\mathrm{Im}\hat{\Sigma}''_{VV}(\overline{m}_V^2)\right] + \ldots\Big\}. \end{aligned} \tag{93}$$

We now introduce the usual on-shell mass m_V:

$$m_V^2 = \hat{\mu}_V - \mathrm{Re}\hat{\Sigma}_{VV}(m_V^2)\,. \tag{94}$$

Eliminating $\hat{\mu}_V$ from Eqs.(92) and (94), $\overline{m}_V^2$ can be calculated from m_V^2 by iteration:

$$\overline{m}_V^2 = m_V^2 - \left[\mathrm{Im}\hat{\Sigma}_{VV}(m_V^2)\right]\left[\mathrm{Im}\hat{\Sigma}'_{VV}(m_V^2)\right] + \ldots\ , \tag{95}$$

i.e., $\overline{m}_V^2$ and m_V^2 coincide at the one-loop level, but differ by two-loop corrections. The on-shell V-boson width is defined as

$$\begin{aligned}\Gamma_V m_V &= \frac{\mathrm{Im}\hat{\Sigma}_{VV}(m_V^2)}{1+\mathrm{Re}\hat{\Sigma}'_{VV}(m_V^2)} \\ &= \mathrm{Im}\hat{\Sigma}_{VV}(m_V^2)\Big\{1-\mathrm{Re}\hat{\Sigma}'_{VV}(m_V^2)+\Big[\mathrm{Re}\hat{\Sigma}'_{VV}(m_V^2)\Big]^2+\ldots\Big\}\,, \end{aligned} \tag{96}$$

giving

$$\begin{aligned}\overline{\Gamma}_V\overline{m}_V &= \Gamma_V m_V - \Big[\mathrm{Im}\hat{\Sigma}_{VV}(m_V^2)\Big]\Big[\mathrm{Im}\hat{\Sigma}'_{VV}(m_V^2)\Big]^2 \\ &\quad -\frac{1}{2}\Big[\mathrm{Im}\hat{\Sigma}_{VV}(m_V^2)\Big]^2\Big[\mathrm{Im}\hat{\Sigma}''_{VV}(m_V^2)\Big]+\ldots\,. \end{aligned} \tag{97}$$

As a next step, we specialize the above formulas to fermionic one-loop corrections and to the case where the V particle decays exclusively into massless fermions. Then, we have the relations

$$\mathrm{Im}\hat{\Sigma}_{VV}\,(s)=\frac{\Gamma_V}{m_V}s\,,\qquad \mathrm{Im}\hat{\Sigma}'_{VV}\,(s)=\frac{\Gamma_V}{m_V}\,, \tag{98}$$

in the vicinity of $s=m_V^2$, so that Eqs.(95) and (97) yield

$$\overline{m}_V^2 = m_V^2-\Gamma_V^2+\ldots\,, \tag{99}$$

$$\frac{\overline{\Gamma}_V}{\overline{m}_V} = \frac{\Gamma_V}{m_V}+\ldots \qquad \text{or} \qquad \overline{\Gamma}_V=\Gamma_V\left(1-\frac{\Gamma_V^2}{2m_V^2}\right)+\ldots\,. \tag{100}$$

Finally, we consider the V-propagator, which can be rewritten as

$$P_V(s)=\Big[s-\hat{\mu}_V+\hat{\Sigma}_{VV}(s)\Big]^{-1}=\Big[s-p_V+\hat{\Sigma}_{VV}(s)-\hat{\Sigma}_{VV}(p_V)\Big]^{-1}_{.} \tag{101}$$

In the vicinity of the resonance it is possible to extract the *running-width contributions* from $\hat{\Sigma}_{VV}(s)$ according to Eq.(98) as follows:

$$\hat{\Sigma}_{VV}(s)=\hat{\Sigma}_V^{\mathrm{rem}}(s)+i\frac{\Gamma_V}{m_V}s. \tag{102}$$

Inserting Eq.(102) into Eq.(101) and using the relations Eqs.(88), (99) and (100), we arrive at

$$P_V(s)=\left[s-m_V^2+i\frac{\Gamma_V}{m_V}s+\hat{\Sigma}_V^{\mathrm{rem}}(s)-\hat{\Sigma}_V^{\mathrm{rem}}(p_V)\right]^{-1}, \tag{103}$$

showing the appearance of the familiar line-shape parameters for the process $e^+e^- \to V \to f\overline{f}$. The *remainder*, $\hat{\Sigma}^{\rm rem}_V(s)$, consists of the complete real part of the self-energy $\hat{\Sigma}_{VV}(s)$ and the contribution of heavy fermions to the imaginary part, such as the top quark contributions for $V = W, Z$, which become relevant at $s > (m_t + m_b)^2$ and $s > 4\,m_t^2$, respectively. As a result of the smallness of the decay widths of the gauge bosons this difference is marginal. Consequently, we continue the discussion in this section by using the line-shape language. Using this definition, M_Z is the zero in the real part of the Z-boson self-energy, we write a third renormalization condition:

$$\frac{M^2}{c_\theta^2} = M_z^2 + \frac{g^2}{16\,\pi^2 c_\theta^2}\,{\rm Re}\Sigma_{zz}\left(p^2 = -M_z^2\right). \tag{104}$$

The extension of this formalism to the full SM is not straightforward since the issue of gauge invariance in the vector boson self-energies must be addressed.

5 Gauge invariance and the width of a vector boson

Our first step will consist in studying Ward identities [1] and, successively, we introduce the correct recipe for handling the imaginary parts induced by the resonances. To illustrate an example of the Ward identity, we consider some specific Green function with an external $\mathcal{C}^a$ source and denote it by $\mathcal{W}$. Validity of the Ward identity, therefore, requires $\mathcal{W} = 0$.

Although the proper treatment of the problem should be discussed in the context of unstable (off-shell) W's, we have already shown all the elements of the method in a much simpler process where we assume, unrealistically, that the W are on-shell particles. The process will be $e^+e^- \to W^+W^-$, and the Z boson will appear in the s-channel, annihilation, diagram. This is already enough to spoil the unitarity cancellation if proper care is not taken.

It is used to be thought that the introduction of a width is a phenomenon peculiar to the region around the resonance and that, away from it, the width effects are safely negligible. Certainly the Z in $e^+e^- \to W^+W^-$ is far from its pole and thus we could avoid introducing a Z-width, especially at very large energies. However, with the simple process $e^+e^- \to W^+W^-$ we intend to mimic the real situation of four-fermion production.

Since a pole on the real axis must be screened, we insist in introducing a (running) width and in looking for the minimal set of additional fermionic loops that have to be included in the calculation for preserving gauge invariance.

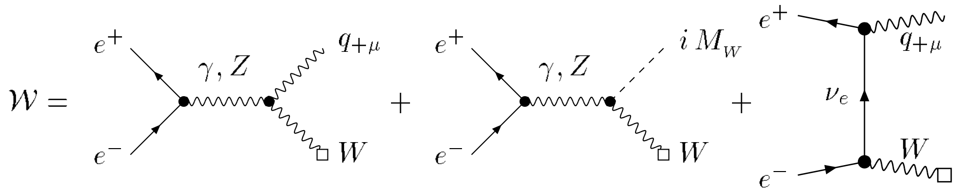


Figure 7: Example of a Ward identity for $e^+e^- \to W^+W^-$.

We now write a Ward identity for the process $e^+(p_+)e^-(p_-) \to W^+(q_+)W^-(q_-)$, considered at the three level. The limit of interest is s well above the WW threshold and the Ward identity is depicted in Fig. 7, where a line terminating with a box indicates multiplication by the corresponding, physical and on-shell, source. It is easy to show that the following identity holds:

$$\mathcal{W} = \frac{1}{4}\,g^2\,\frac{\Gamma_{_Z}(s)}{s - M^2_{_Z} + i\Gamma_{_Z}(s)}\;\bar{u}\,(p_+)\,\gamma_\mu\,\left(4\,s^2_\theta - 1 - \gamma_5\right)v\,(p_-)\,e_\mu(q_-), \quad (105)$$

where we have contracted the W^+-line and added the contributing diagrams with a ϕ^+-line. Moreover, we have introduced

$$\Gamma_{_Z}(s) = \gamma_Z\, s, \qquad \gamma_Z = \frac{g^2}{48\,\pi c^2_\theta}\,\sum_f c_f\,(v^2_f + a^2_f), \quad (106)$$

where the sum is over all the fermions that satisfy the condition $s > 4m^2_f$. It is worth commenting on the width appearing in the Z-propagator: Let $\Sigma_{_{ZZ}}$ be the Z self-energy, then the inverse Z-propagator will become $\Delta\,(s) = s - M^2_{_0} + \Sigma_{_{ZZ}}(s)$. The complex pole $p_{_Z}$ is a solution of the equation $\Delta(p_{_Z}) = 0$. We write $p_{_Z} = m^2 - i\Gamma\, m$ and obtain $m^2 = M^2_{_Z} - \Gamma^2$, $M^2_{_Z} = M^2_{_0} - \mathrm{Re}\Sigma_{_{ZZ}}(M^2_{_Z})$. Whenever fermion masses are ignored, the result of Eq.(96) tells us that

$$\mathrm{Im}\,\Sigma_{_{ZZ}}(s) = \frac{s}{m}\,\Gamma, \qquad \text{for} \quad s < 4\,M^2_{_W}, \tag{107}$$

and that we may introduce a $\Gamma_{_Z} = M_{_Z}\Gamma/m$ with $M_{_Z}\Gamma_{_Z} = \mathrm{Im}\ \Sigma_{_{ZZ}}(M^2_{_Z})$. Therefore, we can write $1/(s - p_{_Z})$ in terms of the familiar line-shape parameters of the resonance,

$$\frac{1}{s - p_{_Z}} = \frac{1 + i\,\Gamma_{_Z}/M_{_Z}}{s - M^2_{_Z} + i\,s\Gamma_{_Z}/M^2_{_Z}}. \tag{108}$$

Thus, the width $\Gamma_{_Z}$ is always computed for $s = m^2 \approx M^2_{_Z}$ and will always contain fermionic contributions but never bosonic ones, if not in some peculiar gauge. In the renormalization of the theory we have encountered terms like

$$\frac{\Sigma_{_{ZZ}}(s) - \mathrm{Re}\ \Sigma_{_{ZZ}}(p_{_Z})}{s - p_{_Z}}, \tag{109}$$

where those contributions to $\mathrm{Im}\Sigma_{_{ZZ}}(s)$ that are due to W-loops must be expanded up to $\mathcal{O}(\alpha)$. In a second step they are combined with the corresponding imaginary parts coming from the ZWW vertex diagram:

$$\begin{aligned}
\frac{1}{s - M^2_{_0} + \Sigma_{_{ZZ}}(s)} &= \frac{1}{s - M^2_{_Z} + i\,s\left(\Gamma_{_Z} M^2_{_Z}\right)}\left[1 + \frac{\Pi(s)}{s - p_{_Z}}\right], \\
\Pi(s) &= \Sigma_{_{ZZ}}(p_{_Z}) - \Sigma_{_{ZZ}}(s) + i\,\frac{\Gamma_{_Z}}{M_{_Z}}\,(s - p_{_Z}). \qquad (110)
\end{aligned}$$

We have $\mathrm{Im}\Sigma_{_{ZZ}}(s) = (\Gamma/m)s = (\Gamma_{_Z}/M_{_Z})s$ and therefore the imaginary part of $W(s)$ will receive contributions from all thresholds that open up because of $s > M^2_{_Z}$. We should write the amplitude in terms of the physical input parameters, i.e. $\alpha(0)$, G_F, $M_{_W}$, $M_{_Z}, \ldots$. Then the question of what to use for s^2_θ will arise and from the simple Ward identity, valid in the limit of zero width ($\gamma_Z = 0$), we learn that gauge invariance is violated unless $c^2_\theta = M^2_{_W}/M^2_{_Z}$.

One can show that a Ward identity is satisfied in all R_ξ gauges when it is satisfied in the renormalizable ($\xi = 1$) gauge. In turn, this means that in all gauges we must respect $c_\theta^2 = M_{_W}^2/M_{_Z}^2$. The reader may wonder about the unitary gauge ($\xi \to \infty$) since there the Higgs-ghost ϕ will disappear, so that the information on the Higgs sector of the theory will remain hidden while we are computing the Ward identity. Of course, there is no problem here, if we remember that the unitary gauge is a well-defined limit of a classes of gauges, the R_ξ one. So by proving a Ward identity in $\xi = 1$ we have proved it in all gauges. We should also note that we are considering only tree-level Ward identities and, therefore, we can use $M = M_{_W}$.

The following point is important: violation of the Ward identity (as a consequence of a non-zero γ_Z) has an immediate counterpart in terms of the asymptotic behaviour of the total cross-section, when $s \to \infty$: unitarity is violated and the violating term is given by $\gamma_Z/(1+\gamma_Z^2)$. This is a rather academic problem since it would be sufficient to ignore the Z width from the beginning. Nevertheless, we insist on the discussion since it is relatively simple and, moreover, the solution has everything in common with the more realistic cases.

There are three additional one-loop diagrams to be considered or, rather, their imaginary parts. The first is given by the Z–γ transition (in this order!), induced by fermionic loops. We easily find

$$-\frac{2}{\pi^2 p^2}\,\mathrm{Im}\,\Sigma_{_{ZA}}(p^2) = -\frac{4}{3}\,\frac{g^2 s_\theta}{c_\theta}\,\sum_f c_f Q_f v_f = -\frac{g^2 s_\theta}{c_\theta}\left(8 - \frac{64}{3}s_\theta^2\right). \qquad (111)$$

In this equation we have assumed that $-p^2 = s \gg 4m_f^2$, so that all fermions are massless. For the case $f = t$ and $s \approx 4\,m_t^2$ these formulas, as well as all the successive ones, must be modified accordingly. We also have the ZW^+W^- vertex, corrected with two fermionic loops of opposite charge flow. The first result is that the imaginary parts do not factorize into the lowest order structure but rather two additional form factors are needed. Indeed, we find

$$-2\,\mathrm{Im}\,V_{_{ZWW}} = 12\,g^3 c_\theta \pi^3 \sum_{i=1,4} v_i W^i_{\mu\alpha\beta}, \tag{112}$$

in terms of four operators:

$$\begin{aligned} V^1_{\mu\alpha\beta} &= 2\,(\delta_{\mu\beta}p_\alpha - \delta_{\mu\alpha}p_\beta - \delta_{\alpha\beta}q_{-\mu}), \qquad & V^2_{\mu\alpha\beta} &= 2\,(\delta_{\mu\beta}p_\alpha - \delta_{\mu\alpha}p_\beta), \\ V^3_{\mu\alpha\beta} &= 2\,(\delta_{\mu\beta}p_\alpha + \delta_{\mu\alpha}p_\beta), & V^4_{\mu\alpha\beta} &= -\frac{2}{s}\,q_{-\mu}p_\alpha p_\beta\,. \end{aligned} \tag{113}$$

The coefficients v_i are easily computed and they are given by

$$\begin{aligned} v_1 &= \frac{2}{3} + \frac{2}{3}\frac{\mu^2}{\beta^2} - 4\,\frac{\mu^4}{\beta^4} + 8\,\frac{\mu^6}{\beta^5}\,l_0, \\ v_2 &= \frac{\mu^2}{\beta^2}\left[1 + 6\,\frac{\mu^2}{\beta^2} - 4\,\frac{\mu^2}{\beta}\left(1 + 3\,\frac{\mu^2}{\beta^2}\right) l_0\right], \qquad v_3 = 0, \\ v_4 &= \frac{\mu^2}{\beta^4}\left[-\frac{8}{3} - 24\,\frac{\mu^2}{\beta^2} + 16\,\frac{\mu^4}{\beta^2} + 16\,\frac{\mu^2}{\beta}\left(1 + 2\,\frac{\mu^2}{\beta^2} + 2\,\frac{\mu^4}{\beta^2}\right) l_0\right], \end{aligned} \tag{114}$$

where we have used some additional variables, $\mu^2 = M^2_{_W}/s$, $\beta^2 = 1 - 4\,\mu^2$ and $l_0 = \ln(1-\beta)/(1+\beta)$. Note that the CP-odd coefficient v_3 is zero. Next we consider the sum $\mathcal{W} + \mathcal{W}_{\rm add}$, where $\mathcal{W}$ represents the tree-level Ward identity with the inclusion of width effects, Eq.(105). $\mathcal{W}_{\rm add}$ contains the additional terms obtained by inserting the loop corrections into the Z-exchange diagram, Eqs.(111)–(112). As a result we find that, for each isospin doublet $d = (f', f)$, the following identity holds:

$$\begin{aligned} \mathcal{W} + \mathcal{W}_{\rm add} &= \frac{1}{4}\,\frac{g^2}{s - M^2_{_Z} + i\,\Gamma_{_Z}(s)}\,\bar{u}\gamma_\mu\left(4s^2_\theta - 1 - \gamma_5\right) v\,\frac{g^2}{96\,\pi c^2_\theta}\sum_{ff'} W_{f'f}\,, \\ W_{f'f} &= 2\sum_{n=ff'}\left(v^2_n + 2\,Q_n v_n s^2_\theta\right) + 1 + (v_f - v_{f'}) + \left[I^{(3)}_f - I^{(3)}_{f'}\right], \end{aligned} \tag{115}$$

where we have introduced $v_f = I^{(3)}_f - 2\,Q_f s^2_\theta$. From the fact that $I^{(3)}_{f'} - I^{(3)}_f = 1$ and $I^{(3)}_f - v_f = 2\,Q_f s^2_\theta$ it follows that

$$\mathcal{W} + \mathcal{W}_{\rm add} = 0. \tag{116}$$

In summary, we have found the minimal set of one-loop diagrams which must be included in order to respect the corresponding Ward identity. This set does not include all fermionic loop insertions into the diagrams of the three level amplitude. For that we should also consider the corrections to the γ–γ and γ–Z transitions and to the γWW vertex. However, we can show that these diagrams satisfy a Ward identity by themselves. We find that the collection of all these diagrams is given by

$$\begin{aligned}
W' &= g^2\, \bar{u}\gamma_\mu v e_\mu \sum_f c_f \left[W'_{\gamma Z}(f) + W'_{\gamma WW}(f) + W'_{\gamma\gamma}(f)\right], \\
W'_{\gamma Z}(f) &= \frac{1}{s - M_Z^2 + i\Gamma_Z(s)}\, \frac{g^2}{24\,\pi}\, Q_f v_f\, [M_Z^2 s_\theta^4 - \left(s - M_W^2\right) s_\theta^2], \\
W'_{\gamma WW}(f) &= \frac{1}{s}\, \frac{g^2}{24\,\pi}\, Q_f I_f^{(3)} s_\theta^2, \qquad W'_{\gamma\gamma}(f) = -\frac{g^2}{12\,\pi}\, Q_f^2 s_\theta^4.
\end{aligned} \tag{117}$$

By collecting the various terms we obtain that for each fermion

$$W' \propto g^4 \sum_f c_f Q_f v_f s_\theta^2 \left[M_W^2 - c_\theta^2 M_Z^2 + \mathcal{O}\left(g^2\right)\right]. \tag{118}$$

Once more the Ward identity $W' = 0$ is satisfied if we have defined from the beginning $c_\theta^2 = M_W^2/M_Z^2$. Strictly speaking, higher-order terms will remain present and their treatment requires a complete analysis of the neutral sector of the theory.

With these results we can rewrite the amplitude for $e^+e^- \to W^+W^-$ as the sum of three contributions. The photon and neutrino diagrams remain unaltered, while the Z diagrams become now

$$\begin{aligned}
A_Z &= A_0 \left(1 + i\, \frac{g^2}{48\,\pi} \gamma^{(1)}\right) + i\, \frac{g^2}{64\,\pi} \left(\gamma^{(2)} A_2 + \gamma^{(4)} A_4\right), \\
A_0 &= -\frac{ig^2}{4}\, \frac{\bar{u}\gamma_\mu \left(4 s_\theta^2 - 1 - \gamma_5\right) v}{s - M_Z^2 + i\gamma_Z s}\, V^1_{\mu\alpha\beta} e_\alpha e_\beta, \\
A_{2,4} &= -\frac{ig^2}{4}\, \frac{\bar{u}\gamma_\mu \left(4 s_\theta^2 - 1 - \gamma_5\right) v}{s - M_Z^2 + i\gamma_Z s}\, V^{2,4}_{\mu\alpha\beta} e_\alpha e_\beta,
\end{aligned} \tag{119}$$

with the γ-coefficients given by

$$\gamma^{(0)} = \frac{1}{c_\theta^2} \sum_f c_f \left(v_f^2 + a_f^2\right), \qquad \gamma^{(1)} = \gamma^{(0)} + \frac{1}{2c_\theta^2} \sum_d c_f \left(1 + v_f - v_{f'}\right) v_1^{\text{rest}},$$

$$\gamma^{(2,4)} = \frac{1}{c_\theta^2}\sum_d c_f\,(1+v_f-v_{f'})\,v_{2,4}, \qquad v_1^{\rm rest}=\frac{3}{2}v_1-1. \tag{120}$$

With Σ_d we denote the sum over doublets, i.e. $(\nu_e, e), \dots, (u, d), \dots$ Within each doublet f' denotes the isospin $+1/2$ fermion and f the isospin $-1/2$ fermion. With this modified A_Z we recover the correct asymptotic behaviour of the total cross-section. Note that $\gamma^{(1)} = \gamma^{(0)} + \mathcal{O}\left(M_W^2/s\right)$ is enough to cancel all the divergent terms of $\mathcal{O}\left(s/M_W^2\right)$ and $\mathcal{O}\left(s^2/M_W^4\right)$. Indeed, the non-factorizable terms in A_Z—i.e. $\gamma^{(2,4)}$—only contribute to the constant term in the asymptotic limit $s \to \infty$. It should be clear enough that the exact Ward identity requires all the terms but, from a numerical point of view, the error that we make in ignoring $\gamma^{(2,4)}$ and in putting $v_1^{\rm rest}$ to zero in Eq.(120) is totally negligible.

An interesting feature emerges when we take into account the top quark mass. For $4\,M_W^2 < s < 4\,m_t^2$ we have that both $\Gamma_Z(s)$ and the Z–γ transition do not receive any contribution from the top quark loop. Conversely, the ZWW vertex will have a diagram where the external Z couples to b-quarks and the remaining fermion in the loop is a top quark. Thus, its imaginary part will be non-zero and, most important, m_t-dependent. In evaluating the Ward identity we must, in this case, take into account the $Z\phi W$ vertex, where ϕ is the Higgs ghost. In computing the cross-section, there will be an intermediate energy region where m_t cannot be ignored and where we have a delayed unitarity cancellation, as described in [2].

It is important to observe that we have defined a minimal set of additional diagrams, needed to restore the Ward identities. The imaginary part of the triangle VWW $(V = Z, \gamma)$ is the sum of all cuts over intermediate states. Thus, two additional contributions have been left out which will be denoted by $E_\pm^{\mu\alpha\beta}$, i.e.

$$E_+^{\mu\alpha\beta} = -2i\,(V+A)\int d^4q\,t_{\mu\rho\alpha\sigma\beta\tau}\,\frac{q_\rho\,(q+q_+)_\sigma\,(q+P)_\tau}{(q+P)^2},$$

$$E_{-}^{\mu\alpha\beta} \;=\; 2i\,(V+A)\int d^4q\, t_{\mu\rho\alpha\sigma\beta\tau}\,\frac{(q+P)_\rho\,(q+q_-)_\sigma\,q_\tau}{(q+P)^2}, \qquad (121)$$

where $V(A)$ denotes the vector (axial–vector) coupling of V to down-fermions and we have not explicitly included all coupling constant factors. Moreover, we have

$$t_{\mu\rho\alpha\sigma\beta\tau} = \mathrm{Tr}\left[\gamma_\mu\gamma_\rho\gamma_\alpha\gamma_\sigma\gamma_\beta\gamma_\tau\right]. \qquad (122)$$

It is easy to observe that these additional imaginary parts have nothing to do with the Z width and a lot to do with the W one. As such, they must certainly be included in any realistic calculation for $e^+e^- \to$ four fermions, where they compensate for the off-shell W's, but not for real W's. Let us consider their contribution to the Ward identity, namely $q_{+\alpha}E_{\pm\mu\alpha\beta}$. It is immediate to find

$$q_{+\alpha}E_{+\mu\alpha\beta} = 0\,, \qquad q_{+\alpha}E_{+\mu\alpha\beta} = -\frac{8}{3}\,i(V+A)M^2_{_W} I_0\delta_{\mu\beta}. \qquad (123)$$

Suppose that this extra term is inserted into the original Ward identity, actually both for the Z and for the γ, and that we sum over all fermions. It is clear that the parts proportional to Γ_Z continue to cancel, while new contributions will appear. These new contributions are exactly cancelled by the following procedure. Keep $M^2_{_W}$ in all vertices as $Z(\gamma)W\phi$ and replace $q^2_\pm = -M^2_{_W}$ with $q^2_\pm = -M^2_{_W} + i\,M_{_W}\Gamma_{_W}$. The presence of the two additional cuts requires the off-shell W's to satisfy the Ward identity or, in other words, only when the W's are off-shell and their width is included in the Feynman diagrams are we bound to consider the additional contributions for restoring the proper behaviour.

Our last comment will be devoted to the correct treatment of the imaginary part of the triangle diagram. One of the results is that this imaginary part does not factorize into the lowest order. In computing the corresponding contribution to be inserted into the Ward identity, particular care must be used in keeping terms proportional to $q_{+\alpha}$. The resulting expression is given by Eq.(112), with the sum extended to six terms and where we introduce the additional operators $W^5_{\mu\alpha\beta} = -2\,\delta_{\mu\beta}q_{+\alpha}$ and $W^6_{\mu\alpha\beta} = 2\,q_{-\mu}q_{+\alpha}P_\beta$, which would give zero when contracted with a physical source. The corresponding contribution to the Ward identity, obtained through a contraction with $q_{+\alpha}$, gives $g^3/(96\,\pi)c_\theta\delta_{\mu\beta}$, which is what the Ward identity requires. If we ignore the $W^{2,4}$ operators, then additional terms will remain, proportional to $q_{-\mu}p_\beta$, and the Ward identity is violated. If we also ignore the $W^{5,6}$ operators, then the right structure $\delta_{\mu\beta}$ is obtained but with the wrong coefficient.

5.1 More Ward identities

It is also interesting to analyse other Ward identities corresponding to situations where their violation corresponds to large numerical errors. We use again 'gedanken processes' which are, however, simple enough to allow for a complete discussion from an analytical point of view. We consider the processes $\gamma W^- \to e^-\overline{\nu}_e$ and $e^+W^- \to e^+\mu^-\overline{\nu}_\mu$.

We begin with the $U(1)$ Ward identity depicted in Fig. 8 where all particles, including the photon, are on their mass shell. Four diagrams suffice in this case, since the γ–Z transition induced by fermions is zero for an on-shell photon. The shaded circle indicates that we take the imaginary part of the two fermion loops, with opposite charge flow. We obtain that the sum of the diagrams is

$$\mathcal{W} = \frac{g^2 s_\theta}{2\sqrt{2}}\,\bar{u}\gamma_\mu\gamma_+ v e_\mu \frac{-\gamma_W + \sum_d \frac{g^2}{48\,\pi}}{s - M^2_W + i\gamma_W s}\,s\,, \tag{124}$$

where the sum is over all doublets $d = (f', f)$. Since $\gamma_W = \sum_d g^2/(48\,\pi)$ we observe that the introduction of the imaginary part of the γWW vertex is enough to guarantee the validity of the Ward identity,, therefore, $\mathcal{W} = 0$. In the Fig. 8 the symbol q_μ means contraction with the external photon momentum and the box indicates an off-shell W.

Consider another $SU(2)$ Ward identity which is given in Fig. 9 and amounts to the statement that $\mathcal{C}^+$ is a free field. Here, we let

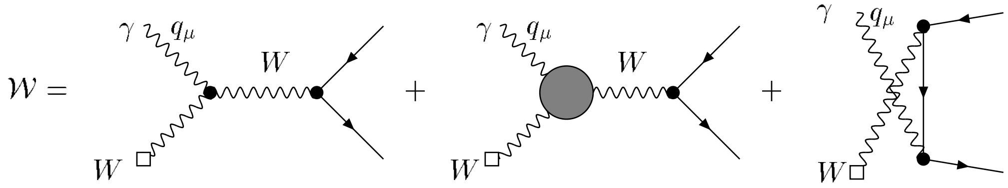


Figure 8: Example of a $U(1)$ Ward identity for $\gamma W^- \to e^- \overline{\nu}_e$.

the photon be off-shell and also allow for a photon propagator in the diagrams. If the vertex correction is not included, then we find

$$\frac{g^2 s_\theta}{2\sqrt{2}}\,\bar{u}\gamma_\mu\gamma_+ v e_\mu \frac{1}{q^2}\,\frac{q^2 - i\gamma_W s}{s - M_W^2 + i\gamma_W s}. \tag{125}$$

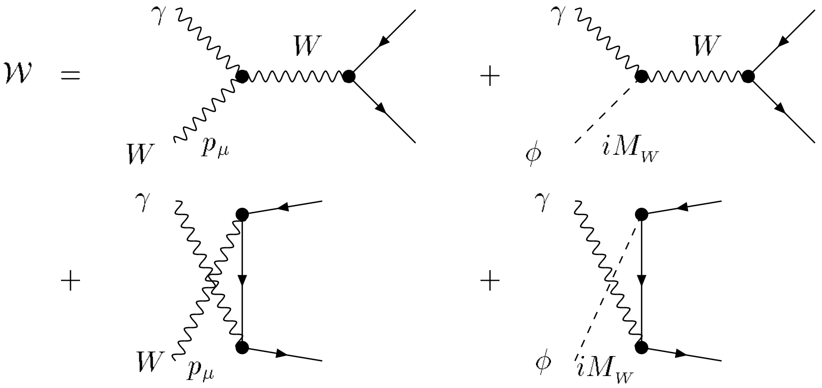


Figure 9: Example of an $SU(2)$ Ward identity for $\gamma W^- \to e^- \overline{\nu}_e$.

The fact that this Ward identity is violated, even for zero width, is well known: as soon as this (sub-)process becomes a part of some realistic process and the photon is off-shell, then other non q^2-resonant diagrams will be present and we expect a constant (w.r.t. q^2) violating term. Alternatively, it should be remembered that for off-shell particles a Ward identity must be compensated by the coupling of the Faddeev–Popov ghosts to the sources. For instance, we consider the corresponding Ward identity with an off-shell photon and a W boson. The three diagrams indicated above give (p is the momentum of the incoming W and $s = -(q+p)^2$)

$$\frac{i\,g^2 s_\theta}{2\sqrt{2}}\,\bar{u}\gamma_\mu\gamma_+ v e_\mu \frac{q^2}{q^2\left(p^2 + M_W^2\right)\left(s - M_W^2\right)}, \tag{126}$$

where we have included the W- and γ-propagators. There is an additional diagram, where the photon source emits both a W-line and a Faddeev–Popov X ghost-line. The rule for this extra vertex can be found following a gauge transformation of the fields:

$$A_\mu \to A_\mu + i\, g s_\theta \left(W^+_\mu X^- - W^-_\mu X^+\right) - \partial_\mu Y^A. \tag{127}$$

From the transformation law we obtain

$$\frac{-i\, g^2 s_\theta}{2\sqrt{2}}\, \bar{u}\gamma_\mu\gamma_+ v e_\mu \frac{1}{\left(p^2 + M^2_{_W}\right)\left(s - M^2_{_W}\right)}, \tag{128}$$

which exactly compensates for the off-shell photon. As for the $U(1)$ off-shell Ward identity, we would obtain

$$\frac{i\, g^2 s_\theta}{2\,\sqrt{2}}\, \bar{u}\gamma_\mu\gamma_+ v\, \frac{p^2 + M^2_{_W}}{q^2\left(p^2 + M^2_{_W}\right)\left(-s + M^2_{_W}\right)}, \tag{129}$$

with an additional diagram where the W source emits both a W-line and a ghost Y^A-line, giving

$$\frac{-i\, g^2 s_\theta}{2\,\sqrt{2}}\, \bar{u}\gamma_\mu\gamma_+ v\, \frac{1}{q^2\left(-s + M^2_{_W}\right)}. \tag{130}$$

Note that the introduction of a W width will violate the identity, with a term proportional to $\Gamma_{_W} s/q^2$. In a realistic process, such as CC20 $e^+e^- \to e^+\nu_e d\overline{u}$, the the presence of a resonant W will require the introduction of the width. At the same time, however, the (almost) on-shell photon will give raise, in certain kinematic regions, to terms proportional to Γ_W/m_e.

As the final step we show an $SU(2)$ Ward identity for the process $e^+W^- \to e^+\mu^-\overline{\nu}_\mu$(eight diagrams): in short, we compute the Green function for $e^+W^- \to e^+\mu^-\overline{\nu}_\mu$, where there are six diagrams, Fig. 10, in lowest order and in the limit of zero fermion masses. Two of these diagrams are exactly $\gamma W^- \to \mu\overline{\nu}_\mu$, embedded with the positron line. We assume a scenario where we are far from the W-resonance, so that the W width can be safely ignored. In this case, we obtain that the diagrams of Fig. 10 add up to ($K = p + q$ is the momentum of the resonant W)

$$\mathcal{W} \;=\; \frac{g^3}{\sqrt{2}} \sum_{i=1,5} W_i,$$

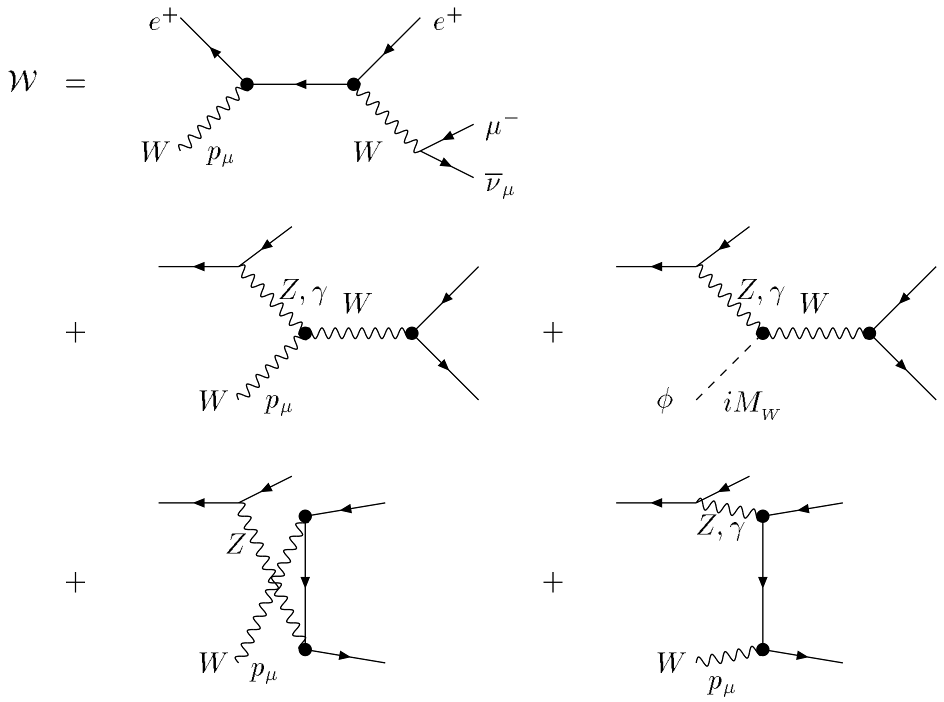


Figure 10: Example of $SU(2)$ Ward identity for $e^+W^- \to e^+\mu^-\overline{\nu}_\mu$.

$$
\begin{aligned}
W_1 &= \frac{1}{2}s_\theta^2\left(\frac{q^2-K^2-M_W^2}{K^2+M_W^2}+1\right)\frac{1}{q^2}\,\gamma_\mu\gamma_+\otimes\gamma_\mu\,,\\
W_2 &= -\frac{1}{8}\frac{1}{K^2+M_W^2}\,\gamma_\mu\gamma_+\otimes\gamma_\mu\gamma_+\,,\\
W_3 &= -\frac{1}{8}\frac{q^2-K^2+s_\theta^2M_Z^2}{\left(K^2+M_W^2\right)\left(q^2+M_Z^2\right)}\,\gamma_\mu\gamma_+\otimes\gamma_\mu\left(4s_\theta^2-1-\gamma_5\right),\\
W_4 &= \frac{1}{16}\frac{2s_\theta^2-1}{c_\theta^2}\,\frac{1}{q^2+M_Z^2}\,\gamma_\mu\gamma_+\otimes\gamma_\mu\left(4s_\theta^2-1-\gamma_5\right),\\
W_5 &= -\frac{1}{16}\,\frac{1}{c_\theta^2}\frac{1}{q^2+M_Z^2}\,\gamma_\mu\gamma_+\otimes\gamma_\mu\left(4s_\theta^2-1-\gamma_5\right). \qquad (131)
\end{aligned}
$$

From $M_W^2 = c_\theta^2 M_Z^2$ it follows that $\mathcal{W} = 0$. Thus, we have verified that:

- The *single* W diagrams are not separately gauge-invariant but, instead, all the *background* must be included, otherwise the Ward identity will give rise to a q^2-independent term.

As shown with the help of the Ward identities, we are allowed to introduce a K^2-dependent width for the W as long as we introduce, at the same time, all the imaginary parts corresponding to all fermionic one-loop corrections: in this case the $\gamma(Z)WW$ vertex, since the Z–Z, γ–γ, Z–γ and γ–Z transitions will not develop an imaginary part (for $q^2 > 0$). The full expression for $\mathrm{Im}\,(V_{\gamma WW})$, with off-shell γ and W's, will not be given here. As soon as the W width is considered, we must include four other diagrams in the Ward identity: they have a fermionic triangle, in the two charge assignments, and the positron line coupled to a γ or a Z. The Ward identity becomes

$$\mathcal{W} = \frac{g^3}{\sqrt{2}} \sum_{i=1,7} W_i, \tag{132}$$

where the $W_i, i = 1, 5$, are the same as before with now $K^2 + M^2_{_W} \to K^2 + M^2_{_W} - i\,\gamma_W K^2$. The two new contributions are

$$\begin{aligned}
W_6 &= -i\,\frac{1}{2}s^2_\theta \frac{K^2}{q^2\left(K^2 + M^2_{_W} - i\gamma_{_W} K^2\right)}\gamma^{(\gamma)}_W\,\gamma_\mu\gamma_+ \otimes \gamma_\mu\,, && (133)\\
W_7 &= i\,\frac{1}{2}s^2_\theta \frac{K^2}{\left(q^2 + M^2_{_Z}\right)\left(K^2 + M^2_{_W} - i\gamma_{_W} K^2\right)}\gamma^{(Z)}_W\,\gamma_\mu\gamma_+ \otimes \gamma_\mu \left(4s^2_\theta - 1 - \gamma_5\right)\,.
\end{aligned}$$

The imaginary parts, for each isospin doublet, are

$$\begin{aligned}
\gamma_W &= \frac{g^2}{48\,\pi}, \qquad \gamma^{(\gamma)}_W = \frac{g^2}{48\,\pi}\,(Q_{f'} - Q_f) = -\gamma_W\,,\\
\gamma^{(Z)}_W &= \frac{g^2}{384\,\pi}\,(v_{f'} - v_f - 1) = -\frac{1}{4}\gamma_W\, c^2_\theta.
\end{aligned} \tag{134}$$

The full Ward identity, i.e. $W = 0$, is now satisfied since both the q^2-dependent terms and the constants cancel. If, instead, $W_{6,7}$ are not included, then we find the wrong result, as already anticipated.

5.2 The Complete Loop framework

The Fermion Loop scheme can be formally extended to include the complete set of corrections.

First, we perform all re-summations of potentially large higher-order terms. There could be a clash between re-summation and gauge parameter independence of the answer. In W-pair production there is an additional complication because of the bad high energy behaviour of individual diagrams. For instance, the amplitude has good high-energy behaviour which is spoiled if we perform a different re-summation for the γ-exchange diagram and for the Z-exchange diagram.

A nice scheme, preserving all Ward identities, and allowing the correct inclusion of radiative corrections in the high-energy limit is the complex-pole scheme. Needless to say, the complex-pole scheme performs a complete re-summation, leading to a gauge parameter dependent answer. For the moment, we proceed without worrying about ξ-dependence.

Let us consider the whole set of self-energy diagrams, so that we may write [1]

$$\{\Sigma_{3Q}(p^2); \Sigma_{33}(p^2); \Sigma_{WW}(p^2)\} = \frac{5}{46}\,\Pi_{\gamma\gamma}(p^2)\,p^2 + \{R_{3Q}(p^2); R_{33}(p^2); R_{WW}(p^2)\}. \tag{135}$$

The explicit form of R_{3Q}, R_{33} and of R_{WW} are of no concern here. We take Eq.(135) as the defining equation. The re-summed propagators for the vector bosons are now

$$\begin{aligned}
G_\gamma(p^2) &= \left\{p^2 - S_{AA}(p^2) - \frac{\left[S_{ZZ}(p^2)\right]^2}{p^2 + M_0^2 - S_{ZZ}(p^2)}\right\}^{-1},\\
G_{Z\gamma}(p^2) &= \frac{S_{ZA}(p^2)}{\left[p^2 - S_{AA}(p^2)\right]\left[p^2 + M_0^2 - S_{ZZ}(p^2)\right] - \left[S_{ZA}(p^2)\right]^2},\\
G_Z(p^2) &= \left\{p^2 + M_0^2 - \frac{\left[S_{ZA}(p^2)\right]^2}{p^2 - S_{AA}(p^2)}\right\}^{-1},\\
G_W(p^2) &= \left[p^2 + M^2 - S_{WW}(p^2)\right]^{-1}.
\end{aligned} \tag{136}$$

The complex poles are determined by the following two equations:

$$\begin{aligned}
p_W &= M^2 - S_{WW}(p_W),\\
p_Z &= M_0^2 - Z(p_Z), \qquad \text{where} \quad Z(s) = S_{ZZ}(s) - \frac{\left[S_{ZA}(s)\right]^2}{s + S_{AA}(s)}.
\end{aligned} \tag{137}$$

Substituting Eq.(137) into the expressions for the propagators, Eq.(136), we see that all ultraviolet divergences not proportional

[1] In the pure FL scheme a similar set of relations holds with $\frac{5}{46}$ replaced by $\frac{1}{8}$.

to p^2 cancel. We obtain

$$
\begin{aligned}
G_{Z}(s) &= \left[-s+p_{Z}-Z(s)+Z(p_{Z})\right]^{-1}, \\
G_{W}(s) &= \left[-s+p_{W}-S_{WW}(s)+S_{WW}(p_{W})\right]^{-1}, \\
G_{Z\gamma}(s) &= -\frac{S_{ZA}(s)}{s+S_{AA}(s)}\,G_{Z}(s), \\
G_{\gamma}(s) &= -\frac{1}{s+S_{AA}(s)}+\left[\frac{S_{ZA}(s)}{s+S_{AA}(s)}\right]^{2}G_{Z}(s). \qquad (138)
\end{aligned}
$$

We now define the running parameters, having the same structure as in the purely fermionic case:

$$
e^2(s)=\frac{e^2}{1-\dfrac{e^2}{16\,\pi^2}\,\Pi_{\gamma\gamma}(s)}, \qquad g^2(s)=\frac{g^2}{1+\dfrac{g^2}{16\,\pi^2}\,\dfrac{\Sigma_{3Q}(s)}{s}}, \qquad (139)
$$

with the following definition of the running weak mixing angle: $s^2(s)=e^2(s)/g^2(s)$. The three running parameters are connected by the same relation which holds for the corresponding bare parameters, which is one of the requested properties. Using the self-energies and the running parameters we can write

$$
\begin{aligned}
1+\frac{Z(s)}{s} &= \frac{g^2}{c_\theta^2}\left[\frac{g^2(s)}{c^2(s)}+\frac{1}{16\,\pi^2}\,\frac{f_{Z}(s)}{s}\right], \\
1+\frac{S_{WW}(s)}{s} &= g^2\left[\frac{1}{g^2(s)}+\frac{1}{16\,\pi^2}\,\frac{f_{W}(s)}{s}\right], \qquad (140)
\end{aligned}
$$

where we have introduced the following two functions:

$$
f_{Z}(s)=R_{33}(s)-R_{3Q}(s), \qquad f_{W}(s)=R_{WW}(s)-R_{3Q}(s). \qquad (141)
$$

As a result, the vector boson propagators are now expressed as

$$
\begin{aligned}
G_{W}(s) &= -\frac{g^2(s)}{g^2}\,\frac{\omega_{W}(s)}{s}, \qquad G_{Z}(s)=-\frac{c_\theta^2}{g^2}\,\frac{g^2(s)}{c^2(s)}\,\frac{\omega_{Z}(s)}{s}, \\
G_{Z\gamma}(s) &= \frac{s_\theta}{c_\theta}\left[1-\frac{s^2(s)}{s_\theta^2}\right]G_{Z}(s), \\
G_{\gamma}(s) &= -\frac{e^2(s)}{e^2\,s}+\frac{s_\theta^2}{c_\theta^2}\left[1-\frac{s^2(s)}{s_\theta^2}\right]^2 G_{Z}(s). \qquad (142)
\end{aligned}
$$

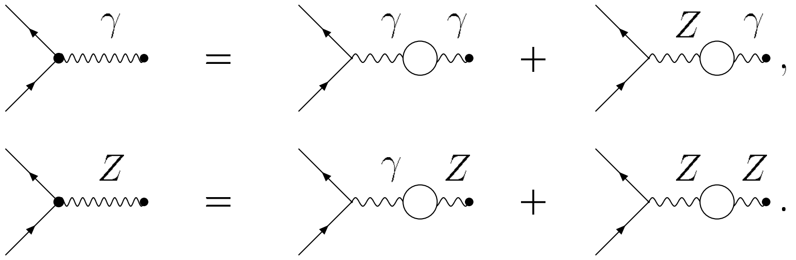
where the propagation functions are

$$
\omega_{W}^{-1}(s) = 1-\frac{g^2(s)}{s}\left\{\frac{p_{W}}{g^2(p_{W})}-\frac{1}{16\,\pi^2}\left[f_{W}(s)-f_{W}(p_{W})\right]\right\},
$$

$$\omega_Z^{-1}(s) \;=\; 1 - \frac{g^2(s)}{c^2(s)s}\left\{\frac{c^2(p_Z)}{g^2(p_Z)}\,p_Z - \frac{1}{16\,\pi^2}\left[f_Z(s) - f_Z(p_Z)\right]\right\}. \tag{143}$$

The ω-functions are ultraviolet finite since the ultraviolet poles in f_Z and f_W do not depend on the scale. Actually, everything is ultraviolet finite, but gauge parameter dependent, since we have re-summed the whole set of self-energy diagrams, not just the fermionic component. The result of including vector boson transitions is illustrated schematically by the following diagrams (Eq.(144)).

First, we write the fermionic corrections to the photon or Z boson coupled to massless fermions as



(144)

Here the open circles denote re-summed propagators and the dot a vertex. The dot on the right-hand side of the diagrams indicates that the corresponding leg is not amputated, i.e. that the propagator is included. Next we introduce the unrenormalized fermion currents:

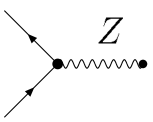
$$\begin{aligned}
\Gamma^f_{A,\mu}(p_+,p_-) &\equiv e\,\bar{v}\,(p_+)\;\gamma_\mu\,(-Q_f)\,u\,(p_-)\,,\\
\Gamma^f_{Z,\mu}(p_+,p_-) &\equiv e\,\bar{v}\,(p_+)\;\gamma_\mu\;(v_f + a_f\gamma_5)\;u\,(p_-)\,.
\end{aligned} \tag{145}$$

The renormalized fermion currents are analogously given by

$$\begin{aligned}
\Gamma^{R,f}_{A,\mu}(p_+,p_-) &\equiv e(s)\;\bar{v}\,(p_+)\;\gamma_\mu\,(-Q_f)\,u\,(p_-)\,,\\
\Gamma^{R,f}_{Z,\mu}(p_+,p_-) &\equiv e(s)\;\bar{v}\,(p_+)\;\gamma_\mu\,[v_f(s) + a_f(s)\gamma_5]\,u\,(p_-)\,,
\end{aligned} \tag{146}$$

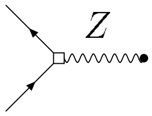
where $P = p_+ + p_-$ and $s = -P^2$. Moreover, $v_f(s)$ and $a_f(s)$ are given in terms of the running couplings.

We now can rewrite the fermion–Z boson coupling including the Z and mixing propagators:



$$\frac{e}{s_\theta c_\theta} \;=\; \frac{e}{s_\theta c_\theta}\left[\Gamma^f_{A,\mu}(p_+,p_-)\;G_{ZA}(s) + \Gamma^f_{Z,\mu}(p_+,p_-)\;G_{ZZ}(s)\right]$$

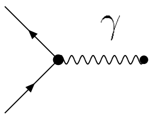
$$= -\Gamma^{R,f}_{Z,\mu}(p_+,p_-)\,\frac{\omega_Z(s)}{s}\,\frac{e(s)}{c(s)s(s)}$$

$\equiv$ 

$$\frac{e(s)}{c(s)s(s)}, \qquad (147)$$

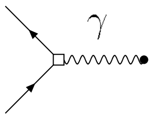
where we denote the renormalized effective Born couplings of the fermions to the vector bosons by a box. The gauge bosons attached to such a box are defined to have their propagators replaced by the corresponding renormalized propagator functions. The effect of the renormalization is apparently to change all bare couplings to renormalized (running) ones and the Z propagator to $-\omega_Z(s)/s$.

Similarly, the fermion–photon interaction can be written as

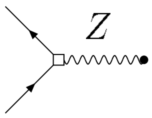


$$e = e\,[\Gamma^{f}_{A,\mu}(p_+,p_-)\,G_{AA}(s) + \Gamma^{f}_{Z,\mu}(p_+,p_-)\,G_{ZA}(s)]$$

$$= -\Gamma^{R,f}_{A,\mu}(p_+,p_-)\,\frac{1}{s}\,e(s)$$
$$-\Gamma^{R,f}_{Z,\mu}(p_+,p_-)\,\frac{\omega_Z(s)}{s}\,\frac{e(s)}{c(s)s(s)}\,[s^2(s) - s_\theta^2]$$

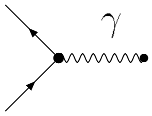
$\equiv$ 

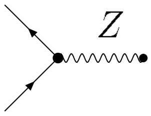
$$e(s)$$



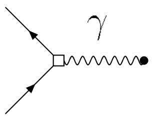
$$\frac{e(s)}{c(s)s(s)}\,[s^2(s) - s_\theta^2]. \qquad (148)$$

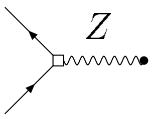
In this case, some bare couplings are still left, but in our specific processes the interactions appear in the following combination:



$$e \; -$$


$$e\,\frac{c_\theta}{s_\theta} =$$

$=$ 

$$e(s) \; -$$


$$e(s)\,\frac{c(s)}{s(s)}. \qquad (149)$$

6 Massive Fermion-Loop.

As we have shown the incorporation of finite-width effects in the theoretical predictions for LEP2 processes and beyond necessitates a careful treatment. Independently of how finite widths of propagating particles are introduced, this requires a re-summation of the vacuum-polarization effects. Furthermore, the principle of gauge invariance must not be violated, i.e., the Ward identities, have to be preserved.

In [3] it was argued that the preferable (Fermion-Loop) scheme consists in the re-summation of the fermionic one-loop corrections to the vector-boson propagators and the inclusion of all remaining fermionic one-loop corrections, in particular those to the Yang–Mills vertices.

In Ref. [3] the main incentive was the discussion of the process $e^+e^- \to e^-\overline{\nu}_e u\overline{d}$ at small scattering angles and LEP2 energies. Naive inclusion of the finite W-boson width breaks U(1) electromagnetic gauge invariance and leads to a totally wrong cross-section in the collinear limit. By taking into account in addition the imaginary parts arising from cutting the massless fermion loops in the triple-gauge-boson vertex, U(1) gauge invariance is restored and a sensible cross-section is obtained.

In Ref. [4] the authors presented the details of the full-fledged Fermion-Loop scheme, taking into account the complete fermionic one-loop corrections including all real and imaginary parts, and all contributions of the massive top quark. A proper treatment of the neutral gauge-boson propagators is performed by solving the Dyson equations for the photon, Z-boson, and mixed photon–Z propagators. This is necessary to guarantee the unitarity cancellations at high energies. The top-quark contributions are particularly important for delayed-unitarity effects. In this respect also terms involving the totally-antisymmetric ε-tensor (originating from vertex corrections) are relevant. While such terms are absent for complete generations of massless fermions owing to the anomaly cancellations, they show up for finite fermion masses. As the ε-dependent terms satisfy the Ward identities by themselves, they can be left out in more minimal treatments like the one used in Ref. [3].

In both papers the external fermionic currents are assumed to be conserved, i.e. one neglects the masses of the external fermions. The effect of including masses is notoriously very small, except in collinear regions like, for instance, the one of single-W production.

In other words, one should remember that the current attached to the photon propagator is strictly conserved, while the ones attached to the (massive) vector boson propagators are not. The missing terms are of order m_f^2 and, therefore, they are negligible if we can show that collinear limits and high-energy limit are not upset by ignoring these terms.

As we have discussed above, the issue of gauge invariance in the CC20 family has been solved by the introduction of the Fermion-Loop scheme but several subtleties remain, connected with the region of vanishing scattering angle of the electron and with the limit of massless final state fermions in a fully extrapolated setup. A satisfactory solution to compute the total cross-section is, therefore, given by the extension of the Fermion-Loop scheme to the case of external, non-conserved currents.

First of all, for single$W$ production one cannot neglect the electron mass, nor in the matrix element, neither in the kinematics of the process. However, keeping a finite electron mass through the calculation is not enough. Massless quarks in the final state induce a singularity, even for finite m_e, if a cut is not imposed on the invariant mass $M(u\overline{d})$.

With a cut on $M(\overline{d}u)$ the singularity at zero momentum transfer ($Q^2 = 0$) is avoided but we still have additional singularities. Indeed, there are two multi-peripheral diagrams contributing to the CC20 process $e^+e^- \to e^-\overline{\nu}_e f_1 \overline{f}_2$, see Fig. 11. When $Q^2 = 0$, i.e. the electron is lost in the beam pipe, and the (massless) $f_1(f_2)$-fermion is emitted parallel to the (quasi-real) photon then the internal fermion propagator will produce an enhancement in the cross section. Taking into account a $\ln m_e^2$ from the photon flux-function, three options follow:

1. to consider massive (f_1/f_2) fermions, giving a result that is proportional to $\ln m_e^2 \ln m_f^2$,
2. to use massless fermions, giving instead $\ln^2 m_e^2$,
3. to introduce an angular cut on the outgoing f_1 and $\overline{f}_2$ fermions with respect to the beam axis, $\theta(f_1, \bar{f}_2) \geq \theta_{\rm cut}$, giving $\ln m_e^2 \ln \theta_{\rm cut}$.

The first option is clean but ambiguous when the final state fermions are light quarks, what to use for m_u, m_d? The second one presents no problems for a fully leptonic CC20 final state but completely fails to describe quarks, as it can be shown by discussing QCD corrections [5]. The last option is also theoretically clean and can be used to give differential distributions for the final state jets. It is, however, disliked by the experimentalists when computing the total sample of events: hadronized jets are seen and not isolated quarks. Even if the quark is parallel to the beam axis the jet could be broad enough and the event selected. These events are also interesting since they correspond to a situation where the electron and one of the quarks are lost in the beam pipe, while the other quark is recoiling against the neutrino, i.e. one has a totally unbalanced mono-jet structure, background to new particle searches[2].

[2] M. Grünewald, private communication.

The singularity induced by massless quarks in $e^+e^- \to e^-\overline{\nu}_e u\overline{d}$ can only be treated within the context of QCD final state corrections and of the photon hadronic structure function (PHSF) scenario. To discuss QCD corrections in the low $(u\overline{d})$ invariant mass region is not the purpose of this lectures, therefore we assume that the light quarks have a mass and that no kinematical cuts are imposed on the process. Our goal will be to formulate a consistent scheme that takes into account all fermion masses.

For many purposes it is enough to introduce a fixed width for the W, both for the s-channel and the t-channel, i.e. the fixed-width scheme. However, there is another important point in favor of adopting the full Fermion-Loop scheme, since it guarantees automatically the correct choice of scale for the running of $\alpha_{\rm QED}$. The latter is particularly relevant in a process that is dominated by a small momentum transfer.

In dealing with theoretical predictions we must distinguish between an Input-Parameter-Set (IPS) and a Renormalization-Scheme (RS). IPS one always has, RS comes only when one starts including loop corrections. The IPS can be made equal in all calculations, RS is the author's choice, very much as the choice of gauge.

Apart from some recent development, each calculation aimed to provide some estimate for $e^+e^- \to 4\,$f production is, at least nominally, a tree level calculation. Among other things it will require the choice of some IPS and of certain relations among the parameters. In the literature, although improperly, this is usually referred to as the choice of the Renormalization Scheme.

Typically we have at our disposal four experimental data point (plus α_s), i.e. the measured vector boson masses $M_{_Z}, M_{_W}$ and the coupling constants, $G_{_F}$ and α. However we only have three bare parameters at our disposal, the charged vector boson mass, the $SU(2)$ coupling constant and the sinus of the weak mixing angle.

While the inclusion of one loop corrections would allow us to fix at least the value of the top quark mass from a consistency relation, this cannot be done at the tree level. Thus, different choices of the basic relations among the input parameters can lead to different results with deviations which, in some case, can be sizeable.

For instance, a possible choice is to fix the coupling constant g as

$$g^2 = \frac{4\pi\alpha}{s^2_{W}}, \quad s^2_{W} = \frac{\pi\alpha}{\sqrt{2}G_{F}M^2_{W}}, \tag{150}$$

where G_F is the Fermi coupling constant. Another possibility would be to use

$$g^2 = 4\sqrt{2}G_{F}M^2_{W}, \tag{151}$$

but, in both cases, we miss the correct running of the coupling. Ad hoc solutions should be avoided, and the running of the parameters must always follow from a fully consistent scheme. Therefore, the only satisfactory solution is in the extension of the full Fermion-Loop to having non zero external masses, or non-conserved currents. Unless, of course, one can compute the full set of corrections.

We will term massive-massless the version of the Fermion-Loop scheme developed in [4]. Our generalization will be denoted as the massive-massive version of the Fermion-Loop scheme.

6.1 A simple recipe for implementing the Fermion-Loop scheme.

There is a simple recipe for implementing the Fermion-Loop scheme: take any process where the external sources are physical and on-shell, eg fermionic currents, then the complete procedure for the re-summation of self-energies and transitions, for the inclusion of running couplings and of re-summed propagators amounts to rewrite the corresponding Born amplitude in terms of re-summed, running, quantities without the inclusion of transitions like the $\gamma - Z$ one. Let us consider one example in more detail. First, we split the 20 Feynman diagrams of the CC20 family into the nine diagrams of Fig. 11, characterized by the presence of a t-channel photon or

Z-boson, and the rest

$$\mathrm{CC20} = \mathrm{CC20}_{\gamma \oplus Z} + \mathrm{CC20}_{\mathrm{R}}. \tag{152}$$

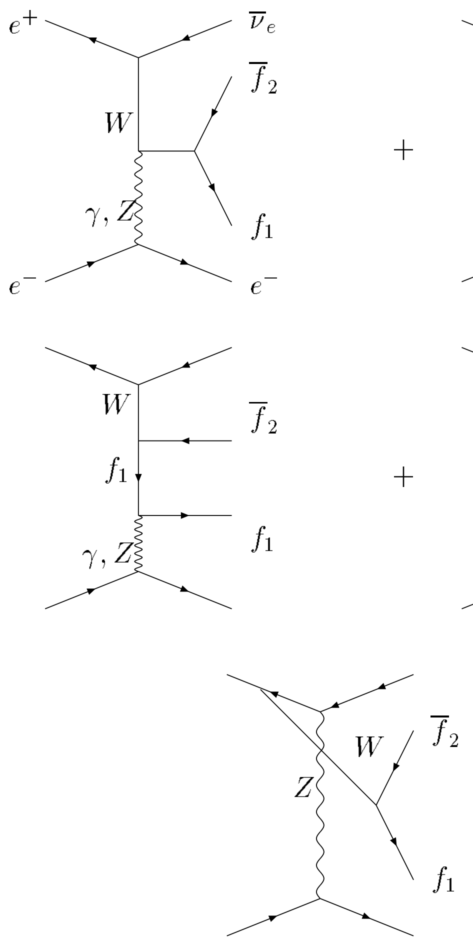


Figure 11: The $\mathrm{CC20}_{\gamma \oplus Z}$ family of diagrams.

Momenta are assigned as follows:

$$e^+(p_+)\, e^-(p_-) \to e^-(q_-)\, \overline{\nu}_e(q_+)\, u(k)\, \overline{d}(\overline{k}), \quad Q_\pm = p_\pm - q_\pm. \tag{153}$$

First we consider the diagram with the non-abelian coupling and generalize it to have re-summed W-boson propagators and $\gamma - \gamma, Z - \gamma$ (1a) transitions and $Z - Z, \gamma - Z$ (1b) transitions. The corresponding amplitudes will then become

$$M_1 = \mathrm{M}_{1\mathrm{a}\mu} \otimes (-igs_\theta)\, \gamma^\mu + \mathrm{M}_{1\mathrm{b}\mu} \otimes \frac{ig}{2\, c_\theta} \gamma^\mu \left(2\, s_\theta^2 - \frac{1}{2}\gamma_+\right), \tag{154}$$

where $\gamma_{\pm} = 1 \pm \gamma_5$ and the sub-amplitudes are

$$
\begin{aligned}
\mathrm{M}_{1\mathrm{a}\mu} &= \frac{1}{8}\,\frac{G_s G_t}{g^3 s_\theta}\,\frac{\omega_W^s \omega_W^t}{(p_- - q_+)^2 (k+\overline{k})^2 Q_-^2}\,\left[e^2(T) - \frac{G_T}{c^2(T)}\,\omega_Z^T \Delta_\gamma\right] \\
&\times\ \gamma^\alpha \gamma_+ \otimes \gamma^\beta \gamma_+ \, V^0_{\mu\alpha\beta},
\end{aligned}
$$

$$
\begin{aligned}
\mathrm{M}_{1\mathrm{b}\mu} &= -\frac{1}{8}\,\frac{G_s G_t}{g^3}\,\frac{\omega_W^s \omega_W^t}{(p_- - q_+)^2 (k+\overline{k})^2 Q_-^2}\,G_T \omega_Z^T \\
&\times\ \gamma^\alpha \gamma_+ \otimes \gamma^\beta \gamma_+ \, V^0_{\mu\alpha\beta},
\end{aligned} \tag{155}
$$

and where $V^0_{\mu\alpha\beta}$ is the tree-level non-abelian coupling. Furthermore,

$$
\Delta_\gamma = s_\theta^4 - 2\,s^2(T) s_\theta^2 + s^4(T) + s_\theta^2 c_\theta^2 - s^2(T) c_\theta^2. \tag{156}
$$

In the previous equations we have introduced the following quantities:

$$
\begin{aligned}
T &= -(p_- - q_-)^2, \\
G_T &= g^2(T), \quad G_t = g^2((p_+ - q_+)^2), \quad G_s = g^2((k+\overline{k})^2).
\end{aligned} \tag{157}
$$

Our sign convention is such that

$$
g^2(s) \equiv g^2(p^2)\mid_{p^2=-s}. \tag{158}
$$

In the same way we have introduced ω_W^s, the propagation function for a W-boson with $p^2 = -(k+\overline{k})^2$. When we combine the two results we obtain

$$
\begin{aligned}
M &= -\frac{i}{8}\,\frac{G_s G_t}{g^2}\,\frac{\omega_W^s \omega_W^t}{(p_- - q_+)^2 (k+\overline{k})^2 Q_-^2}\,\Big\{ e^2(T)\gamma^\mu \\
&+\ \frac{1}{4} G_T \omega_Z^T \gamma^\mu \left[4\,s^2(T) - \gamma_+\right]\Big\} \otimes \gamma^\alpha \gamma_+ \otimes \gamma^\beta \gamma_+ \, V^0_{\mu\alpha\beta}.
\end{aligned} \tag{159}
$$

The only remaining bare quantity is an overall $1/g^2$ factor which is essential in performing the renormalization of the amplitude. The rest is exactly the sum of two Born-like diagrams with γ and Z exchange and with running parameters instead of bare ones. In the following we will denote by Q_f the charge of the fermion. Similarly, for the other diagrams we obtain

$$
\begin{aligned}
M_2 &= \mathrm{M}_{2\mathrm{a}\mu} \otimes (-igs_\theta)\,\gamma^\mu + \mathrm{M}_{2\mathrm{b}\mu} \otimes \frac{ig}{2\,c_\theta}\gamma^\mu \left(2\,s_\theta^2 - \frac{1}{2}\gamma_+\right) \\
&= \frac{i}{8}\,\frac{G_s e^2(T)}{(p_- + Q_-)^2 (k + \overline{k})^2 Q_-^2}\,\omega_{\scriptscriptstyle W}^s\,\gamma^\mu\,(\not{p}_- + \not{Q}_-) \otimes \gamma^\alpha\gamma_+ \otimes \gamma^\alpha\gamma_+ \otimes \gamma^\mu \\
&- \frac{i}{8}\,\frac{G_s G_{\scriptscriptstyle T}}{(p_- + Q_-)^2 (k + \overline{k})^2 Q_-^2}\,\omega_{\scriptscriptstyle W}^s \omega_{\scriptscriptstyle Z}^{\scriptscriptstyle T}\,\frac{1}{16\,c^2(T)} \\
&\times\ \gamma^\mu \left(4\,s^2(T) - \gamma_+\right)(\not{p}_- + \not{Q}_-)\,\gamma^\alpha\gamma_+ \otimes \gamma^\alpha\gamma_+ \otimes \gamma^\mu\ \left(4\,s^2(T) - \gamma_+\right), \\
\\
M_3 &= \mathrm{M}_{3\mathrm{a}\mu} \otimes (-igs_\theta)\,\gamma^\mu + \mathrm{M}_{3\mathrm{b}\mu} \otimes \frac{ig}{2\,c_\theta}\gamma^\mu \left(2\,s_\theta^2 - \frac{1}{2}\gamma_+\right) \\
&= \frac{i}{8}\,\frac{G_t e^2(T)}{Q_+^2 Q_-^2}\,Q_u\,\omega_{\scriptscriptstyle W}^t\,\gamma^\alpha\gamma_+ \otimes \gamma^\mu\,\frac{\not{Q}_- - \not{k} - i\,m_u}{(Q_- - k)^2 + m_u^2}\,\gamma^\alpha\gamma_+ \otimes \gamma^\mu \\
&+ \frac{i}{8}\,\frac{G_t G_{\scriptscriptstyle T}}{16\,c^2(T)\,Q_+^2 Q_-^2}\,\omega_{\scriptscriptstyle W}^t \omega_{\scriptscriptstyle Z}^{\scriptscriptstyle T}\,\gamma^\alpha\gamma_+ \\
&\otimes\ \gamma^\mu \left(\gamma_+ - 4\,Q_u s^2(T)\right)\frac{\not{Q}_- - \not{k} - i\,m_u}{(Q_- - k)^2 + m_u^2}\,\gamma^\alpha\gamma_+\ \otimes \gamma^\mu\ \left(4\,s^2(T) - \gamma_+\right), \\
M_4 &= \mathrm{M}_{4\mathrm{a}\mu} \otimes (-igs_\theta)\,\gamma^\mu + \mathrm{M}_{4\mathrm{b}\mu} \otimes \frac{ig}{2\,c_\theta}\gamma^\mu \left(2\,s_\theta^2 - \frac{1}{2}\gamma_+\right) \\
&= \frac{i}{8}\,\frac{G_t e^2(T)}{Q_+^2 Q_-^2}\,Q_d\,\omega_{\scriptscriptstyle W}^t\,\gamma^\alpha\gamma_+ \otimes \gamma^\mu\,\frac{\not{Q}_- - \overline{\not{k}} + i\,m_d}{(Q_- - \overline{k})^2 + m_d^2}\,\gamma^\alpha\gamma_+ \otimes \gamma^\mu \\
&+ \frac{i}{8}\,\frac{G_t G_{\scriptscriptstyle T}}{16\,c^2(T)\,Q_+^2 Q_-^2}\,\omega_{\scriptscriptstyle W}^t \omega_{\scriptscriptstyle Z}^{\scriptscriptstyle T}\,\gamma^\alpha\gamma_+ \\
&\otimes\ \gamma^\mu \left(-\gamma_+ - 4\,Q_d s^2(T)\right)\frac{\not{Q}_- - \overline{\not{k}} + i\,m_d}{(Q_- - \overline{k})^2 + m_d^2}\,\gamma^\alpha\gamma_+\ \otimes \gamma^\mu\ \left(4\,s^2(T) - \gamma_+\right), \\
M_5 &= \mathrm{M}_{5\mathrm{a}\mu} \otimes (-igs_\theta)\,\gamma^\mu + \mathrm{M}_{5\mathrm{b}\mu} \otimes \frac{ig}{2\,c_\theta}\gamma^\mu \left(2\,s_\theta^2 - \frac{1}{2}\gamma_+\right) \\
&= -\frac{i}{8}\,\frac{G_s G_{\scriptscriptstyle T}}{16\,c^2(T)\,(k + \overline{k})^2 (Q_- - q_+)^2 Q_-^2}\,\omega_{\scriptscriptstyle W}^s \omega_{\scriptscriptstyle Z}^{\scriptscriptstyle T} \\
&\times\ \gamma^\alpha\gamma_+\,(\not{Q}_- - \not{q}_+)\,\gamma^\mu\gamma_+ \otimes \gamma^\mu\ \left(4\,s^2(T) - \gamma_+\right) \otimes \gamma^\alpha\gamma_+. \qquad (160)
\end{aligned}
$$

This set of equations proves the assertion that Fermion-Loop is exactly an improved, gauge preserving, Born approximation.

To complete the construction of the Fermion-Loop scheme one must include the one-loop fermionic vertices. At the Born level the γW^+W^- and the ZW^+W^- vertices are the same, once we have factorized s_θ and c_θ in front of them. For one-loop corrected vertices this is no longer true and one may wonder whether this fact spoils the transition from bare quantities to re-summed ones.

The lowest order interaction for $V(P) \to W^+(q_+)W^-(q_-)$ is specified by the tensor

$$V^0_{\mu\alpha\beta}\left(P;q_+,q_-\right) = \delta_{\mu\beta}\left(P-q_-\right)_\alpha + \delta_{\alpha\beta}\left(q_- - q_+\right)_\mu + \delta_{\mu\alpha}\left(q_+ - P\right)_\beta . \tag{161}$$

At the one-loop level we need seven independent form-factors, if the external sources are physical; they are as follows:

$$\begin{aligned}
V^1_{\mu\alpha\beta} &= \frac{g^2 s_\theta}{16\,\pi^2}\sqrt{s}\sum_{i=1,7} I_i\,W^i_{\mu\alpha\beta},\\
W^1_{\mu\alpha\beta} &= \frac{4}{\sqrt{s}}\left[\delta_{\alpha\beta}q_{-\mu} + \delta_{\mu\beta}P_\alpha + \delta_{\mu\alpha}q_{+\beta}\right],\\
W^{2,3}_{\mu\alpha\beta} &= \frac{2}{\sqrt{s}}\left[\delta_{\mu\beta}P_\alpha \pm \delta_{\mu\alpha}q_{+\beta}\right],\\
W^4_{\mu\alpha\beta} &= \frac{2}{s\sqrt{s}}\,q_{-\mu}P_\alpha q_{+\beta},\\
W^{5,6}_{\mu\alpha\beta} &= \frac{1}{\sqrt{s}}\left[\varepsilon\left(q_+,\mu,\alpha,\beta\right) \pm \varepsilon\left(q_-,\mu,\alpha,\beta\right)\right],\\
W^7_{\mu\alpha\beta} &= \frac{1}{s\sqrt{s}}\,\varepsilon\left(q_-,q_+,\alpha,\beta\right)\,q_{+\mu},
\end{aligned} \tag{162}$$

where we have introduced a special notation,

$$\varepsilon(a,b,c,d) = \varepsilon^{\mu\nu\alpha\beta}\,a_\mu b_\nu c_\alpha d_\beta. \tag{163}$$

For a massless fermion generation there is no difference between the Z and γ coefficients once we have factorized $c_\theta(s_\theta)$ in front of the full vertex. For the third generation we have the same for all loops except the $(m_t,0,m_t)$ one, where the difference between Z and γ is given by the following relations:

$$I_i^{ZWW} = I_i^{\gamma WW} + \frac{1}{c_\theta^2}\,\Delta I_i, \qquad i=1,\dots,7 \tag{164}$$

where the explicit expression for the extra term, ΔI_i, is of no concern here. The amplitude M_1, containing the non-abelian coupling, retains its structure when we substitute $V^0_{\gamma WW}$ with $V^1_{\gamma WW}$ and we have additional contributions

$$
\begin{aligned}
\delta_i &= \frac{g^3 \Delta I_i}{c^2_\theta}\, c_\theta\, G_Z \left[\frac{s_\theta}{c_\theta} \left(1 - \frac{s^2(T)}{s^2_\theta} \right) (-igs_\theta) \otimes \gamma^\mu \right. \\
&+ \left. \frac{ig}{2\, c_\theta} \otimes \gamma^\mu \left(2\, s^2_\theta - \frac{1}{2}\, \gamma_+ \right) \right] \\
&= +\frac{i}{4} g^2\, \Delta_i\, \frac{G_T}{c^2(T)} \frac{\omega^T_Z}{Q^2_-} \otimes \gamma^\mu \left(4\, s^2(T) - \gamma_+ \right). \qquad (165)
\end{aligned}
$$

Therefore, also for one-loop vertices we can write a Born-like amplitude and promote all bare quantities to running ones, i.e. we take the full one-loop corrected vertex of Eq.(162) with factors $s_\theta(c_\theta)$ factorized and also replace $1/c^2_\theta$ with $1/c^2(T)$ in the extra term of Eq.(164).

6.2 Re-summed propagators in the charged sector.

We now proceed to the construction of the Fermion-Loop scheme for non-conserved currents. The first step consists in re-deriving the propagators in a situation where all external fermion masses are kept. We work in the 't Hooft-Feynman gauge and compute the following transitions:

- The $W - W$ transition, $S^{\mu\nu}_W$ with

$$
S^{\mu\nu}_W = \frac{g^2}{16\,\pi^2}\, \Sigma^{\mu\nu}_W, \quad \Sigma^{\mu\nu}_W = \Sigma^0_W\, \delta^{\mu\nu} + \Sigma^1_W\, p^\mu p^\nu. \qquad (166)
$$

  We also introduce a special notation,

$$
\Sigma^T_W = \Sigma^0_W + p^2\, \Sigma^1_W. \qquad (167)
$$

- The $\phi - \phi$ transition, S_ϕ with

$$
S_\phi = \frac{g^2}{16\,\pi^2}\, \Sigma_\phi. \qquad (168)
$$

- The $W - \phi$ and $\phi - W$ transitions,

$$
S^\mu_{W\phi} = +\frac{g^2}{16\,\pi^2}\, \Sigma_{W\phi}\, ip^\mu, \quad S^\mu_{\phi W} = -\frac{g^2}{16\,\pi^2}\, \Sigma_{W\phi}\, ip^\mu. \qquad (169)
$$

Let us introduce indices $a, b, \ldots = 1, \ldots, 5$; the re-summation amounts to write the following equation:

$$\begin{aligned}
\bar{\Delta}_{ab} &= \Delta_{ab} + \Delta_{ac}\, S_{cd}\, \Delta_{db} + \ldots \\
&= \delta_{ac}\, \left(\Delta_{cb} + S_{cd}\, \Delta_{db} + \ldots\right) = \Delta_{ac}\, X_{cb}, \\
X_{ab} &= \left(1 - S\, \Delta\right)^{-1}_{ab}.
\end{aligned} \tag{170}$$

Here we have made use of Born propagators given, in the 't Hooft-Feynman gauge, by

$$\Delta^{\mu\nu}_{WW} = \frac{\delta_{\mu\nu}}{p^2 + M^2}, \qquad \Delta_{\phi\phi} = \frac{1}{p^2 + M^2}. \tag{171}$$

Examples of Dyson re-summation are as follows:

$$\begin{aligned}
\bar{\Delta}^{\mu\nu}_{WW} &= \Delta^{\mu\nu}_{WW} + \Delta^{\mu\alpha}_{WW}\, S^{\alpha\beta}_{W}\, \delta^{\beta\nu}_{WW} + \ldots, \\
\bar{\Delta}_{\phi\phi} &= \Delta_{\phi\phi} + \Delta_{\phi\phi}\, S_{\phi}\, \Delta_{\phi\phi} + \ldots, \\
\bar{\Delta}^{\mu}_{W\phi} &= \Delta^{\mu\alpha}_{WW}\, S^{\alpha}_{W\phi}\, \Delta_{\phi\phi} + \ldots
\end{aligned} \tag{172}$$

After performing the inversion of the matrix in Eq.(170), we obtain

$$\begin{aligned}
\bar{\Delta}^{\mu\nu}_{WW} &= \frac{1}{p^2 + M^2 - S^0_W} \\
&\times \quad \Big[\delta^{\mu\nu} + \frac{S^1_W + \left(S_{W\phi}\right)^2 / \left(p^2 + M^2 - S_\phi\right)}{p^2 + M^2 - S^T_W - p^2 \left(S_{W\phi}\right)^2 / \left(p^2 + M^2 - S_\phi\right)}\, p^\mu p^\nu\Big]
\end{aligned} \tag{173}$$

The previous result can be cast into a simpler form when we use some important relation originating from Ward identities applied to two-point functions. The Ward identities for transitions in the charged sector are shown in Fig. 12 where we used the symbol =, attached to a vector boson line, to indicate multiplication by $i\, p_\mu$.

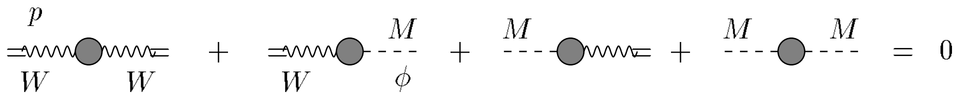


Figure 12: Example of Ward identities for transitions in the charged sector.

From the explicit expressions of these Ward identities we derive the following results:

$$S_{W\phi} = \frac{M}{p^2}\, S_\phi, \qquad S^T_W = \frac{M^2}{p^2}\, S_\phi,$$

$$S_W^1 \;=\; \frac{1}{p^2}\left(\frac{M^2}{p^2}\,S_\phi - S_W^0\right). \tag{174}$$

With their help the re-summed W propagator becomes

$$\begin{aligned}
\bar{\Delta}_{WW}^{\mu\nu} &= \frac{1}{p^2 + M^2 - S_W^0}\,\left(\delta^{\mu\nu} + \Delta_L\,p^\mu p^\nu\right), \\
\Delta_L &= \frac{1}{p^2}\,[M^2\,S_\phi - p^2\,S_W^0 + \frac{M^2\,(S_\phi)^2}{p^2 + M^2 - S_\phi}] \\
&\times\;[p^2\,\left(p^2 + M^2\right) - M^2\,S_\phi - \frac{M^2\,(S_\phi)^2}{p^2 + M^2 - S_\phi}]^{-1}.
\end{aligned} \tag{175}$$

The re-summed W propagator satisfies the following identity:

$$p_\mu\,\bar{\Delta}_{WW}^{\mu\nu} = \frac{p^\nu}{p^2 + M^2 - \frac{M^2}{p^2}\,S_\phi\,[1 + S_\phi/\left(p^2 + M^2 - S_\phi\right)]}. \tag{176}$$

Similarly, we obtain the $W - \phi$ re-summed transition,

$$\bar{\Delta}_{W\phi}^{\mu} = i\,\frac{M p^\mu}{p^2}\,\frac{S_\phi}{p^2 + M^2 - S_\phi}\,\frac{1}{p^2 + M^2 - \frac{M^2}{p^2}\,S_\phi\,[1 + S_\phi/\left(p^2 + M^2 - S_\phi\right)]}, \tag{177}$$

and the $\phi - \phi$ re-summed transition,

$$\bar{\Delta}_{\phi\phi} = \frac{p^2 + M^2 - M^2/p^2\,S_\phi}{\left(p^2 + M^2 - S_\phi\right)\,\left(p^2 + M^2 - M^2/p^2\,S_\phi\right) - M^2/p^2\,\left(S_\phi\right)^2}. \tag{178}$$

Before continuing we define some auxiliary quantities:

$$\begin{aligned}
\bar{\Delta}_{WW}^{\mu\nu} &= \frac{1}{p^2 + M^2 - S_W^0}\,\left(\delta^{\mu\nu} + \frac{N_v}{p^4\,D}\,p^\mu p^\nu\right), \\
\bar{\Delta}_{W\phi}^{\mu} &= i\,\frac{M p^\mu}{p^2}\,\frac{S_\phi}{\left(p^2 + M^2 - S_\phi\right)\,D}, \\
\bar{\Delta}_{\phi\phi} &= \frac{1}{p^2 + M^2 - S_\phi}\,\frac{N_s}{D},
\end{aligned} \tag{179}$$

where we have introduced

$$\begin{aligned}
D &= p^2 + M^2 - \frac{M^2}{p^2}\,S_\phi\,[1 + \frac{S_\phi}{p^2 + M^2 - S_\phi}], \\
N_v &= M^2\,S_\phi - p^2\,S_W^0 + \frac{M^2\,(S_\phi)^2}{p^2 + M^2 - S_\phi}, \\
N_s &= p^2 + M^2 - \frac{M^2}{p^2}\,S_\phi.
\end{aligned} \tag{180}$$

We have seen that, for zero external masses, the re-summation of transitions has a very simple effect on the Born amplitude, it promotes all quantities to running ones and all bare couplings disappear from the amplitude itself. Is it possible to have a generalization of this phenomenon that accounts for non zero external masses? The answer to this question will be the subject of the next section.

6.3 The running W-boson mass.

Running couplings appear naturally in the one-loop corrected amplitude when we consider an $\mathcal{S}$-element, i.e. a transition between physical sources. Therefore, we start with some simple example to illustrate the generalization of the Fermion-Loop scheme. Consider the process $\nu_\mu \mu \to \overline{\nu}_e e$, as given in Fig. 13.

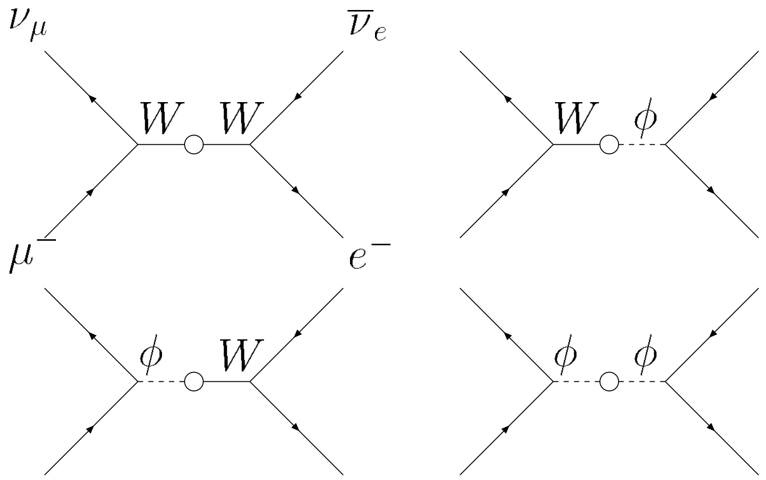


Figure 13: The process $\mu\nu_\mu \to e\overline{\nu}_e$.

For our purposes only four diagrams are relevant. They correspond to the inclusion of re-summed $W-W, W-\phi, \phi-W$ and $\phi-\phi$ transitions. We obtain

$$M \;=\; \left(\frac{ig}{2\sqrt{2}}\right)^2 \mathcal{M},$$

$$
\begin{aligned}
\mathcal{M} &= \gamma^\mu\gamma_+ \otimes \gamma^\nu\gamma_+\,\bar{\Delta}^{\mu\nu}_{WW} + \frac{m_e}{M}\,\gamma^\mu\gamma_+ \otimes \gamma_+\,\bar{\Delta}^{\mu}_{W\phi} \\
&- \frac{m_\mu}{M}\,\gamma_- \otimes \gamma^\mu\gamma_+\,\bar{\Delta}^{\mu}_{\phi W} - \frac{m_e m_\mu}{M^2}\,\gamma_- \otimes \gamma_+\,\bar{\Delta}_{\phi\phi}.
\end{aligned} \tag{181}
$$

The amplitude can we written as the sum of two terms, a familiar one where $m_e = m_\mu = 0$ and an *extra* contribution given by:

$$
\mathcal{M} = \frac{1}{p^2 + M^2 - S^0_{W}}\gamma^\mu\gamma_+ \otimes \gamma^\mu\gamma_+ + \mathcal{M}_{\rm extra}. \tag{182}
$$

After some lengthy but straightforward algebra, making use of the Dirac equation and of the relation,

$$
D\,\left(p^2 + M^2 - S_\phi\right) = \left(p^2 + M^2\right)^2\left(1 - \frac{S_\phi}{p^2}\right), \tag{183}
$$

the second term in Eq.(182) can be written as follows:

$$
\mathcal{M}_{\rm extra} = m_e m_\mu\,\gamma_- \otimes \gamma_+\,\frac{1}{p^2 + M^2 - S^0_{W}}\,\frac{S^0_{W} - p^2 - \frac{M^2}{p^2}\,S_\phi}{M^2\,\left(p^2 - S_\phi\right)}. \tag{184}
$$

There are two ingredients that we need to continue our construction of the Fermion-Loop scheme. First the complex W-pole. We start with the relation

$$
-g^2\Delta_{W} = -g^2\,[p^2 + M^2 - S^0_{W}(p^2)]^{-1} = [\frac{s - M^2}{g^2} + \frac{1}{16\,\pi^2}\,\Sigma^0_{W}]^{-1}, \tag{185}
$$

and define the complex pole as a solution of

$$
\frac{p_{W} - M^2}{g^2} + \frac{1}{16\,\pi^2}\,\Sigma^0_{W}\,(p_{W}) = 0. \tag{186}
$$

Therefore, for the W propagator, we obtain

$$
-g^2\,\Delta_{W} = \{\frac{s - p_{W}}{g^2(s)} + \frac{p_{W}}{16\,\pi^2}\,[\Pi_{3Q}(p_{W}) - \Pi_{3Q}(s)] + \frac{1}{16\,\pi^2}\,[f_{W}(s) - f_{W}(p_{W})]\}^{-1}. \tag{187}
$$

As expected, the position of the complex W-boson pole is solely fixed by the S^0_{W}-component of its self-energy. Next, we need a second ingredient, the running W-boson mass.

Definition 1 *The W-boson running mass is defined by the following equation:*

$$
\frac{1}{M^2(p^2)} = \frac{1}{M^2}\,\frac{p^2 - S^0_{W} + \frac{M^2}{p^2}\,S_\phi}{p^2 - S_\phi}. \tag{188}
$$

It is not an independent quantity but, instead, it is related to the complex pole. To give a simple illustration of this relation, let us consider the case of massless fermions in the one-loop transitions. This means, in particular, $S_\phi = 0$. Therefore, we have

$$\frac{1}{M^2(p^2)} = \left(1 - \frac{S^0_{_W}}{p^2}\right)\frac{1}{M^2}. \tag{189}$$

Using the fact that the bare mass and the complex pole are related by

$$M^2 = p_{_W} + S^0_{_W}(p_{_W}), \tag{190}$$

and combining it with the following two relations,

$$S^0_{_W}(p^2) = p^2\,\frac{g^2}{16\,\pi^2}\,\Pi_{_{3Q}}(p^2), \qquad \frac{1}{g^2(p^2)} = \frac{1}{g^2} - \frac{1}{16\,\pi^2}\,\Pi_{_{3Q}}(p^2), \tag{191}$$

for a massless internal world we obtain

$$M^2(p^2) = \frac{g^2(p^2)}{g^2(p_{_W})}\,p_{_W}, \qquad M^2(p_{_W}) = p_{_W}. \tag{192}$$

Equipped with this result, we can write $\mathcal{M}_{\rm extra}$ as

$$\mathcal{M}_{\rm extra} = \gamma^\mu\gamma_+ \otimes \gamma^\nu\gamma_+\,\frac{1}{p^2 + M^2 - S^0_{_W}}\,\frac{p^\mu p^\nu}{M^2(p^2)}. \tag{193}$$

To summarize our findings, the complete one-loop re-summation in the 't Hooft-Feynman gauge is equivalent to some *effective* unitary-gauge W-propagator. The whole amplitude can be written in terms of a W-boson exchange diagram, if we make use of the following effective propagator:

$$\Delta^{\mu\nu}_{\rm eff} = \frac{1}{p^2 + M^2 - S^0_{_W}}\,[\delta^{\mu\nu} + \frac{p^\mu p^\nu}{M^2(p^2)}]. \tag{194}$$

we obtain a similar, although a little more complicated, result also when the top quark mass is not neglected in loop corrections. We start with

$$S^0_{_W} = \frac{g^2}{16\,\pi^2}\,[p^2\,\Pi_{_{3Q}}(p^2) + f_{_W}(p^2)], \tag{195}$$

and derive

$$S_\phi = \frac{g^2}{16\,\pi^2}\,\frac{m_t^2}{M^2}\,f_\phi(p^2). \tag{196}$$

With $f_i(p^2) = p^2 \sigma_i(p^2)$, we end up with the following result for the running mass:

$$\begin{aligned}\frac{p_W}{M^2(p^2)} &= \frac{g^2(p_W)}{g^2(p^2)}\,\{1 - \frac{g^2(p^2)}{16\,\pi^2}\,[\sigma_W(p^2) - \frac{m_t^2}{p^2}\,\sigma_\phi(p^2)]\} \\ &\times\ \{1 - \frac{g^2(p_W)}{16\,\pi^2}\,[\sigma_W(p_W) + \frac{m_t^2}{p_W}\,\sigma_\phi(p^2)]\}^{-1}. \end{aligned} \tag{197}$$

The above result is ultraviolet finite. Indeed we obtain

$$f_W(p^2)\mid_{\rm UV} = -\frac{3}{2}\,m_t^2\,\frac{1}{\bar{\varepsilon}}, \tag{198}$$

where $\bar{\varepsilon}$ is the ultraviolet regulator,

$$\frac{1}{\bar{\varepsilon}} = \frac{2}{\varepsilon} - \gamma - \ln\pi, \tag{199}$$

and $\gamma = 0.577216$ is the Euler constant. From the explicit expression for f_ϕ, i.e.

$$f_\phi(p^2) = -\frac{3}{2}\,[A_0\,(m_t) + \left(p^2 + m_t^2\right)\,B_0\left(p^2; m_t, 0\right)], \tag{200}$$

we get the ultraviolet part of f_ϕ,

$$f_\phi(p^2)\mid_{\rm UV} = -\frac{3}{2}\,p^2\,\frac{1}{\bar{\varepsilon}}, \tag{201}$$

giving a cancellation of the ultraviolet divergent term, $1/\bar{\varepsilon}$, inside Eq.(197).

There are several examples where one can show that external fermion masses can be easily included in the Fermion-Loop scheme. We simply promote all quantities to be running ones and use the unitary-gauge expression for the W-boson propagator, but with a running mass. Some of these examples are very instructive, since they clearly show how the strategy works only for $\mathcal{S}$-matrix elements, i.e. for amputated Green's function with on-shell and properly renormalized external sources.

Consider $W^+(q) + \gamma(k) \to u\overline{d}$, a component of the single-W process. We have an amplitude that can be written as follows:

$$M_{\mu\alpha} = V^0_{\mu\alpha\beta}\,\frac{\gamma^\beta\gamma_+}{p^2 + M^2 - S^0_W} + i\,\mathcal{M}_{\mu\alpha}\,\left(m_u\gamma_+ - m_d\gamma_-\right). \tag{202}$$

The first result that we obtain is for a situation where the incoming particles are physical, i.e.

$$k^2 = 0, \quad q^2 = -M^2, \qquad k \cdot e(k) = 0, \quad q \cdot \varepsilon(q) = 0, \tag{203}$$

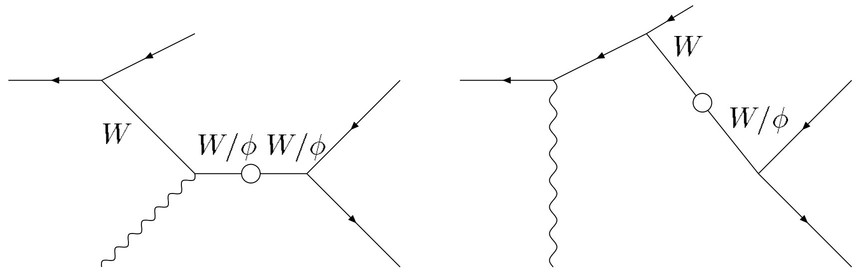
where $e_\mu(k)$ and $\varepsilon_\alpha(q)$ are the photon and the W polarization vectors. In this case we find

$$\mathcal{M}_{\mu\alpha} = \frac{V^0_{\mu\alpha\beta}}{p^2 + M^2 - S^0_{W}} \frac{p^\beta}{M^2(p^2)}, \tag{204}$$

where $p = q + k$. To go further, we only require conservation of the e.m. current, $k \cdot e(k) = 0$, but not the mass-shell condition $k^2 = 0$. In this case, even for $m_e = 0$, a residual term remains:

$$\delta_{\mu\alpha}\, p^2 \frac{M^2 + q^2 - k^2}{M^2\ (p^2 + M^2)\ (p^2 - S_\phi)}. \tag{205}$$

To understand the mechanism of cancellation we embed the subprocess (four diagrams) corresponding to the annihilation $W^+\gamma \to u\overline{d}$ into $e^+\gamma \to \overline{\nu}_e u\overline{d}$. We obtain the following result:



annihilation **bremsstrahlung**

Figure 14: Different topologies for the process $e^+\gamma \to \overline{\nu}_e u\overline{d}$.

$$\begin{aligned} M^{\rm ann}_\mu &= \frac{V^0_{\mu\alpha\beta}}{p^2 + M^2 - S^0_{W}} [\delta^{\beta\lambda} + \frac{p^\beta p^\lambda}{M^2(p^2)} \gamma^\alpha\gamma_+ \otimes \gamma^\lambda\gamma_+ \\ &+ \delta_{\mu\alpha} \frac{p^2}{M^2} \frac{M^2 + q^2 - k^2}{(p^2 + M^2)\ (p^2 - S_\phi)} \gamma^\alpha\gamma_+ \otimes \not{p}\gamma_+. \end{aligned} \tag{206}$$

Some unwanted term remains. However, there is another contribution, where the $u\overline{d}$ pair is emitted by a W-boson that, in turn, is coming from the splitting $e \to \nu_e W$, see Fig. 14. Here, in the limit $m_e = 0$, two diagrams contribute, a $W-W$ propagator and a $W-\phi$ propagator. We find

$$\begin{aligned} M_\mu^{\rm brem} &= \frac{1}{p^2+M^2-S_W^0}\,[\delta_{\alpha\beta}+\frac{p_\alpha p_\beta}{M^2(p^2)}]\,\gamma^\mu\,\frac{i\,(\not p_+ + \not k)}{(p_+ + k)^2}\,\gamma^\alpha\gamma_+\otimes\gamma^\beta\gamma_+ \\ &- \frac{p^2}{M^2}\,\frac{1}{(p^2+M^2)\,(p^2-S_\phi)}\,\gamma^\mu\,\frac{i\,(\not p_+ + \not k)}{(p_+ + k)^2}\,(\not p_+ + \not k)\;\gamma_+\otimes\not p\gamma_+ \end{aligned} \tag{207}$$

The two extra terms coming from Eqs.(206)–(207) add up to something proportional to k^2, therefore vanishing for $k^2=0$. This line of arguments is correct only as long as we neglect re-summation in the t-channel W-propagator.

6.4 Ward identities for single-W.

Before proving the relevant U(1) Ward identity for the single-W amplitude we recall that the total of 20 Feynman diagrams is, first of all, split into a 10 s-channel part and a 10 t-channel part. In the latter part, one diagram has a W-exchange, five a Z-exchange and four a γ-exchange. As we have shown, this picture is not changed by the inclusion of one-loop fermionic corrections: when all transitions are properly taken into account, we still end up with the $1-5-4$ subdivision described above, as long as the fermion–anti-fermion-vector–boson couplings are described in terms of the re-summed expressions.

This $s\oplus t$ splitting is a gauge invariant one, although we can further restrict the number of diagrams. The argument is as follows: Take $e^+\mu^-\to\overline{\nu}_e\mu^- u\overline{d}$. Only the CC20 t-channel diagrams contribute and, since these are all diagrams that we need, this set is gauge invariant. Next, take $e^+\nu_\mu\to\overline{\nu}_e\nu_\mu u\overline{d}$. Its is, again, only t-channel but the photon does not contribute, so that we have $10-4=6$ diagrams. Moreover, if one writes any Z current as $J=J_Q+J_L$ (with J_L proportional to γ_+), only J_L contributes here, because of the neutrinos, so that we have five J_L^Z diagrams plus one W diagram that form a gauge invariant set. Since the whole t-channel is gauge invariant, the eight diagrams, four with photons and four with J^Z_Q, must form a gauge invariant sub-set. This remains true for one-loop corrections when writing everything in terms of running objects and including vertices.

The gauge invariance property that we are referring to is the full SU(2) one. However, the first step is to prove the simpler U(1) gauge invariance or, stated differently, we write the amplitude squared for the four photon-mediated t-channel diagrams as the product of a leptonic tensor $L_{\mu\nu}$ and of a single-W tensor, $W_{\mu\nu}$. Successively, we must be able to prove that $Q^{-\mu} W_{\mu\nu} = Q^{-\nu} W_{\mu\nu} = 0$ (where Q_- is the photon momentum), so that the only terms that survive in the product $L_{\mu\nu} W_{\mu\nu}$ are those proportional to Q^2_- or to m^2_e, giving rise to the familiar contributions to the cross-section. Either logarithmically enhanced, $1/Q^2_-$, or the sub-leading, constant one, m^2_e/Q^4_-.

To be more specific, we give the explicit expression for $L_{\mu\nu}$. With momenta assignment $e^+(p_+) e^-(p_-) \to e^-(q_-) \overline{\nu}_e(q_+) u(k) \overline{d}(\overline{k})$, we have

$$L_{\mu\nu} = 2\, Q^2_- \delta_{\mu\nu} + 4\, \left(p_{-\mu} q_{-\nu} + q_{-\mu} p_{-\nu}\right). \tag{208}$$

U(1) gauge invariance, yet to be proven, requires that the following decomposition holds for $W_{\mu\nu}$,

$$\begin{aligned} W_{\mu\nu} &= W_1 \,[-\delta_{\mu\nu} + \frac{Q_{-\mu} Q_\nu}{Q^2_-}] - W_2 \,\frac{Q^2_-}{(p_+\cdot Q_-)^2} \left(p_{+\mu} - \frac{p_+\cdot Q_-}{Q^2_-} Q_{-\mu}\right) \\ &\times \left(p_{+\nu} - \frac{p_+\cdot Q_-}{Q^2_-} Q_{-\nu}\right). \end{aligned} \tag{209}$$

After contraction, and for $X \to 0$, we obtain

$$\frac{1}{4} L_{\mu\nu} W^{\mu\nu} = W_1 \,[2 \,\frac{m^2_e}{(Xys)^2} - \frac{1}{Xys}] + 2\, W_2 \,\frac{y-1}{X y^3 s} + \mathcal{O}(X), \tag{210}$$

where we have introduced the variable y, equivalent to the fraction of the electron energy carried by the photon, and

$$p_+ \cdot Q_- = \frac{1}{2} [m^2_e - (X+1)\, ys],$$

$$Q_-^2 \;=\; X\,ys, \qquad (p_+ + p_-)^2 = -ys. \tag{211}$$

The decomposition of Eq.(209) requires that we can prove a U(1) Ward identity. This we will do without performing any approximation, within the framework of the Fermion-Loop scheme. One has to be fully aware that the seven tensor structures, introduced in Eq.(162), are not enough to describe the situation, simply because currents are not-conserved.

Before giving the most general structure for the vertex, we will introduce the invariants needed to describe the sub-process,

$$e^+(p_+)\gamma(Q_-) \to \overline{\nu}_e(q_+)u(k)\overline{d}(\overline{k}). \tag{212}$$

They are specified in the following list:

$$\begin{aligned}
p_+ \cdot Q_- &= \frac{1}{2}\left(m_e^2 - Q_-^2 - y\,s\right), \\
k \cdot \overline{k} &= \frac{1}{2}\left(-s' + m_u^2 + m_d^2\right), \\
Q_- \cdot k &= \frac{1}{2}\left(Q_-^2 - m_u^2 + t\right), \\
p_+ \cdot \overline{k} &= \frac{1}{2}\left(-m_e^2 - m_d^2 + t'\right), \\
Q_- \cdot \overline{k} &= \frac{1}{2}\left(Q_-^2 - m_d^2 + u\right), \\
p_+ \cdot k &= \frac{1}{2}\left(-m_e^2 - m_u^2 + u'\right), \\
p_+ \cdot q_+ &= \frac{1}{2}\left(\kappa_+ - m_e^2\right), \\
Q_- \cdot q_+ &= \frac{1}{2}\left(Q_-^2 + \kappa_-\right), \\
\overline{k} \cdot q_+ &= \frac{1}{2}\left(-m_d^2 + \zeta_+\right), \\
k \cdot q_+ &= \frac{1}{2}\left(m_u^2 + \zeta_-\right),
\end{aligned} \tag{213}$$

The linearly independent invariants are:

$$\begin{aligned}
t &= \tau ys, \quad s' = x_2 ys, \quad \kappa_- = zys, \\
\zeta_- &= (x_2 - x_1)\,ys, \quad Q_-^2 = Xys,
\end{aligned} \tag{214}$$

with the following solution for the remaining ones:

$$
\begin{aligned}
\kappa_+ &= m_e^2 + y\,s(-1 - X + x_2 - z),\\
u &= m_e^2 + m_u^2 + m_d^2 + y\,s(-1 - \tau - 2X - z),\\
u' &= m_e^2 + 2\,m_u^2 + m_d^2 + y\,s(-\tau - X - x_1),\\
t' &= -m_u^2 + y\,s(\tau + X + x_1 - x_2 + z),\\
\zeta_+ &= -m_u^2 + m_d^2 + y\,s(-1 + x_1).
\end{aligned} \tag{215}
$$

We introduce another auxiliary quantity

$$Q_+^2 = -\,Yys, \tag{216}$$

and also the vector $Q = Q_+ + Q_- = k + \overline{k}$. In the one-loop corrected γW^+W^- vertex of Fig. 15 all vector-boson lines are off mass-shell and non-conserved.

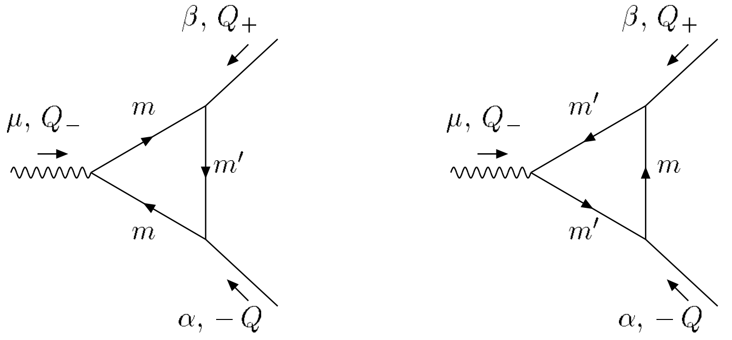


Figure 15: The γW^+W^- vertex.

For the e.m. current this is a consequence of the fact that we are computing a Ward identity and not an amplitude. As a consequence, several more structures are needed and we list them below:

$$
\begin{aligned}
W^1_{\mu\alpha\beta} &= 4\,\mathcal{N}\,[\delta_{\alpha\beta}Q_{+\mu} + \delta_{\mu\beta}Q_{-\alpha} - \delta_{\mu\alpha}Q_\beta],\\
W^{2,3}_{\mu\alpha\beta} &= 2\,\mathcal{N}\,[\delta_{\mu\beta}Q_\alpha \mp \delta_{\mu\alpha}Q_\beta],\\
W^4_{\mu\alpha\beta} &= 2\,\mathcal{N}\,\delta_{\alpha\beta}Q_{-\mu},\\
W^5_{\mu\alpha\beta} &= 2\,\mathcal{N}\,\delta_{\mu\alpha}Q_{+\beta},\\
W^6_{\mu\alpha\beta} &= 2\,\mathcal{N}\,\delta_{\mu\beta}Q_{+\alpha},\\
W^7_{\mu\alpha\beta} &= -2\,\mathcal{N}\,\delta_{\mu\beta}Q_\alpha,\\
W^8_{\mu\alpha\beta} &= -2\,\mathcal{N}\,\delta_{\alpha\beta}Q_\mu,
\end{aligned}
$$

$$W^9_{\mu\alpha\beta} = 2\,\mathcal{N}^3 Q_{+\mu}Q_{-\alpha}Q_\beta,$$

$$
\begin{aligned}
W^{10}_{\mu\alpha\beta} &= 2\,\mathcal{N}^3\,Q_{-\mu}Q_{-\alpha}Q_{\beta},\\
W^{11}_{\mu\alpha\beta} &= -2\,\mathcal{N}^3\,Q_{+\mu}Q_{+\alpha}Q_{+\beta},\\
W^{12}_{\mu\alpha\beta} &= 2\,\mathcal{N}^3\,Q_{+\mu}Q_{+\alpha}Q_{\beta},\\
W^{13}_{\mu\alpha\beta} &= 2\,\mathcal{N}^3\,Q_{+\mu}Q_{\alpha}Q_{+\beta},\\
W^{14}_{\mu\alpha\beta} &= -2\,\mathcal{N}^3\,Q_{+\mu}Q_{\alpha}Q_{\beta},\\
W^{15}_{\mu\alpha\beta} &= 2\,\mathcal{N}^3\,Q_{\mu}Q_{+\alpha}Q_{+\beta},\\
W^{16}_{\mu\alpha\beta} &= -2\,\mathcal{N}^3\,Q_{\mu}Q_{+\alpha}Q_{\beta},\\
W^{17}_{\mu\alpha\beta} &= -2\,\mathcal{N}^3\,Q_{\mu}Q_{\alpha}Q_{+\beta},\\
W^{18}_{\mu\alpha\beta} &= 2\,\mathcal{N}^3\,Q_{\mu}Q_{\alpha}Q_{\beta},\\
\\
W^{19}_{\mu\alpha\beta} &= -2\,\mathcal{N}\,\varepsilon(Q_-,\mu,\alpha,\beta),\\
W^{20}_{\mu\alpha\beta} &= -2\,\mathcal{N}\,[\varepsilon(Q_-,\mu,\alpha,\beta)+2\,\varepsilon(Q_+,\mu,\alpha,\beta)],\\
\\
W^{21}_{\mu\alpha\beta} &= -\mathcal{N}^3\,\varepsilon(Q_+,Q_-,\alpha,\beta)Q_{\mu},\\
W^{22}_{\mu\alpha\beta} &= \mathcal{N}^3\,\varepsilon(Q_+,Q_-,\alpha,\beta)Q_{+\mu},\\
W^{23}_{\mu\alpha\beta} &= \mathcal{N}^3\,[\varepsilon(Q_+,Q_-,\mu,\alpha)Q_{+\beta}-\varepsilon(Q_+,Q_-,\mu,\beta)Q_{+\alpha}],\\
W^{24}_{\mu\alpha\beta} &= -\,\mathcal{N}^3\,\varepsilon(Q_+,Q_-,\mu,\beta)Q_{\alpha}. \qquad (217)
\end{aligned}
$$

The normalization factor is $\mathcal{N} = (Xys)^{-1/2}$. In presenting the vertices we face the usual problem of introducing a full *scalarization* of the result; that is, all results should be presented in terms of one-loop scalar-integral coefficient functions. This procedure is mandatory since the symmetry of the vertices has to be verified. However, there is no need to show the scalarized version of the full vertices, only the corresponding contraction entering into the Ward identity is needed and, as a matter of fact, the latter is considerably simpler and much shorter. To summarize, we will not present explicitly the full vertex

$$
V^1_{\mu\alpha\beta} = \frac{g^3 s_\theta}{16\,\pi^2}\,(Xys)^{1/2}\sum_{i=1,24} I_i\,W^i_{\mu\alpha\beta}, \qquad (218)
$$

but only the contraction. Actually, we cannot limit our analysis to the γW^+W^- vertex but we must include also the $\gamma W^+\phi^-$, $\gamma\phi^+W^-$ and $\gamma\phi^+\phi^-$ vertices. For $\gamma W^+\phi^-$ the operators are as follows:

$$
W^1_{\mu\beta} = \delta_{\mu\beta},
$$

$$
\begin{aligned}
W^2_{\mu\beta} &= \mathcal{N}^2\, Q_{+\mu} Q_{\beta}, \\
W^3_{\mu\beta} &= -\mathcal{N}^2\, Q_{+\mu} Q_{+\beta}, \\
W^4_{\mu\beta} &= \mathcal{N}^2\, Q_{\mu} Q_{+\beta}, \\
W^5_{\mu\beta} &= -\mathcal{N}^2\, Q_{\mu} Q_{\beta}, \\
W^6_{\mu\beta} &= \mathcal{N}^2\, \varepsilon(Q_+, Q_-, \mu, \beta).
\end{aligned} \tag{219}
$$

For $\gamma\phi^+W^-$ the operators are as follows:

$$
\begin{aligned}
W^1_{\mu\alpha} &= \delta_{\mu\alpha}, \\
W^2_{\mu\alpha} &= \mathcal{N}^2\, Q_{+\mu} Q_{\alpha}, \\
W^3_{\mu\alpha} &= -\mathcal{N}^2\, Q_{+\mu} Q_{+\alpha}, \\
W^4_{\mu\alpha} &= \mathcal{N}^2\, Q_{+\mu} Q_{\alpha}, \\
W^5_{\mu\alpha} &= -\mathcal{N}^2\, Q_{\mu} Q_{\alpha}, \\
W^6_{\mu\alpha} &= \mathcal{N}^2\, \varepsilon(Q_+, Q_-, \mu, \alpha).
\end{aligned} \tag{220}
$$

Finally for $\gamma\phi^+\phi^-$ we have

$$
W^1_{\mu} = Q_{+\mu}, \qquad W^2_{\mu} = Q_{\mu}. \tag{221}
$$

For γW^+W^- we will, therefore, present the contraction

$$
\begin{aligned}
V^{ww}_{\alpha\beta} &= Q^{\mu}_{-} V^1_{\mu\alpha\beta} = \frac{g^3 s_\theta}{16\,\pi^2}\, [W^{ww}_0\, \delta_{\alpha\beta} + W^{ww}_1\, (Q_{+\alpha}Q_{-\beta} + Q_{+\beta}Q_{-\alpha} \\
&+ Q_{-\alpha}Q_{-\beta}) + W^{ww}_2\, Q_{+\alpha}Q_{+\beta}].
\end{aligned} \tag{222}
$$

This, is all what we need for a massless internal world, i.e. $m_t = 0$. Otherwise, additional contractions are needed. They are:

$$
\begin{aligned}
V^{w\phi}_{\beta} &= \frac{i g^3 s_\theta}{16\,\pi^2 M}\, [W^{w\phi}_1\, Q_{+\beta} + W^{w\phi}_2\, Q_{-\beta}], \\
V^{\phi w}_{\alpha} &= \frac{i g^3 s_\theta}{16\,\pi^2 M}\, [W^{\phi w}_1\, Q_{+\alpha} + W^{\phi w}_2\, Q_{-\alpha}], \\
V^{\phi\phi} &= \frac{g^3 s_\theta}{16\,\pi^2 M^2}\, W^{\phi\phi}.
\end{aligned} \tag{223}
$$

The Ward identity that we want to prove can be written as follows:

$$
\mathrm{WI} = Q^{\mu}_{-} \sum_{i=1,4} D^i_{\mu} + D^1_V, \tag{224}
$$

where the D^i are the four diagrams of t-channel with a photon line and the electron line removed and D^1_V is D^1 with the inclusion of the all one-loop vertices and with the saturation already performed.

The sum of all diagrams, needed for our Ward identity, is as follows:

$$\begin{aligned}
D^1_\mu &= \left(\frac{ig}{2\sqrt{2}}\right)^2 gs_\theta[\bar v\,(p_+)\,S^V_{l\beta}(1)v\,(q_+)\,\bar u\,(k)\,S^V_{q\alpha}(1)v\left(\overline{k}\right)V^0_{WW\mu\lambda\rho}\bar\Delta^{\alpha\lambda}_{WW}(s)\bar\Delta^{\beta\rho}_{WW}(t)\\
&+ \bar v\,(p_+)\,S^V_{l\beta}(1)v\,(q_+)\,\bar u\,(k)\,S^S_{q}(1)v\left(\overline{k}\right)V^0_{WW\mu\lambda\rho}\bar\Delta^{\lambda}_{W\phi}(s)\bar\Delta^{\beta\rho}_{WW}(t)\\
&+ \bar v\,(p_+)\,S^V_{l\beta}(1)v\,(q_+)\,\bar u\,(k)\,S^V_{q\alpha}(1)v\left(\overline{k}\right)V^0_{W\phi\mu\rho}\bar\Delta^{\alpha}_{\phi W}(s)\bar\Delta^{\beta\rho}_{WW}(t)\\
&+ \bar v\,(p_+)\,S^V_{l\beta}(1)v\,(q_+)\,\bar u\,(k)\,S^S_{q}(1)v\left(\overline{k}\right)V^0_{W\phi\mu\rho}\bar\Delta_{\phi\phi}(s)\bar\Delta^{\beta\rho}_{WW}(t)\\
&+ \bar v\,(p_+)\,S^V_{l\beta}(1)v\,(q_+)\,\bar u\,(k)\,S^V_{q\alpha}(1)v\left(\overline{k}\right)V^0_{\phi W\mu\lambda}\bar\Delta^{\alpha\lambda}_{WW}(s)\bar\Delta^{\beta}_{W\phi}(t)\\
&+ \bar v\,(p_+)\,S^V_{l\beta}(1)v\,(q_+)\,\bar u\,(k)\,S^S_{q}(1)v\left(\overline{k}\right)V^0_{\phi W\mu\lambda}\bar\Delta^{\lambda}_{W\phi}(s)\bar\Delta^{\beta}_{W\phi}(t)\\
&+ \bar v\,(p_+)\,S^V_{l\beta}(1)v\,(q_+)\,\bar u\,(k)\,S^V_{q\alpha}(1)v\left(\overline{k}\right)V^0_{\phi\phi\mu}\bar\Delta^{\alpha}_{\phi W}(s)\bar\Delta^{\beta}_{W\phi}(t)\\
&+ \bar v\,(p_+)\,S^V_{l\beta}(1)v\,(q_+)\,\bar u\,(k)\,S^S_{q}(1)v\left(\overline{k}\right)V^0_{\phi\phi\mu}\bar\Delta_{\phi\phi}(s)\bar\Delta^{\beta}_{W\phi}(t)\\
&+ \bar v\,(p_+)\,S^S_{l}(1)v\,(q_+)\,\bar u\,(k)\,S^V_{q\alpha}(1)v\left(\overline{k}\right)V^0_{WW\mu\lambda\rho}\bar\Delta^{\alpha\lambda}_{WW}(s)\bar\Delta^{\rho}_{\phi W}(t)\\
&+ \bar v\,(p_+)\,S^S_{l}(1)v\,(q_+)\,\bar u\,(k)\,S^S_{q}(1)v\left(\overline{k}\right)V^0_{WW\mu\lambda\rho}\bar\Delta^{\lambda}_{W\phi}(s)\bar\Delta^{\rho}_{\phi W}(t)\\
&+ \bar v\,(p_+)\,S^S_{l}(1)v\,(q_+)\,\bar u\,(k)\,S^V_{q\alpha}(1)v\left(\overline{k}\right)V^0_{W\phi\mu\rho}\bar\Delta^{\alpha}_{\phi W}(s)\bar\Delta^{\rho}_{\phi W}(t)\\
&+ \bar v\,(p_+)\,S^S_{l}(1)v\,(q_+)\,\bar u\,(k)\,S^S_{q}(1)v\left(\overline{k}\right)V^0_{W\phi\mu\rho}\bar\Delta_{\phi\phi}(s)\bar\Delta^{\rho}_{\phi W}(t)\\
&+ \bar v\,(p_+)\,S^S_{l}(1)v\,(q_+)\,\bar u\,(k)\,S^V_{q\alpha}(1)v\left(\overline{k}\right)V^0_{\phi W\mu\lambda}\bar\Delta^{\alpha\lambda}_{WW}(s)\bar\Delta_{\phi\phi}(t)\\
&+ \bar v\,(p_+)\,S^S_{l}(1)v\,(q_+)\,\bar u\,(k)\,S^S_{q}(1)v\left(\overline{k}\right)V^0_{\phi W\mu\lambda}\bar\Delta^{\lambda}_{W\phi}(s)\bar\Delta_{\phi\phi}(t)\\
&+ \bar v\,(p_+)\,S^S_{l}(1)v\,(q_+)\,\bar u\,(k)\,S^V_{q\alpha}(1)v\left(\overline{k}\right)V^0_{\phi\phi\mu}\bar\Delta^{\alpha}_{\phi W}(s)\bar\Delta_{\phi\phi}(t)\\
&+ \bar v\,(p_+)\,S^S_{l}(1)v\,(q_+)\,\bar u\,(k)\,S^S_{q}(1)v\left(\overline{k}\right)V^0_{\phi\phi\mu}\bar\Delta_{\phi\phi}(s)\bar\Delta_{\phi\phi}(t)]
\end{aligned}\tag{225}$$

$$\begin{aligned}
D^2_\mu &= \left(\frac{ig}{2\sqrt{2}}\right)^2(-igs_\theta)\;[\bar v\,(p_+)\,S^V_{l\mu\beta}(2)v\,(q_+)\,\bar u\,(k)\,S^V_{q\alpha}(2)v\left(\overline{k}\right)\bar\Delta^{\beta\alpha}_{WW}(s)\\
&+ \bar v\,(p_+)\,S^V_{l\mu\beta}(2)v\,(q_+)\,\bar u\,(k)\,S^S_{q}(2)v\left(\overline{k}\right)\bar\Delta^{\beta}_{W\phi}(s)\\
&+ \bar v\,(p_+)\,S^S_{l\mu}(s)v\,(q_+)\,\bar u\,(k)\,S^V_{q\alpha}(2)v\left(\overline{k}\right)\bar\Delta^{\alpha}_{\phi W}(s)\\
&+ \bar v\,(p_+)\,S^S_{l\mu}(s)v\,(q_+)\,\bar u\,(k)\,S^S_{q}(2)v\left(\overline{k}\right)\bar\Delta_{\phi\phi}(s)]
\end{aligned}\tag{226}$$

$$\begin{aligned}
D^3_\mu &= \left(\frac{ig}{2\sqrt{2}}\right)^2(igQ_us_\theta)\;[\bar v\,(p_+)\,S^V_{l\beta}(3)v\,(q_+)\,\bar u\,(k)\,S^V_{q\mu\alpha}(3)v\left(\overline{k}\right)\bar\Delta^{\beta\alpha}_{WW}(t)\\
&+ \bar v\,(p_+)\,S^V_{l\beta}(3)v\,(q_+)\,\bar u\,(k)\,S^S_{q\mu}(3)v\left(\overline{k}\right)\bar\Delta^{\beta}_{W\phi}(t)\\
&+ \bar v\,(p_+)\,S^S_{l}(3)v\,(q_+)\,\bar u\,(k)\,S^V_{q\mu\alpha}(3)v\left(\overline{k}\right)\bar\Delta^{\alpha}_{\phi W}(t)\\
&+ \bar v\,(p_+)\,S^S_{l}(3)v\,(q_+)\,\bar u\,(k)\,S^S_{q\mu}(3)v\left(\overline{k}\right)\bar\Delta_{\phi\phi}(t)]
\end{aligned}\tag{227}$$

$$D^4_\mu = \left(\frac{ig}{2\sqrt{2}}\right)^2(igQ_ds_\theta)\;[\bar v\,(p_+)\,S^V_{l\beta}(4)v\,(q_+)\,\bar u\,(k)\,S^V_{q\mu\alpha}(4)v\left(\overline{k}\right)\bar\Delta^{\beta\alpha}_{WW}(t)$$

$$
\begin{aligned}
&+ \ \bar{v}\left(p_{+}\right) S_{l\beta}^{V}(4) v\left(q_{+}\right) \bar{u}\left(k\right) S_{q\mu}^{S}(4) v\left(\overline{k}\right) \bar{\Delta}_{W\phi}^{\beta}(t) \\
&+ \ \bar{v}\left(p_{+}\right) S_{l}^{s}(4) v\left(q_{+}\right) \bar{u}\left(k\right) S_{q\mu\alpha}^{V}(4) v\left(\overline{k}\right) \bar{\Delta}_{\phi W}^{\alpha}(t) \\
&+ \ \bar{v}\left(p_{+}\right) S_{l}^{s}(4) v\left(q_{+}\right) \bar{u}\left(k\right) S_{q\mu}^{S}(4) v\left(\overline{k}\right) \bar{\Delta}_{\phi\phi}(t)]
\end{aligned} \tag{228}
$$

Here, the argument s or t for propagators denotes the variable s' and κ_+, of Eq.(215), respectively. Furthermore,

$$
Q_{\pm} = p_{\pm} - q_{\pm}, \qquad Q = Q_{+} + Q_{-}. \tag{229}
$$

The strings of gamma-matrices are given by

$$
\begin{aligned}
S_{l\mu}^{V}(1) &= \gamma_\mu \gamma_+, \quad S_l^S(1) = \frac{m_e}{M}\gamma_+, \quad S_{q\mu}^V(1) = \gamma_\mu\gamma_+, \\
S_q^S(1) &= \frac{1}{2}[\frac{m_u - m_d}{M}(\gamma_+ + \gamma_-) + \frac{m_u + m_d}{M}(\gamma_+ - \gamma_-)].
\end{aligned} \tag{230}
$$

$$
\begin{aligned}
S_{l\mu\beta}^{V}(2) &= \gamma_\mu \frac{i(\not{q}_+ + \not{Q}) + m_e}{(q_+ + Q)^2 + m_e^2}\gamma_\beta\gamma_+, \\
S_{l\mu}^{S}(2) &= \gamma_\mu \frac{i(\not{q}_+ + \not{Q}) + m_e}{(q_+ + Q)^2 + m_e^2}\frac{m_e}{M}\gamma_+, \quad S_{q\mu}^V(2) = \gamma_\mu\gamma_+, \\
S_q^S(2) &= \frac{1}{2}[\frac{m_u - m_d}{M}(\gamma_+ + \gamma_-) + \frac{m_u + m_d}{M}(\gamma_+ - \gamma_-)].
\end{aligned} \tag{231}
$$

$$
\begin{aligned}
S_{l\mu}^{V}(3) &= \gamma_\mu \gamma_+, \quad S_l^S(3) = \frac{m_e}{M}\gamma_+, \\
S_{q\mu\alpha}^{V}(3) &= \gamma_\mu \frac{i(\not{Q}_- - \not{k}) + m_u}{(Q_- - k)^2 + m_u^2}\gamma_\alpha\gamma_+, \\
S_{q\mu}^{S}(3) &= \gamma_\mu \frac{i(\not{Q}_- - \not{k}) + m_u}{(Q_- - k)^2 + m_u^2}(\frac{m_u}{M}\gamma_+ - \frac{m_d}{M}\gamma_-).
\end{aligned} \tag{232}
$$

$$
\begin{aligned}
S_{l\mu}^{V}(4) &= \gamma_\mu \gamma_+, \quad S_l^S(4) = \frac{m_e}{M}\gamma_+, \\
S_{q\mu\alpha}^{V}(4) &= \gamma_\alpha\gamma_+ \frac{-i(\not{Q}_- - \overline{\not{k}}) + m_d}{(Q_- - \overline{k})^2 + m_d^2}\gamma_\mu, \\
S_{q\mu}^{S}(4) &= (\frac{m_u}{M}\gamma_+ - \frac{m_d}{M}\gamma_-)\frac{-i(\not{Q}_- - \overline{\not{k}}) + m_d}{(Q_- - \overline{k})^2 + m_d^2}\gamma_\mu.
\end{aligned} \tag{233}
$$

The remaining tree-level vertices, with gs_θ factorized out, are given by

$$
V_{W\phi\mu\alpha}^0 = iM\delta_{\mu\alpha}, \quad V_{\phi W\mu\alpha}^0 = -iM\delta_{\mu\alpha}, \quad V_{\phi\phi\mu}^0 = Q_{+\mu} + Q_\mu. \tag{234}
$$

The D^1_V one-loop contraction of Eq.(224) is obtained from D^1_μ of Eq.(225) after multiplication by $Q_{-\mu}$ and after replacing the lowest order triple vertices (γW^+W^-, $\gamma W^+\phi^-$, $\gamma\phi^+W^-$ and $\gamma\phi^+\phi^-$) with the one-loop corrected ones. The corresponding contractions are given in Eqs.(222)–(223).

6.5 The massless internal world.

To prove the Ward identity is particularly simple if we neglect all fermion masses in loops. To make the final result more compact we introduce the following notations:

$$\begin{aligned}
\Gamma^\pm_l(\mu) &= \bar{v}\,(p_+)\,\gamma_\mu\gamma_\pm v\,(q_+)\,,\\
\Gamma^\pm_q(\mu) &= \bar{u}\,(k)\,\gamma_\mu\gamma_\pm v\,\left(\overline{k}\right),\\
\Gamma^\pm_l &= \bar{v}\,(p_+)\,\gamma_\pm v\,(q_+)\,,\\
\Gamma^\pm_q &= \bar{u}\,(k)\,\gamma_\pm v\,\left(\overline{k}\right),
\end{aligned}\tag{235}$$

and also,

$$\begin{aligned}
\Gamma\,(Q_-,+,1,\pm) &= m_u\,\Gamma^+_l(Q_-)\,\Gamma^+_q - m_d\,\Gamma^+_l(Q_-)\,\Gamma^-_q,\\
\Gamma\,(1,+,1,\pm) &= m_u\,\Gamma^+_l\,\Gamma^+_q - m_d\,\Gamma^+_l\,\Gamma^-_q.
\end{aligned}\tag{236}$$

For the non-vertex part of the Ward identity, after using the Dirac equation, we get the following result:

$$\frac{1}{g s_\theta}\,Q^\mu_-\sum_{i=1,4} D^i_\mu = \mathrm{WI}_{\rm nv},\tag{237}$$

where, with $\mu^2_y = M^2/ys$, the contractions is

$$\begin{aligned}
\mathrm{WI}_{\rm nv} &= \frac{i\,g^2}{16}\,\Gamma\,(Q_-,+,1,\pm)\,\frac{R_s}{x_2-\mu^2_y}\,\frac{1}{P_s}\left(\frac{1}{ys}+\frac{Y-x_2}{P_t}\right),\\
&+ \frac{g^2 m_e}{16}\,\Gamma\,(1,+,1,\pm)\,\frac{R_sR_t}{P_sP_t}\,[\frac{ys}{M^2}\,(x_2-Y)+\frac{1}{2}\,\frac{x_2+X-Y}{x_2-\mu^2_y}\\
&+ \frac{1}{2}\,\frac{x_2-X-Y}{Y-\mu^2_y}],\\
&+ \frac{g^2 m_e}{16\,M^2}\,\Gamma\,(1,+,1,\pm)\,\frac{R_s}{P_s}\,[\frac{ys}{P_t}\,(x_2-Y)-1],\\
&+ \frac{g^2 m_e}{16\,M^2}\,\Gamma\,(1,+,1,\pm)\,\frac{R_t}{P_t}\,[\frac{ys}{P_s}\,(x_2-Y)+1],
\end{aligned}$$

$$
\begin{aligned}
&+ \frac{g^2 m_e}{16\,M^2}\,\Gamma\,(1,+,1,\pm)\;[\frac{x_2-Y}{P_s P_t}+\frac{1}{P_t}-\frac{1}{P_s}] \\
&+ \frac{i\,g^2 m_e}{16}\,\Gamma\,(1,+,Q_-,+)\;\frac{R_t}{Y-\mu_y^2}\,\frac{1}{P_t}\,[\frac{Y-x_2}{P_s}-1], \\
&+ \frac{g^2}{16}\,\Gamma_l^+(\mu)\,\Gamma_q^+(\mu)\,[\frac{ys}{P_s P_t}\,(x_2-Y)+\frac{1}{P_t}-\frac{1}{P_s}].
\end{aligned}
\tag{238}
$$

The propagators are expressed as

$$
P_s = [\,-s' + M^2 - S_W^0(s')]^{-1}, \qquad P_t = [\,-\kappa_+ + M^2 - S_W^0(\kappa_+)]^{-1}. \tag{239}
$$

Moreover we have ratios of bare to running masses,

$$
R(p^2) = \frac{M^2}{M^2(p^2)} - 1, \qquad R_s = R(s'), \quad R_t = R(\kappa_+). \tag{240}
$$

This case is relatively simple and, therefore, we give the derivation in some details. The expression that we have obtained is still written in terms of bare quantities and renormalization can be achieved by using the following set of relations:

$$
\begin{aligned}
S_\phi(p^2) &= 0, \qquad S_W^0(p^2) = p^2\,[1-\frac{g^2}{g^2(p^2)}], \\
M^2 &= M^2(p^2)\,\frac{g^2}{g^2(p^2)} = \frac{g^2}{g^2(p_W)}\,p_W, \qquad R(p^2) = \frac{g^2}{g^2(p^2)} - 1, \\
p^2 + M^2 &= \frac{g^2}{g^2(p_W)}\,p_W\,\{[1+\frac{g^2(p_W)}{16\,\pi^2}\,\Pi_{3Q}(p_W)]\,\frac{p^2}{p_W}+1\}
\end{aligned}
\tag{241}
$$

Furthermore, the propagators are rewritten as

$$
\begin{aligned}
P_s^{-1} &= -\frac{G_s}{g^2}\,\frac{1}{p_s}, \quad p_s = x_2 ys - p_W + \left(1-\frac{G_s}{G_W}\right)\,p_W, \\
P_t^{-1} &= -\frac{G_t}{g^2}\,\frac{1}{p_t}, \quad p_t = Y ys - p_W + \left(1-\frac{G_t}{G_W}\right)\,p_W,
\end{aligned}
\tag{242}
$$

where $G_W = g^2(p_W)$. We introduce an auxiliary function

$$
\Phi_W(x) = [\,\left(1+\frac{G_W}{16\,\pi^2}\,\Pi_W\right)\,\frac{xys}{p_W} - 1]^{-1}, \qquad \Pi_W = \Pi_{3Q}(p_W). \tag{243}
$$

In this way, the non-vertex part of the Ward identity will become

$$
\frac{p_s p_t}{G_s G_t}\,\mathrm{WI}_{\mathrm{nv}} \;=\; \frac{i}{16}\,\Gamma\,(Q_-,+,1,\pm)\,\frac{\Pi_s}{16\,\pi^2}
$$

$$
\begin{aligned}
&\times\;\; [1 - \frac{Y ys}{p_{W}}\,\frac{\Pi_t}{16\,\pi^2}\,G_{W}\,\Phi_{W}(x_2)] \\
&+\;\; \frac{im_e}{16}\,\Gamma\,(1,+,1,\pm)\;\{\frac{\Pi_s-\Pi_t}{16\,\pi^2} + G_{W}\,\frac{\Pi_s\Pi_t}{256\,\pi^4}\,\frac{ys}{2\,p_{W}} \\
&\times\;\; [\,(x_2+X-y)\;\Phi_{W}(x_2) + (x_2-X-Y)\;\Phi_{W}(Y)]\} \\
&+\;\; \frac{im_e}{16}\,\Gamma_l^+\,\Gamma_q^+(Q_-)\,\frac{\Pi_t}{16\,\pi^2}\,[\frac{x_2 ys}{p_{W}}\,\frac{\Pi_s}{16\,\pi^2}\,\Phi_{W}(Y)-1] \\
&+\;\; \Gamma_l^+(\mu)\,\Gamma_q^+(\mu)\,\frac{x_2\Pi_s - Y\Pi_t}{16\,\pi^2}. \qquad (244)
\end{aligned}
$$

As usual, we have introduced subscripts s, t to denote $\Pi_s = \Pi_{3Q}(s')$ and $\Pi_t = \Pi_{3Q}(\kappa_+)$. Also for the vertices we get a contribution to the Ward identity that we write as

$$
\frac{1}{gs_\theta}\,D_V^1 = \mathrm{WI_v}. \qquad (245)
$$

Two additional quantities are needed,

$$
\mu_y^2 = \frac{M^2}{ys}, \qquad \Phi(x) = (x-\mu_y^2)^{-1}. \qquad (246)
$$

The function Φ is related to Φ_W by

$$
\begin{aligned}
\Phi(x) &= \frac{G_{W}}{g^2}\,\frac{ys}{p_{W}}\,\Phi_{W}(x) = (\Phi_{W}+1)\;\frac{\Phi_{W}(x)}{\Phi_{W}}, \\
\Phi_{W} &\equiv \Phi_{W}(1). \qquad (247)
\end{aligned}
$$

The explicit expression for this part of the Ward identity is

$$
16\,\pi^2\,\frac{p_s p_t}{G_s G_t}\,\mathrm{WI_v} = \mathrm{WI'_v}, \qquad (248)
$$

$$
\begin{aligned}
\mathrm{Wi'_v} &= \frac{i}{16}\,\Gamma\,(Q_-,+,1,\pm)\,[R_s\,\Phi(x_2)\,\left(\frac{1}{ys}W_0^{WW} - x_2\,W_1^{WW}\right) - W_1^{WW}] \\
&+\;\; \frac{m_e}{32}\,\Gamma\,(1,+,1,\pm)\;\{R_sR_t\,\Phi(x_2)\,\Phi(Y)[\frac{2\,x_2+X}{ys}\,W_0^{WW} - x_2\,(X+2\,Y)\;W_2^{WW}] \\
&+\;\; R_sR_t\,\Phi(x_2)[\frac{1}{ys}\,W_0^{WW} + (x_2+X+Y)\;W_1^{WW} - (X+Y)\;W_2^{WW}]
\end{aligned}
$$

$$
\begin{aligned}
&+ \quad R_s R_t\, \Phi(Y)[-\frac{1}{ys} W_0^{WW} - (x_2 + X + Y)\, W_1^{WW} + (X+Y)\, W_2^{WW}] \\
&+ \quad R_s\, \Phi(x_2)[\frac{2}{ys} W_0^{WW} + (-x_2 + X + Y)\, W_1^{WW} - (x_2 + X + Y)\, W_2^{WW}] \\
&+ \quad R_t\, \Phi(Y)[\frac{2}{ys} W_0^{WW} + (-x_2 - X + Y)\, W_1^{WW} - 2\,Y\, W_2^{WW}] - 2\, W_2^{WW}\} \\
&+ \quad \frac{im_e}{16} \Gamma_l^+\, \Gamma_q^+(Q_-)\, \{R_t\, \Phi(Y)[\frac{1}{ys} W_0^{WW} + Y\, W_1^{WW} - Y\, W_2^{WW}] \\
&+ \quad W_1^{WW} - W_2^{WW}\} - \frac{1}{16} \Gamma_l^+(\mu)\, \Gamma_q^+(\mu)\, W_0^{WW} \qquad (249)
\end{aligned}
$$

To go one step further, to this expression we apply the renormalization procedure of Eqs.(241)–(242). By inspection, its is seen that the Ward identity

$$
\mathrm{WI_{nv}} + \mathrm{WI_{v}} = 0, \qquad (250)
$$

is satisfied if the following conditions hold:

$$
\begin{aligned}
W_0^{WW} &= (x_2\, \Pi_s - Y\, \Pi_t)\; ys, \\
W_1^{WW} &= \Pi_s, \\
W_2^{WW} &= \Pi_s - \Pi_t. \qquad (251)
\end{aligned}
$$

An explicit calculation shows that

$$
\begin{aligned}
\Pi_s &= \frac{4}{9} - \frac{4}{3} B_0\,(-s';0,0)\,, \\
\Pi_t &= \frac{4}{9} - \frac{4}{3} B_0\,(-\kappa_+;0,0)\,, \\
W_0^{WW} &= \{\frac{4}{9}\,(x_2 - Y) + \frac{4}{3}[Y\, B_0\,(-\kappa_+;0,0) - x_2\, B_0\,(-s';0,0)\,]\}, \\
W_1^{WW} &= \frac{4}{9} - \frac{4}{3} B_0\,(-s';0,0)\,, \\
W_2^{WW} &= \frac{4}{3}\,[B_0\,(-\kappa_+;0,0) - B_0\,(-s';0,0)\,], \qquad (252)
\end{aligned}
$$

which represents the solution of Eq.(250).

6.6 The case of non-zero m_t.

When the internal world is not massless the derivation of the Ward identity, although straightforward, is considerably lengthy. First, we have to change the normalization equations, Eqs.(241)–(242). Now they can be written as

$$
\begin{aligned}
P_s^{-1} &= -\frac{G_s}{g^2}\frac{1}{p_s}, \quad p_s = x_2 ys - p_W + \left(1 - \frac{G_s}{G_W}\right) p_W + G_s\,[f_W(s') - f_W(p_W)],\\
P_t^{-1} &= -\frac{G_t}{g^2}\frac{1}{p_t}, \quad p_t = Yys - p_W + \left(1 - \frac{G_t}{G_W}\right) p_W + G_t\,[f_W(\kappa_+) - f_W(p_W)],
\end{aligned}
\tag{253}
$$

Furthermore, the relations between bare and running quantities is now modified into the following form:

$$
\begin{aligned}
\frac{p_W}{M^2(p^2)} &= \frac{g^2(p_W)}{g^2(p^2)}\,\eta(p^2),\\
\eta(p^2) &= \{1 - \frac{g^2(p^2)}{16\,\pi^2}\,[\sigma_W(p^2) - \frac{m_t^2}{p^2}\,\sigma_\phi(p^2)]\}\\
&\times \{1 - \frac{g^2(p_W)}{16\,\pi^2}\,[\sigma_W(p_W) + \frac{m_t^2}{p_W}\,\sigma_\phi(p^2)]\}^{-1},\\
M^2 &= M^2(p^2)\,\frac{g^2}{g^2(p^2)}\,\eta(p^2) + \frac{g^2}{16\,\pi^2}\,f_W(p_W),\\
R(p^2) &= \frac{g^2}{g^2(p^2)}\,\eta(p^2)\,[1 - \frac{g^2(p_W)}{16\,\pi^2}\,\sigma_W(p_W)] - 1.
\end{aligned}
\tag{254}
$$

Moreover, all vertices contribute. We have created a FORM program that computes the massive-massive Ward identity [3] and have found that the latter is satisfied if

$$
\begin{aligned}
W_0^{WW} &= x_2 ys\,\left(\Pi_s + \sigma_W^s\right) - Yys\,\left(\Pi_t + \sigma_W^t\right),\\
W_1^{WW} &= \Pi_s + \sigma_s^W + \frac{m_t^2}{x_2 ys}\,\sigma_s^\phi,\\
W_2^{WW} &= \Pi_s - \Pi_t + \sigma_W^s - \sigma_W^t + \frac{m_t^2}{x_2 ys}\,\sigma_\phi^s - \frac{m_t^2}{Yys}\,\sigma_\phi^t,\\
W_1^{W\phi} &= m_t^2\,\left(\sigma_\phi^t - \sigma_\phi^s\right),\\
W_2^{W\phi} &= -m_t^2\,\sigma_\phi^s,\\
W^{\phi\phi} &= m_t^2\,\left(x_2\sigma_\phi^s - Y\sigma_\phi^t\right)\,ys.
\end{aligned}
\tag{255}
$$

We have also written a FORM program for the evaluation of vertices [4] derived from a more general one that computes all one-loop

[3] available from http://www.to.infn.it/giampier/form/mflwi.f
[4] available from http://www.to.infn.it/giampier/form/vertsw.f

diagrams in the standard model. From it we derive

$$W_i^{\phi w} = -W_i^{w\phi}, \tag{256}$$

and, with N_g generations of fermions, the result is as follows:

$$\begin{aligned}
W_0^{ww} &= N_g ys\{\frac{4}{9}(x_2 - Y) + \frac{4}{3}[Y B_0(-\kappa_+;0,0) - x_2 B_0(-s';0,0)]\} \\
&+ \frac{1}{3}ys(x_2 - Y) - \frac{1}{2}\frac{m_t^4}{Yys}B_0(-\kappa_+;0,m_t) + YysB_0(-\kappa_+;0,m_t) \\
&- \frac{1}{2}m_t^2 B_0(-\kappa_+;0,m_t) + \frac{1}{2}\frac{m_t^4}{x_2 ys}B_0(-s';0,m_t) \\
&- x_2 ysB_0(-s';0,m_t) + \frac{1}{2}m_t^2 B_0(-s';0,m_t) + \frac{1}{2}(\frac{1}{x_2} - \frac{1}{Y})\frac{m_t^2}{ys}A_0(m_t) \\
&- \frac{1}{3}ys(x_2 - Y) - YysB_0(-\kappa_+;0,0) + x_2 ysB_0(-s';0,0)\,,
\end{aligned}$$

$$\begin{aligned}
W_1^{ww} &= N_g[\frac{4}{9} - \frac{4}{3}B_0(-s';0,0)] - \frac{m_t^2}{x_2 ys} + 2\left(\frac{m_t^2}{x_2 ys}\right)^2 B_0(-s';0,m_t) \\
&- B_0(-s';0,m_t)\frac{m_t^2}{x_2 ys} - B_0(-s';0,m_t) + 2\frac{m_t^2}{(x_2 ys)^2}A_0(m_t) \\
&- \frac{1}{x_2 ys}A_0(m_t) + B_0(-s';0,0)\,,
\end{aligned}$$

$$\begin{aligned}
W_2^{ww} &= \frac{4}{3}N_g[B_0(-\kappa_+;0,0) - B_0(-s';0,0)] \\
&+ \frac{m_t^2}{ys}(-\frac{1}{x_2} + \frac{1}{Y}) - 2\left(\frac{m_t^2}{Yys}\right)^2 B_0(-\kappa_+;0,m_t) \\
&+ \frac{m_t^2}{Yys}B_0(-\kappa_+;0,m_t) + B_0(-\kappa_+;0,m_t) + 2\left(\frac{m_t^2}{x_2 ys}\right)^2 B_0(-s';0,m_t) \\
&- \frac{m_t^2}{x_2 ys}B_0(-s';0,m_t) - B_0(-s';0,m_t) + 2\frac{m_t^2}{(ys)^2}(\frac{1}{x_2^2} - \frac{1}{Y^2})B_0(-s';0,m_t) \\
&+ (-\frac{1}{x_2 ys} + \frac{1}{Yys})A_0(m_t) - B_0(-\kappa_+;0,0) + B_0(-s';0,0)\,,
\end{aligned}$$

$$\begin{aligned}
W_1^{w\phi} &= \frac{3}{2}\frac{m_t^4}{Yys}B_0(-\kappa_+;0,m_t) - \frac{3}{2}m_t^2 B_0(-\kappa_+;0,m_t) \\
&- \frac{3}{2}\frac{m_t^4}{x_2 ys}B_0(-s';0,m_t) + B_0(-s';0,m_t)\,m_t^2(\frac{3}{2}) - \frac{3}{2}\frac{m_t^2}{ys}(\frac{1}{x_2} - \frac{1}{Y})A_0(m_t)\,,
\end{aligned}$$

$$
\begin{aligned}
W_2^{W\phi} &= -\frac{3}{2}\frac{m_t^4}{x_2 ys} B_0\left(-s';0,m_t\right) \\
&+ \frac{3}{2} m_t^2 B_0\left(-s';0,m_t\right) - \frac{3}{2}\frac{m_t^2}{x_2 ys} A_0\left(m_t\right), \\
W^{\phi\phi} &= \frac{3}{2} Yys\, m_t^2 B_0\left(-\kappa_+;0,m_t\right) - \frac{3}{2} m_t^4 B_0\left(-\kappa_+;0,m_t\right) \\
&- \frac{3}{2} x_2 ys\, m_t^2 B_0\left(-s';0,m_t\right) + \frac{3}{2} m_t^4 B_0\left(-s';0,m_t\right)
\end{aligned} \tag{257}
$$

The remaining quantities have already been given, but we repeat them, for completeness, in their scalarized form:

$$
\begin{aligned}
f_W(s') &= -\frac{1}{3} m_t^2 - \frac{4}{3} B_0\left(-s';m_t,m_t\right) m_t^2 \\
&- \frac{2}{3} x_2 ys B_0\left(-s';m_t,m_t\right) - \frac{1}{2} B_0\left(-s';0,m_t\right) m_t^2 - \frac{1}{2}\frac{m_t^4}{x_2 ys} B_0\left(-s';0,m_t\right) \\
&+ x_2 ys B_0\left(-s';0,m_t\right) - \frac{1}{3} x_2 ys B_0\left(-s';0,0\right) - \frac{1}{2}\frac{m_t^2}{x_2 ys} A_0\left(m_t\right) - \frac{1}{3} A_0\left(m_t\right), \\
f_W(\kappa_+) &= -\frac{1}{3} m_t^2 - \frac{4}{3} B_0\left(-\kappa_+;m_t,m_t\right) m_t^2 \\
&- \frac{2}{3} Yys B_0\left(-\kappa_+;m_t,m_t\right) - \frac{1}{2} B_0\left(-\kappa_+;0,m_t\right) m_t^2 - \frac{1}{2}\frac{m_t^4}{Yys} B_0\left(-\kappa_+;0,m_t\right) \\
&+ Yys B_0\left(-\kappa_+;0,m_t\right) - \frac{1}{3} Yys B_0\left(-\kappa_+;0,0\right) - \frac{1}{2}\frac{m_t^2}{Yys} A_0\left(m_t\right) - \frac{1}{3} A_0\left(m_t\right).
\end{aligned} \tag{258}
$$

Furthermore, we have

$$
\begin{aligned}
\sigma_\phi^s &= \frac{3}{2}\frac{1}{x_2 ys}\left[A_0\left(m_t\right) + \left(m_t^2 - x_2 ys\right) B_0\left(-s';0,m_t\right)\right], \\
\sigma_\phi^t &= \frac{3}{2}\frac{1}{Yys}\left[A_0\left(m_t\right) + \left(m_t^2 - Yys\right) B_0\left(-\kappa_+;0,m_t\right)\right].
\end{aligned} \tag{259}
$$

The explicit expressions shown in Eqs.(257)–(259) are solutions of Eq.(255), therefore proving the validity of the U(1) massive-massive Ward identity. The proof is straightforward, although the FORM program that derives WI $= 0$ generates of the order of $100,000$ in some intermediate step.

6.7 Fixed-width scheme

In the previous section we have shown that the massive-massive Ward identity is satisfied in the Fermion-Loop scheme when vertices are added. In many numerical evaluations, where one would like to save as much as possible of CPU time, the implementation of the Fermion-Loop is very time-consuming. Therefore, one would like to have a gauge-preserving scheme which is as simple as possible from the point of view of building an event generator. For conserved currents this scheme is given by the use of a fixed width for vector bosons in all channel, not only annihilation but also scattering. It is a nonsense, from the point of view of a full-fledged field-theoretical formulation, but it has been shown [4] that the numerical differences, with respect to the Fermion-Loop scheme, are tiny at LEP 2 energies and also for small, but not vanishing, electron's scattering angle.

We have taken the complete, massive-massive, Ward identity and have neglected all vertex corrections. In this case one can show that the Ward identity is satisfied with the following choice:

$$
\begin{aligned}
P_s^{-1} &= -x_2 y s + M^2 - i\, M\Gamma_W, \\
P_t^{-1} &= -Y y s + M^2 - i\, M\Gamma_W, \\
M^2(p^2) &= M^2, \quad \mathit{for} \quad p^2 = -s' \quad \text{and} \quad p^2 = -\kappa_+,
\end{aligned} \tag{260}
$$

where Γ_W is the W, on-shell, total width. Therefore, we have a formulation of the fixed-width scheme that works also for non-zero external fermion masses.

6.8 Cancellation of the ultraviolet divergences.

When we assume that the W-currents are treated as conserved ones, then ten operators are enough to describe the γWW vertex, even for proving the U(1) Ward identity. We will use

$$
\begin{aligned}
W^1_{\mu\alpha\beta} &= 4\,\mathcal{N}\,[\delta_{\alpha\beta}\,Q_{+\mu} + \delta_{\mu\beta}\,Q_{-\alpha} - \delta_{\mu\alpha}\,Q_\beta], \\
W^{2,3}_{\mu\alpha\beta} &= 2\,\mathcal{N}\,[\delta_{\mu\beta}\,Q_{-\alpha} \mp \delta_{\mu\alpha}\,Q_\beta],
\end{aligned}
$$

$$
\begin{aligned}
W^4_{\mu\alpha\beta} &= 2/, \mathcal{N}^3\, Q_{+\mu}\, Q_{-\alpha}\, Q_\beta, \\
W^5_{\mu\alpha\beta} &= \mathcal{N}\, [\varepsilon(Q_+, \mu, \alpha, \beta) - \varepsilon(Q_-, \mu, \alpha, \beta)], \\
W^6_{\mu\alpha\beta} &= -\mathcal{N}\, [\varepsilon(Q_+, \mu, \alpha, \beta) + \varepsilon(Q_-, \mu, \alpha, \beta)], \\
W^7_{\mu\alpha\beta} &= -\mathcal{N}^3\, \varepsilon(Q_+, Q-, \alpha, \beta)\, Q_\mu, \\
W^8_{\mu\alpha\beta} &= 2\, \mathcal{N}^3\, Q_{-\mu}\, Q_{-\alpha}\, Q_\beta, \\
W^9_{\mu\alpha\beta} &= 2\, \mathcal{N}\, \delta_{\alpha\beta}\, Q_{-\mu}, \\
W^{10}_{\mu\alpha\beta} &= \mathcal{N}^3\, \varepsilon(Q_+, Q_-, \alpha, \beta)\, Q_{+\mu}. \qquad (261)
\end{aligned}
$$

We will write the vertex as

$$
W^{\gamma ww}_{\mu\alpha\beta} = \frac{g^3 s_\theta}{16\, \pi^2}\, (X y s)^{1/2} \sum_{i=1,10} I^{\gamma ww}_i\, W^i_{\mu\alpha\beta}. \qquad (262)
$$

For one generation, l, ν_l, u, d, of massless fermions we obtain

$$
\begin{aligned}
I^1 &= x_2\, y\, s\, (-C_{32} - C_{33} + 2\, C_{34}) \\
&+ X\, y\, s\, (C_{12} + C_{22} - C_{33} + C_{34}) + Y\, y\, s\, (C_{21} + C_{22} - 2\, C_{23} + C_{31} \\
&- 2\, C_{33} + C_{34}) - \frac{2}{3} + 2\, C_{24} - 2\, C_{35} + 2\, C_{36},
\end{aligned}
$$

$$
\begin{aligned}
I^2 &= x_2\, y\, s\, (-2\, C_{22} + 2\, C_{23} + 3\, C_{32} + 3\, C_{33} - 6\, C_{34}) \\
&+ X\, y\, s\, (-C_{12} - 3\, C_{22} + 2\, C_{23} + 3\, C_{33} - 3\, C_{34}) \\
&+ Y\, y\, s\, (-2\, C_{11} + 2\, C_{12} - 5\, C_{21} - 3\, C_{22} + 8\, C_{23}) \\
&- 3\, C_{31} + 6\, C_{33} - 3\, C_{34} + 2\, C_{24} + 6\, C_{35} - 6\, C_{36},
\end{aligned}
$$

$$
\begin{aligned}
I^3 &= x_2\, y\, s\, (-2\, C_{22} + 2\, C_{23} - C_{32} + C_{33}) \\
&+ X\, y\, s\, (C_{12} + C_{22} + 2\, C_{23} + C_{33} + C_{34}) + Y\, y\, s\, (-C_{21} + C_{22} \\
&- C_{31} + C_{34}) + 2\, C_{24} + 2\, C_{35} + 2\, C_{36},
\end{aligned}
$$

$$
I^4 = 8\, X\, y\, s\, (-\, C_{22} +\, C_{23} +\, C_{33} -\, C_{34})\,,
$$

$$
I^{5,6,7} = 0,
$$

$$
I^8 = X\, y\, s\, (-4\, C_{12} - 8\, C_{22} - 4\, C_{23} - 8\, C_{34})\,,
$$

$$
\begin{aligned}
I^9 &= x_2\, y\, s\, (2\, C_{12} + 2\, C_{23} - 2\, C_{32} + 2\, C_{34}) \\
&+ X\, y\, s\, (2\, C_{12} + 2\, C_{22} + 2\, C_{23} + 2\, C_{34})
\end{aligned}
$$

$$
\begin{aligned}
&+ \; Y\, y\, s\, \left(-2\, C_{11} - 2\, C_{21} + 2\, C_{22} - 2\, C_{23} - 2\, C_{33} + 2\, C_{34}\right) \\
&- \; \frac{2}{3} + 4\, C_{24} + 4\, C_{36}
\end{aligned}
$$

$$
I^{10} \;=\; 0, \tag{263}
$$

where all C_{ij}-functions are computed with zero internal masses. If we compute the ultraviolet divergent part of the vertex, for one massless generation, then the following results will emerge:

$$
W^{\gamma WW}_{\mu\alpha\beta} \mid_{\rm UV} = \frac{g^3 s_\theta}{16\,\pi^2}\, (X y s)^{1/2}\, \frac{2}{3}\, \frac{1}{\bar{\varepsilon}}\, [W^1_{\mu\alpha\beta} + W^9_{\mu\alpha\beta}], \tag{264}
$$

but, for conserved currents, i.e. $Q_{+\beta} = 0, Q_{+\alpha} = -Q_{-\alpha}$, we get

$$
W^1_{\mu\alpha\beta} + W^9_{\mu\alpha\beta} = 2\,\mathcal{N}\, V^0_{\mu\alpha\beta}. \tag{265}
$$

Therefore, the ultraviolet divergent part of the vertex is proportional to the lowest order. This has an important consequence, in view of the fact that inside Eq.(159) there is a remaining factor $1/g^2$. The combination of bare quantities and of one-loop corrections,

$$
\frac{1}{g^2} + \frac{1}{8\,\pi^2}\, (I_1 + I_9) \mid_{\rm UV}, \tag{266}
$$

appearing in the total amplitude, is ultraviolet finite.

For the $t - b$ doublet result we need two different configurations of internal masses, $(m_t, 0, m_t)$ and $(0, m_t, 0)$. With

$$
C_{ij}\,(t) = C_{ij}\,(0, m_t, 0) \qquad C_{ij}\,(tt) = C_{ij}\,(m_t, 0, m_t)\,, \tag{267}
$$

and

$$
C^+_{ij} = C_{ij}\,(tt) + \frac{1}{2}\, C_{ij}\,(t)\,, \qquad C^-_{ij} = C_{ij}\,(tt) - \frac{1}{2}\, C_{ij}\,(t)\,, \tag{268}
$$

the resulting expressions for the coefficients are as follows:

$$
\begin{aligned}
I_1 \;=&\; x_2 y s\, [\, -\frac{1}{2} C^+_{32} - \frac{1}{2} C^+_{33} + C^+_{34}] \\
&+ \; \frac{1}{2} X y s\, [C^+_{12} + C^+_{22} - C^+_{33} + C^+_{34}] \\
&+ \; Y y s\, [\frac{1}{2} C^+_{21} + \frac{1}{2} C^+_{22} - C^+_{23} + \frac{1}{2} C^+_{31} - C^+_{33} + \frac{1}{2} C^+_{34}] \\
&+ \; \frac{1}{2} m^2_t\, [C_{11}\,(tt) - C_{12}\,(tt)\,] + \frac{1}{2} m^2_t C_0\,(tt)
\end{aligned}
$$

$$
\begin{aligned}
&- && \frac{1}{2} + C^+_{24} - C^+_{35} + C^+_{36}, \\
I_2 &= && x_2ys\,[-C^+_{22} + C^+_{23} + \frac{3}{2}C^+_{32} + \frac{3}{2}C^+_{33} - 3C^+_{34}] \\
&+ && Xys\,[-\frac{1}{2}C^+_{12} - \frac{3}{2}C^+_{22} + C^+_{23} + \frac{3}{2}C^+_{33} - \frac{3}{2}C^+_{34}] \\
&+ && Yys\,[-C^+_{11} + C^+_{12} - \frac{5}{2}C^+_{21} - \frac{3}{2}C^+_{22} \\
&+ && 4C^+_{23} - \frac{3}{2}C^+_{31} + 3C^+_{33} - \frac{3}{2}C^+_{34}] \\
&- && \frac{1}{2}m_t^2\,[C_{11}\,(tt) - C_{12}\,(tt)\,] - \frac{1}{2}m_t^2 C_0\,(tt) + C^+_{24} + 3C^+_{35} - 3C^+_{36} \\
I_3 &= && x_2ys\,[-C^+_{22} + C^+_{23} - \frac{1}{2}C^+_{32} + \frac{1}{2}C^+_{33}] \\
&+ && Xys\,[\frac{1}{2}C^+_{12} + \frac{1}{2}C^+_{22} + C^+_{23} + \frac{1}{2}C^+_{33} + \frac{1}{2}C^+_{34}] \\
&+ && \frac{1}{2}Yys\,[-C^+_{21} + C^+_{22} - C^+_{31} + C^+_{34}] + \frac{1}{2}m_t^2\,[C_{11}\,(tt) + C_{12}\,(tt)\,] \\
&+ && \frac{1}{2}m_t^2 C_0\,(tt) + C^+_{24} + C^+_{35} + C^+_{36} \\
I_4 &= && 4Xys\,[-C^+_{22} + C^+_{23} + C^+_{33} - C^+_{34}] \\
I_5 &= && x_2ys\,[-C^-_{32} + C^-_{33}] + Xys\,[C^-_{12} + C^-_{22} + 2C^-_{23} + C^-_{33} + C^-_{34}] \\
&+ && Yys\,[-2C^-_{11} + 2C^-_{12} - 3C^-_{21} + C^-_{22} + 2C^-_{23} - C^-_{31} + C^-_{34}] \\
&+ && m_t^2\,[C_{11}\,(tt) + C_{12}\,(tt)\,] + m_t^2 C_0\,(tt) + 6C^-_{24} + 6C^-_{35} + 6C^-_{36} \\
I_6 &= && x_2ys\,[-C^-_{32} - C^-_{33} + 2C^-_{34}] \\
&+ && Xys\,[-C^-_{12} + C^-_{22} - 2C^-_{23} - C^-_{33} + C^-_{34}] \\
&+ && Yys\,[C^-_{21} + C^-_{22} - 2C^-_{23} + C^-_{31} - 2C^-_{33} + C^-_{34}] \\
&- && m_t^2\,[C_{11}\,(tt) - C_{12}\,(tt)\,] - m_t^2 C_0\,(tt) \\
&- && \frac{1}{3} - 2C^-_{24} - 6C^-_{35} + 6C^-_{36} \\
I_7 &= && 4Xys\,[C^-_{12} + C^-_{23}] \\
I_8 &= && 2Xys\,[-C^+_{12} - 2C^+_{22} - C^+_{23} - 2C^+_{34}]
\end{aligned}
$$

$$
\begin{aligned}
I_9 &= x_2 ys\,[C^+_{12} + C^+_{23} - C^+_{32} + C^+_{34}] \\
&+ Xys\,[C^+_{12} + C^+_{22} + C^+_{23} + C^+_{34}] \\
&+ Yys\,[-C^+_{11} - C^+_{21} + C^+_{22} - C^+_{23} - C^+_{33} + C^+_{34}] \\
&- m_t^2 C_{12}\,(tt) - \frac{1}{2} + 2C^+_{24} + 2C^+_{36} \\
\\
I_{10} &= 4Xys\,[C^-_{12} + C^-_{23}] \qquad (269)
\end{aligned}
$$

When computing the $t-b$ contribution to the vertex we find that the ultraviolet divergence is m_t-independent. One should remember that the whole calculation is organized in the following way: if we denote the vertex schematically by V, then

$$
V = \sum_f V_f(\text{massless}) + [V_t(\text{massive}) - V_t(\text{massless})]. \qquad (270)
$$

The sum in Eq.(270) combines with the $1/g^2$-term to cancel the ultraviolet divergent term while the rest is a subtracted term that is ultraviolet finite. When we consider the massive-massive case, there are 24 form-factors. Computing the divergent part we find

$$
\begin{aligned}
W^{\gamma WW}_{\mu\alpha\beta}\,\Big|_{\text{UV}} &= \frac{g^3 s_\theta}{16\,\pi^2}\,(Xys)^{1/2}\,\frac{1}{3}\,\frac{1}{\bar{\varepsilon}}\,[W^1_{\mu\alpha\beta} - 2\,W^3_{\mu\alpha\beta} \\
&+ 2\,W^5_{\mu\alpha\beta} - 4\,W^6_{\mu\alpha\beta} - 2\,W^7_{\mu\alpha\beta} - 2\,W^8_{\mu\alpha\beta}], \qquad (271)
\end{aligned}
$$

where the operators are given in Eq.(217). However, we easily derive the following relation

$$
\begin{aligned}
4\,\mathcal{N}\,V^0_{\mu\alpha\beta} &= W^1_{\mu\alpha\beta} - 2\,W^3_{\mu\alpha\beta} + 2\,W^5_{\mu\alpha\beta} \\
&- 4\,W^6_{\mu\alpha\beta} - 2\,W^7_{\mu\alpha\beta} - 2\,W^8_{\mu\alpha\beta}, \qquad (272)
\end{aligned}
$$

showing, again, factorization of the divergence into the lowest order. For a massive internal world there are also $\gamma W\phi$ and $\gamma\phi\phi$ vertices. One finds again factorization of ultraviolet divergences, for instance

$$
\begin{aligned}
W^{\gamma W\phi}_{\mu\beta} &= i\,g^3 s_\theta\,\mathcal{W}^{\gamma W\phi}_{\mu\beta}, \\
\mathcal{W}^{\gamma W\phi}_{\mu\beta}\,\Big|_{\text{UV}} &= \frac{i\,g^3 s_\theta}{16\,\pi^2}\,\frac{m_t^2}{M}\,\frac{3}{2}\,\frac{1}{\bar{\varepsilon}}\,\delta_{\mu\beta}. \qquad (273)
\end{aligned}
$$

The corresponding lowest order vertex gives $i\, g s_\theta M \delta_{\mu\beta}$. The two combine into the following expression:

$$C_{\mu\beta} = \frac{M^2}{g^2}\,\delta_{\mu\beta} + M\,\mathcal{W}^{\gamma W \phi}_{\mu\beta}, \tag{274}$$

and

$$\begin{aligned} C_{\mu\beta}\,|_{\rm UV} &= p_{_W}\,[\frac{1}{G_{_W}} + \frac{1}{16\,\pi^2}\,\frac{f_{_W}(p_{_W})}{p_{_W}}]_{\rm UV} + \frac{1}{16\,\pi^2}\,m_t^2\,\frac{3}{2}\,\frac{1}{\bar{\varepsilon}} \\ &= \frac{1}{16\,\pi^2}\,[f_{_W}(p_{_W})_{\rm UV} + \frac{3}{2}\,m_t^2\,\frac{1}{\bar{\varepsilon}}] = 0. \end{aligned} \tag{275}$$

Similar results hold for the other combinations, $\gamma\phi W$ and $\gamma\phi\phi$.

These observations, completing the argument shown in Eq.(266), prove that the amplitude is ultraviolet finite in the Fermion-Loop scheme.

As we have shown, the ε-terms do not cancel in Eq.(269). They only do for a complete massless generation, $l\,\nu_l\,u\,d$. Indeed, it is very easy to show that the ε-terms are proportional to the hypercharge so that in the total they cancel, but they cancel only if we add the full content of the three fermionic generations while keeping all the fermions (including the top quark) at zero mass. With a massive top quark they do not cancel and give an m_t-dependent contribution.

6.9 Re-summations in the neutral sector.

Dyson re-summation of transitions in the neutral sector are also available and we will proceed to their construction. First, we define indices $i, j, \ldots = 1, 2$ with $1 \equiv Z$ and $2 \equiv \gamma$. The transitions are given by

$$S = \frac{g^2}{16\,\pi^2}\,\Sigma, \qquad \Sigma^{ij}_{\mu\nu} = \Sigma^{ij}_0\,\delta_{\mu\nu} + \Sigma^{ij}_1\,p_\mu p_\nu. \tag{276}$$

The re-summed propagators are obtained in the usual way, namely

$$\begin{aligned} \bar{\Delta}^{ij}_{\mu\nu} &= \Delta^{ij}\,\delta_{\mu\nu} + \Delta^{il}\,S^{lk}_{\mu\nu}\,\Delta^{kj} + \ldots \\ &= \Delta^{il}\,\delta_{\mu\alpha}\,X^{lj}_{\alpha\nu}, \quad X = (1 - S\,\Delta)^{-1}. \end{aligned} \tag{277}$$

It is convenient to write

$$(1 - S\,\Delta)_{\mu\nu} = K\,\delta_{\mu\nu} + H\,p_\mu p_\nu, \tag{278}$$

with K and H matrices defined by

$$K = \begin{pmatrix} 1 - S_0^{zz}\Delta_z & -S_0^{z\gamma}\Delta_\gamma \\ -S_0^{\gamma z}\Delta_z & 1 - S_0^{\gamma\gamma}\Delta_\gamma \end{pmatrix} \qquad H = -\begin{pmatrix} S_1^{zz}\Delta_z & S_1^{z\gamma}\Delta_\gamma \\ S_1^{\gamma z}\Delta_z & S_1^{\gamma\gamma}\Delta_\gamma \end{pmatrix} \tag{279}$$

Let $X = I\,\delta_{\mu\nu} + J\,p_\mu p_\nu$, then one can easily find a solution of the following form:

$$\begin{aligned} I &= K^{-1}, \qquad J = -\left(K + p^2\,H\right)^{-1}\,H\,K^{-1}, \\ X_{\mu\nu} &= \left[\delta_{\mu\nu} - \left(K + p^2\,H\right)^{-1}\,H\,p_\mu p_\nu\right] K^{-1}. \end{aligned} \tag{280}$$

The re-summed propagators are

$$\bar{\Delta}^{ij}_{\mu\nu} = \Delta^i\,X^{ij}_{\mu\nu}, \tag{281}$$

where we have found that

$$X_{\mu\nu} = K^{-1}\,P_{\mu\nu} + T^{-1}\,L_{\mu\nu}. \tag{282}$$

The Lorentz-tensors appearing in the previous equation are defined by

$$P_{\mu\nu} = \delta_{\mu\nu} - \frac{p_\mu p_\nu}{p^2}, \qquad L_{\mu\nu} = \frac{p_\mu p_\nu}{p^2}, \tag{283}$$

the remaining matrices being expressed as

$$\begin{aligned} K^{ij} &= \delta^{ij} - S_0^{il}\,\Delta_{lj}, \quad H^{ij} = -S_1^{il}\,\Delta_{lj}, \\ T^{ij} &= K^{ij} + p^2\,H^{ij} = \delta^{ij} - t^{il}\,\Delta_{lj}. \end{aligned} \tag{284}$$

These results can be simplified, as we have done in the charged sector, by using Ward identities for two-point functions, i.e.

$$\begin{aligned} i\,p^\mu\,S^{\gamma\gamma}_{\mu\nu} &= i\,t^{\gamma\gamma}\,p_\nu = 0, \\ i\,p^\mu\,S^{\gamma z}_{\mu\nu} &= i\,t^{\gamma z}\,p_\nu = 0, \\ i\,p^\mu\,S^{z\gamma}_{\mu\nu} &= i\,t^{z\gamma}\,p_\nu = 0, \end{aligned} \tag{285}$$

giving $t^{\gamma\gamma} = t^{\gamma Z} = t^{Z\gamma} = 0$. Therefore the matrix T^{-1} is diagonal, with entries

$$T^{-1}_{zz} = \frac{1}{1 - t^{zz}\,\Delta_z}, \qquad T^{-1}_{\gamma\gamma} = 1. \tag{286}$$

The re-summed photon propagator becomes

$$\bar{\Delta}^{\gamma\gamma}_{\mu\nu} = [p^2 - S^{\gamma\gamma}_0 - \frac{\left(S^{\gamma z}_0\right)^2}{p^2 + M^2_{_0} - S^{zz}_0}]\, P_{\mu\nu} + \frac{1}{p^2}\, L_{\mu\nu}. \tag{287}$$

The re-summed photon propagator, in the 't Hooft-Feynman gauge, satisfies the same relation as the bare propagator,

$$p^\mu\, \Delta^{\gamma\gamma}_{\mu\nu} = p^\mu\, \bar{\Delta}^{\gamma\gamma}_{\mu\nu} = \frac{p_\nu}{p^2}. \tag{288}$$

The $Z - \gamma$ re-summed transition is

$$\bar{\Delta}^{\gamma z}_{\mu\nu} = \frac{S^{\gamma z}_0}{\left(p^2 + M^2_{_0} - S^{zz}_0\right), \left(p^2 - S^{\gamma\gamma}_0\right) - \left(S^{\gamma z}_0\right)^2}\, P_{\mu\nu}. \tag{289}$$

It is convenient to re-express the results of Eqs.(287)–(289) in terms of running couplings. We obtain

$$\begin{aligned}
\bar{\Delta}^{\gamma\gamma}_{\mu\nu} &= \{ -\frac{e^2(p^2)}{e^2\, p^2} + \frac{s^2_\theta}{c^2_\theta}[1 - \frac{s^2(p^2)}{s^2_\theta}]^2\, G_z(p^2)\}\, P_{\mu\nu} + \frac{L_{\mu\nu}}{p^2}, \\
\bar{\Delta}^{\gamma z}_{\mu\nu} &= \frac{s_\theta}{c_\theta}[1 - \frac{s^2(p^2)}{s^2_\theta}]\, G_z(p^2)\, P_{\mu\nu}.
\end{aligned} \tag{290}$$

For fermionic loops there is no transition $\gamma - \phi^0$, while

$$S^\mu_{\phi z} = \frac{g^2}{16\,\pi^2 c^2_\theta}\, \Sigma_{\phi z}\, ip^\mu, \quad \Sigma_{\phi z} = -\frac{1}{2}\,\frac{m^2_t}{M}\, B_0\left(p^2; m_t, m_t\right). \tag{291}$$

In our conventions, we denote with a subscript ϕ the ϕ^0-field. Therefore, for non-zero m_t, we have to complete the diagonalization. Let us introduce indices $a, b, \ldots = 1, \ldots, 5$ with

$$S_{55} = S_{\phi\phi}, \quad S_{\mu 5} = S^\mu_{z\phi}, \quad S_{5\nu} = S^\nu_{\phi z}. \tag{292}$$

One should remember that the transitions S_{VV} have already been re-summed in $\bar{\Delta}^{ZZ}_{\mu\nu}$ etc, therefore the additional re-summation gives

$$\begin{aligned}
\tilde{\Delta}_{ab} &= \Delta_{ab} + \Delta_{ac}\, S_{cd}\, \Delta_{db} + \ldots = \Delta_{ac}\, X_{cb}, \\
X_{ab} &= (1 - S\,\Delta)^{-1}_{ab}\,.
\end{aligned} \tag{293}$$

In this equation one should use

$$\Delta_{\mu\nu} \equiv \bar{\Delta}^{zz}_{\mu\nu}, \qquad \Delta_{\phi\phi} \equiv \Delta_\phi. \tag{294}$$

The matrix to be inverted is

$$\begin{pmatrix} \delta_{\mu\nu} & +i\, S_{Z\phi}\, \Delta_\phi\, p^\mu \\ -i\, S_{Z\phi}\, p^\mu\, \bar{\Delta}^{ZZ}_{\mu\nu} & 1 - S_{\phi\phi}\, \Delta_\phi \end{pmatrix} \tag{295}$$

however, it is easily found that the following property holds:

$$p^\mu\, \bar{\Delta}^{ZZ}_{\mu\nu} = \Delta_Z\, T^{-1}_{ZZ}\, p^\nu = \frac{1}{p^2 + M_{_0}^2 - t^{ZZ}}\, p^\nu. \tag{296}$$

There are additional Ward identities in the neutral sector, corresponding to two-point functions, that give

$$i\, p^\mu\, S^{\mu\nu}_{ZZ} + M_{_0}\, S_{\phi Z}\, i\, p^\nu = \left(t^{ZZ} + M_{_0}\, S_{\phi Z}\right)\, i\, p^\nu = 0, \tag{297}$$

$$\begin{aligned} & i\, p_\mu\, S^{\mu\nu}_{ZZ}\, (-i\, p_\nu) + i\, p_\mu\, M_{_0}\, (-S_{\phi Z})\, i\, p^\mu \\ - & \; i\, p_\mu\, M_{_0}\, (S_{\phi Z})\, i\, p^\mu + M_{_0}^2\, S_{\phi\phi} \\ = & \; p^2\, t^{ZZ} + 2\, p^2\, M_{_0}\, S_{\phi Z} + M_{_0}^2\, S_{\phi\phi} = 0. \end{aligned} \tag{298}$$

The solution to these conditions is

$$S_{\phi Z} = -\frac{M_{_0}}{p^2}\, S_{\phi\phi}, \qquad t^{ZZ} = \frac{M_{_0}^2}{p^2}\, S_{\phi\phi}, \tag{299}$$

allowing to express everything in terms of the $\phi^0 - \phi^0$ transition. In this way our matrix becomes

$$\begin{pmatrix} \delta_{\mu\nu} & i\, b\, p_\mu \\ -i\, b'\, p_\nu & c \end{pmatrix} \tag{300}$$

with entries given by

$$\begin{aligned} b &= -\frac{M_{_0}}{p^2}\, S_{Z\phi}\, \Delta_Z, \qquad b' = -\frac{M_{_0}}{p^2}\, \frac{S_{Z\phi}}{p^2 + M_{_0}^2 - t^{ZZ}}, \\ c &= 1 - S_{\phi\phi}\, \Delta_Z, \end{aligned} \tag{301}$$

where we have used the fact that in the 't Hooft-Feynman gauge $\Delta_Z = \Delta_\phi = 1/(p^2 + M_{_0}^2)$. After inversion of the matrix we obtain

$$\frac{1}{c - bb'p^2} \begin{pmatrix} \left(c - bb'p^2\right)\, \delta_{\mu\nu} + bb'\, p_\mu p_\nu & -i\, b\, p_\mu \\ +i\, b'\, p_\nu & 1 \end{pmatrix} \tag{302}$$

This result allows us to write down the remaining re-summed propagators. They are:

$$
\begin{aligned}
\tilde{\Delta}^{zz}_{\mu\nu} &= [p^2 + M^2_{_0} - S^{zz}_0 - \frac{\left(S^{z\gamma}_0\right)^2}{p^2 - S^{\gamma\gamma}_0}]^{-1}\, P_{\mu\nu} \\
&+ \frac{p^2\left(p^2 + M^2_{_0} - S_{\phi\phi}\right)}{\left(p^2 + M^2_{_0}\right)^2\,\left(p^2 - S_{\phi\phi}\right)}\, L_{\mu\nu}, \\
\tilde{\Delta}_{\phi\phi} &= \frac{p^2\,\left(p^2 + M^2_{_0}\right) - M^2_{_0}\, S_{\phi\phi}}{\left(p^2 + M^2_{_0}\right)^2\,\left(p^2 - S_{\phi\phi}\right)} \\
\tilde{\Delta}^{\mu}_{z\phi} &= -\tilde{\Delta}^{\mu}_{\phi z} = i\, p^\mu\, M_{_0}\, \frac{S_{\phi\phi}}{\left(p^2 + M^2_{_0}\right)^2\,\left(p^2 - S_{\phi\phi}\right)}. \qquad (303)
\end{aligned}
$$

6.10 The running Z mass.

With the results for transitions in the neutral sector, we can compute the amplitude for $e^+(p_+)\, e^-(p_-) \to f(q_-)\, \overline{f}(q_+)$. There are seven diagrams contributing, when we do not neglect the fermion masses. Accordingly, the amplitude is

$$
\begin{aligned}
M &= g^2 Q_f s^2_\theta\, \gamma^\mu \otimes \gamma^\mu\, \bar{\Delta}^{\gamma\gamma}_{\mu\nu} \\
&+ \frac{g^2 s_\theta}{2\, c_\theta}\, \gamma^\mu \otimes \gamma^\nu\, \left(I^{(3)}_f\, \gamma_+ - 2\, Q_f\, s^2_\theta\right)\, \bar{\Delta}^{\gamma z}_{\mu\nu} \\
&- \frac{g^2 Q_f\, s_\theta}{2\, c_\theta}\, \gamma^\mu\, \left(-\frac{1}{2}\, \gamma_+ + 2\, s^2_\theta\right) \otimes \gamma^\nu\, \bar{\Delta}^{z\gamma}_{\mu\nu} \\
&- \frac{g^2}{4\, c^2_\theta}\, \gamma^\mu\, \left(-\frac{1}{2}\, \gamma_+ + 2\, s^2_\theta\right) \otimes \gamma^\nu\, \left(I^{(3)}_f\, \gamma_+ - 2\, Q_f\, s^2_\theta\right)\, \tilde{\Delta}^{zz}_{\mu\nu} \\
&- \frac{g^2}{2\, c_\theta}\, I^{(3)}_f\, \frac{m_f}{M}\, \gamma^\mu\, \left(-\frac{1}{2}\, \gamma_+ + 2\, s^2_\theta\right) \otimes \gamma_5\, \bar{\Delta}^{z\phi}_\mu \\
&+ \frac{g^2}{2\, c_\theta}\, \frac{m_e}{2\, M}\, \gamma_5 \otimes \gamma^\mu\, \left(I^{(3)}_f\, \gamma_+ - 2\, Q_f\, s^2_\theta\right)\, \tilde{\Delta}^{\phi z}_\mu \\
&+ \frac{g^2}{2}\, I^{(3)}_f\, \frac{m_e m_f}{M^2}\, \gamma_5 \otimes \gamma_5\, \tilde{\Delta}^{\phi\phi}, \qquad (304)
\end{aligned}
$$

where $I^{(3)}_f$ is the third component of isospin. We extract from the first four terms the part of the vector-vector propagators proportional to $\delta_{\mu\nu}$ and obtain the familiar $\gamma \oplus Z$ exchanges with running couplings. This part of the amplitude will be denoted by $M_{\delta\delta}$ and we write

$$M = M_{\delta\delta} + M_{\rm extra}. \tag{305}$$

From the use of the Dirac equation we find

$$\begin{aligned}
\not p \;&\otimes\; \not p = \not p \otimes \not p\,\gamma_+ = \not p\,\gamma_+ \otimes \not p = 0,\\
\not p\,\gamma_+ \;&\otimes\; \not p\,\gamma_+ = 4\,m_e m_f\,\gamma_5 \otimes \gamma_5,\\
\not p\,\gamma_+ \;&\otimes\; \gamma_5 = -2\,i\,m_e\,\gamma_5 \otimes \gamma_5,\\
\gamma_5 \;&\otimes\; \not p\,\gamma_+ = 2\,i\,m_f\,\gamma_5 \otimes \gamma_5.
\end{aligned} \tag{306}$$

Therefore, collecting the various terms, we obtain the following expression

$$\begin{aligned}
M_{\rm extra} &= g^2\,m_e m_f\,I_f^{(3)}\,\gamma_5 \otimes \gamma_5\\
&\times\;\Big\{\frac{1}{2\,c_\theta^2}\Big[-\,G_z + \frac{p^2\left(p^2 + M_0^2 - S_{\phi\phi}\right)}{\left(p^2 + M_0^2\right)^2\,(p^2 - S_{\phi\phi})}\Big]\frac{1}{p^2}\\
&+\;\frac{1}{c_\theta}\,\frac{M_0}{M}\,\frac{S_{\phi\phi}}{\left(p^2 + M_0^2\right)^2\,(p^2 - S_{\phi\phi})} + \frac{1}{2\,M^2}\,\frac{p^2\left(p^2 + M_0^2 - \frac{M_0^2}{p^2}\,S_{\phi\phi}\right)}{\left(p^2 + M_0^2\right)^2\,(p^2 - S_{\phi\phi})}\Big\}.
\end{aligned} \tag{307}$$

Definition 2 *This result can be simplified considerably if we introduce a running Z-mass. We define*

$$M_{\rm extra} = \frac{g^2}{2\,c_\theta^2}\,m_e m_f\,I_f^{(3)}\,\gamma_5 \otimes \gamma_5\,\frac{G_z}{M_0^2(p^2)}, \tag{308}$$

so that the entire amplitude can be written as the Born-like sum of γ and Z exchange, but with $g^2 \to g^2(p^2)$, $s_\theta^2 \to s^2(p^2)$ etc and with an *effective* Z propagator given by

$$\Delta_{\mu\nu}^{ZZ,{\rm eff}} = G_z(p^2)\,[\delta_{\mu\nu} + \frac{p_\mu p_\nu}{M_0^2(p^2)}]. \tag{309}$$

For a massless internal world we obtain a rather simple result:

$$\begin{aligned}
\frac{1}{M_0^2(p^2)} &= \frac{1}{p^2}\,[\frac{1}{M_0^2\,G_z(p^2)} - 1]\\
&= \frac{g^2(p_z)}{g^2(p^2)}\,\frac{c^2(p^2)}{c^2(p_z)}\,\frac{1}{p_z}.
\end{aligned} \tag{310}$$

which satisfies the relation $M_0^2(p_Z) = p_Z$.

As expected the Z and W running masses satisfy the same relation as the bare masses,

$$\frac{1}{M_{_0}^2(p^2)} = \frac{c^2(p^2)}{M^2(p^2)}, \tag{311}$$

while complex poles satisfy

$$p_{_Z} = \frac{g^2(p_{_Z})}{g^2(p_{_W})}\,\frac{p_{_W}}{c^2(p_{_Z})}. \tag{312}$$

The extension to non-zero m_t is straightforward.

6.11 Projection into the Imaginary Fermion-Loop.

All relevant Ward identities are linear in the vertex functions and the inverse propagators. As a consequence the real and imaginary parts fulfill the Ward identities separately. Therefore, a simplified minimal approach to incorporate the finite width while ensuring both U(1) and SU(2) gauge invariance consists in only taking into account the imaginary parts of fermionic corrections.

Having the full Fermion-Loop at our disposal we can project the result into this minimal version. The relevant formulas are listed below. First, the two-point functions. With $s = -p^2$ we write:

$$\begin{aligned}
\mathrm{Im}B_0\left(p^2;0,0\right) &= \pi\theta\left(s\right), \\
\mathrm{Im}B_0\left(p^2;0,m\right) &= \pi\left(1-\frac{m^2}{s}\right)\,\theta\left(s-m^2\right), \\
\mathrm{Im}B_0\left(p^2;m,m\right) &= \pi\left(1-4\frac{m^2}{s}\right)^{1/2}\,\theta\left(s-4m^2\right),
\end{aligned} \tag{313}$$

Furthermore, when $p^2 = -s_p$ and s_p is a complex pole, the relevant functions become as follows:

$$\begin{aligned}
B_0\left(-s_p;0,0\right) &= \frac{1}{\bar{\varepsilon}} + 2 - [\ln\left(\frac{-s_p}{\mu^2}\right) - 2\,i\,\pi], \\
B_0\left(-s_p;0,m\right) &= \frac{1}{\bar{\varepsilon}} + 2 - \ln\frac{m^2}{\mu^2}\left(1-\frac{m^2}{s_p}\right)\,[\ln\left(1-\frac{s_p}{m^2}\right) \\
&- \;2\,i\,\pi\,\theta\left(M^2-m^2\right)],
\end{aligned}$$

$$B_0\left(-s_p; m, m\right) = \frac{1}{\bar{\varepsilon}} + 2 - \ln\frac{m^2}{\mu^2} - \beta\left[\ln\frac{\beta+1}{\beta-1} - 2\,i\,\pi\,\theta\left(M^2 - 4\,m^2\right)\right], \tag{314}$$

where $M^2 = \mathrm{Re}\left(s_p\right)$ and $\beta^2 = 1 - 4\,m^2/s_p$. After scalarization the vertices are combinations of B_0-functions and of C_0-functions. The imaginary part of a C_0-function, in the general case, is not particularly simple but it is obtained from its complete expression [6]. An alternative way of deriving it is to remember that the imaginary part follows from the sum over all cuts of the scalar three-point function. In the generalization of the Fermion-Loop and in proving the Ward identity one should remember to sum over the three cuts, i.e., the imaginary part of the triangle VWW ($V = Z, \gamma$) is the sum of all cuts over intermediate states. Here we give one explicit example where we consider the function

$$C_0\left(p_1^2, p_2^2, P^2; m_1, m_2, m_3\right). \tag{315}$$

The cuts give

$$\begin{aligned}
C_{13} &= \int d^4q\,\theta\left(-q_0\right)\,\theta\left(q_0 + P_0\right) \\
&\times \frac{\delta\left(q^2 + m_1^2\right)\,\delta\left(\left(q+P\right)^2 + m_3^2\right)}{\left(q+p_1\right)^2 + m_2^2}, \\
C_{23} &= \int d^4q\,\theta\left(q_0 + p_{10}\right)\,\theta\left(-q_0 - P_0\right) \\
&\times \frac{\delta\left(\left(q+p_1\right)^2 + m_2^2\right)\,\delta\left(\left(q+P\right)^2 + m_3^2\right)}{q^2 + m_1^2}, \\
C_{12} &= \int d^4q\,\theta\left(q_0\right)\,\theta\left(-q_0 - p_{10}\right) \\
&\times \frac{\delta\left(q^2 + m_1^2\right)\,\delta\left(\left(q+p_1\right)^2 + m_2^2\right)}{\left(q+P\right)^2 + m_3^2},
\end{aligned} \tag{316}$$

where the subscript ij specify the cut: C_{ij} is the diagram where the cut propagators correspond to internal masses m_i, m_j. If we introduce the following quantities:

$$\begin{aligned}
a^2 &= \frac{1}{4\,s}\,\lambda\left(-s, p_1^2, p_2^2\right), \\
b &= -\frac{1}{2}\frac{-s + p_1^2 - p_2^2}{\sqrt{s}}, \\
c &= \frac{1}{2\,\sqrt{s}}\left(m_3^2 - m_1^2 - s\right),
\end{aligned}$$

$$d^2 \;=\; \frac{1}{4\,s}\,\lambda\left(s, m_1^2, m_3^2\right), \tag{317}$$

with $P^2 = -s$ and λ the Kállen λ-function, then we obtain

$$C_{13} = \frac{\pi}{4\,a\sqrt{s}}\,\theta\left(s - (m_1 + m_3)^2\right)\,\ln\frac{m_2^2 - m_1^2 + p_1^2 - 2\,cb + 2\,ad}{m_2^2 - m_1^2 + p_1^2 - 2\,cb - 2\,ad}, \tag{318}$$

where the θ-function reflects the positivity of $\lambda(s, m_1^2, m_3^2)$ and the constraint $s\ \geq \mid\ m_1^2 - m_3^2\ \mid$. Similar results can be derived for the other two cuts. Note, however, that once the massive top contributions are included this scheme is in fact not much easier than the complete Fermion-Loop scheme.

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