MultiLoopLand
a mini-series of lectures representing
a personal view on the subject

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Prolegomena: the need for Multi - Loops

- the program for **EW - NLO** accurate predictions is incomplete:

  a) **NLO** for $e^+e^- \to 4\,\mathbf{f}$ (LC)

  b) **NLO** for $pp \to WW + \mathbf{jets}$ (LHC) $\equiv$ (almost) model independent study of EWSB.

  c) etc

- the program for **EW - NNLO** accurate predictions is in its early infancy:

  a) **NNLO** for $\sin^2\theta^l_{\rm eff}$. There is evidence for tiny anomalies in LEP physics and a measurement at LHC may very well be the only additional experimental information in many years.

  a) etc

- **NLO/NNLO** cannot be a standalone piece of calculation and must be designed as an essential part of an **event generator**.

A Structural Difference: NLO/NNLO, QCD vs EW

QCD is in a much better shape, **cfr. E. W. Glover, Progress in NNLO calculations for scattering processes, arXiv:hep-ph/0211412**.

There are structural reasons,

– less particles,

– less scales,

– no γ^5

although the **infrared/collinear** structure is far from trivial. Instead, in the full sector

SM $\equiv$ 11 lines, 57 vertices, **modulus** f-multiplicity

and, moreover,

$$\mathbf{QED} + \mathbf{QCD} \in \mathbf{SM}$$

so that **infrared/collinear** is still there, even more complex.

Basics: the problem made explicit

No matter how long you turn it around the problem in any realistic multi-loop, multi-leg calculation is connected with a simple fact:

$$\text{(non-)scalar } G = \frac{\text{something}^{\dagger}}{f(\text{parameters})},$$

$$\dagger \equiv \text{HTF or a smooth integrand.}$$

Examples of f are

– Gram determinants in the standard tensor reduction;

– denominators in the **IBP** - techmique.

Integration-by-parts identities are a popular and quite succesful tool. For one-loop diagrams they can be written as

$$\int d^n q \, \frac{\partial}{\partial q_\mu} \left[v_\mu \, F(q, p, m_1 \cdots) \right] = 0,$$

where $v = q, p$.

By careful examination of the IBPI one can show that

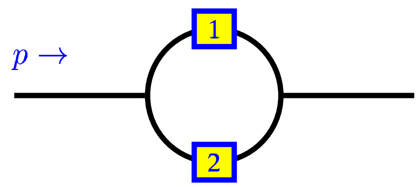
all one-loop diagrams can be reduced to a limited set of master integrals (MI)

For two-loops we can write

$$\int d^n q_1\, d^n q_2 \,\frac{\partial}{\partial a_\mu}\left[b^\mu\, F(q_1, q_2, \{p\}, m_1 \cdots)\right] \;=\; 0,$$

$$a_\mu \;=\; q_{i\mu}, \quad b_\mu = q_{i\mu}, p_{1\mu} \cdots$$

Again, using IBPI, arbitrary two-loop integrals can be written in terms of a restricted number of MI.



Consider, for instance, the following solution

$$B_0(1,2\,;\,p,m_1,m_2) = \frac{1}{\lambda(-p^2,m_1^2,m_2^2)}$$
$$\times\;\; \left[(n-3)\,(m_1^2 - m_2^2 - p^2)\,B_0(1,1\,;\,p,m_1,m_2)\right.$$
$$\left. +\;\; (n-2)\,A_0(1,m_1) - (n-2)\,\frac{p^2+m_1^2+m_2^2}{2\,m_2^2}\,A_0(1,m_2)\right].$$

The factor in front of the square bracket is exactly zero at the normal threshold.

However a general analysis tells us that at the normal threshold the leading behavior of $B_0(1,2)$ is

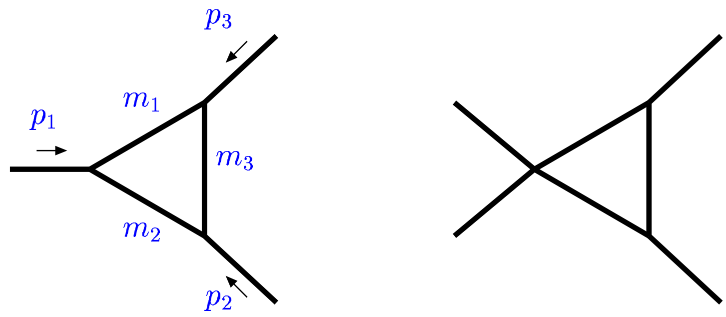
$$\lambda^{-1/2}$$

so the reduction to MI overestimate the singular behavior.

Of course, by a carefull examination of the square bracket, one can derive the right expansion but the result, as it stands, is again a source of **cancellation/instabilities**.

(possibly) Singular behavior

When we are around the region where $f = 0$, an alternative derivation is needed. Actually there are two sub-cases to be discussed. Consider C_0 **that can be written as**



$$C_0 = \int dS_2 \left[(x - X)^t \, H \, (x - X) + B_3\right]^{-1-\epsilon/2}.$$

If $B_3 = 0$ **the integral is singular or regular depending on the values of** $X_{1,2}$**.**

If $B_3 = 0$**, but the condition**

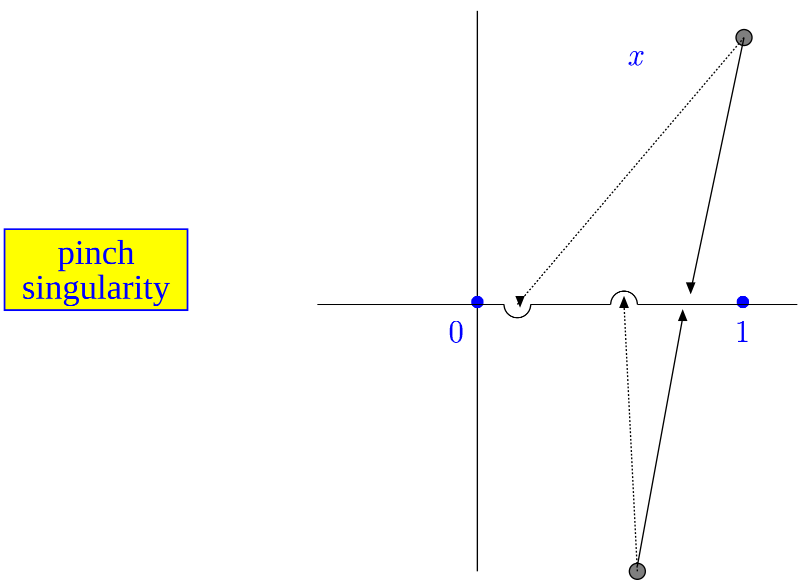
$$0 \leq X_2 \leq X_1 \leq 1$$

is not fulfilled, there is no real singularity inside the integration domain; still we cannot apply standard techniques.

If the condition is fulfilled then there is a pinch and the integral is singular. We write

$$
\begin{aligned}
C_0 &= \int dS_2\, V^{-1-\epsilon/2}(x_1,x_2) = \int dC_2\, V^{-1-\epsilon/2}(x_1,x_2) \\
&- \int_0^1 dx_1 \int_{x_1}^1 dx_2\, V^{-1-\epsilon/2}(x_1,x_2) \\
&= \int dC_2\, V^{-1-\epsilon/2}(x_1,x_2) - \int dS_2\, V^{-1-\epsilon/2}(x_2,x_1) \\
&= C_0^{\rm square} - C_0^{\rm comp}.
\end{aligned}
$$

Since the point $x_i = X_i$ is now internal to the integration domain the complementary C_0 will be regular.



G. Passarino, Calc03

$C_0^{\rm square}$ **is the integral over** $[0,1]^2$ **which we rewrite by changing variables,** $x'_i = x_i - X_i$, **as**

$$C_0^{\rm square} = \sum_{i=1}^{4} \alpha_i\,\beta_i \int dC_2 [Q_i(x_1,x_2) + B_3]^{-1-\epsilon/2},$$

where

$$\alpha_1 = \alpha_2 = 1 - X_1, \quad \alpha_3 = \alpha_4 = X_1,$$
$$\beta_1 = \beta_3 = 1 - X_2, \quad \beta_2 = \beta_4 = X_2,$$

and where the new quadrics are defined by

$$\begin{aligned} Q_1 &= Q((1-X_1)x_1\,,\,(1-X_2)x_2), \\ Q_2 &= Q((1-X_1)x_1\,,\,-X_2x_2), \\ Q_3 &= Q(-X_1x_1\,,\,(1-X_2)x_2), \\ Q_4 &= Q(-X_1x_1\,,\,-X_2x_2), \end{aligned}$$

with $Q = x^t\,H\,x$. **In general we define**

$$Q_i(x_1,x_2) = A_i\,x_1^2 + B_i\,x_2^2 + C_i\,x_1\,x_2.$$

In order the derive a Laurent expansion around $B_3 = 0$ **we introduce** $\rho_3 = 1/B_3$ **and perform a Mellin-Barnes splitting**

$$\begin{aligned} C_0^{\rm square} &= \sum_{i=1}^{4} \frac{\alpha_i\,\beta_i}{2\,\pi\,i} \int_{-i\,\infty}^{+i\,\infty} ds\,B(s,1-s)\,\rho_3^{1-s}\,\mathcal{C}_i(s), \\ \mathcal{C}_i(s) &= \int dC_2\,Q_i^{-s}(x_1,x_2). \end{aligned}$$

Let us consider in detail the $\mathcal{C}_i$**-functions. We use a simple sector decomposition to obtain**

$$
\begin{aligned}
\mathcal{C}_i &= [\int dS_2 + \int d\bar{S}_2]\, Q_i^{-s}(x_1,x_2) \\
&= \int dC_2\, x_1^{1-2\,s}\,(A_i + C_i\,x_2 + B_i\,x_2^2)^{-s} \\
&+ \int dC_2\, x_2^{1-2\,s}\,(B_i + C_i\,x_1 + A_i\,x_1^2)^{-s} \\
&= \frac{1}{2\,(1-s)}\,\sum_{j=1}^{2}\,\mathcal{C}_{i,j}(s).
\end{aligned}
$$

For each of the $\mathcal{C}_{ij}$-functions we have a reduced quadratic form in one variable. Let us postpone for a moment the problem of their evaluation:

$$
\begin{aligned}
C_0^{\rm square} &= \sum_{i=1}^{4}\,\sum_{j=1}^{2}\,\frac{\alpha_i\,\beta_i}{4\,\pi\,i}\,\int_{-i\,\infty}^{+i\,\infty} ds\,\frac{\Gamma\,(s)\;\Gamma\,(1-s)}{1-s}\,\rho_3^{1-s}\,\mathcal{C}_{ij}(s), \\
\mathcal{C}_{ij}(s) &= \int_0^1 dx\,(h_{ij}\,x^2 + 2\,k_{ij}\,x + l_{ij})^{-s},
\end{aligned}
$$

and $0 < {\rm Re}\,s < 1$. **Suppose that the factors,**

$$
b_{ij} \;=\; l_{ij} - k_{ij}^2/h_{ij} = \alpha_i^2\,\beta_i^2\,\det(H)/h_{ij}
$$

are not zero and that we are interested in the region of large $|\rho_3|$. Then we close the integration contour over the right-hand complex half-plane at infinity. The poles are at $s = 1$ (double) and at $s = k+1$ (single) where $k \ge 1$ is an integer. For $\mathcal{C}_{ij}$ we use

$$
\begin{aligned}
\mathcal{C}_{ij}(s) &= \frac{1}{b_{ij}}\int_0^1 dx\,[1+\frac{1}{2}\,\frac{x-X_{ij}}{s-1}\,\frac{d}{dx}]\,Q_{ij}^{1-s}(x),\\
Q_{ij}(x) &= h_{ij}\,x^2+2\,k_{ij}\,x+l_{ij},
\end{aligned}
$$

where X_{ij} is the co-factor for Q_{ij}. In the limit $s\to 1$ we obtain

$$
\begin{aligned}
\mathcal{C}_{ij}(s) &= \mathcal{C}_{ij}(1)+(s-1)\,\mathcal{C}'_{ij}(1)+\mathcal{O}\left((s-1)^2\right),\\
\mathcal{C}_{ij}(1) &= \frac{1}{b_{ij}}\int_0^1 dx\,[1-\frac{1}{2}\,(x-X_{ij})\,\frac{d}{dx}\,\ln\,Q_{ij}(x)],\\
\mathcal{C}'_{ij}(1) &= \frac{1}{b_{ij}}\int_0^1 dx\,[-\,\ln\,Q_{ij}(x)+\frac{1}{4}\,(x-X_{ij})\,\frac{d}{dx}\,\ln^2\,Q_{ij}(x)].
\end{aligned}
$$

Therefore the residue of the double pole at $s=1$ is

$$
\mathcal{R}_{ij}\,|_{s=1} \;=\; \mathcal{C}_{ij}(1)\,\ln\rho_3-\mathcal{C}'{}_{ij}(1).
$$

For the single poles at $s=k+1, k\geq 1$ we find that the residues are

$$
\mathcal{R}_{ij}\,|_{s=k+1} \;=\; -\frac{(-1)^k}{k}\,B_0(k+1)\,\rho_3^{-k},
$$

where $B_0(k{+}1)$ is a generalized two-point function.

IBP: QCD vs EW

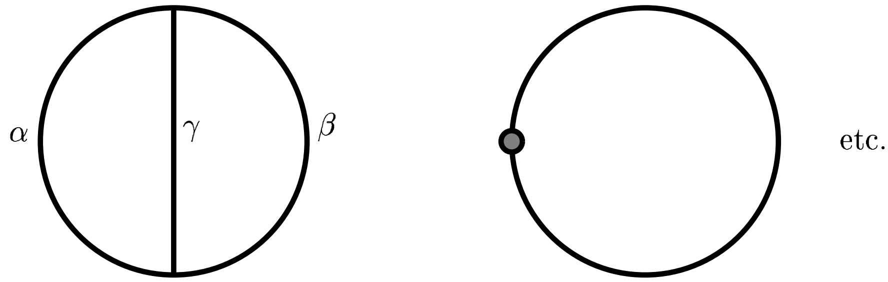


Figure 1: The arbitrary two-loop diagram $G_L^{\alpha\beta\gamma}$ and one of the associated subtraction sub-diagrams.

An arbitrary two-loop diagram has the following expression:

$$(\alpha; m_1, \dots, m_\alpha; \eta_1 | \gamma; m_{\alpha+1}, \dots, m_{\alpha+\gamma}; \eta_{12} | \beta; m_{\alpha+\gamma+1}, \dots, m_{\alpha+\gamma+\beta}; \eta_2) =$$

$$\frac{\mu^{2\epsilon}}{\pi^4} \int d^n q_1\, d^n q_2 \prod_{i=1}^{\alpha} (k_i^2 + m_i^2)^{-1} \prod_{j=\alpha+1}^{\alpha+\gamma} (k_j^2 + m_j^2)^{-1} \prod_{l=\alpha+\gamma+1}^{\alpha+\gamma+\beta} (k_l^2 + m_l^2)^{-1},$$

where $n = 4 - \epsilon$, with n being the space-time dimension, and where α, β and γ give the number of lines in the q_1, q_2 and $q_1 - q_2$ loops respectively. Furthermore we have

$$\begin{aligned} k_i &= q_1 + \Sigma_{j=1}^N\, \eta^1_{ij}\, p_j, & i &= 1, \dots, \alpha \\ k_i &= q_1 - q_2 + \Sigma_{j=1}^N\, \eta^{12}_{ij}\, p_j, & i &= \alpha+1, \dots, \alpha+\gamma \\ k_i &= q_2 + \Sigma_{j=1}^N\, \eta^2_{ij}\, p_j, & i &= \alpha+\gamma+1, \dots, \alpha+\gamma+\beta, \end{aligned}$$

N being the number of vertices, $\eta^a = \pm 1$, or 0 and $\{p\}$ the set of external momenta. Furthermore, μ is the arbitrary unit of mass.

For tensor integrals with L external momenta and $I = \alpha + \beta + \gamma$ propagators we have $3 + 2\,L$ scalar products containing either q_1 and/or q_2 and

$$r_{\max} = 2\,L - I + 3$$ **irreducible scalar products**

Therefore we will write IBPs

$$\int d^n q_1\, d^n q_2\, \frac{\partial}{\partial a_\mu}\, [b^\mu \prod_{r=1}^{r_{\max}} s_r^{n_r}\, \mathcal{G}\,] = 0,$$

$$\mathcal{G} = \prod_{i=1}^{\alpha} (k_i^2 + m_i^2)^{-1} \prod_{j=\alpha+1}^{\alpha+\gamma} (k_j^2 + m_j^2)^{-1} \prod_{l=\alpha+\gamma+1}^{\alpha+\gamma+\beta} (k_l^2 + m_l^2)^{-1}.$$

Writing all equations for increasing values of n_r the number of equations grows faster than the number of unknowns. However,

- **In QCD the coefficients are *relatively* simple;**
- **in EW the number of scales makes the coefficients ugly higher order polynomials in several variables for which a factorization is hard to attempt.**

An algorithm for all one-loop diagrams

Any one-loop Feynman diagram G, irrespective of the number N ofvertices, can be expressed as

$$G = \int_S dx\, Q(x)\, V^{\mu}(x),$$

where the integration region is $x_j \geq 0, \Sigma_j x_j \leq 1$, with $j = 1, \dots, N-1$, and $V(x)$ is a quadratic polynomial in x,

$$V(x) = x^t H x + 2 K^t x + L,$$

and $Q(x)$ is also a polynomial that accounts for parametrized tensor integrals. The solution to the problem of determining the polynomial $\mathcal{P}$ is as follows:

$$\mathcal{P} = 1 - \frac{(x - X)^t\, \partial_x}{2\,(\mu + 1)},$$
$$X^t = -K^t H^{-1}, \qquad B = L - K^t H^{-1} K,$$

where the matrix H is symmetric.

Smoothness of the integrand vs # of terms

The generic, scalar, pentagon is given by

$$E_0 = \int dS_4\, V^{-3-\epsilon/2}(x_1, x_2, x_3, x_4),$$
$$V(x) = x^t\, H\, x + 2\, K^t\, x + L,$$

where $H_{ij} = -\, p_i \cdot p_j$ **with** $i, j = 1, \dots, 4$**,** $L = m_1^2$**, and where**

$$K_1 = -\frac{1}{2}\,(m_1^2 - m_2^2 - p_1 \cdot p_1),$$
$$K_2 = -\frac{1}{2}\,(m_2^2 - m_3^2 - 2\, p_1 \cdot p_2 - p_2 \cdot p_2),$$
$$K_3 = -\frac{1}{2}\,(m_3^2 - m_4^2 - 2\, p_1 \cdot p_3 - 2\, p_2 \cdot p_3 - p_3 \cdot p_3),$$
$$K_4 = -\frac{1}{2}\,(m_4^2 - m_5^2 - 2\, p_1 \cdot p_4 - 2\, p_2 \cdot p_4 - 2\, p_3 \cdot p_4 - p_4 \cdot p_4).$$

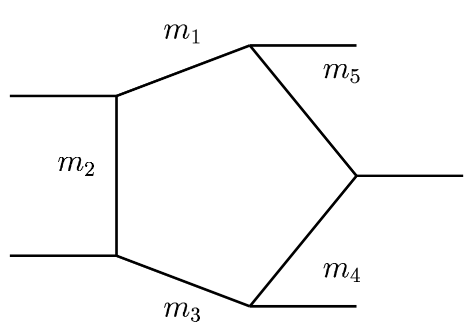


Figure 2: The one-loop, five-point Green function. Propagators are $q^2 + m_1^2 \cdots\ (q + p_1 + \cdots + p_4)^2 + m_5^2$.

Evaluation of E_0 when $B_5 \neq 0$

It is a virtue of the BT algorithm that we can show the decomposition of E_0 in five boxes (in $d = 4$) with just one iteration, as depictedin Fig. 3. We obtain

$$E_0 \;=\; \frac{1}{4\,B_5}\,\sum_{i=0}^{4}\,w_i\,D_0^{(i)},$$

where the weights are

$$w_i \;=\; X_i - X_{i+1}, \qquad X_0 = 1, \quad X_5 = 0,$$

and where $B_5 = L - K^t\,H^{-1}\,K$ and $X = -K^t\,H^{-1}$. The further advantage in this derivation is that the nature of the weights is transparent since $B_5 = 0$ corresponds to a Landau singularity of the pentagon. Furthermore the boxes are specified by

$$D_0^{(i)} \;=\; \int dS_3\,V^{-2-\epsilon/2}(\widehat{i\;i+1}),$$

where the contractions are

$$\widehat{0\;1} \;=\; (1, x_1, x_2, x_3), \qquad \widehat{4\;5} = (x_1, x_2, x_3, 0),$$

etc. As long as B_5 is not around zero the derivation for the pentagon is completed since we know how to deal with boxes.

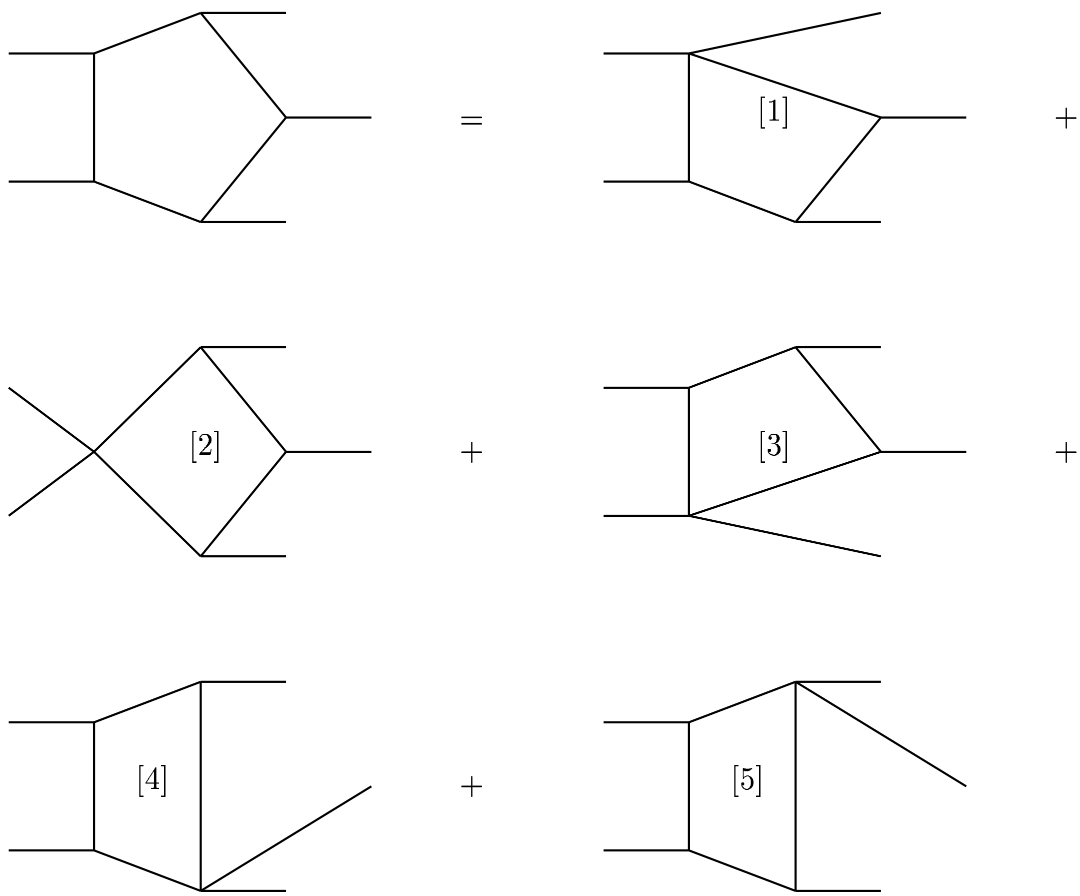


Figure 3: Diagrammatical representation of the BT algorithm for the pentagon. The symbol $[i]$ denotes multiplication of the corresponding box by a factor $w_i/(4\,B_5)$.

Form factors in the E-family

The nice feature of the scalar pentagon is that fourfold integrals disappear in the final answer. The same is not immediately true for form factors, E_{11} etc. We can use a new identity,

$$(x_i - X_i)\, V^{\mu}(x_1, x_2, x_3, x_4) \;=\; \frac{H^{-1}_{ij}}{2\,(\mu+1)}\,\partial_j\, V^{\mu+1}(x_1, x_2, x_3, x_4).$$

With the help of the identity and after integration-by-parts we are able to remove again all fourfold integrals in the form factors of the E_{1i}-series. To give an example we consider

$$\begin{aligned} E_{11} &= \int dS_4\, x_1\, V^{-3-\epsilon/2}(x_1, x_2, x_3, x_4) \\ &= \frac{1}{4\,B_5}\int dS_3\,(E^{-2}_{11} - \frac{1}{2}\,E^{-1}_{11}). \end{aligned}$$

Secondary quadrics are defined as:

$$\begin{aligned} V_a(x_1, x_2, x_3) &= V(x_1, x_1, x_2, x_3), \\ V_b(x_1, x_2, x_3) &= V(x_1, x_2, x_2, x_3), \\ V_c(x_1, x_2, x_3) &= V(x_1, x_2, x_3, x_3), \\ V_d(x_1, x_2, x_3) &= V(x_1, x_2, x_3, 0), \\ V_e(x_1, x_2, x_3) &= V(1, x_1, x_2, x_3). \end{aligned}$$

The E_{11} form factor is fully specified by

$$
\begin{aligned}
E_{11}^{-2} &= (X_1 - X_2)\, x_1\, V_a^{-2-\epsilon/2} + (X_2 - X_3)\, x_1\, V_b^{-2-\epsilon/2} + (X_3 - X_4)\, x_1\, V_c^{-2-\epsilon/2} \\
&+ X_4\, x_1\, V_d^{-2-\epsilon/2} + (1 - X_1)\, V_e^{-2-\epsilon/2},
\end{aligned}
$$

$$
\begin{aligned}
E_{11}^{-1} &= (H_{11}^{-1} - H_{12}^{-1})\, V_a^{-1-\epsilon/2} + (H_{12}^{-1} - H_{13}^{-1})\, V_b^{-1-\epsilon/2} + (H_{13}^{-1} - H_{14}^{-1})\, V_c^{-1-\epsilon/2} \\
&+ H_{14}^{-1}\, V_d^{-1-\epsilon/2} - H_{11}^{-1}\, V_e^{-1-\epsilon/2}.
\end{aligned}
$$

Similarly we obtain

$$
\begin{aligned}
E_{i1}^{-1} &= (H_{i1}^{-1} - H_{i2}^{-1})\, V_a^{-1-\epsilon/2} + (H_{i2}^{-1} - H_{i3}^{-1})\, V_b^{-1-\epsilon/2} + (H_{i3}^{-1} - H_{i4}^{-1})\, V_c^{-1-\epsilon/2} \\
&+ H_{i4}^{-1}\, V_d^{-1-\epsilon/2} - H_{i1}^{-1}\, V_e^{-1-\epsilon/2},
\end{aligned}
$$

for $i = 1, \dots, 4$ and

$$
\begin{aligned}
E_{21}^{-2} &= (X_1 - X_2)\, x_1\, V_a^{-2-\epsilon/2} + (X_2 - X_3)\, x_2\, V_b^{-2-\epsilon/2} + (X_3 - X_4)\, x_2\, V_c^{-2-\epsilon/2} \\
&+ X_4\, x_2\, V_d^{-2-\epsilon/2} + (1 - X_1)\, x_1\, V_e^{-2-\epsilon/2}, \\
E_{31}^{-2} &= (X_1 - X_2)\, x_2\, V_a^{-2-\epsilon/2} + (X_2 - X_3)\, x_2\, V_b^{-2-\epsilon/2} + (X_3 - X_4)\, x_3\, V_c^{-2-\epsilon/2} \\
&+ X_4\, x_3\, V_d^{-2-\epsilon/2} + (1 - X_1)\, x_2\, V_e^{-2-\epsilon/2}, \\
E_{41}^{-2} &= (X_1 - X_2)\, x_3\, V_a^{-2-\epsilon/2} + (X_2 - X_3)\, x_3\, V_b^{-2-\epsilon/2} + (X_3 - X_4)\, x_3\, V_c^{-2-\epsilon/2} \\
&+ (1 - X_1)\, x_3\, V_e^{-2-\epsilon/2}.
\end{aligned}
$$

Note that there are also delicate points connected with the decomposition into objects belonging to the E-family since the decomposition itself is strictly defined in 4 dimensions. Furthermore, starting from six powers of momenta in the numerator (R_ξ-gauge), we will encounter UV divergent terms so that some care is needed.

With two momenta in the numerator we define form factors as

$$
\begin{aligned}
E_{2;i} &= E_2(ii), \qquad i,j = 1,\cdots,4,\\
E_{2;5} &= E_2(12), \quad E_{2;6} = E_2(13), \quad E_{2;7} = E_2(14),\\
E_{2;8} &= E_2(23), \quad E_{2;9} = E_2(24), \quad E_{2;10} = E_2(34),
\end{aligned}
$$

where the auxiliary function $E_2(ij)$ is

$$E_2(ij) = \int dS_4\, x_i\, x_j\, V^{-3-\epsilon/2}(x_1,x_2,x_3,x_4).$$

However, a new form factor arises, the one proportional to $\delta_{\mu\nu}$,

$$E_{2;11} = \frac{1}{4}\int dS_4\, V^{-2-\epsilon/2}(x_1,x_2,x_3,x_4).$$

$E_{2;11}$ can still be reduced to form factors of the E-family by writing

$$E_{2;11} = \frac{1}{4}\Big[\sum_{i,j=1}^{4} H_{ij}\, E_2(ij) + 2\sum_{i=1}^{4} K_i\, E_{1i} + L\, E_0\Big].$$

Let us give the complete expressions for the form factors of the E_2-series.

$$
\begin{aligned}
4\, B_5\, E_2(ij) &= -\int dS_4\, H^{-1}_{ij}\, V^{-1-\epsilon/2}(x_1,x_2,x_3,x_4)\\
&\quad + \int dS_3\,(E^{-2}_{2;ij} - E^{-1}_{2;ij}),
\end{aligned}
$$

where the various coefficients are:

$$
\begin{aligned}
E^{-2}_{2;11} &= (X_1 - X_2)\, x_1^2\, V_a^{-2-\epsilon/2} + (X_2 - X_3)\, x_1^2\, V_b^{-2-\epsilon/2}\\
&+ (X_3 - X_4)\, x_1^2\, V_c^{-2-\epsilon/2} + X_4\, x_1^2\, V_d^{-2-\epsilon/2} + (1 - X_1)\, V_e^{-2-\epsilon/2},
\end{aligned}
$$

$$
\begin{aligned}
E^{-2}_{2;22} &= (X_1 - X_2)\, x_1^2\, V_a^{-2-\epsilon/2} + (X_2 - X_3)\, x_2^2\, V_b^{-2-\epsilon/2} \\
&+ (X_3 - X_4)\, x_2^2\, V_c^{-2-\epsilon/2} + X_4\, x_2^2\, V_d^{-2-\epsilon/2} + (1 - X_1)\, x_1^2\, V_e^{-2-\epsilon/2}, \\
E^{-2}_{2;33} &= (X_1 - X_2)\, x_2^2\, V_a^{-2-\epsilon/2} + (X_2 - X_3)\, x_2^2\, V_b^{-2-\epsilon/2} \\
&+ (X_3 - X_4)\, x_3^2\, V_c^{-2-\epsilon/2} + X_4\, x_3^2\, V_d^{-2-\epsilon/2} + (1 - X_1)\, x_2^2\, V_e^{-2-\epsilon/2}, \\
E^{-2}_{2;44} &= (X_1 - X_2)\, x_3^2\, V_a^{-2-\epsilon/2} + (X_2 - X_3)\, x_3^2\, V_b^{-2-\epsilon/2} \\
&+ (X_3 - X_4)\, x_3^2\, V_c^{-2-\epsilon/2} + (1 - X_1)\, x_3^2\, V_e^{-2-\epsilon/2}, \\
E^{-2}_{2;12} &= (X_1 - X_2)\, x_1^2\, V_a^{-2-\epsilon/2} + (X_2 - X_3)\, x_1\, x_2\, V_b^{-2-\epsilon/2} \\
&+ (X_3 - X_4)\, x_1\, x_2\, V_c^{-2-\epsilon/2} + X_4\, x_1\, x_2\, V_d^{-2-\epsilon/2} + (1 - X_1)\, x_1\, V_e^{-2-\epsilon/2},
\end{aligned}
$$

$$
\begin{aligned}
E^{-1}_{2;11} &= (H^{-1}_{11} - H^{-1}_{12})\, x_1\, V_a^{-1-\epsilon/2} + (H^{-1}_{12} - H^{-1}_{13})\, x_1\, V_b^{-1-\epsilon/2} \\
&+ (H^{-1}_{13} - H^{-1}_{14})\, x_1\, V_c^{-1-\epsilon/2} + H^{-1}_{14}\, x_1\, V_d^{-1-\epsilon/2} - H^{-1}_{11}\, V_e^{-1-\epsilon/2}, \\
E^{-1}_{2;22} &= (H^{-1}_{21} - H^{-1}_{22})\, x_1\, V_a^{-1-\epsilon/2} + (H^{-1}_{22} - H^{-1}_{23})\, x_2\, V_b^{-1-\epsilon/2} \\
&+ (H^{-1}_{23} - H^{-1}_{24})\, x_2\, V_c^{-1-\epsilon/2} + H^{-1}_{24}\, x_2\, V_d^{-1-\epsilon/2} - H^{-1}_{21}\, x_1\, V_e^{-1-\epsilon/2}, \\
E^{-1}_{2;33} &= (H^{-1}_{31} - H^{-1}_{32})\, x_2\, V_a^{-1-\epsilon/2} + (H^{-1}_{32} - H^{-1}_{33})\, x_2\, V_b^{-1-\epsilon/2} \\
&+ (H^{-1}_{33} - H^{-1}_{34})\, x_3\, V_c^{-1-\epsilon/2} + H^{-1}_{34}\, x_3\, V_d^{-1-\epsilon/2} - H^{-1}_{31}\, x_2\, V_e^{-1-\epsilon/2}, \\
E^{-1}_{2;44} &= (H^{-1}_{41} - H^{-1}_{42})\, x_3\, V_a^{-1-\epsilon/2} + (H^{-1}_{42} - H^{-1}_{43})\, x_3\, V_b^{-1-\epsilon/2} \\
&+ (H^{-1}_{43} - H^{-1}_{44})\, x_3\, V_c^{-1-\epsilon/2} - H^{-1}_{41}\, x_3\, V_e^{-1-\epsilon/2}, \\
2\, E^{-1}_{2;12} &= (H^{-1}_{11} - H^{-1}_{22})\, x_1\, V_a^{-1-\epsilon/2} + \Big[(H^{-1}_{12} - H^{-1}_{13})\, x_2 + (H^{-1}_{22} - H^{-1}_{23})\, x_1 \Big]\, V_b^{-1-\epsilon/2} \\
&+ \Big[(H^{-1}_{13} - H^{-1}_{14})\, x_2 + (H^{-1}_{23} - H^{-1}_{24})\, x_1 \Big]\, V_c^{-1-\epsilon/2} \\
&+ (H^{-1}_{14}\, x_2 + H^{-1}_{24}\, x_1)\, V_d^{-1-\epsilon/2} - (H^{-1}_{11}\, x_1 + H^{-1}_{21})\, V_e^{-1-\epsilon/2}.
\end{aligned}
$$

All functions E_{ij} contain a term which is proportional to $E_0(d = 8)$. Consider $E_{2;11}$:

$$
\begin{aligned}
4\, B_5\, E_{2;11} &= -E_0(d = 8) + \frac{1}{2} \int dS_3 \big[(X_1 - X_2)\, V_a^{-1-\epsilon/2} \\
&+ (X_2 - X_3)\, V_b^{-1-\epsilon/2} + (X_3 - X_4)\, V_c^{-1-\epsilon/2} \\
&+ X_4\, V_d^{-1-\epsilon/2} + (1 - X_1)\, V_e^{-1-\epsilon/2} \big].
\end{aligned}
$$

When we consider

$$E_{\mu\nu} \;=\; \frac{1}{i\pi^2\,\Gamma\,(3)}\int d^dq\,\frac{q_\mu\,q_\nu}{(q^2+m_1^2)\,\cdots\,((q+p_1+\,\cdots\,+p_4)^2+m_5^2)},$$

the contribution proportional to $E_0(d=8)$ is

$$E_{\mu\nu} \;=\; -\,\frac{E_0(d=8)}{4\,B_5}\,[\delta_{\mu\nu}+\sum_{i,j=1}^{4}\,H^{-1}_{ij}\,p_{i\mu}\,p_{j\nu}] \\ +\quad \textbf{form factors}\quad N<5.$$

Since the four-vectors $p_{i\mu}$ span $d=4$ space-time the term proportional to $E_0(d=8)$ disappears and, therefore, there is no need to compute it. Indeed let us contract with $p_l, l=1,\dots,4$ to obtain

$$p_{l\nu}\,(\delta_{\mu\nu}+\sum_{i,j=1}^{4}\,H^{-1}_{ij}\,p_{i\mu}\,p_{j\nu}) = p_{l\mu}-\sum_{i,j=1}^{4}\,H^{-1}_{ij}\,p_{i\mu}\,H_{il}=0.$$

When we go to three powers of q in the numerator the following result is valid:

$$E_{\mu\nu\alpha} \;=\; -\,\frac{1}{2\,B_5}\int dS_4\,V^{-1-\epsilon/2}\,[\sum_i x_i\,\{\delta p_i\}_{\mu\nu\alpha} \\ +\;\sum_{i\leq j\leq l} e^3_{ijl}\,\{p_ip_jp_l\}_{\mu\nu\alpha}]\;+\;\textbf{form factors}\;N<5.$$

where the coefficients e^3 are

$$e^3_{ijl} \;=\; H^{-1}_{ij}\,x_l \quad + \quad \textbf{cyclic},$$

and where we introduced

$$\{\delta p\}_{\mu\nu\alpha} = \delta_{\mu\nu}\, p_\alpha + \delta_{\mu\alpha}\, p_\nu + \delta_{\nu\alpha}\, p_\mu,$$
$$\{pqk\}_{\mu\nu\alpha} = p_\mu\, q_\nu\, k_\alpha \quad + \quad \textbf{cyclic},$$

showing that P$E_0(d = 8)$ disappears again. It is straightforward to extend the demonstration up to five powers of loop momentum in the numerator. This is the maximum of non-contracted powers that we can have in a renormalizable theory and starting from six powers we will have numerators of the following structure,

$$\{q \cdot p_i\,,\, q^2\}\, q_{\mu_1} \cdots q_{\mu_5},$$

where scalar products can be simplified according to $q^2 = [1] - m_1^2$ etc. There is only one case where the argument fails: suppose that one external line of momentum p_i splits into two lines of momenta $p_{ij}, j = 1, 2$. Then scalar products $q \cdot p_{ij}$ do not occur in any of the propagators and in the final answer we would end up with evanescent operators like

$$\frac{1}{\epsilon}\, \delta^{\mu\nu}_{[-\epsilon]}.$$

Evaluation of E_0 when $B_5 \approx 0$ and E_0 is regular

If $B_5 \approx 0$ and the condition $0 < X_i < X_{i+1} < 1, i = 1, \cdots, 4$ is not satisfied E_0 is regular but the decomposition into a sum of boxes fails. We write

$$E_0 \; = \; \int dS_4(X) \, (x^t \, H \, x + B_5)^{-3-\epsilon/2}.$$

Since E_0 is regular at $B_5 = 0$ we perform a Taylor expansion in B_5 with coefficients $\mathcal{E}(n)$,

$$E_0 \; = \; \frac{1}{2} \sum_{n=0}^{\infty} (n+1)\,(n+2)\,\mathcal{E}(n+3)\,(-\,B_5)^n,$$

$$\mathcal{E}(n) \; = \; \int dS_4(X) \, (x^t \, H \, x)^{-n-\epsilon/2},$$

with $n \geq 3$. Using

$$\int dS_4(X) \, [1 + \frac{x \, \partial_x}{2\,n+\epsilon}] \, (x^t \, H \, x)^{-n-\epsilon/2} \; = \; 0,$$

we easily obtain

$$\mathcal{E}(n) \; = \; \frac{1}{4-2\,n} \sum_{i=0}^{4} \int dS_3 \, (X_i - X_{i+1}) \, Q^{-n-\epsilon/2}(T \, i \, \widehat{i+1}),$$

where, as usual, we introduced shifted arguments. If we denote $x^t \, H \, x$ with $Q(x_1, x_2, x_3, x_4)$ the secondary quadrics and the corresponding coefficients are

$$
\begin{aligned}
Q(T\,\widehat{0\,1}) &\equiv Q_1 = Q(1 - X_1, x_1 - X_2, x_2 - X_3, x_3 - X_4),\\
Q(T\,\widehat{1\,2}) &\equiv Q_2 = Q(x_1 - X_1, x_1 - X_2, x_2 - X_3, x_3 - X_4),\\
Q(T\,\widehat{2\,3}) &\equiv Q_3 = Q(x_1 - X_1, x_2 - X_2, x_2 - X_3, x_3 - X_4),\\
Q(T\,\widehat{3\,4}) &\equiv Q_4 = Q(x_1 - X_1, x_2 - X_2, x_3 - X_3, x_3 - X_4),\\
Q(T\,\widehat{4\,5}) &\equiv Q_5 = Q(x_1 - X_1, x_2 - X_2, x_3 - X_3, -X_4),
\end{aligned}
$$

and $X_0 = 1, X_5 = 0$. **Again, each coefficient in the Taylor expansion is written as a combination of threefold integrals which can be evaluated with standard BT techniques. This will introduce sub-leading quadrics, i.e.**

$$
Q_{i1} = Q_i(\widehat{0\,1}) = Q_i(1, x_1, x_2),
$$

etc, and sub-subleading quadrics, i.e.

$$
Q_{ij1} = Q_{ij}(\widehat{0\,1}) = Q_{ij}(1, x_1),
$$

etc, and also constant terms, e.g. $Q_{ij1}(1,1)$ **etc. At each step non-leading BT factors are introduced and the procedure fails when one of the sub-leading BT factors is zero. In this case, since** E_0 **is the sum of** 5 **terms of the form**

$$
E_0^i = -\frac{1}{4}\sum_{n=0}^{\infty}(n+2)\int dS_2\, q_i\, Q_i^{-n-3}\,(-B_5)^n,
$$

with q_i constant, we rewrite the sum as

$$\frac{1}{4}\sum_{n=0}^{\infty}(n+2)\,Q_i^{-n-3}\,(-\,B_5)^n \;=\; \frac{1}{2}\int_0^1 dz\,(Q_i+B_5\,z)^{-3}.$$

Likewise, by changing variables we obtain

$$\int_0^1 dz\int dS_3\,q_i\,(Q_i+B_5\,z)^{-3}$$
$$=\int dS_3(X)\int_0^1 dz\,q_i\,(x^t\,H_i\,x+B_{4i}+B_5\,z)^{-3},$$

where B_{4i} is the relevant sub-leading BT factor and we can apply a Mellin-Barnes splitting, followed by a sector decomposition, to $x^t\,H_i\,x$ and $B_{4i}+B_5\,z^2$. For the Mellin-Barnes anti-transform the leading contribution comes from the pole at $s=3/2$ giving

$$\int_0^1 dz\,(B_{4i}+B_5\,z)^{-3/2}=-\,\frac{2}{B_5}\,[(B_{4i}+B_5)^{-1/2}-B_{4i}^{-1/2}].$$

Alternatively we define $V_0(x_1,\dots,x_4)=V(x_1,\dots,x_4)-B_5$ and write down the BT relations corresponding to $B_5\neq 0$ and $B_5=0$:

$$[1+\frac{1}{4+\epsilon}\,(x-X)\,\partial_x]\,[V_0(x_1,\dots,x_4)+B_5]^{-2-\epsilon/2}$$
$$=\;B_5\,[V_0(x_1,\dots,x_4)+B_5]^{-3-\epsilon/2},$$
$$[1+\frac{1}{4+\epsilon}\,(x-X)\,\partial_x]\,V_0^{-2-\epsilon/2}(x_1,\dots,x_4)=0.$$

Then, after having summed each side of the two equations, we integrate by parts and set $\epsilon = 0$, obtaining:

$$E_0 = \frac{1}{4} \int dS_2 \sum_{i=0}^{4} (X_i - X_{i+1})\, B_5^{-1}\, V^{-2}(i\ \widehat{i+1})\ |_{\rm sub}\ .$$

Hence the study of the scalar pentagon reduces again to the study of four-point functions. The tensor pentagons too can be treated analogously to the case $B_5 \neq 0$.

Algorithms for IR

Consider diagrams belonging to the G^{1N1} family. The parametrization that we use is

$$
\begin{aligned}
G^{1N1} &= -\left(\frac{\mu^2}{\pi}\right)^{\epsilon} \Gamma(N-2+\epsilon) \int_0^1 dx \int dS_N(y, u_1, \cdots, u_{N-1}) \\
&\times \left[x(1-x)\right]^{-\epsilon/2} (1-y)^{\epsilon/2-1} \chi_{1N1}^{2-N-\epsilon}, \\
\chi_{1N1} &= u^t \mathcal{H} u + 2\mathcal{K}^t u + (m_x^2 - m_3^2)(1-y) + m_3^2,
\end{aligned}
$$

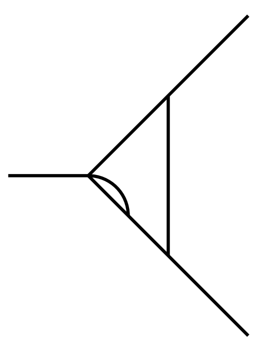
where

$$
\begin{aligned}
\mathcal{H}_{ij} &= -p_i \cdot p_j, \\
\mathcal{K}_i &= \frac{1}{2}(k_i^2 - k_{i-1}^2 + m_{i+3}^2 - m_{i+2}^2), \qquad i, j = 1, \ldots, N-1
\end{aligned}
$$

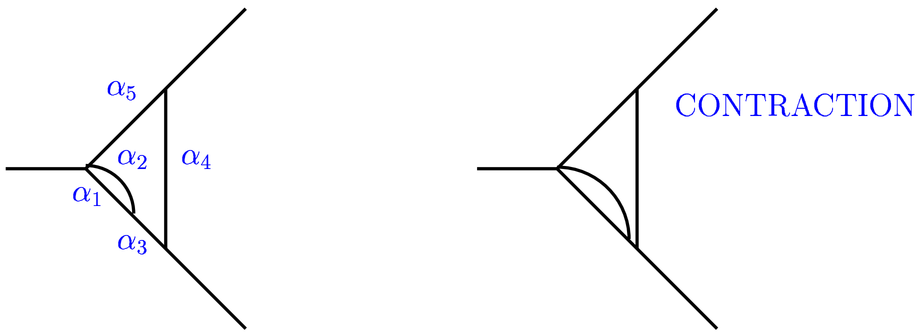
and where we have introduced a x-dependent mass

$$
m_x^2 = \frac{(1-x)\, m_1^2 + x\, m_2^2}{x\,(1-x)}.
$$

131 vertex



The leading Landau singularity for V^{131} requires $m_3 = m_1 + m_2$. Therefore we look for IR configurations corresponding to sub-leading Landau singularites. Since a two-point functions does not develop any IR pole we limit our analysis to reduced three-point functions.

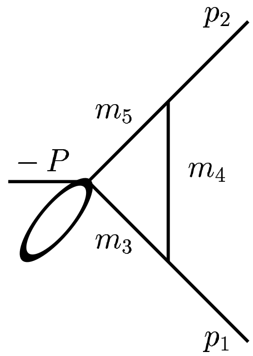


$- \alpha_3 = 0$

The reduced diagram, i.e. the one where the third propagator is shrunk to a point, corresponds to a V^{121} topology which does not develop IR poles.

$- \alpha_1 = \alpha_2 = 0$

In this case the reduced diagram is a one loop three-point function and the classification of the infrared singularites is simpler. We obtain



$$
\begin{array}{llll}
\mathbf{1)} & m_3 = 0, & P^2 = -m_5^2, & p_1^2 = -m_4^2, \\
\mathbf{2)} & m_4 = 0, & p_1^2 = -m_3^2, & p_2^2 = -m_5^2, \\
\mathbf{3)} & m_5 = 0, & P^2 = -m_3^2, & p_2^2 = -m_4^2.
\end{array}
$$

The corresponding quadratic form are

$$
\mathbf{1)} \quad \chi_I = m_1^2 \frac{1-y}{x} + m_2^2 \frac{1-y}{1-x} + m_4^2 z_1^2 - p_2^2 z_2^2 + (p_2^2 + m_5^2 - m_4^2) z_1 z_2,
$$

$$
\mathbf{2)} \quad \chi_I = m_1^2 \frac{1-y}{x} + m_2^2 \frac{1-y}{1-x} - m_3^2 (1-y) + m_3^2 (1-z_1)^2 + m_5^2 z_2^2 + (P^2 + m_3^2 + m_5^2) (1-z_1) z_2,
$$

$$
\begin{aligned}
\mathbf{3)} \quad \chi_I &= m_1^2 \frac{1-y}{x} + m_2^2 \frac{1-y}{1-x} - m_3^2 (1-y) - p_1^2 (1-z_1)^2 + m_4^2 (1-z_2)^2 \\
&+ (p_1^2 + m_3^2 - m_4^2) (1-z_1) (1-z_2),
\end{aligned}
$$

showing a zero at

$$
\begin{array}{lll}
\mathbf{1)} & y = 1, & z_1 = z_2 = 0, \\
\mathbf{2)} & y = z_1 = 1, & z_2 = 0, \\
\mathbf{3)} & y = z_1 = z_2 = 1. &
\end{array}
$$

In each of the three cases, after transforming $y = 1 - y'$, the diagram reads as follows:

$$V_i^{131} = -\left(\frac{\mu^2}{\pi}\right)^{\epsilon} \Gamma(1+\epsilon) \int_0^1 dx \int_0^1 dz_1 \int_0^{z_1} dz_2 \int_0^{1-z_1} dy \times [x(1-x)]^{-\epsilon/2} y^{\epsilon/2-1} (a_i y + Z_i)^{-1-\epsilon}$$

where $i = 1, 2, 3$ **and**

$$\begin{aligned} Z_1 &= m_4^2 z_1^2 - p_2^2 z_2^2 + (p_2^2 + m_5^2 - m_4^2) z_1 z_2, \qquad a_1 = m_x^2; \\ Z_2 &= m_3^2 (1-z_1)^2 + m_5^2 z_2^2 \\ &+ (P^2 + m_3^2 + m_5^2)(1-z_1) z_2, \qquad a_2 = m_x^2 - m_3^2; \\ Z_3 &= -p_1^2 (1-z_1)^2 + m_4^2 (1-z_2)^2 \\ &+ (p_1^2 + m_3^2 - m_4^2)(1-z_1)(1-z_2), \qquad a_3 = m_x^2 - m_3^2. \end{aligned}$$

We are looking for a procedure that extracts, first of all, the ultraviolet pole. Consider the innermost integral

$$\mathcal{Y}_i = Z_i^{-1-\epsilon} \int_0^{1-z_1} dy\, y^{\epsilon/2-1} (1 + \frac{a_i}{Z_i} y)^{-1-\epsilon}.$$

It is convenient to evaluate this integral in terms of an hyper-geometric function,

$$\mathcal{Y}_i = \frac{2}{\epsilon} Z_i^{-1-\epsilon} z_1^{\epsilon/2} {}_2F_1(1+\epsilon, \frac{\epsilon}{2}; 1 + \frac{\epsilon}{2}; -\frac{a_i(1-z_1)}{Z_i}),$$

and to use well-known properties of hyper-geometric functions to obtain

$$
\begin{aligned}
\mathcal{Y}_i &= \frac{2}{\epsilon}\frac{\Gamma^2(1+\epsilon/2)}{\Gamma(1+\epsilon)}\, a_i^{-\epsilon/2}\, Z_i^{-1-\epsilon/2} - \frac{2}{2+\epsilon}\, a_i^{-1-\epsilon}\,(1-z_1)^{-1-\epsilon/2} \\
&\times\ {}_2F_1(1+\epsilon\,,\,1+\frac{\epsilon}{2}\,;\,2+\frac{\epsilon}{2}\,;\,-\frac{Z_i}{a_i\,(1-z_1)}) \\
&= \frac{2}{\epsilon}\frac{\Gamma^2(1+\epsilon/2)}{\Gamma(1+\epsilon)}\, a_i^{-\epsilon/2}\, Z_i^{-1-\epsilon/2} \\
&- \frac{1}{Z_i}\ln(1+\frac{Z_i}{a_i\,(1-z_1)}) + \mathcal{O}(\epsilon)\,.
\end{aligned}
$$

In this way we have extracted the ultraviolet pole. The second term is well-behaved when integrated over x, z_1 and z_2, while for the first term we have to compute:

$$
\begin{aligned}
\mathcal{X}_i &= \int_0^1 dx\,[x(1-x)]^{-\epsilon/2}\, a_i^{-\epsilon/2}, \\
\mathcal{Z}_i &= \int_0^1 dz_1 \int_0^{z_1} dz_2\, Z_i^{-1-\epsilon/2}.
\end{aligned}
$$

It is easily seen that the first one gives:

$$
\mathcal{X}_i = 1 - \frac{\epsilon}{2}\int_0^1 dx\ln V_i + \frac{\epsilon^2}{8}\int_0^1 dx\ln^2 V_i
$$

where

$$
\begin{aligned}
V_1 &= m_1^2\,(1-x) + m_2^2\, x, \\
V_2 &= V_3 = m_1^2\,(1-x) + m_2^2\, x - m_3^2\, x\,(1-x).
\end{aligned}
$$

For the other one we have to distinguish the three cases. As an example consider case 1**. First of all we map the integration domaine into** $[0,1]^2$ **obtaining:**

$$\begin{aligned}\mathcal{Z}_1 &= \int dS_2\,(A_1\,z_1^2 + B_1\,z_2^2 + \;C_1\,z_1\,z_2)^{-1-\epsilon/2} \\ &= \int dC_2\,z_1^{-1-\epsilon}\,(B_1\,z_2^2 + C_1\,z_2 + \;A_1)^{-1-\epsilon/2} \\ &= -\frac{1}{\epsilon}\int dC_1\,(B_1\,z_2^2 + C_1\,z_2 + \;A_1)^{-1-\epsilon/2}.\end{aligned}$$

where the coefficients are

$$A_1 \;=\; m_4^2,\; B_1 = -p_2^2,\; C_1 = p_2^2 + m_5^2 - m_4^2.$$

For the z_2**-integral we use one BT iteration, integrate by parts and expand in** ϵ **obtaining:**

$$\mathcal{Z}_1 \;=\; \frac{1}{\lambda_1}(\mathcal{R}_1^{-1}\,\epsilon^{-1} + \mathcal{R}_1^0 + \mathcal{R}_1^1\,\epsilon)$$

where

$$\lambda_1 \;=\; \lambda(B_1, A_1, A_1 + B_1 + C_1) = \lambda(-p_2^2, m_4^2, m_5^2).$$

and where residue and finite parts, up to $\mathcal{O}\left(\epsilon\right)$ **are**

$$
\begin{aligned}
\mathcal{R}_1^{-1} &= 2\,B_1\,[\,\int_0^1 dz\;\ln\eta_1(z) - \ln\eta_1(1) + 2] \\
&- C_1\,[\,\ln\eta_1(1) - \ln\eta_1(0)]; \\
\mathcal{R}_1^{0} &= -\frac{1}{2}\,B_1\,[\,\int_0^1 dz\;\ln\eta_1(z)\,(4 + \ln\eta_1(z)) - \ln^2\eta_1(1)] \\
&+ \frac{1}{4}\,C_1\,[\,\ln^2\eta_1(1) - \ln^2\eta_1(0)]; \\
\mathcal{R}_1^{1} &= \frac{1}{12}\,B_1\,[\,\int_0^1 dz\;\ln^2\eta_1(z)\,(6 + \ln\eta_1(z)) - \ln^3\eta_1(1)] \\
&- \frac{1}{24}\,C_1\,[\,\ln^3\eta_1(1) - \ln^3\eta_1(0)].
\end{aligned}
$$

where we have defined $\eta_1(z) = B_1 z^2 + C_1 z + A_1$**.**

A Novel Approach

Given a two-loop diagram $G(n_1, \cdots n_I)$ we define the associated δ-diagram

$$G_\delta(0\,;\,n_1,\,\cdots\,n_I) = \frac{\mu^{2\epsilon}}{\pi^4}\int d^nq_1\,d^nq_2\,\prod_{i=1}^{\alpha}(k_i^2+m_i^2)^{-n_i-\delta}$$

$$\times\,\prod_{j=\alpha+1}^{\alpha+\gamma}(k_j^2+m_j^2)^{-n_j-\delta}\,\prod_{l=\alpha+\gamma+1}^{\alpha+\gamma+\beta}(k_l^2+m_l^2)^{-n_k-\delta},$$

and are interested in

$$G(0\,;\,n_1,\,\cdots\,n_I)\;=\;\lim_{\delta\to 0}G_\delta(0\,;\,n_1,\,\cdots\,n_I).$$

Furthermore, a two-loop diagram is carachterized

- **by a degree, $\mathcal{D} = \Sigma\, n_i - 4 + \epsilon$;**
- **by a rank, given by the power of irreducible scalar products present in the numerator.**

Clearly, diagrams with $\mathcal{D} = \epsilon$ are simple to evaluate since, after a Laurent expansion they contain at most integrals of logarithms. The idea is to lower $\mathcal{D}$ by

- **writing IBP (8) and Lorentz identities (1) for**

$$G_\delta(\mathbf{rank}\,;\,0\,,\,-1\,,\,\cdots\,,\,-1)\quad\textbf{and}$$
$$G_\delta(\mathbf{rank}\,;\,+1\,,\,-1\,,\,\cdots\,,\,-1),$$

with increasing values for the power of irreducible scalar products till we can solve for the δ-scalar diagram of degree $\mathcal{D}$ in favor of (possibly) scalar δ-diagrams of degree $\mathcal{D}-1$ or less. The whole procedure is then iterated till we reach a decomposition in terms of δ-diagrams of degree $\mathcal{D} = \epsilon\,(\mathbf{mod}\,\delta)$. At this point we take the limit $\delta \to 0$ and obtain an expression for the original diagram. The solution will be of the following form:

$$G_\delta(-1\,,\,-1\,,\,\cdots\,,\,-1) \;=\; \delta^{-1} \sum_{\{\mathcal{C}\}} \sum_{\mathbf{rank}} P_{\mathcal{C},\mathbf{r}}\, G_\delta(\mathcal{C}),$$

where $\mathcal{C}$ represents a *contraction*, i.e. at least one power $-1-\delta$ is replaced by $-\delta$ and the P are polynomials in the external parameters. Note that positive powers should be also eliminated since, e.g.

$$G_\delta(0\,;\,1-\delta\,,\,-1-\delta\,,\,-2-\delta)$$

has always to be understood as

$$G_\delta(q_1^2\,;\,-\delta\,,\,-1-\delta\,,\,-2-\delta) + m_1^2\, G_\delta(0\,;\,-\delta\,,\,-1-\delta\,,\,-2-\delta),$$

i.e. with $\mathcal{D} = -1\,(\mathbf{mod}\,\delta)$.

Solution for C_0

– Equations:

$$\int d^n q \, \frac{\partial}{\partial q_\mu} \{ v_\mu \, [1]^{-\delta} \, [2]^{-1-\delta} \, [3]^{-1-\delta} \} = 0,$$

$$\int d^n q \, \frac{\partial}{\partial q_\mu} \{ v_\mu \, [1]^{-1-\delta} \, [2]^{-\delta} \, [3]^{-1-\delta} \} = 0$$

$$\int d^n q \, \frac{\partial}{\partial q_\mu} \{ v_\mu \, [1]^{-1-\delta} \, [2]^{-1-\delta} \, [3]^{-\delta} \} = 0$$

$$\int d^n q \, \frac{\partial}{\partial q_\mu} \{ v_\mu \, [1]^{1-\delta} \, [2]^{-1-\delta} \, [3]^{-1-\delta} \} = 0$$

$$\int d^n q \, \{ p_1 \cdot p_2 \, (p_{1\mu} \frac{\partial}{\partial p_{1\mu}} - p_{2\mu} \frac{\partial}{\partial p_{2\mu}})$$
$$+ \; p_2^2 \, p_{1\mu} \frac{\partial}{\partial p_{2\mu}} - p_1^2 \, p_{2\mu} \frac{\partial}{\partial p_{1\mu}} \} \, [1]^{-\delta} \, [2]^{-1-\delta} \, [3]^{-1-\delta} = 0,$$

$$\int d^n q \, \{ p_1 \cdot p_2 \, (p_{1\mu} \frac{\partial}{\partial p_{1\mu}} - p_{2\mu} \frac{\partial}{\partial p_{2\mu}})$$
$$+ \; p_2^2 \, p_{1\mu} \frac{\partial}{\partial p_{2\mu}} - p_1^2 \, p_{2\mu} \frac{\partial}{\partial p_{1\mu}} \} \, [1]^{-1-\delta} \, [2]^{-\delta} \, [3]^{-1-\delta} = 0,$$

$$\int d^n q \, \{ p_1 \cdot p_2 \, (p_{1\mu} \frac{\partial}{\partial p_{1\mu}} - p_{2\mu} \frac{\partial}{\partial p_{2\mu}})$$
$$+ \; p_2^2 \, p_{1\mu} \frac{\partial}{\partial p_{2\mu}} - p_1^2 \, p_{2\mu} \frac{\partial}{\partial p_{1\mu}} \} \, [1]^{-\delta} \, [2]^{-1-\delta} \, [3]^{-1-\delta} = 0,$$

$$\int d^n q \, \{ p_1 \cdot p_2 \, (p_{1\mu} \frac{\partial}{\partial p_{1\mu}} - p_{2\mu} \frac{\partial}{\partial p_{2\mu}})$$
$$+ \; p_2^2 \, p_{1\mu} \frac{\partial}{\partial p_{2\mu}} - p_1^2 \, p_{2\mu} \frac{\partial}{\partial p_{1\mu}} \} \, [1]^{-\delta} \, [2]^{-1-\delta} \, [3]^{-\delta} = 0,$$

$$\int d^n q \, \{ p_1 \cdot p_2 \, (p_{1\mu} \frac{\partial}{\partial p_{1\mu}} - p_{2\mu} \frac{\partial}{\partial p_{2\mu}})$$

$$+ \; p_2^2\, p_{1\mu}\, \frac{\partial}{\partial p_{2\mu}} - p_1^2\, p_{2\mu}\, \frac{\partial}{\partial p_{1\mu}} \} \, [1]^{1-\delta}\, [2]^{-1-\delta}\, [3]^{-1-\delta} = 0,$$

– Solution

$$C_{0\delta}(-1,-1,-1) \;=\; \delta^{-1}\, \frac{m^2}{M^2\, s\, (s + M^2 - 4\, m^2)}\, C,$$

$$\begin{aligned}
C \;=\; & M^2\, (1 - \frac{s}{m^2})\, C_{0\delta}(-1,-1,0)\, \delta \\
+ \; & \frac{sM^2}{m^2}\, C_\delta(-1,0,-1)\, \delta \\
+ \; & 2\, C_{0\delta}(1,-1,-1)\, (1 - 2\, \delta) \\
- \; & C_{0\delta}(1,0,-2)\, (\delta + 1) \\
+ \; & (2 - \frac{s}{m^2})\, C_{0\delta}(0,-2,0)\, (\delta + 1) \\
+ \; & (1 - 2\, \frac{s}{m^2})\, M^2\, C_{0\delta}(0,-1,-1)\, \delta \\
+ \; & \frac{sM^2}{m^2}\, C_{0\delta}(0,-1,-1) \\
+ \; & C_{0\delta}(0,0,-1)\, (1 - \delta)).
\end{aligned}$$

After this we only have to compute δ-integrals in parametric space with $\Sigma\, \delta_i = -2 - 3\,\delta$,

$$I(\delta_1\,,\,\delta_2\,,\,\delta_3) = \delta^{-1}\,\frac{\Gamma\,(3\,\delta)}{\Gamma^3(\delta)}$$

$$\times\,\int_0^1 dx\,\int_0^x dy\,(x-y)^{-1-\delta_1}\,y^{-1-\delta_2}\,(1-x)^{-1-\delta_3}\,\chi^{-3\,\delta}(x,y)$$

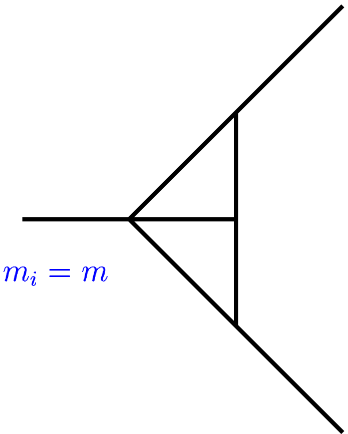
An example:

$$I(-1\,,\,-1\,,\,0) = \frac{\Gamma\,(3\,\delta)}{\delta^2\,\Gamma^3(\delta)}\,[I_{-2}\,\delta^{-2} + I_{-1}\,\delta^{-1} + I_0 + \mathcal{O}\,(\delta)],$$

$$\begin{aligned}
I_{-2} &= 1, \\
I_{-1} &= -3 - 3\,\int_0^1 dx\,\ln\chi(1,x), \\
I_0 &= -3\,\int_0^1 dx\,\int_0^x dy\,[\frac{\ln\chi(x,y)}{1-x}]_+ \\
&+ \int_0^1 dx\,[\frac{\ln(x-y)}{1-x}]_+ \\
&- 6\,\int_0^1 dx\,\ln(1-x)\,\ln\chi(1,x) - 3\,\int_0^1 dx\,\ln x\,\ln\chi(1,x) \\
&+ \frac{9}{2}\,\int_0^1 dx\,\ln^2\chi(1,x) + 9 - 2\,\zeta(2).
\end{aligned}$$

An important check on the calculation is that all poles in δ cancel in the total.

At the two-loop level the # of terms increases considerably.



$V^{221}\; m_i = m$

There are 2 **completely irreducible scalar product that we choose to be** $q_2 \cdot q_2 - q_1 \cdot p_2$ **and one** δ **-irreducible one,** q_1^2**. It follows**

$$V_\delta^{221} = \delta^{-1}\, \frac{1}{(s-4\,m^2)\,(s-m^2)}\, \sum_{i,j,k=0}^{2}$$

$$\times\; \sum_{\mathcal{C}=S}^{D} \sum_{\overline{\mathcal{C}}} P^{ijk}_{\mathcal{C}\,,\overline{\mathcal{C}}}\, V^{221}_{\mathcal{C}\,,\overline{\mathcal{C}}},$$

where

- $\mathcal{C}$ **denotes simple** (S) **and double** (D) **contractions,**
- $\overline{\mathcal{C}}$ **sums over the remaining powers such that**

$$\mathcal{D}\,[V^{221}_{\mathcal{C}\,,\overline{\mathcal{C}}}] = \epsilon\,(\mathbf{mod}\,\delta).$$

– **the remaining sum is over powers of** $q_2 \cdot q_2, q_1 \cdot p_2$ **and of** q_1^2**. Example of** D **is**

$$
\begin{aligned}
& \frac{7}{6}\, m^2\, V_\delta^{221}(0,0;0,0,-1,-1,-1) \\
- & \frac{5}{6}\, m^2\, V_\delta^{221}(0,0;0,0,-1,-1,-1)\,\delta + \frac{\delta+1}{m^2}\,[\\
+ & (-\frac{23}{6}\, sm^2 + \frac{5}{6}\, s^2 + 5\, m^4)\, V_\delta^{221}(0,0;0,0,-2,-1,-1) \\
- & \frac{1}{6}\, m^2\,(s - m^2)\, V_\delta^{221}(0,0;0,0,-1,-2,-1) \\
- & \frac{1}{6}\, m^2\,(s - m^2)\, V_\delta^{221}(0,0;0,0,-1,-1,-2) \\
- & (\frac{7}{3}\, s - 6\, m^2)\, V_\delta^{221}(0,1;0,0,-2,-1,-1) \\
+ & \frac{4}{3}\, V_\delta^{221}(0,2;0,0,-2,-1,-1) \\
- & (\frac{1}{6}\, s - m^2)\, V_\delta^{221}(1,0;0,0,-2,-1,-1) \\
- & \frac{1}{3}\, m^2\, V_\delta^{221}(1,0;0,0,-1,-2,-1) \\
- & \frac{1}{3}\, m^2\, V_\delta^{221}(1,0;0,0,-1,-1,-2) \\
+ & \frac{1}{3}\, V_\delta^{221}(1,1;0,0,-2,-1,-1)]
\end{aligned}
$$

with a grand total of 103 terms therefore proving a sort of unspoken law

Any algorithm aimed to *reduce* the analytical complexity of a multi - loop Feynman diagrams is generally bound to

- replace the original integral with a sum of many simpler diagrams,
- introducing denominators that show zeros.

An algorithm is optimal when

- there is a minimal number of terms,
- zeros of denominators correspond to solutions of Landau equations and
- the nature of the singularities is not badly over-estimated.

Index Saturation is not Reduction

In any realistic calculation one has to get rid of indices and to perform algebraic reduction of obvious. Still, this is not yet getting rid of tensor reduction.

Example: $V \to \bar{f} f$ vertex

Instead of immediately indroducing

$$
\begin{aligned}
& V^{\alpha\beta\gamma}(\mu\,|\,,0\,;\,p_-,p_+,m_1,\cdots,m_I) = \\
& V^{\alpha\beta\gamma}_{11}(p_-,p_+,m_1,\cdots,m_I)\,p_{1\mu} + \\
& V^{\alpha\beta\gamma}_{12}(p_-,p_+,m_1,\cdots,m_I)\,p_{2\mu},
\end{aligned}
$$

etc, we first decompose the vertex into

$$
\begin{aligned}
\mathcal{M} &= \bar{v}(p_-)\,\epsilon(P_+)\cdot V\,u(p_+) \\
&= \bar{v}(p_-)\,[F_S\,P_-\cdot\epsilon(P_+) + F_V\,\not{\epsilon}(P_+) + F_A\,\not{\epsilon}(P_+)\,\gamma_5]\,u(p_+)
\end{aligned}
$$

and introduce projectors such that all form factors are extracted

$$
\sum_{\text{spin}} P_I\,\mathcal{M} = F_I,
$$

with solution

$$P_V = -\frac{1}{2\,(2-n)\,s}\,[\bar{u}(p_+)\,\not{\epsilon}(P_+)\,v(p_-) - 2\,i\,\frac{m}{s-4\,m^2}\,\not{P}_-\epsilon(P_+)\,\bar{u}(p_+)\,v(p_-)],$$

etc, giving

$$F_V = \mathbf{Tr}\,\mathcal{F}_V,$$

$$\mathcal{F}_V = \frac{1}{(2-n)\,s}\,[\frac{m}{s-4\,m^2}\,P_-\cdot V - \frac{1}{2}\,\not{V}]\,(-i\,\not{p}_+ + m)\,(i\,\not{p}_- + m).$$

Index saturation allows us to consider only integrals with irreducible scalar products. Then reduction follows.

Define vector integrals for the S^{111}-family, where we have $5-3$ irreducible scalar products,

$$S^{111}(\mu\,|\,0\,;\,p,m_1,m_2,m_3) = \frac{\mu^{2\epsilon}}{\pi^4}\int d^nq_1\,d^nq_2 \times \frac{q_{1\mu}}{(q_1^2+m_1^2)((q_1-q_2+p)^2+m_2^2)(q_2^2+m_3^2)}.$$

we obtain

$$
\begin{aligned}
S^{111}(0\,|\,0\,;\,p,\{m\}) &= S^{111}_0, \\
S^{111}(\mu\,|\,0\,;\,p,\{m\}) &= S^{111}_1\,p_\mu \\
S^{111}(0\,|\,\mu\,;\,p,\{m\}) &= S^{111}_2\,p_\mu,
\end{aligned}
$$

$$
\begin{aligned}
S^{111}_i &= (\frac{\mu^2}{\pi})^\epsilon\,\Gamma\,(\epsilon-1)\,\int_0^1 dx\,\int_0^1 dy\,[x\,(1-x)]^{-\epsilon/2}\,y^{\epsilon/2-1}, \\
&\times\; P_i(x,y)\,\chi_E^{1-\epsilon}(x,y),
\end{aligned}
$$

where we have polynomials

$$
P_0 = -1, \qquad P_1 = x\,(1-y), \qquad P_2 = -\,y.
$$

– **Solution I: solve the integrals as they stand;**

– **Solution II: reduce them.**

If we define generalized functions

$$
\begin{aligned}
&\mathcal{S}^{\alpha_1\alpha_3\alpha_2}(n\,;\,p,m_1,m_2,m_3) \\
&= \int d^nq_1\,d^nq_2\,\prod_{i=1}^{3}\,[i]^{-\alpha_i},
\end{aligned}
$$

with arbitrary space-time dimension n and select $n=\Sigma_i\,\alpha_1+1-\epsilon$ we obtain

$$\mathcal{S}^{\alpha_1\alpha_3\alpha_2}(n\,;\,p,m_1,m_2,m_3) = -\,\frac{\Gamma\left(\epsilon-1\right)}{\Pi_i\;\Gamma\left(\alpha_i\right)}$$

$$\times\;\int_0^1 dx\,\int_0^1 dy\,(1-x)^{-(1-\alpha_1+\alpha_2+\alpha_3+\epsilon)/2}$$

$$\times\;x^{-(1+\alpha_1+\alpha_2-\alpha_3+\epsilon)/2}\,(1-y)^{\alpha_2-1}\,y^{(\alpha_1-\alpha_2+\alpha_3-3+\epsilon)/2}\,\chi_E^{1-\epsilon}(x,y).$$

Obviously a scalar integral in shifted space-time dimension and with arbitrary powers of propagators is exactly as the scalar integral in $4-\epsilon$ dimensions with all powers equal to one and with Feynman parameters in the numerator.

We will write

$$\mathcal{S}_i^{111} = \sum_{j=1}^{3} k_{ij}\,\mathcal{S}^{\alpha_j,\beta_j\,\gamma_j}(n_{ij}\,;\,p,m_1,m_2,m_3),$$

and fix coefficients and exponents in order to match. A solution is therefore given by

$$\alpha_1 = 2,\qquad \beta_1 = 2,\qquad \gamma_1 = 1,$$
$$\alpha_2 = 2,\qquad \beta_2 = 1,\qquad \gamma_2 = 2,$$
$$\alpha_1 = 1,\qquad \beta_1 = 1,\qquad \gamma_1 = 1,$$

$$k_{11} = \frac{1}{2},\qquad k_{12} = \frac{1}{6},\qquad k_{13} = -1,$$
$$k_{21} = 0,\qquad k_{22} = \frac{1}{6},\qquad k_{23} = 0.$$

Terefore one can adopt a reduction to scalar integrals in shifted dimensions, followed by a solution of the recursion relations, or to relate the form factors to truly scalar integrals in the same number of dimensions plus integrals with contracted and irreducible numerators. The two procedures are algebraically equivalent.

Summary:

Any Feynman integral, independently from its tensorial structure, can be written as a combination of

$$G_{\mu_1 \cdots \mu_N} = \sum_{i=1}^{i_{\max}} G^i_S \, F_{i\,;\,\mu_1 \cdots \mu_N},$$

where the Fs are tensorial structures made of

- **external momenta,**
- **Kroneker's delta-functions,**
- **elements of the Dirac algebra.**

□ **the scalar projections G^i_S admit a parametric representation which differ from the one in the original scalar diagram (S) only because of polynomials of Feynman parameters in the numerator.**

- **Once we have an integral representation for S with the property of**

$$S \;=\; \int_S dx\, \mathcal{G}(x) = \frac{1}{B_G} \int_S dx\, \mathcal{F}\{\mathcal{G}(x)\},$$

analogous representations, with the same properties, follow also for the G^i_S. Therefore, from the point of view of numerical evaluation, there is really little difference between scalar and tensor integrals.

However, there is a problem which arises as a consequence of the fact that we are dealing with gauge theories with inherent gauge cancellations.

Consider the one-loop photon self-energy in QED and express the result in terms of scalar one-loop form factors

$$\Pi^f_{\mu\nu} \;=\; \Pi^f_1\, \delta_{\mu\nu} + \Pi^f_2\, p_\mu p_\nu,$$

$$\begin{aligned}
\Pi^f_0 \;&=\; -4\, e^2 \left[(2-n)\, B_{22}(p^2\,;\, m_f, m_f)\right.\\
&-\; p^2\, B_{21}(p^2\,;\, m_f, m_f) - p^2\, B_1(p^2\,;\, m_f, m_f)\\
&-\; \left. m^2_f\, B_0(p^2\,;\, m_f, m_f)\right],\\
\Pi^f_1 \;&=\; -8\, e^2 [B_{21}(p^2\,;\, m_f, m_f)\\
&+\; B_1(p^2\,;\, m_f, m_f)].
\end{aligned}$$

Gauge invariance of the theory is controlled by a set of **Ward identities** among which one requires $\Pi_{\mu\nu}$ to be transverse and

– this hardly follows from expressing the form factors in parametric space followed by some numerical integration.

– Rather, it follows from a set of identities that one can write among the form factors directly in momentum space, the so-called procedure of scalarization.

The same procedure is plagued by the occurence of inverse powers of Gram's determinants whose zeros are unphysical but sometimes dangerous for the numerical stability.

There is another place where gauge cancellations play a crucial role: the expected gauge-parameter independence is seen at the level of S-matrix elements and not for individual contributions to Green's functions. From this point of view any procedure that computes single diagrams and sums the corresponding numerical results, without controlling gauge cancellations analytically, is bound to have its own troubles.

Procedure:

- impose Ward - Slavnov - Taylor identities (WSTI) and see that they are satisfied,
- at this point organize the calculation according to building blocks that are, by construction, gauge-parameter independent.

The first step requires some form of scalarization which is only needed to prove that certain combinations of form factors are zero, therefore any occurence of *denominators* do not pose a problem.

In the second step we need to control the ξ behavior of individual Green's functions; the tool is represented by the use of Nielsen's identities. Typically we will consider the transverse propagator of a gauge field:

$$D_{\mu\nu} = \frac{1}{(2\pi)^4 i} \frac{\delta_{\mu\nu} + p_\mu p_\nu / s}{s - M_0^2 - \Pi(\xi, s)},$$

where $p^2 = -s$, M_0 is the bare mass and ξ is a generic gauge parameter. The NI is

$$\frac{\partial}{\partial \xi} \Pi(\xi, s) = \Lambda(\xi, s) \, \Pi(\xi, s),$$

where Λ is a complex, amputated, 1**PI**, two-point Green function. Let us introduce the complex pole defined by

$$\bar{s} - M^2 + 0 - \Pi(\xi, \bar{s}) = 0, \qquad \partial_\xi \, \bar{s} \; = \; 0.$$

Let us consider now the amplitude for $i \to V \to f$ where V is an unstable gauge boson and i/f are initial/final states. The overall amplitude becomes

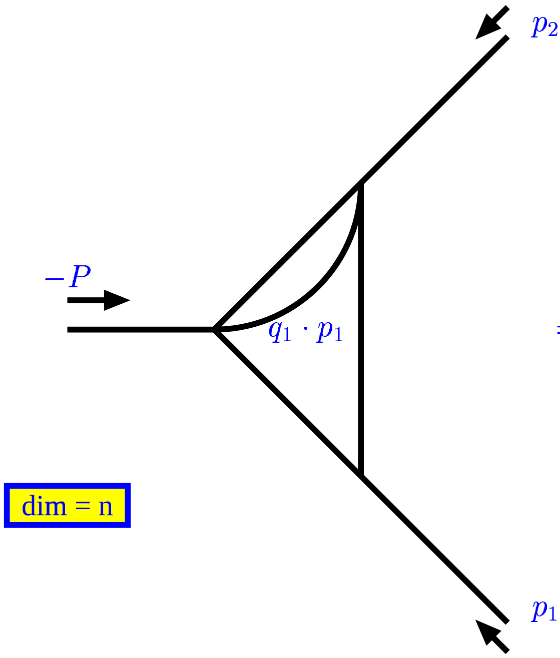
$$A_{fi}(s) \; = \; \frac{d_{\mu\nu}}{s - \bar{s}} \, \frac{V_f^\mu(\bar{s}) \, V_i^\nu(\bar{s})}{1 - \Pi'(\bar{s})} + \quad \textbf{non-resonant},$$

where it is understood that the V's include wave-function renormalization factors for the external, on-shell, particles. It has been proved that

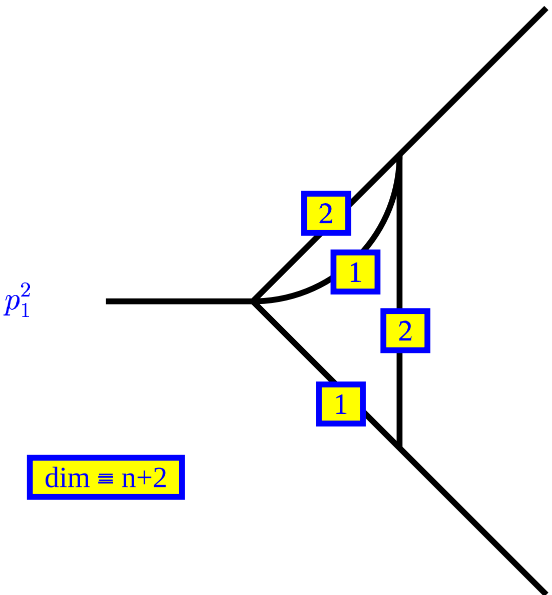
$$\frac{d}{d\,\xi} \left[1 - \Pi'(\bar{s})\right]^{-1/2} V_f^\mu(\bar{s}) \; = \; 0,$$

etc, and this combination is the prototype of one of the gauge-parameter independent building blocks that are needed in assembling our calculation for some physical observables. All gauge-parameter independent blocks will then be mapped into one (multi-dimensional) integral to be evaluated numerically.

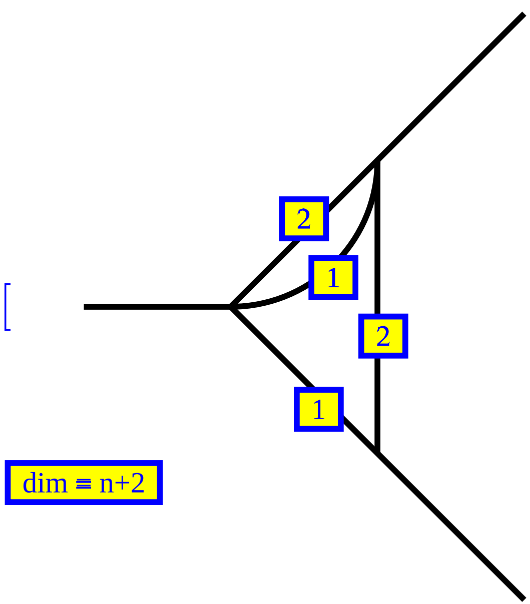
Examples of *scalarization*



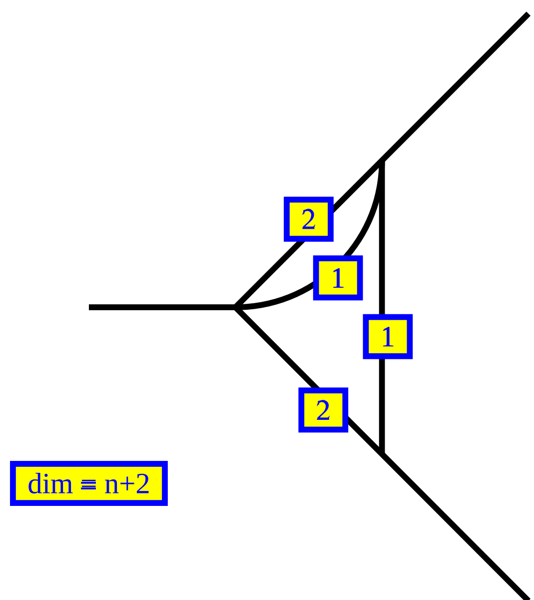
$=$ p_1^2



$p_1 \cdot p_2$ [



$+$



]

or

As usual the still is the problem of reducing all these integrals to MI. There is another solution; given

$$\mathcal{V}^{121}(\mu\,|\,,0\,;\,p_2,P,m_1,\cdots,m_4) = \mathcal{V}^{121}_{11}(p_2,P,m_1,\cdots,m_4)\,p_{1\mu} + \mathcal{V}^{121}_{12}(p_2,P,m_1,\cdots,m_4)\,p_{2\mu},$$
$$\mathcal{V}^{121}(0\,|\,\mu\,;\,p_2,P,m_1,\cdots,m_4) = \mathcal{V}^{121}_{21}(p_2,P,m_1,\cdots,m_4)\,p_{1\mu} + \mathcal{V}^{121}_{22}(p_2,P,m_1,\cdots,m_4)\,p_{2\mu}.$$

let us start with $\mathcal{V}^{121}(\mu,0)$, where

$$\int d^nq_1\,\frac{q_{1\mu}}{[1][2]} = X\,q_{2\mu},$$

and where X by standard methods of reduction is computed to be

$$X = \frac{1}{2}\int d^n_q 1\,\{\frac{1}{[1][2]} + \frac{1}{[0]}\,[\frac{m_2^2-m_1^2}{[1][2]} - \frac{1}{[1]} + \frac{1}{[2]}]\},$$

with $[0]=q_2^2$. As a consequence of this result we obtain

$$\mathcal{V}^{121}(\mu\,|\,,0\,;\,p_2,P,m_1,\cdots,m_4) = \frac{1}{2}\,(m_2^2-m_1^2)\,\mathcal{V}^{131}(0\,|\,\mu\,;\,p_2,P,m_1,m_2,0,m_3,m_3)$$

$$
\begin{aligned}
&+ \ \frac{1}{2}\,\mathcal{V}^{121}(0\,|\,\mu\,;\,p_2,P,m_1,\cdots,m_4)\\
&+ \ \frac{1}{2}\,C_\mu(p_2,P;0,m3,m4)\,[A_0(m_2)-A_0(m_1)],
\end{aligned}
$$

so that the $q_{1\mu}$ vector integral is related to the $q_{2\mu}$ vector integral of two families. For the V^{121}-family we only have partial reducibility:

$$
\begin{aligned}
&\mathcal{V}^{121}(0\,|\,p_1\,;\,p_2,P,m_1,\cdots,m_4)\\
&= \ \frac{1}{2}\,(m_3^2-m_4^2-p_1^2-2\,p_1\cdot p_2)\,V_0^{121}(p_2,P,m_1,m_2,m_3,m_4)\\
&+ \ \frac{1}{2}\,S_0^{111}(p_2,m_1,m_2,m_3)-\frac{1}{2}\,S_0^{111}(P,m_1,m_2,m_4).
\end{aligned}
$$

Thus we can write

$$
\begin{aligned}
&\mathcal{V}^{121}(0\,|\,p_1\,;\,p_2,P,m_1,\cdots,m_4)\\
&= \ p_1^2\,V_{21}^{121}-p_1\cdot P\,\mathcal{V}^{2|1+1|2}(6-\epsilon)+\mathcal{V}_0^{121}(p_2,P,m_1,\cdots,m_4).
\end{aligned}
$$

Assuming that $p_1^2\neq 0$ we can eliminate one component, although we can evaluate $\mathcal{V}^{121}(0,p_2)$ directly in $n=4-\epsilon$ dimensions.