

Do we need NLO SMEFT?

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Yes, we do...

virtual, undercover, unequivocal, abridged, c. June, 2015



This short note is about **why NLO SMEFT¹**, it is not

X how NLO²

X what NLO³

however, see backup material [▶ go now](#)

X why POs⁴

fuel for discussion ... nothing more

¹What can be said at all should be said clearly and whereof one cannot speak thereof one must be silent

²Covered in “ATLAS Higgs (N)NLO MC and Tools Workshop for LHC RUN-2”,
<https://indico.cern.ch/event/345455/>, see also <https://indico.desy.de/conferenceDisplay.py?confId=476>

³same as above

⁴Covered in “Pseudo-observables: from LEP to LHC”, <https://indico.cern.ch/event/373667/>



No NP yet?

A study of SM-deviations: here the reference process is

$$H \rightarrow \gamma\gamma$$

✓ κ -approach: write the amplitude as

$$A = \sum_{i=t,b,w} \kappa_i \mathcal{A}^i + \kappa_c$$

\mathcal{A}^t being the SM t -loop etc. The contact term (which is the LO SMEFT) is given by κ_c . Furthermore

$$\kappa_i = 1 + \Delta\kappa_i \quad i \neq c$$

✓ For the sake of simplicity assume

$$\kappa_b = \kappa_w = 1 \quad \left(\kappa_w^{\text{exp}} = 0.95_{-0.13}^{+0.14} \text{ATLAS } 0.96_{-16}^{+35} \text{CMS} \right)$$

and compute


$$\kappa_\gamma \mapsto R = \Gamma(\kappa_t, \kappa_c) / \Gamma_{\text{SM}} - 1 \quad [\%]$$

In LO SMEFT κ_c is non-zero and $\kappa_t = 1$ ⁵. You measure a deviation and you get a value for κ_c . However, at NLO $\Delta\kappa_t$ is non zero and you get a degeneracy. The interpretation in terms of κ_c^{LO} or in terms of $\{\kappa_c^{\text{NLO}}, \Delta\kappa_t^{\text{NLO}}\}$ could be rather different (suggested by Michael).

⁵Certainly true in the linear realization

Fitting is not interpreting

Of course, depending on what you measure, the corresponding interpretation could tell us that the required kappas or Wilson coefficients are too large to allow for a meaningful interpretation in terms of a weakly coupled UV completion⁶



Caveat: SMEFT interpretation should include LO SMEFT and (at least) RGE modified predictions ([arXiv:1301.2588](https://arxiv.org/abs/1301.2588)); furthermore, full one-loop SMEFT gives you (new) logarithmic and constant terms that are not small compared to the one from RGE, see [arXiv:1505.02646](https://arxiv.org/abs/1505.02646), [arXiv:1505.03706](https://arxiv.org/abs/1505.03706)



For interpretations other than weakly coupled renormalizable, see [arXiv:1305.0017](https://arxiv.org/abs/1305.0017)

EFT purist: there is no model independent EFT statement on some operators being big and other small ([arXiv:1305.0017](https://arxiv.org/abs/1305.0017))

⁶Simpler theories are preferable to more complex ones because they are better testable and falsifiable

Going interpretational

$$\mathbf{A}_{\text{SMEFT}} = \frac{g^2 s_\theta^2}{8\pi^2} \left[\sum_{i=t,b,w} \kappa_i \mathcal{A}^i + \frac{g_6}{g^2 s_\theta^2} \frac{M_H^2}{M_W^2} 8\pi^2 \mathbf{a}_{AA} \right]$$

- ✓ **Assumption:** use [arXiv:1505.03706](https://arxiv.org/abs/1505.03706), work in the Einhorn-Wudka PTG scenario ([arXiv:1307.0478](https://arxiv.org/abs/1307.0478)), adopt Warsaw basis ([arXiv:1008.4884](https://arxiv.org/abs/1008.4884))
- ① LO SMEFT: $\kappa_i = 1$ and \mathbf{a}_{AA} is scaled by $1/16\pi^2$ being LG
- ② NLO PTG-SMEFT: $\kappa_i \neq 1$ but only PTG operators inserted in loops (non-factorizable terms absent), \mathbf{a}_{AA} scaled as above

At NLO, $\Delta\kappa = \mathbf{g}_6 \boldsymbol{\rho}$ and $\mathbf{a}_{AA} = s_\theta^2 \mathbf{a}_{\phi W} + c_\theta^2 \mathbf{a}_{\phi B} + s_\theta c_\theta \mathbf{a}_{\phi WB}$

$$\mathcal{A}_{\text{SMEFT}} = \sum_{i=t,b,w} (1 + \mathbf{g}_6 \boldsymbol{\rho}_i) \mathcal{A}^i + \mathbf{g}_c \mathbf{a}_{AA}$$

$$g_6^{-1} = \sqrt{2} G_F \Lambda^2$$

$$g_c = \frac{1}{2} \frac{g_6}{g^2 s_\theta^2} \frac{M_H^2}{M_W^2}$$

$$\rho_t = -\frac{1}{2} \left[a_{\phi D} - 2 s_\theta^2 (a_{t\phi} + 2 a_{\phi\Box}) \right] \frac{1}{s_\theta^2}$$

$$\rho_b = -\frac{1}{2} \left[a_{\phi D} + 2 s_\theta^2 (a_{b\phi} - 2 a_{\phi\Box}) \right] \frac{1}{s_\theta^2}$$

$$\rho_W = -\frac{1}{2} \left[a_{\phi D} - 4 s_\theta^2 a_{\phi\Box} \right] \frac{1}{s_\theta^2}$$

$$\Gamma_{\text{SMEFT}} = \frac{\alpha^2 G_F M_H^3}{32 \sqrt{2} \pi^3} \frac{M_W^4}{M_H^4} \left| \mathcal{A}_{\text{SMEFT}} \right|^2 \quad \Gamma_{\text{SM}} = \Gamma_{\text{SMEFT}} \Big|_{\Delta\kappa_j=0, \kappa_c=0}$$



Relaxing the PTG assumption introduces non-factorizable sub-amplitudes proportional to

$a_{tW}, a_{tB}, a_{bW}, a_{bB}, a_{\phi W}, a_{\phi B}, a_{\phi WB}$ with a mixing among $\{a_{\phi W}, a_{\phi B}, a_{\phi WB}\}$. Meanwhile, renormalization has made one-loop SMEFT finite, e.g. in the G_F -scheme, with a residual μ_R -dependence

Appendix C. Dimension-Six Basis Operators for the SM²².

X^3 (LG)		φ^6 and $\varphi^4 D^2$ (PTG)		$\psi^2 \varphi^3$ (PTG)	
Q_G	$f^{ABC} G_\mu^{Av} G_\nu^{B\rho} G_\rho^{C\mu}$	Q_φ	$(\varphi^\dagger \varphi)^3$	$Q_{e\varphi}$	$(\varphi^\dagger \varphi)(\bar{l}_p e_r \varphi)$
$Q_{\tilde{G}}$	$f^{ABC} \tilde{G}_\mu^{Av} G_\nu^{B\rho} G_\rho^{C\mu}$	$Q_{\varphi\Box}$	$(\varphi^\dagger \varphi)\Box(\varphi^\dagger \varphi)$	$Q_{u\varphi}$	$(\varphi^\dagger \varphi)(\bar{q}_p u_r \tilde{\varphi})$
Q_W	$\varepsilon^{IJK} W_\mu^{I\nu} W_\nu^{J\rho} W_\rho^{K\mu}$	$Q_{\varphi D}$	$(\varphi^\dagger D^\mu \varphi)^* (\varphi^\dagger D_\mu \varphi)$	$Q_{d\varphi}$	$(\varphi^\dagger \varphi)(\bar{q}_p d_r \varphi)$
$Q_{\tilde{W}}$	$\varepsilon^{IJK} \tilde{W}_\mu^{I\nu} W_\nu^{J\rho} W_\rho^{K\mu}$				
$X^2 \varphi^2$ (LG)		$\psi^2 X \varphi$ (LG)		$\psi^2 \varphi^2 D$ (PTG)	
$Q_{\varphi G}$	$\varphi^\dagger \varphi G_{\mu\nu}^A G^{A\mu\nu}$	Q_{eW}	$(\bar{l}_p \sigma^{\mu\nu} e_r) \tau^I \varphi W_{\mu\nu}^I$	$Q_{\varphi l}^{(1)}$	$(\varphi^\dagger i \overset{\leftrightarrow}{D}_\mu \varphi)(\bar{l}_p \gamma^\mu l_r)$
$Q_{\varphi \tilde{G}}$	$\varphi^\dagger \varphi \tilde{G}_{\mu\nu}^A G^{A\mu\nu}$	Q_{eB}	$(\bar{l}_p \sigma^{\mu\nu} e_r) \varphi B_{\mu\nu}$	$Q_{\varphi l}^{(3)}$	$(\varphi^\dagger i \overset{\leftrightarrow}{D}_\mu^I \varphi)(\bar{l}_p \tau^I \gamma^\mu l_r)$
$Q_{\varphi W}$	$\varphi^\dagger \varphi W_{\mu\nu}^I W^{I\mu\nu}$	Q_{uG}	$(\bar{q}_p \sigma^{\mu\nu} T^A u_r) \tilde{\varphi} G_{\mu\nu}^A$	$Q_{\varphi e}$	$(\varphi^\dagger i \overset{\leftrightarrow}{D}_\mu \varphi)(\bar{e}_p \gamma^\mu e_r)$
$Q_{\varphi \tilde{W}}$	$\varphi^\dagger \varphi \tilde{W}_{\mu\nu}^I W^{I\mu\nu}$	Q_{uW}	$(\bar{q}_p \sigma^{\mu\nu} u_r) \tau^I \tilde{\varphi} W_{\mu\nu}^I$	$Q_{\varphi q}^{(1)}$	$(\varphi^\dagger i \overset{\leftrightarrow}{D}_\mu \varphi)(\bar{q}_p \gamma^\mu q_r)$
$Q_{\varphi B}$	$\varphi^\dagger \varphi B_{\mu\nu} B^{\mu\nu}$	Q_{uB}	$(\bar{q}_p \sigma^{\mu\nu} u_r) \tilde{\varphi} B_{\mu\nu}$	$Q_{\varphi q}^{(3)}$	$(\varphi^\dagger i \overset{\leftrightarrow}{D}_\mu^I \varphi)(\bar{q}_p \tau^I \gamma^\mu q_r)$
$Q_{\varphi \tilde{B}}$	$\varphi^\dagger \varphi \tilde{B}_{\mu\nu} B^{\mu\nu}$	Q_{dG}	$(\bar{q}_p \sigma^{\mu\nu} T^A d_r) \varphi G_{\mu\nu}^A$	$Q_{\varphi u}$	$(\varphi^\dagger i \overset{\leftrightarrow}{D}_\mu \varphi)(\bar{u}_p \gamma^\mu u_r)$
$Q_{\varphi WB}$	$\varphi^\dagger \tau^I \varphi W_{\mu\nu}^I B^{\mu\nu}$	Q_{dW}	$(\bar{q}_p \sigma^{\mu\nu} d_r) \tau^I \varphi W_{\mu\nu}^I$	$Q_{\varphi d}$	$(\varphi^\dagger i \overset{\leftrightarrow}{D}_\mu \varphi)(\bar{d}_p \gamma^\mu d_r)$
$Q_{\varphi \tilde{W}B}$	$\varphi^\dagger \tau^I \varphi \tilde{W}_{\mu\nu}^I B^{\mu\nu}$	Q_{dB}	$(\bar{q}_p \sigma^{\mu\nu} d_r) \varphi B_{\mu\nu}$	$Q_{\varphi ud}$	$i(\tilde{\varphi}^\dagger D_\mu \varphi)(\bar{u}_p \gamma^\mu d_r)$

Table C.1: Dimension-six operators other than the four-fermion ones.

²²These tables are taken from [5], by permission of the authors.

Einhorn, Wudka

Grzadkowski, Iskrzynski, Misiak, Rosiek

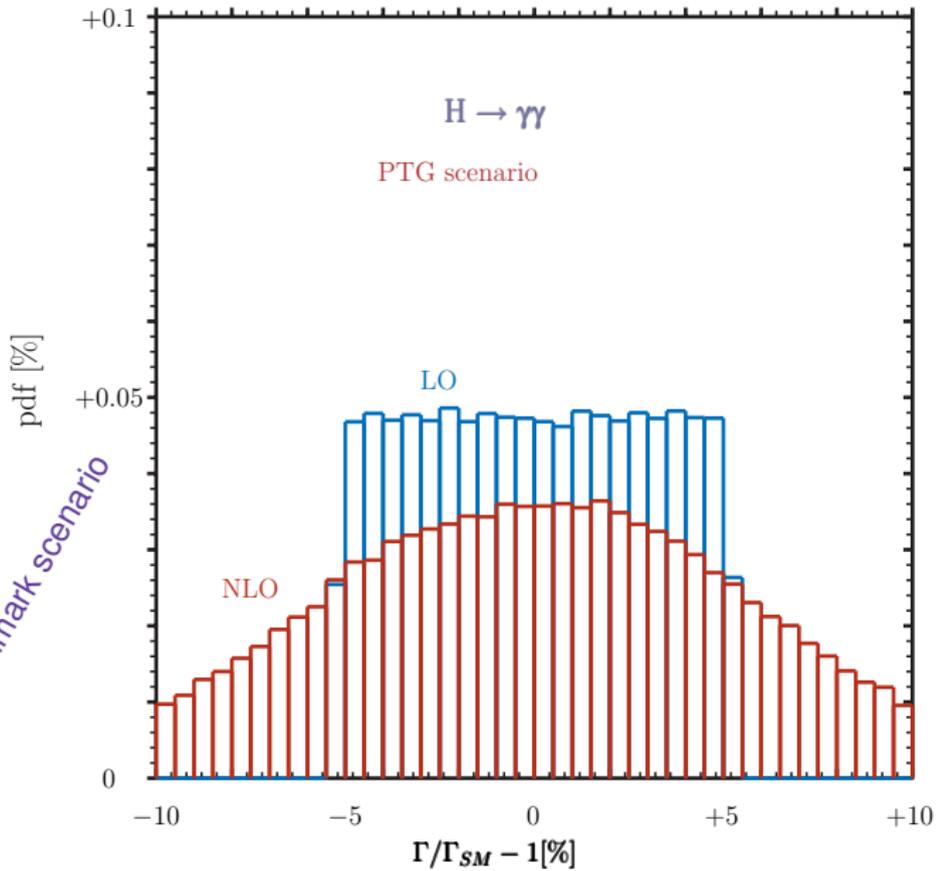
✓ Demonstration strategy:

- ① Allow each Wilson coefficient to vary in the interval $\mathbf{I}_2 = [-2, +2]$ (naturalness⁷; put $\Lambda = 3 \text{ TeV}$ (conventional point))
- ② LO: generate points from \mathbf{I}_2 for \mathbf{a}_{AA} with uniform probability and calculate \mathbf{R}_{LO}
- ② NLO: generate points from \mathbf{I}_2^5 for $\{\mathbf{a}_{\phi D}, \mathbf{a}_{\phi \square}, \mathbf{a}_{t\phi}, \mathbf{a}_{b\phi}, \mathbf{a}_{AA}\}$ with uniform probability and calculate \mathbf{R}_{NLO}
- ③ Calculate the \mathbf{R} pdf

N.B. $|\mathbf{a}_{AA}| < 1$ is equivalent to $|\mathbf{g}_c \mathbf{a}_{AA}| < 8.6 \cdot 10^{-2}$

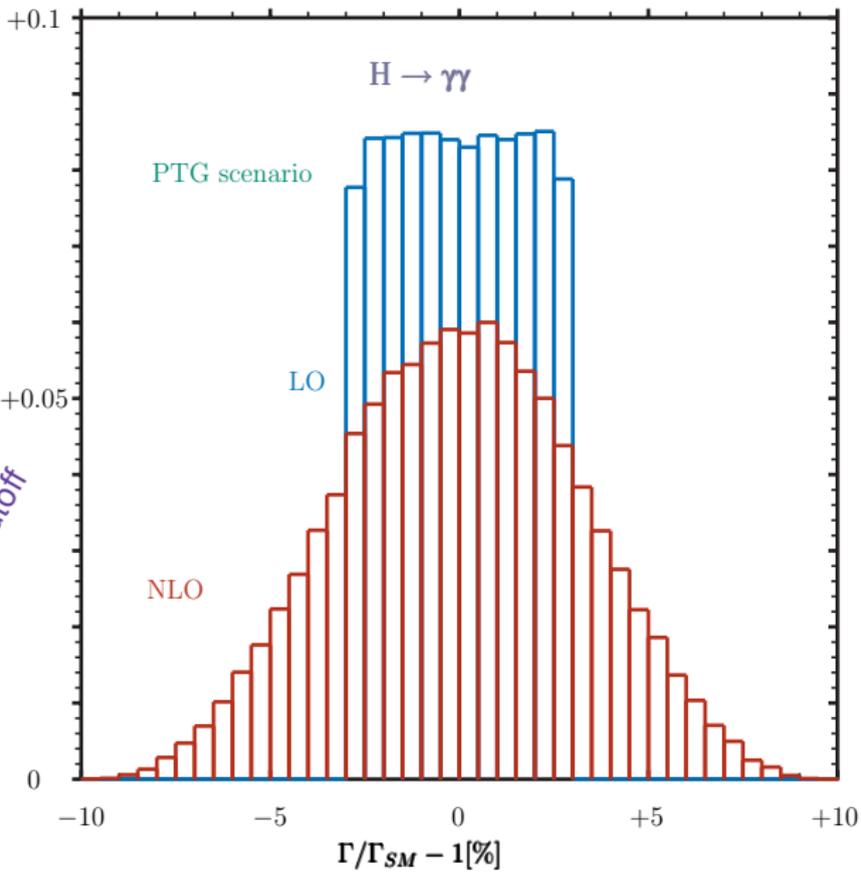
⁷Disregarding TH bias for the sign (Sect. D of [arXiv:0907.5413](https://arxiv.org/abs/0907.5413))

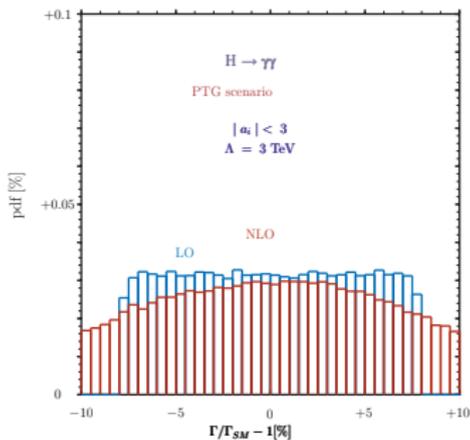
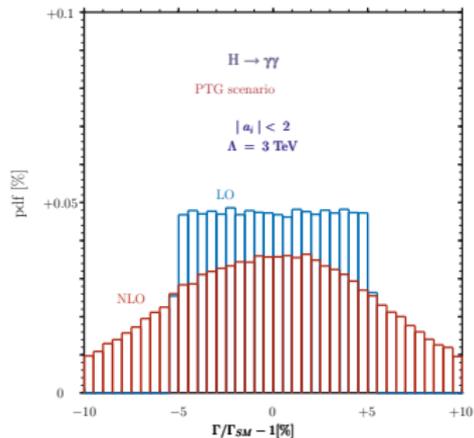
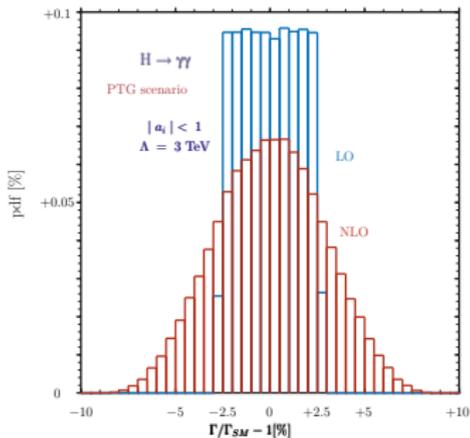
Benchmark scenario



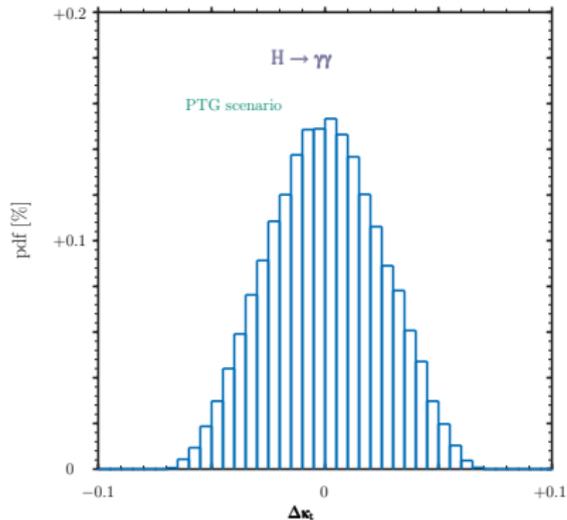
$\Lambda = 4 \text{ TeV}$

Changing the cutoff





Changing the interval



ATLAS $\kappa^t = 1.28 \pm 0.35$

CMS $\kappa^t = 1.60^{+0.34}_{-0.32}, 2\sigma?$

other couplings $< 10^{-2}$, MAGA note

<https://cds.cern.ch/record/2001958/files/LHCHXSWG-INT-2015-001-2.pdf>

Is κ^t the only window? Relax bounds compared to LO analysis (arXiv:1502.02570)?

Correctly define kappas? $\kappa_t^{tH} \neq \kappa_t^{H\gamma\gamma}$ etc.

From Wilson coefficients (a) to κ

$\Lambda = 3 \text{ TeV}$

$-2 \leq a_j \leq +2$

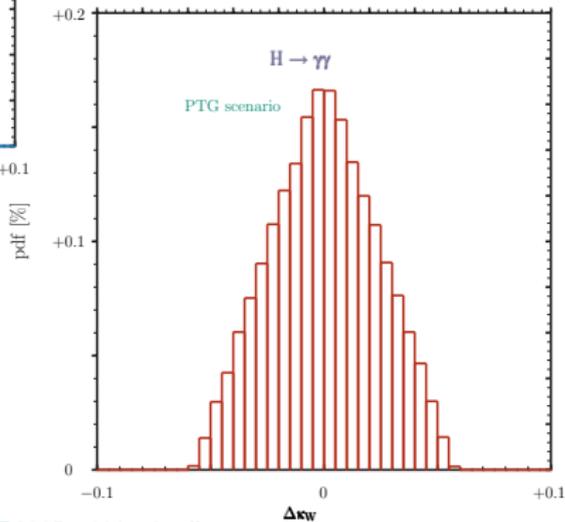
ATLAS: $\kappa_\gamma = 0.90^{+0.16}_{-0.14}$

[CONFNOTES/ATLAS-CONF-2015-007/tab-08.png](https://cds.cern.ch/record/2015007/files/CONFNOTES-2015-007-tab-08.png)

CMS: $\kappa_\gamma = 1.14^{+0.12}_{-0.13}$

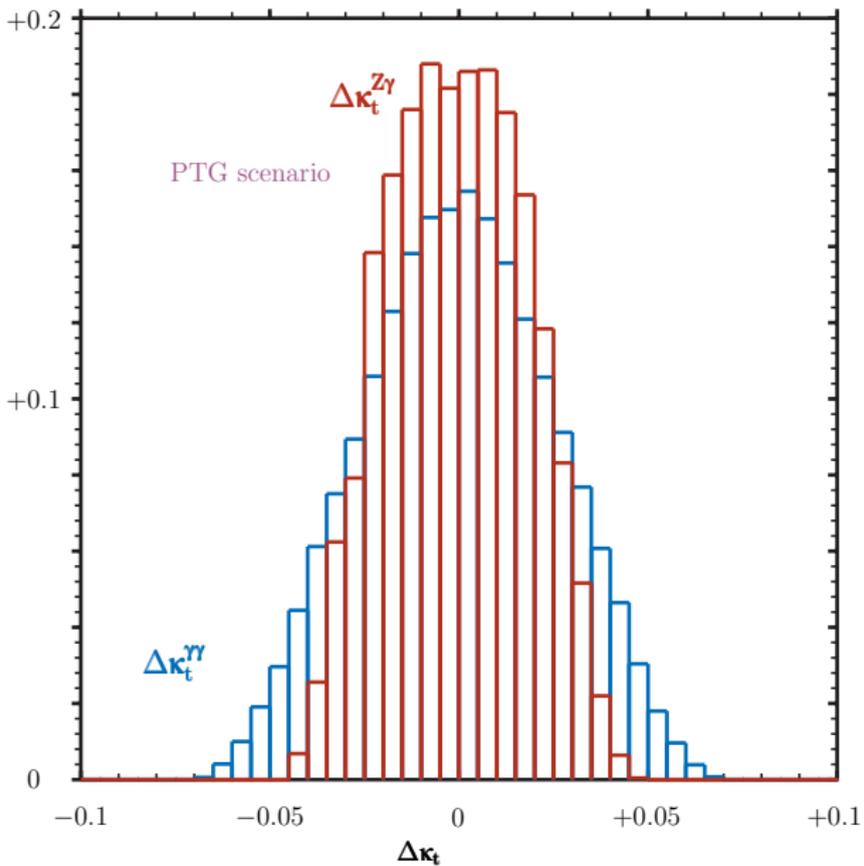
<http://arxiv.org/pdf/1412.8662.pdf>

these unc. cannot be underestimated



$$\kappa_t^{Z\gamma} \neq \kappa_t^{\gamma\gamma}$$

pdf [%]



Conclusions:

- ① For the SMEFT, (almost) regardless of the κ_C , to have more than **5%** deviation (at $\Lambda = 3 \text{ TeV}$) you have to go NLO, or unnatural⁸ (Wilson coefficients not $\mathcal{O}(1)$)
- ② The LO, NLO pdfs are different, therefore interpretation is different, how to reweight once your analysis was LO interpreted? It all depends on the new central value for κ_Y^{exp}

presently ATLAS: $a_{AA}^{\text{LO}} = +3.79^{+5.31}_{-6.06}$ CMS: $a_{AA}^{\text{LO}} = -5.31^{+4.93}_{-4.55}$
naive dimensional estimate $a_{AA} \approx 1$

- ③ *Chi ha avuto, ha avuto, ha avuto ... chi ha dato, ha dato, ha dato ...
scurdammoce o ppassato*
Those who've taken, taken, taken ... Those who've given, given, given
... Let's forget about the past

⁸from the point of view of a weakly coupled UV completion

Other than Higgs (just one example): if we neglect LG operators in loops, the following result holds for vacuum polarization:

$$\Pi_{AA}^{(\text{dim}=6)}(0) = -8 \frac{c_\theta^2}{s_\theta^2} a_{\phi D} \Pi_{AA}^{(\text{dim}=4)}(0)$$

One of the key ingredients in computing precision (pseudo-)observables is α_{QED} at the mass of the Z. Define

$$\alpha(M_Z) = \frac{\alpha(0)}{1 - \Delta\alpha^{(5)}(M_Z) - \Delta\alpha_t(M_Z) - \Delta\alpha_t^{\alpha\alpha_s}(M_Z)}$$

$$\Delta\alpha^{(5)}(M_Z) = \Delta\alpha_l(M_Z) + \Delta\alpha_{\text{had}}^{(5)}(M_Z)$$

$$\begin{aligned}
\Delta\alpha_{\text{had}}^{(5)}(M_Z) &= 0.0280398 \\
10^4 \times \Delta\alpha_1(M_Z) &= 0.0314976 \\
10^4 \times \Delta\alpha_t(M_Z) &\approx [-0.62, -0.55] \\
10^4 \times \Delta\alpha_t^{\alpha\alpha_s}(M_Z) &\approx [-0.114, -0.095]
\end{aligned}$$

The SMEFT effect is equivalent to replace

$$\Delta\alpha_1(M_Z) + \Delta\alpha_t(M_Z) \rightarrow (1 - \kappa_\alpha) [\Delta\alpha_1(M_Z) + \Delta\alpha_t(M_Z)]$$

$$\kappa_\alpha = 8g_6 \frac{c_\theta^2}{s_\theta^2} a_{\phi D} = 0.188 a_{\phi D} \quad \text{at } \Lambda = 3 \text{ TeV}$$



$$\begin{aligned}
|\kappa_\alpha \Delta\alpha_t| &> \Delta\alpha_1 \\
|\kappa_\alpha \Delta\alpha_t| &\approx |\Delta\alpha_t^{\alpha\alpha_s}|
\end{aligned}$$



interpretation: POs à la LEP

<https://indico.cern.ch/event/373667/>

arXiv:1504.04018

$$H \rightarrow \gamma\gamma (\gamma Z) \mapsto \rho_H^{\gamma\gamma(Z)} \frac{p_1 \cdot p_2 g^{\mu\nu} - p_2^\mu p_1^\nu}{M_H}$$

$$H \rightarrow VV \mapsto \rho_H^V \left(M_H g^{\mu\nu} + \frac{\mathcal{G}_L^V}{M_H} p_2^\mu p_1^\nu \right)$$

$$H \rightarrow \bar{b}b \mapsto \rho_H^b \bar{u}v$$

etc.



a middle way language
wolf, goat, and cabbage



POs (container) at LHC: summary table

- ① external layer (similar to LEP σ_f^{peak})

$$\Gamma_{\nu\nu} \quad A_{\text{FB}}^{\text{ZZ}} \quad N_{\text{off}}^{41} \quad \text{etc}$$

- ② intermediate layer (similar to LEP g_{VA}^e)

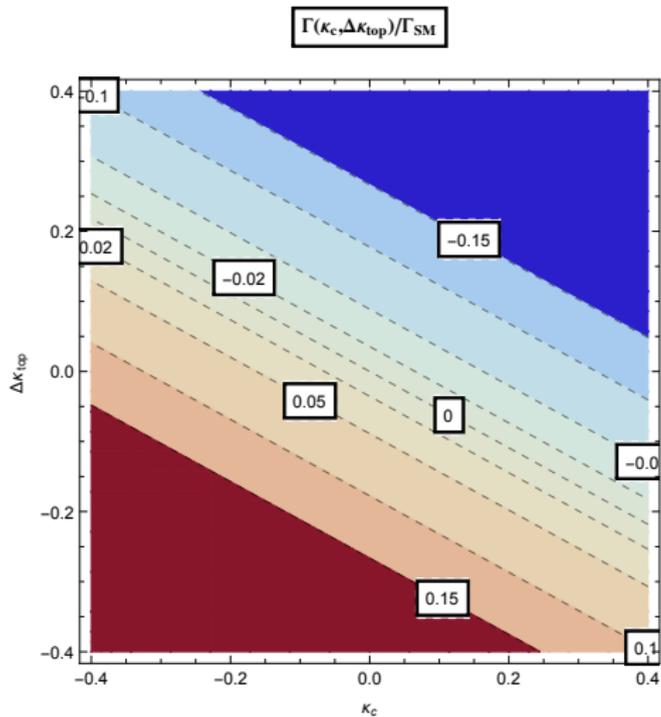
$$\rho_H^V \quad g_L^V \quad \rho_H^{\gamma\gamma}, \rho_H^{\gamma Z} \quad \rho_H^f$$

- ③ internal layer: the kappas

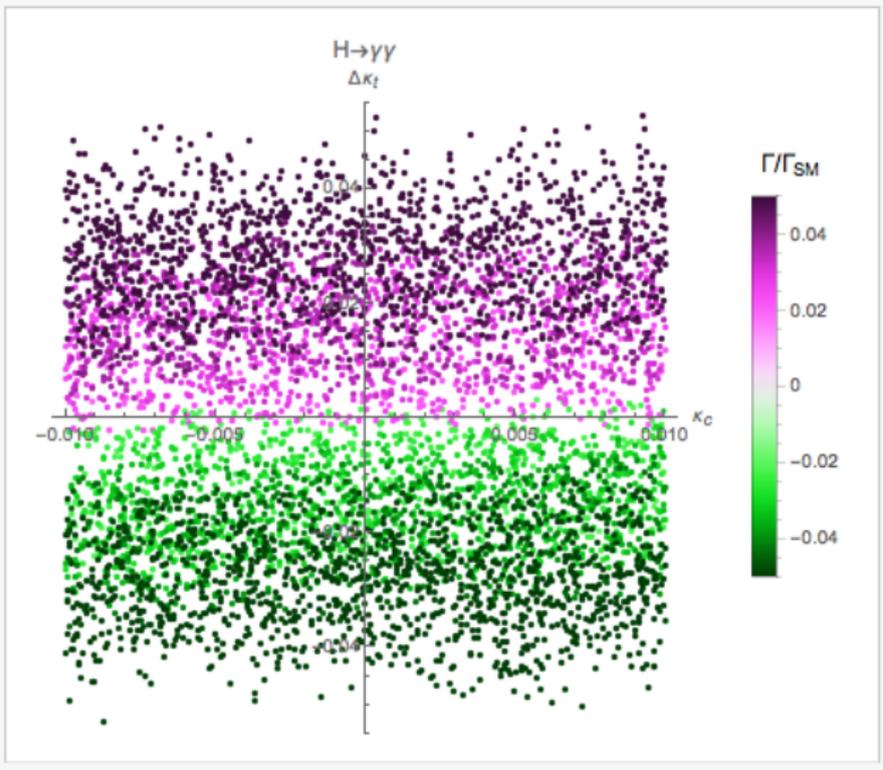
$$\kappa_f^{\gamma\gamma} \quad \kappa_W^{\gamma\gamma} \quad \kappa_i^{\gamma\gamma \text{NF}} \quad \text{etc}$$

- ④ innermost layer: Wilson coeff. or non-SM parameters in BSM (e.g. $\alpha, \beta, M_{\text{sb}}$ etc. in THDMs)

More professional plots by André



$|\Gamma/\Gamma_{SM}-1| > x$



Backup Slides (moving backward)

▶ Return

How/what NLO?

- ✓ Start with Warsaw basis, full set, write down Lagrangian and Feynman rules ■
- ✓ Normalize the quadratic part of the Lagrangian and pay due attention to the FP ghost sector ■
- ✓ Compute (all) self-energies (up to one $\mathcal{O}_{\text{dim}=6}$ insertion), write down counterterms, make self-energies UV finite ■
- ✓ Compute the set of processes you like/want (don't forget non-SM topologies), mix Wilson coefficients to make them UV finite, check closure under renormalization ■
- ✓ Perform finite renormalization, selecting a scheme (better the \mathbf{G}_F -scheme), introduce wave-function factors, get the answer ■
- ✓ Start making approximations now (if you like), e.g. neglecting operators etc. ■

How/what NLO? (cont.)

- ✓ Transform the answer in terms of κ -shifted SM sub-amplitudes and non SM factorizable sub-amplitudes ■
- ✓ Derive κ -parameters in terms of Wilson coefficients ■
- ✓ Write Pseudo-Observables in terms of κ -parameters ■
- ✓ Decide about strategy for including EWPD ■
- ✓ Claim you invented the whole procedure □

NLO is like bicycling, you learn it when you are a kid

■ Fade Out ■ Round House ■ Fast Pace □ Cooked Pistol

SMEFT evolution

LO $\mathcal{A}^{\text{SMEFT}} = \mathcal{A}^{\text{SM}} + \mathbf{a}_i$, where $\mathbf{a}_i \in \mathbf{V}_6$ and \mathbf{V}_6 is the set of $\text{dim} = 6$ Wilson coefficients

RGE $\mathbf{a}_i \rightarrow \mathbf{Z}_{ij}(\mathbf{L}) \mathbf{a}^j$, where $\mathbf{L} = \ln(\Lambda/M_{\text{H}})$ and $\mathbf{j} \in \mathbf{H}_6 \subset \mathbf{V}_6$

NLO $\mathcal{A}^{\text{SMEFT}} = \mathcal{A}^{\text{SM}} + \mathcal{A}_k(\mathbf{L}, \text{const}) \mathbf{a}^k$, where $\mathbf{k} \in \mathbf{S}_6$ and $\mathbf{H}_6 \subset \mathbf{S}_6 \subset \mathbf{V}_6$

How/what NLO? FAQ

- ✓ Are there some pieces that contain the dominant NLO effects
- ✓ It depends on the TH bias:
 - ① For EFT purists there is no model independent EFT statement on some operators being big and other small
 - ② Remember, logarithms are not large, constants matter too (see Mike Trott talk)
- ✓ which could be easily incorporated in other calculations/tools?
- ✓ Well, Well, Well, its certainly a compelling provocative exciting to think about idea

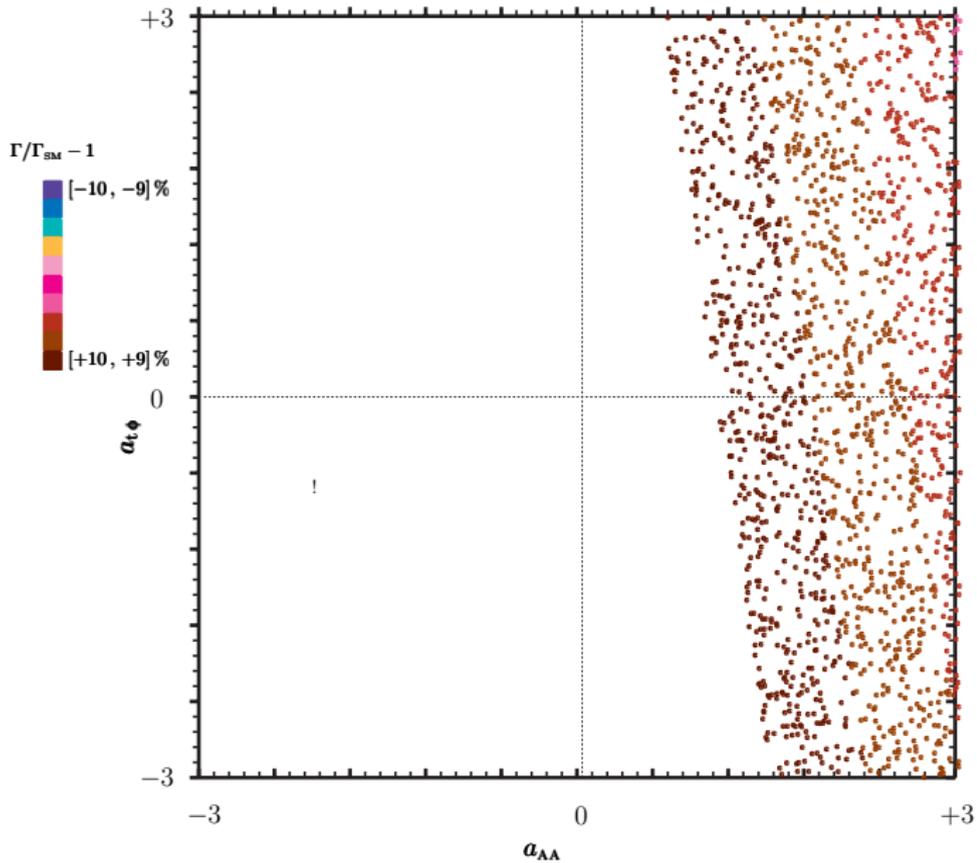
How/what NLO? FAQ

✓ NLO SMEFT availability? From [arXiv:1505.03706](https://arxiv.org/abs/1505.03706) ▶ Return

- ① Counterterms (SM fields and parameters): all
- ② Mixing: those entries related to $\mathbf{H} \rightarrow \gamma\gamma, \mathbf{Z}\gamma, \mathbf{Z}\mathbf{Z}, \mathbf{W}\mathbf{W}$
- ③ Self-energies, complete and at $\mathbf{p}^2 = \mathbf{0}$: all
- ④ Amplitudes, sub-amplitudes (both SM and non-factorizable, full PTG + LG scenario)
 - ① $\mathbf{H} \rightarrow \gamma\gamma$ ② $\mathbf{H} \rightarrow \mathbf{Z}\gamma$ ③ $\mathbf{H} \rightarrow \mathbf{Z}\mathbf{Z}, \mathbf{W}\mathbf{W}$ ⁹ ④ $\mathbf{H} \rightarrow \bar{f}f$ (the latter available, although not public)
- ⑤ EWPD, \mathbf{M}_W , \mathbf{T} -parameter; $\mathbf{Z} \rightarrow \bar{f}f$ available, although not public.

⁹Green's functions in well-defined kinematic limit, i.e. residue of the poles after extracting the parts which are 1P reducible

Backup Plots (the role of κ_w)



$\Lambda = 3 \text{ TeV}$ $\kappa_W = 0.95$ $\kappa_b = 0$



Slides for the next talk
(moving forward with work)

The κ -parameters have an extra index, specifying the process,
e.g.

$$\left(\Delta \kappa_t^{\text{HAA}}, \Delta \kappa_t^{\text{HAZ}} \right) = g_6 \begin{pmatrix} -\frac{1}{2s_\theta^2} & 4 & 2 \\ -\frac{1}{2} & 2 & 1 \end{pmatrix} \begin{pmatrix} a_{\phi D} \\ a_{\phi \square} \\ a_{t\phi} \end{pmatrix}$$

$$\left(\Delta \kappa_W^{\text{HAA}}, \Delta \kappa_W^{\text{HAZ}} \right) = g_6 \begin{pmatrix} -\frac{1}{2s_\theta^2} & 2 \\ -\frac{1}{4} \frac{1+4c_\theta^2}{c_\theta^2} & \frac{1+6c_\theta^2}{c_\theta^2} \end{pmatrix} \begin{pmatrix} a_{\phi D} \\ a_{\phi \square} \end{pmatrix}$$

There are correlations among different observables, and constraints too, e.g.

$$\begin{aligned} \Delta\kappa_b^{\text{HAZ}} - \Delta\kappa_t^{\text{HAZ}} &= \Delta\kappa_b^{\text{HAA}} - \Delta\kappa_t^{\text{HAA}} \\ c_\theta^2 \Delta\kappa_w^{\text{HAZ}} + \left(\frac{3}{2} + 2c_\theta^2\right) \Delta\kappa_t^{\text{HAZ}} &= \left(\frac{3}{2} + 2c_\theta^2\right) \Delta\kappa_t^{\text{HAA}} - \left(\frac{1}{2} + 3c_\theta^2\right) \Delta\kappa_w^{\text{HAA}} \end{aligned}$$

$$a_{t\phi} = \frac{1}{2s_{\theta}^2} a_{\phi D} - 2 a_{\phi\Box} + \Delta\kappa_t^{\text{HAA}}$$

$$a_{b\phi} = -\frac{1}{2s_{\theta}^2} a_{\phi D} + 2 a_{\phi\Box} - \Delta\kappa_b^{\text{HAA}}$$

$$a_{\phi\Box} = \frac{1}{4s_{\theta}^2} a_{\phi D} + \frac{1}{2} \Delta\kappa_w^{\text{HAA}}$$

$$2c_{\theta}^2 a_{\phi D} = s_{\theta}^2 \left(\Delta\kappa_b^{\text{HAZ}} - \Delta\kappa_b^{\text{HAA}} \right)$$