The Frontier of Multi-Loop Feynman Diagrams: (Semi-) Numerical Techniques

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Fast and efficient ways to compute multi-loop and multi-leg Feynman diagrams are discussed.

Why do we need fast and efficient ways to deal with multi-loop, multi-leg Feynman diagrams? Because computing higher order corrections is complex and the complexity is reflected by the fact that already in the Standard Model (SM) there are 11lines and 57 vertices (modulo fermion multiplicity) but a full two-loop electroweak calculation is needed for GigaZ ($10^9\, Z$ events). There is more, even for LEP physics the so-called theoretical uncertainty is not yet at a satisfactory level and also requires a full two-loop analysis of the SM (or Beyond).

The main mesage is that we have been moving from a mediaeval architecture of radiative corrections to the renaissance of LEP period and now one should not indulge in deciphering the intimate meaning of the last digit in the last fit but prepare for the new era (hopefully not baroque), although many auxiliary features are far from satisfactory, e.g. diagrammatica versus numerica, new algorithms versus standardization and the growth in complexity versus various software problems .

Analytically (quasi) solvable cases are: QCD, all masses $\equiv 0$; QED, only $m_e \neq 0$ and 0 or mass shell for external scales; SM $\oplus$ simplifying conditions, e.g. $m/m_{\rm top} \ll 1$. For the general case (arbitray int/ext masses) the analytical way (AW) is not feasible and, although a lot can be done, what is so special about it? Of course the analytical properties are explicit and expansions are available, cf.

$$\mathrm{Li}_2(z) \;=\; \sum_{n=0}^{\infty} (-1)^{n+1}\, \frac{B_n}{(n+1)\,!}\, \ln^{n+1}(1-z),$$

where the expansion parameter has the same cut of the special function itself. However, beyond one-loop and for the general mass case polylogarithms do not suffice. The AW is, in general, possible only for small number of scales; for instance the sunset topology is a sum of generalized hypergeometric series i.e. is accessible for small or large p^2 but not in the intermediate region of thresholds.

An alternative strategy is represented by the numerical way (NW). Here we summarize the main contributions available in the literature.

In ref. [1] the use of dispersion relations produces one-dimensional integral representation for two-loop self-energies. In re. [2] all diagrams are reduced to a limited number of master integrals to be solved via differential equations in p_i^2. Results are available for the sunset, the difficulty is in determining the initial conditions that are to be provided by other methods. Ref. [3] uses fully numerical integration below thresholds. Above thresholds, numerical integration is done after symmetric methods and/or hybrid methods, i.e. integrals for a few variables analytically until an

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integrable function is obtained. In the approach of ref. [4] calculations (in QCD) are performed using numerical integration over FS particles and analytical integration over virtual particles. It requires to deform some of the integration contours into the complex plane. In ref. [5] all diagrams are reduced to generalized sunsets and a one-dimensional integral representation is produced for the 10 generalized sunset master integrals needed in a renormalizable theory, with numerical integration over the external Feynman parameters.

Our conclusion is that, because of the size of the problem, full numerical calculability can hardly be avoided.

A generic (scalar) N_L - eg, n_l - loop diagram is

$$\begin{aligned} G &= \left(i\,\pi^{n/2}\right)^{n_l}\,\Gamma\,\left(N_L - \frac{n}{2}\,n_l\right) \\ &\times \int \frac{dx_G\,\delta\,(1-x_G)}{U^{n/2}\,(V-i\,\delta)^{N_L - n n_l/2}}, \\ &\equiv \int_S \prod_{i=1}^{N_l - 1} dx_i\,I_G(\{x_i\}), \\ dx_G &= \prod_{i=1}^{N_l} dx_i, \qquad x_G = \sum_{i=1}^{N_l} x_i. \end{aligned} \tag{1}$$

Note that the parametric function I_G has not a definite sign, that is poles inside the integration region that hinder the numerical integration. Example:

$$\begin{aligned} I(\alpha) &= \int_0^1 dx\,\Big[\chi_B(x) - i\,\delta\Big]^{-\alpha} \\ \chi_B(x) &= a\,(x+\frac{b}{2\,a})^2 - \frac{b^2-4\,a\,c}{4a}. \end{aligned} \tag{2}$$

When the roots of $\chi_B - i\,\delta = 0$ are real and $\in [0,1]$ the integral can be computed by distorting the integration contour unless a pinch will occur; in this case the integral is singular.

A novel numerical approach has been introduced by the TOPSIDE Collaboration [6]: the method makes a search for algorithms of smoothness that avoid spurious singularites and can be summarized in the following transformation:

$$G = \frac{1}{B_G}\int_S dx\,\mathcal{F}\Big[I_G\Big], \tag{3}$$

where $\mathcal{F}$ is a functional of I_G that makes the whole integrand smooth in parametric space (from poles to cuts) and B_G is a function of masses and momenta that contains the true singularities of G (if any).

The prototype of all algorithms of smothness is represented by the Bernstein-Tkachov (BT) theorem [7]: for any finite set of polynomials $V_i(x)$, $x = (x_1, \dots, x_N)$ there exists an identity of the following form:

$$\mathcal{P}\,(x,\partial)\prod_i V_i^{\mu_i+1}(x) \quad = \quad B\prod_i V_i^{\mu_i}(x). \tag{4}$$

where $\mathcal{P}$ is a polynomial of x, $\partial_i = \partial/\partial x_i$; B and all coefficients of $\mathcal{P}$ are polynomials of μ_i and of the coefficients of $V_i(x)$. Iterative applicationsof the functional relations, followed by integration by parts, allow us to express the integrand in parametric space as a combination of $P^N(x)\times\ln P(x)$. Let us illustrate the strategy in a simple example: fermionic self-energy in QED. First we introduce a parametric representation

$$\begin{aligned} \Sigma &= -\,ie^2\,\pi^{2-\epsilon/2}\,\mu^\epsilon\,\Gamma\left(\frac{\epsilon}{2}\right) \\ &\times \int_0^1 dx\,\Big[i\,(2-\epsilon)\,x\not{p} + (4-\epsilon)\,m_e\Big]\,\chi^{-\epsilon/2}, \\ \chi(x) &= (1-x)\,\left(x\,p^2 + m_e^2\right). \end{aligned} \tag{5}$$

Then we perform a reduction to form - factors

$$\Sigma = -\,i\pi^2 e^2\,\Big[\Sigma_p\,i\not{p} + \Sigma_m\,m_e\Big], \tag{6}$$

followed by BT - iteration and integration - by - parts

$$\begin{aligned} \Sigma_p &= \frac{1}{\bar{\varepsilon}} + \frac{p^2}{P_e^2}\,\Big[\frac{\gamma}{2}\,\frac{P_e^2}{p^2} - \frac{1}{3}\,m_e^2 - \frac{1}{6}\,p^2 \\ &+ \int_0^1 dx\,\left(\frac{1}{2} - \frac{1}{2}\,\frac{m_e^2}{p^2} - 4\,x\right)\,\chi\,\ln\frac{\chi}{\mu^2}\Big], \\ \Sigma_m &= \frac{4}{\bar{\varepsilon}} + \frac{p^2}{P_e^2}\,\Big[\left(2\,\gamma - \ln\frac{m_e^2}{\mu^2} + \frac{1}{2}\right)\,\frac{P_e^2}{p^2} \\ &+ p^2\,\ln\frac{m_e^2}{\mu^2} - \frac{2}{3}\,P_e\,\left(3\,\ln\frac{m_e^2}{\mu^2} - 2\right) \\ &- 6\int_0^1 dx\,\chi\,\ln\frac{\chi}{\mu^2}\Big], \end{aligned} \tag{7}$$

where $P_e = p^2 + m_e^2$.

The solution for the BT - polynomial $\mathcal{P}$ is known $\forall$ all quadrics, i.e. for all $1l$ & nL diagrams. However, moving to multi-loop, we see that already the sunset topology (the easiest one) requires degree = 4. Furthermore, around - threshold behavior remains unsolved in BT approach. We propose a minimal BT - approach where

$$G \;=\; \int \frac{dx_S}{B(x_S)} \int dx_L\, \mathcal{F}\Big[I_G(x_S,x_L)\Big]. \qquad (8)$$

The analytical structure of G_S (the sub-diagram with the smallest number of internal lines) is simpler, the x_S integration can be performed with contour - distortion which is always possible unless a pinch (or an end-point) singularity occur but this is a real Landau singularity. Our analysis has shown that, $\forall$ two-loop two-point functions 2l&2L and $\forall$ two-loop three-point functions 2l&3L (but one) contour - distortion is not needed because a parametrization can be found that allows for a BT - procedure with $B \equiv B(\text{parameters})$. These $2l$ diagrams are no more involved that a $1l$ diagram and require normal BT $\oplus$ integration by parts.

Clearly, numerical evaluation requires a precise knowledge of the analytical structure of the diagram, i.e. a solution of the corresponding et of Landau equations. For instance, for the scalar one-loop vertex (C_0) the leading singularity $\equiv B_3 = 0 \equiv$ anomalous threshold (AT).

We now addres the question of what to do around $B_G = 0$. Consider the simple C_0 example: if $B_3 \neq 0$, then we use $V = (x-X)^t\, H\,(x-X) + B_3$ to obtain

$$\begin{aligned} C_0 \;=\; & \frac{1}{B_3}\Big\{\frac{1}{2} + \int_0^1 dx_1\,\Big[\int_0^{x_1} dx_2\,\ln V(x_1,x_2) \\ +\; & \frac{1}{2}(X_2 - X_1)\,\ln V(x_1,x_1) \\ -\; & \frac{1}{2} X_2\,\ln V(x_1,0) \\ -\; & \frac{1}{2}(1-X_1)\,\ln V(1,x_1)\Big]\Big\}. \qquad (9) \end{aligned}$$

If $B_3 \approx 0$ and $0 < X_2 < X_1 < 1$ not fulfilled: C_0 regular, standard BT not applicable. Solution: Taylor expansion plus special BT - procedure.

If $B_3 \approx 0$ and $0 < X_2 < X_1 < 1$ fulfilled: C_0 singular, standard BT not applicable. Solution: Mellin - Barnes splitting plus sector decomposition plus BT - algorithm. For more general cases there is a tree of B coefficients: for the scalar box we obtain

$$D_0(n=4) \;=\; \frac{1}{2\,B_4}\Big\{D_0^{n=6} \oplus \sum C_0\Big\}. \qquad (10)$$

At the next iteration each reduced C_0 brings a sub-leading B_3 and the full chain $B_{\text{leading}} = B_3, B_{i,\text{subl}} = B_{3,i}$ must be treated.

To summarize the Mellin - Barnes technique we consider a simple example:

$$I(\alpha) \;=\; \int_0^1 dx\,\Big[\chi_B(x) - i\,\delta\Big]^{-\alpha}. \qquad (11)$$

$$\begin{aligned} \chi_B(x) \;&=\; -P^2\,(x-a)^2 + \frac{\lambda}{4\,P^2} - i\,\delta, \\ a \;&=\; \frac{1}{2}\,\frac{P^2 + m_2^2 - m_1^2}{P^2}. \qquad (12) \end{aligned}$$

A pinch will occur for $\lambda(-P^2, m_1^2, m_2^2) = 0$. Assume $P^2 = -s$, $s \geq 0$ and $0 \leq a \leq 1$ and obtain

$$\begin{aligned} I(\alpha) \;&=\; \Big(\frac{\rho}{s}\Big)^{\alpha} \sum_{X=a}^{1-a} \int_0^X dx\,(1+\rho\,x^2)^{-\alpha} \\ &=\; \Big(\frac{\rho}{s}\Big)^{\alpha} \sum_{X=a}^{1-a} J_X(\alpha,\rho), \qquad (13) \end{aligned}$$

where $\rho = -4\,s^2/\lambda$. We are now interested in the behavior for $|\rho| \to \infty$. Consider the case $\lambda < 0$:

$$\begin{aligned} j_X(\alpha,s) \;&=\; \int_0^{\infty} d\rho\,\rho^{s-1}\,J_X(\alpha,\rho) \\ &=\; \frac{B(s,\alpha-s)}{1-2\,s}\,X^{1-2\,s}, \qquad (14) \end{aligned}$$

which is defined for $0 < \mathrm{Re}\,s < \min\{\frac{1}{2}, \mathrm{Re}\,\alpha\}$ and $|\arg x^2| < \pi$. The inverse Mellin transform will be

$$J_X(\alpha,\rho) \;=\; \frac{1}{2\,\pi\,i} \int_{-i\,\infty}^{+i\,\infty} ds\,\rho^{-s}\,j_X(\alpha,s). \qquad (15)$$

The integration contour will be closed over the right-hand complex half-plane at infinity and leading behavior comes from the pole at $s = 1/2$,

$$J_X(\alpha,\rho) \;=\; \frac{1}{2}\,B(\frac{1}{2}, \alpha - \frac{1}{2})\,\rho^{-1/2} + \dots, \qquad (16)$$

with $1 < \rho < \infty$.

A complete project must be more ambitious: any realistic numerical strategy must be able to deal with infrared (IR) configurations, collinear configurations and, possibly, Sudakov configurations. Our numerical way to IR configurations is as follows. A necessary condition for the presence of infrared divergences is that the Landau equations are fulfilled. We regularize in n-dimensions and isolate the infrared poles by using the technique of iterated sector decomposition [8] (or defoliation). A k/n defoliation is defined by the following relation

$$\begin{aligned} & \int_0^1 \prod_{i=1}^{k} dx_i \prod_{j=k+1}^{n} dy_j\, F(\{x\}\,;\,\{y\}) \\ = & \sum_{l=1}^{k} \int_0^1 \prod_{i=1}^{k} dx_i\, x_l^{k-1} \prod_{j=k+1}^{n} dy_j \\ \times & \quad F(x_l\, x_1, \dots, x_l, \dots x_l\, x_k\,;\,\{y\}). \end{aligned} \tag{17}$$

A chain of defoliations will stop when all the singularites of F which are located at the vertices of the simplex $[0,1]^k$ have been factorized,

$$\begin{aligned} & F(x_l\, x_1, \dots, x_l, \dots x_l\, x_k\,;\,\{y\}) \\ = & \ x_l^{\mu_l}\, F(x_1, \dots, 1, \dots x_k\,;\,\{y\}). \end{aligned} \tag{18}$$

To find the sectors, although far from unique, is not the difficult part of the evaluation. The resulting expression will contain infrared poles as well as an infrared finite part. The latter and the residues are given by integrals of inverse powers of polynomials that are not (usually) positive defined. BT - iterations or Mellin - Barnes splittings are still required.

Finally, the standard procedure in dealing with loop calculations is based on the introduction of scalar functions and of tensor integrals that are reduced to linear combinations of scalar ones. The new procedure is different since the BT functional relation can be written in a more general form starting from

$$\int_S dx\, Q(x) \prod_i V_i^{\mu_i}(x),$$

$$Q(x) \;=\; \text{an arbitrary polynomial of}\, x. \tag{19}$$

Hence we are led to consider all integrals in parametric space with no pre-reduction in momentum space. The final strategy will be to map as many integrals as possible in one integral,

$$\text{Process} \;\equiv\; \int_{\text{momenta}} \sum \text{diagrams} \equiv I \otimes FF$$

$$I \;=\; \int_{S_{\rm mx}} \sum_{ij} Q_{ij}(X)\, V_i^{\mu_i}(X), \tag{20}$$

where $X = x_{\text{sub}_i} \in \{x_{\rm mx}\}$ and where I evaluation is performed in one stroke.

In conclusion we stress that, at the moment, our results allow us to deal with one-loop (up to six legs) and two-loop (up to three legs) Feynman diagrams with up to 9 different scales, 6 normal thresholds (plus pseudo and anomalous ones); the kinematics is unconstrained and IR - configurations are accessible.

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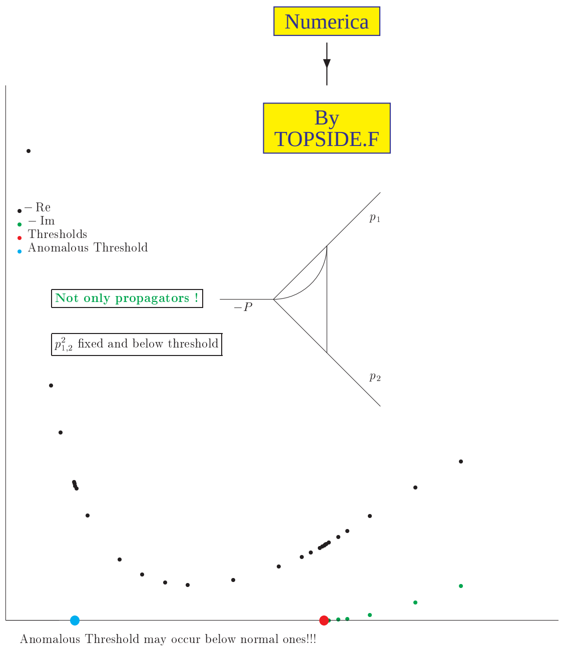

Figure 1: Evaluation of a two-loop vertex function.

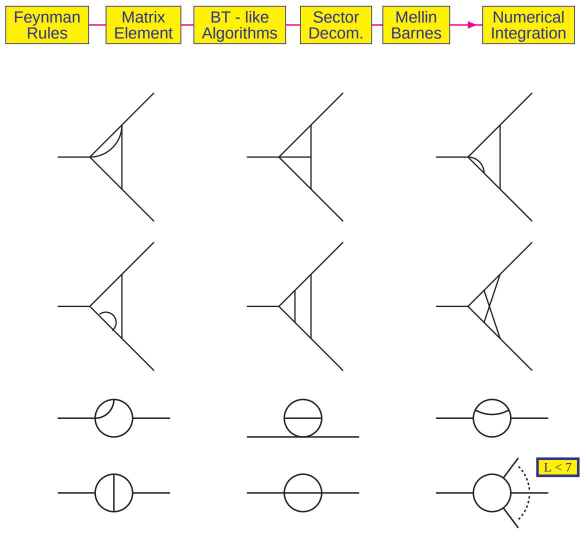

Figure 2: Availability in TOPSIDE.

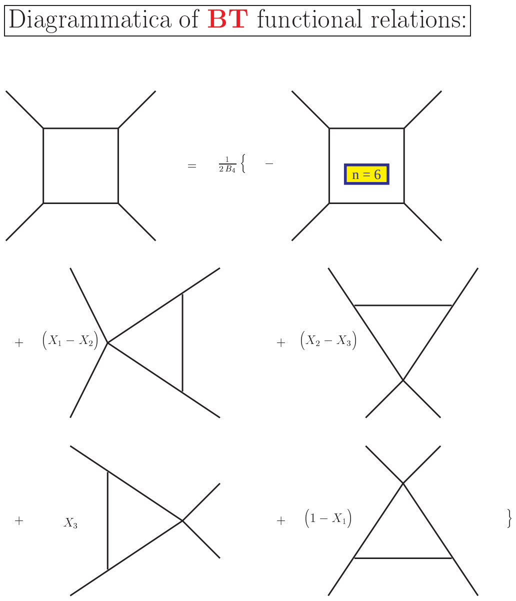

Figure 3: *Diagrammatica* of BT relations.