Public Vices and Private Virtues of Future High Precision Physics

Giampiero PASSARINO

Dipartimento di Fisica Teorica, Università di Torino, Italy
INFN, Sezione di Torino, Italy

A personal (and technical) perspective

Outlines

(1, 2,)

1. *The present of two loop calculus*

   A probable decision about its usefulness is possible inductively by studying its success (verifiable consequences)

2. *The future of two loop calculus*

   A prospective case study, per aspera ad astra

Outlines

(1, 2,)

1. *The present of two loop calculus*

   A probable decision about its usefulness is possible inductively by studying its success (verifiable consequences)

2. *The future of two loop calculus*

   A prospective case study, per aspera ad astra

Outlines

(1, 2,)

1. *The present of two loop calculus*

   A probable decision about its usefulness is possible inductively by studying its success (verifiable consequences)

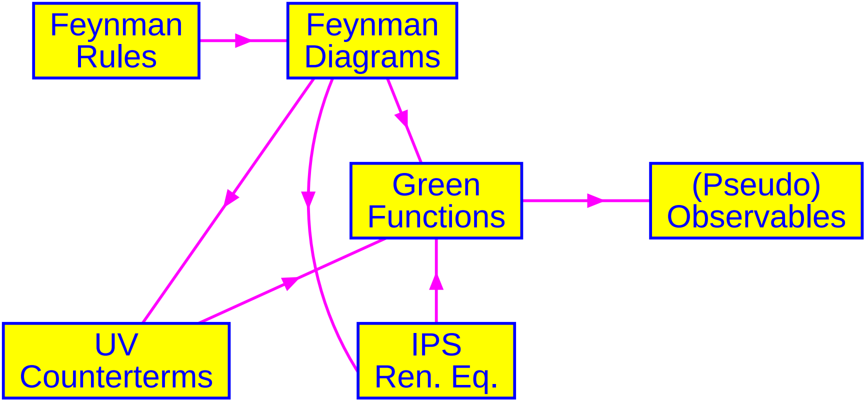
2. *The future of two loop calculus*

   A prospective case study, per aspera ad astra

Part I

The loop tree: embedded case study

flow-chart


Feynman Rules
Feynman Diagrams
Green Functions
(Pseudo) Observables
UV Counterterms
IPS Ren. Eq.

Loop calculus in a nutshell

Theorem

Any *algorithm* *aimed at reducing the analytical complexity of a (multi - loop) Feynman diagram is generally bound to*

- *replace the* *original integral* *with a sum of* *many simpler diagrams*,
- *introducing* *denominators* *that show zeros.*

Definition

An algorithm is *optimal* when

- there is a minimal number of terms,
- zeros of denominators correspond to solutions of Landau equations
- the nature of the singularities is not badly overestimated.

Sunny-side up

Progress

In the past years an enormous progress in the field of $2\,L$ integrals for massless $2 \rightarrow 2$ scattering; $gg \rightarrow gg\,,qg \rightarrow qg$ and $qQ \rightarrow qQ$ as well as Bhabha scattering.

Achievements

- basic $2\,L$ integrals have been evaluated
- e.g. analytic expressions for the two loop planar and non-planar box
- master integrals connected with the tensor integrals have been determined.

Sunny-side up

Progress

In the past years an enormous progress in the field of $2\,L$ integrals for massless $2 \to 2$ scattering; $gg \to gg\,,qg \to qg$ and $qQ \to qQ$ as well as Bhabha scattering.

Achievements

- basic $2\,L$ integrals have been evaluated
- e.g. analytic expressions for the two loop planar and non-planar box
- master integrals connected with the tensor integrals have been determined.

Status of HO loop calculations

zero or one

Impressive calculations (up to four loops) for zero or one kinematical variable, e.g. $g - 2$, R, β-function

> 1

Computations involving more than one kin. var. is a new art

Example

We would like to have $n = 4$ Green functions to all loop orders, from maximally supersymmetric YM amplitudes to real life it's a long way

Status of HO loop calculations

zero or one

Impressive calculations (up to four loops) for zero or one kinematical variable, e.g. $g-2$, R, β-function

> 1

Computations involving more than one kin. var. is a new art

Example

We would like to have $n = 4$ Green functions to all loop orders, from maximally supersymmetric YM amplitudes to real life it's a long way

Status of HO loop calculations

zero or one

Impressive calculations (up to four loops) for zero or one kinematical variable, e.g. $g-2$, R, β-function

>1

Computations involving more than one kin. var. is a new art

Example

We would like to have $n=4$ Green functions to all loop orders, from maximally supersymmetric YM amplitudes to real life it’s a long way

Main road

Step 1

reduce reducible integrals

Step 2

construct systems of IBP or Lorentz invariance identities

Step 3

reduce irreducible integrals to generalized scalar integrals

Step 4

solve systems of eqns in terms of MI

Step 5

evaluate MI, e.g. differential eqns, MB representations, nested sums, etc

Main road

Step 1

reduce reducible integrals

Step 2

construct systems of IBP or Lorentz invariance identities

Step 3

reduce irreducible integrals to generalized scalar integrals

Step 4

solve systems of eqns in terms of MI

Step 5

evaluate MI, e.g. differential eqns, MB representations, nested sums, etc

Main road

Step 1

reduce reducible integrals

Step 2

construct systems of IBP or Lorentz invariance identities

Step 3

reduce irreducible integrals to generalized scalar integrals

Step 4

solve systems of eqns in terms of MI

Step 5

evaluate MI, e.g. differential eqns, MB representations, nested sums, etc

Main road

Step 1

reduce reducible integrals

Step 2

construct systems of IBP or Lorentz invariance identities

Step 3

reduce irreducible integrals to generalized scalar integrals

Step 4

solve systems of eqns in terms of MI

Step 5

evaluate MI, e.g. differential eqns, MB representations, nested sums, etc

Main road

Step 1

reduce reducible integrals

Step 2

construct systems of IBP or Lorentz invariance identities

Step 3

reduce irreducible integrals to generalized scalar integrals

Step 4

solve systems of eqns in terms of MI

Step 5

evaluate MI, e.g. differential eqns, MB representations, nested sums, etc

But, for the real problem

Loop integrals are not enough

$+$

assemblage of scattering amplitudes

$+$

infrared divergenges

$+$

collinear divergenges

$+$

numerical programs

IBP and LI

Tools

A popular and quite successful tool in dealing with multi-loop diagrams is represented by the IBPI and LII. Arbitrary integrals can be reduced to an handful of Master Integrals (MI)

Let us point out one drawback of this solution. Consider, for instance, the following result,

IBP example

Example

$$
\begin{aligned}
B_0(1,2\,;\,p,m_1,m_2) &= \frac{1}{\lambda(-p^2,m_1^2,m_2^2)} \\
&\times\ (n-3)\,(m_1^2-m_2^2-p^2)\,B_0(p,m_1,m_2) \\
&+\ (n-2)\,\Bigg[A_0(m_1)-\frac{p^2+m_1^2+m_2^2}{2\,m_2^2} \\
&\times\ A_0(m_2)\Bigg]\Bigg\},
\end{aligned}
$$

IBP example

Around threshold

We know that at the normal threshold the leading behavior of $B_0(1,2)$ is $\lambda^{-1/2}$,

Conclusion:

reduction to MI apparently overestimates the singular behavior; of course one can derive the right expansion at threshold, but the result is again a source of cancellations/instabilities.

Two-loop conceptual problems

WSTI vs LSZ

- Two loop *à la* LSZ
- The LSZ formalism is unambiguously defined only for stable particles, and it requires some care when external unstable particles appear

Unstable internal

Unphysical behaviors induced by self-energy insertions into 1 L diagrams; they signal the presence of an unstable particle and are the consequence of a misleading organization of PT.

Around thresholds

These regions are not accessible with approximations, e.g. expansions

Two-loop conceptual problems

WSTI vs LSZ

- Two loop *à la* LSZ
- The LSZ formalism is unambiguously defined only for stable particles, and it requires some care when external unstable particles appear

Unstable internal

Unphysical behaviors induced by self-energy insertions into 1 L diagrams; they signal the presence of an unstable particle and are the consequence of a misleading organization of PT.

Around thresholds

These regions are not accessible with approximations, e.g. expansions

Two-loop conceptual problems

WSTI vs LSZ

- Two loop *à la* LSZ
- The LSZ formalism is unambiguously defined only for stable particles, and it requires some care when external unstable particles appear

Unstable internal

Unphysical behaviors induced by self-energy insertions into $1\,L$ diagrams; they signal the presence of an unstable particle and are the consequence of a misleading organization of PT.

Around thresholds

These regions are not accessible with approximations, e.g. expansions

Technical problems I

Reduction to MI

Algebraic problem,

- Buchberger algorithm to construct Gröbner bases seems to be inefficient

It remains

- to generalize to more than few scales
- to compute the MI

New bases?

Technical problems I

Reduction to MI

Algebraic problem,

- Buchberger algorithm to construct Gröbner bases seems to be inefficient

New bases?

It remains

- to generalize to more than few scales
- to compute the MI

Technical problems I

Reduction to MI

Algebraic problem,

- Buchberger algorithm to construct Gröbner bases seems to be inefficient

It remains

- to generalize to more than few scales
- to compute the MI

New bases?

Technical problems II

Although

HTF (usually) have nice properties,

- expansions are often available with good properties of convergence
- the expansion parameter has the same cut of the function

where is the limit?

- One - loop, Nielsen - Goncharov
- Two - loop, one scale ($s = 0, m^2$ cuts) harmonic polylogarithms
- Two - loop, two scales ($s = 4\,m^2$ cuts) generalized harmonic polylogarithms
- next? New higher transcendental functions?

Part II

Future of $2L$ calc: exploratory case study

From modern $1\,L$ to $2\,L$

$1\,L$ in a nutshell

$$
\begin{aligned}
S_{n\,;\,N}(f) &= \frac{\mu^{\epsilon}}{i\,\pi^2}\int d^n q\,\frac{f\,(q\,,\,\{p\})}{\prod_{i=0,N-1}(i)},\\
(i) &= (q+p_0+\cdots+p_i)^2+m_i^2.
\end{aligned}
$$

$$
\begin{aligned}
S_{n\,;\,N}(f) &= \sum_i b_i\,B_0(P_i^2)+\sum_{ij} c_{ij}\,C_0(P_i^2,P_j^2)\\
&+ \sum_{ijk} d_{ijk}\,D_0(P_i^2,P_j^2,P_k^2)+R,
\end{aligned}
$$

From modern $1\,L$ to $2\,L$

$1\,L$ in a nutshell

$$
\begin{aligned}
S_{n\,;\,N}(f) &= \frac{\mu^{\epsilon}}{i\,\pi^2}\int d^n q\,\frac{f\,(q\,,\,\{p\})}{\prod_{i=0,N-1}(i)},\\
(i) &= (q+p_0+\cdots+p_i)^2+m_i^2.
\end{aligned}
$$

$$
\begin{aligned}
S_{n\,;\,N}(f) &= \sum_i b_i\,B_0(P_i^2)+\sum_{ij} c_{ij}\,C_0(P_i^2,P_j^2)\\
&+\sum_{ijk} d_{ijk}\,D_0(P_i^2,P_j^2,P_k^2)+R,
\end{aligned}
$$

The multi facets of QFT

Popular wisdom

- Tree is nirvana
- $1\,L$ is limbo
- $2\,L$ is samsara

$1\,L \rightarrow$

$1\,L$ will be nirvana when general consensus on reduction is reached

$1\,L \rightarrow\rightarrow$

Which is the most efficient way of computing the coefficients?

The multi facets of QFT

Popular wisdom

- Tree is nirvana
- 1 L is limbo
- 2 L is samsara

1 L $\longrightarrow$

1 L will be nirvana when general consensus on reduction is reached

1 L $\longrightarrow\longrightarrow$

Which is the most efficient way of computing the coefficients?

The multi facets of QFT

Popular wisdom

- Tree is nirvana
- $1\,L$ is limbo
- $2\,L$ is samsara

$1\,L \rightarrow$

$1\,L$ will be nirvana when general consensus on reduction is reached

$1\,L \longrightarrow\longrightarrow$

Which is the most efficient way of computing the coefficients?

Reduction at $2\,L$

Problem

At $2\,L$ reduction is different since irreducible scalar products are present

Master Integrals

One way or the other a basis of generalized scalar functions is selected (MI)

Which MI are present?

Some care should be payed in avoiding MIs that do not occur in the actual calculation. This fact is especially significant when the MI itself is divergent and the singularity must be extracted analytically

Reduction at $2\,L$

Problem

At $2\,L$ reduction is different since irreducible scalar products are present

Master Integrals

One way or the other a basis of generalized scalar functions is selected (MI)

Which MI are present?

Some care should be payed in avoiding MIs that do not occur in the actual calculation. This fact is especially significant when the MI itself is divergent and the singularity must be extracted analytically

Reduction at $2\,L$

Problem

At $2\,L$ reduction is different since irreducible scalar products are present

Master Integrals

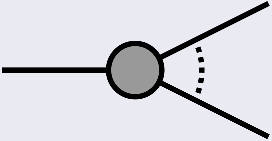
One way or the other a basis of generalized scalar functions is selected (MI)

Which MI are present?

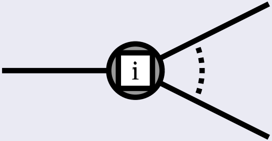
Some care should be payed in avoiding MIs that do not occur in the actual calculation. This fact is especially significant when the MI itself is divergent and the singularity must be extracted analytically

Stadard reduction? Unitarity based?

I



$$= \int d^n q \, \frac{1}{\prod_{i=0,N-1} [i]},$$



$$= \int d^n q \, \frac{q \cdot p_i}{\prod_{i=0,N-1} [i]}.$$

Figure: Convention for Feynman diagrams.

Stadard reduction? Unitarity based?

Example

$$\frac{\mu^{\epsilon}}{i\,\pi^2}\int d^n q\,\frac{q\cdot p_1}{\prod_{i=0,3}[i]} = \sum_{i=1}^{3} D_{1i}\,p_1\cdot p_i$$

$$= -\sum_{i=1}^{3} D_{1i}\,H_{1i}.$$

$H_{ij} = -p_i \cdot p_j$; $G = \det H$ is the Gram determinant.

Stadard reduction?

naive

In naive SR $D_{1i} \to D_0$ and $\to$ three-point functions, with inverse powers of G_3 etc.

revised

$$D_{1i} \;=\; -\frac{1}{2} H^{-1}_{ij}\, d_j, \qquad d_i = D_0^{(i+1)} - D_0^{(i)} - 2\, K_i\, D_0,$$

where $D_0^{(i)}$ is the scalar triangle obtained by removing propagator i from the box.

Stadard reduction?

Therefore we obtain

$$\frac{\mu^\epsilon}{i\,\pi^2}\int d^n q\,\frac{q\cdot p_1}{\prod_{i=0,3}[i]} = \frac{1}{2}\sum_{i,j=1}^{3} H_{ij}^{-1} H_{1i}\, d_j$$
$$= \frac{1}{2}\, d_1,$$

without explicit factors involving G_3.

Stadard reduction?

Furthermore

The coefficient of D_0 in the reduction is

$$\frac{1}{2}\left(m_0^2 - m_1^2 - p_1^2\right),$$

. At the leading Landau singularity of the box we must have

$$q^2 + m_0^2 \quad = \quad 0, \qquad (q + p_1)^2 + m_1^2 = 0, \qquad \text{etc.}$$

Therefore

the coefficient of D_0 is fixed by

$$2\, q \cdot p_1\Big|_{AT} \quad = \quad m_0^2 - m_1^2 - p_1^2,$$

which is what a careful application of standard reduction gives.

Reduction is telling us that

Anomalous threshold behavior $\equiv$ standard reduction of a tensor box easily shows if the corresponding scalar box has to be considered, e.g.

$$\frac{\mu^{\epsilon}}{i\,\pi^2}\int d^n q\,\frac{q\cdot p_1}{\prod_{i=0,3}[i]} \quad \not\to \quad D_0$$

$$\text{iff} \qquad p_1^2 \;=\; m_0^2 - m_1^2,$$

Two loop extension?

Embedding

The N-point, $1\,L$, function is a sub-diagram (q_2) of a $2\,L$ diagram (q_1, q_2) with I internal legs. The numerator contains red $\oplus$ irr scalar products

if after reduction $N \to N-1$ the coeff of the S, V or T $1\,L$ diagram are zero then the $2\,L$ - I-prop - diagram will not appear, only its $(I-1)$-daughters

In particular,

if the original two-loop diagram is (e.g. collinear) divergent the singular behavior can be read off its daughters which is a simpler problem because one propagator less is involved.

Two loop extension?

Embedding

The N-point, $1\,L$, function is a sub-diagram (q_2) of a $2\,L$ diagram (q_1, q_2) with l internal legs. The numerator contains red $\oplus$ irr scalar products

if after reduction $N \rightarrow N-1$ the coeff of the S, V or T $1\,L$ diagram are zero then the $2\,L$ - l-prop - diagram will not appear, only its $(l-1)$-daughters

In particular,

if the original two-loop diagram is (e.g. collinear) divergent the singular behavior can be read off its daughters which is a simpler problem because one propagator less is involved.

Two loop extension?

Embedding

The N -point, $1\,L$, function is a sub-diagram (q_2) of a $2\,L$ diagram (q_1, q_2) with I internal legs. The numerator contains red $\oplus$ irr scalar products

if after reduction $N \to N-1$ the coeff of the S, V or T $1\,L$ diagram are zero then the $2\,L$ - I -prop - diagram will not appear, only its $(I-1)$ -daughters

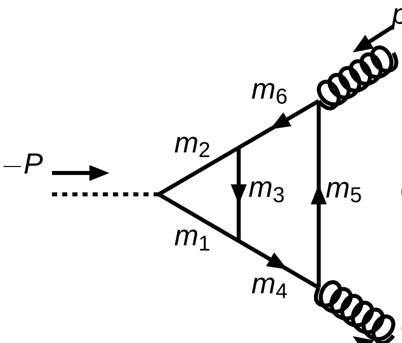
In particular,

if the original two-loop diagram is (e.g. collinear) divergent the singular behavior can be read off its daughters which is a simpler problem because one propagator less is involved.

Example: I

Example

Consider now the V^{K} -configuration projected with P_{D}



$\otimes\ P_D^{\mu\nu}$

Example: II

After decomposition $6 \to 5$ the 6 -propagator terms disappear from the projected V^{κ} if $m_4 = m_5 = m_6 = 0$, for arbitrary m_1, m_2 and m_3.

massive case

When all fermion lines in the V^{κ} -configuration have a mass m, we obtain

$$\left[32\left(v_+^2 + v_-^2\right) m^2 \left(p_1 \cdot p_2 - M^2 + 2\,m^2\right)\right.$$
$$\left. - \quad 128\, v_+ v_-\, m^2 \left(p_1 \cdot p_2 + 2\,m^2\right)\right] \int d^n q \, \frac{1}{\prod_{i=1,6} [i]_{\kappa}}$$
$$+ \quad \leq 5\text{- propagator contractions.}$$

As a consequence only the scalar V^{κ} is present.

Example: II

After decomposition $6 \rightarrow 5$ the 6-propagator terms disappear from the projected V^K if $m_4 = m_5 = m_6 = 0$, for arbitrary m_1, m_2 and m_3.

massive case

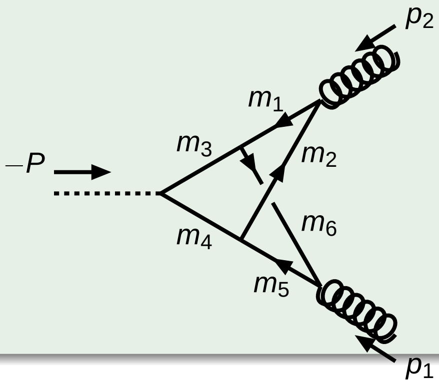
When all fermion lines in the V^K-configuration have a mass m, we obtain

$$\begin{aligned} &\Big[32\,\left(v_+^2 + v_-^2\right))\, m^2\,\left(p_1 \cdot p_2 - M^2 + 2\,m^2\right) \\ - \quad & 128\, v_+ v_-\, m^2\,\left(p_1 \cdot p_2 + 2\,m^2\right)\Big] \int d^n q\, \frac{1}{\prod_{i=1,6}[i]_K} \\ + \quad & \leq 5\text{-propagator contractions.} \end{aligned}$$

As a consequence only the scalar V^K is present.

Example: III

Example



Example: IV

all fermion massless

$$
\begin{aligned}
& \left\{16\,\left(v_+^2 + v_-^2\right)\,\left(p_1\cdot p_2 + M^2\right)\right. \\
\times\quad & \left.\left[M^2 + 2\,p_1\cdot q_1\left(1 - \frac{p_1\cdot q_1}{p_1\cdot p_2}\right)\right]\right\} \\
\times\quad & \int d^n q\,\frac{1}{\prod_{i=1,6}[i]_H} \\
+\quad & \leq 5\text{ - propagator contractions,}
\end{aligned}
$$

i.e. one combination of S, V and T V^H is the MI

Part III

Computing MI

Beyond Nielsen - Goncharov

New Approach

New integral representations for diagrams

Theorem

Diagrams $\equiv$

$$\int dC_k(\{x\})\frac{1}{A}\ln\left(1+\frac{A}{B}\right) \quad \text{or} \quad \int dC_k(\{x\})\frac{1}{A}\mathrm{Li}_n\left(\frac{A}{B}\right)$$

where A, B are multivariate polynomials in the Feynman parameters. One-(Two-) loop diagrams are always reducible to combinations of integrals of this type where the usual monomials that appear in the integral representation of Nielsen - Goncharov generalized polylogarithms are replaced by multivariate polynomials of arbitrary degree.

Example

General C_0: definitions

$$
\begin{aligned}
C_0 &= \int dS_2\, V^{-1-\epsilon/2}(x_1, x_2), \\
V(x_1, x_2) &= x^t\, H\, x + 2\, K^t\, x + L = Q(x_1, x_2) + B, \\
H_{ij} &= -p_i \cdot p_j, \quad L = m_1^2, \\
K_1 &= \frac{1}{2}(p_1 \cdot p_1 + m_2^2 - m_1^2), \\
K_2 &= \frac{1}{2}(P \cdot P - p_1 \cdot p_1 + m_3^2 - m_2^2),
\end{aligned}
$$

General C_0: result

$$C_0 = \frac{1}{2}\sum_{i=0}^{2}(X_i - X_{i+1}) \times \int_0^1 \frac{dx}{Q(\widehat{i\ i+1})}\ln\left(1+\frac{Q(\widehat{i\ i+1})}{B}\right)$$

$$Q(\widehat{0\ 1}) = Q(1,x),\ Q(\widehat{1\ 2}) = Q(x,x),\ Q(\widehat{2\ 3}) = Q(x,0)$$

$$X^t = -K^t H^{-1},\ X_0 = 1,\ X_3 = 0$$

Basics

Define

$$\mathcal{L}_n(z) = z^n L_n(z) = z^n \int dC_n \left(\prod_{i=1}^{n} y_i\right)^{n-1} \left[1 + \prod_{j=1}^{n} y_j\, z\right]^{-n}$$
$$= \left(\frac{z}{n}\right)^n {}_{n+1}F_n\left((n)_{n+1}\,;\,(n+1)_n\,;\,-z\right),$$

$$\mathcal{L}_1(z) = -S_{0,1}(-z),$$
$$\mathcal{L}_2(z) = S_{0,1}(-z) - S_{1,1}(-z),$$
$$\mathcal{L}_3(z) = -\frac{1}{2} S_{0,1}(-z) + \frac{3}{2} S_{1,1}(-z) - S_{2,1}(-z),$$

Problem

- For any quadratic form in n-variables

$$V(x) \;=\; (x-X)^t\, H\, (x-X) + B = Q(x) + B,$$

- we want to compute

$$I(n,\mu) \;=\; \int dC_n\, V^{-\mu} = \int dC_n \left[Q(x)+B\right]^{-\mu}.$$

Definition

- Consider the operator

$$\mathcal{P} \;=\; (x-X)^t\,\partial, \quad \text{satisfying} \quad \mathcal{P}\,Q = 2\,Q$$

Solution

Introduce

$$J(\beta,\mu) \;=\; \int_0^1 dy\, y^{\beta-1}\, W^{-\mu}(y), \quad W(y) = Q(x)\, y + B.$$

Use

$$\left(\frac{1}{2}\mathcal{P} - y\,\partial_y\right) W^{-\mu} = 0 \;\;\rightarrow\;\; V^{-\mu} = \left(\beta + \frac{1}{2}\mathcal{P}\right) J(\beta,\mu),$$

$$I(n,\mu) \;=\; \int dC_n \left(\beta + \frac{1}{2}\mathcal{P}\right) J(\beta,\mu),$$

Further definitions

Define

$$
\begin{aligned}
f([x]) &= f(x_1\,, \cdots\,, x_n),\\
f({}_i[x]) &= f(x_1\,, \cdots\,, x_i = 0\,, x_n),\\
f([x]_i) &= f(x_1\,, \cdots\,, x_i = 1\,, x_n),\\
\int dC_n &= \int_0^1 \prod_{i=1}^{n} dx_i\,, \qquad \int dC_{n,j} = \int_0^1 \prod_{i=1, i\neq j}^{n} dx_i\,.
\end{aligned}
$$

Results I

Example

- For $\mu = 1$ it is convenient to choose $\beta = 1$, to obtain

$$
\begin{aligned}
I(n,1) &= \left(\frac{n}{2} - 1\right) \int dC_n \, L_1([x]) \\
&- \frac{1}{2} \sum_{i=1}^{n} \int dC_{n,i} \left\{ X_i \, L_1({}_i[x]) - (1 - X_i) \, L_1([x]_i) \right\}
\end{aligned}
$$

Results II

Example

- For $\mu = 2$ it is more convenient to write

$$V^{-2} = \left(2 + \frac{1}{2}\mathcal{P}\right)^2 J(2,2) = \left(2 + \frac{1}{2}\mathcal{P}\right)^2 L_2.$$

- integration-by-parts follows
- additional work (along the same lines) is needed to deal with surface terms ...

Challenge

The challenge remains: unprecedented precision needed in high energy QCD and electroweak radiative corrections with more than a single kinematical invariant. Don't miss the forest (complete calculation) for the trees (Feynman diagrams).