On the Mass and Width of the Z-boson and Other Relativistic Quasistable Particles

Arno R. Bohm[a] **and N. L. Harshman**[b]

The University of Texas at Austin
Austin, Texas 78712

Abstract

The ambiguity in the definition for the mass and width of relativistic resonances is discussed, in particular for the case of the Z-boson. This ambiguity can be removed by requiring that a resonance's width Γ (defined by a Breit-Wigner lineshape) and lifetime τ (defined by the exponential law) always and exactly fulfill the relation $\Gamma = \hbar/\tau$. To justify this one needs relativistic Gamow vectors which in turn define the resonance's mass M_R as the real part of the square root $\mathrm{Re}\sqrt{\mathrm{s_R}}$ of the S-matrix pole position s_R. For the Z-boson this means that $M_R \approx M_Z - 26\mathrm{MeV}$ and $\Gamma_R \approx \Gamma_Z - 1.2\mathrm{MeV}$ where M_Z and Γ_Z are the values reported in the particle data tables.

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[a] E-mail: bohm@physics.utexas.edu
[b] E-mail: harshman@physics.utexas.edu

1 Introduction

The subject of this paper can be looked at as an example of time asymmetric quantum theory [1–4]; then one should read it (or write it) from the end to the beginning, starting with the "axiom" of Appendix A for the relativistic case. Then, using the Poincaré semigroup representations, one will be led to a particular form for the S-matrix Breit-Wigner and will arrive at the conclusion that the Z-boson mass and width must be the particular parameterization (M_R, Γ_R) or else the generally desired width-lifetime relation $\tau = \hbar/\Gamma_R$ will be violated. We have chosen instead to present the paper in a heuristic way, starting from the problems of defining the resonance amplitude and the mass and width of a quasi-stable particle for the analysis of experimental data—in particular for relativistic particles and focusing on the lineshape of the Z-boson. This presentation corresponds more closely to the historical process, where the extraordinary accuracy of the LEP data [5–7] set into motion developments that questioned the standard definitions of a resonance [8–14], though for a long time it had been known that there was no consistent theory of resonance and decay phenomena [15].

Relativistic resonances and quasi-stable particles are listed along with stable, elementary particles [5] and are not considered qualitatively different from the latter. Both are characterized by species labels or internal quantum numbers, spin-parity j^P and the center of mass energy squared s. The only difference between the characterizations is that stable particles have a real value for $s = m^2 \geq 0$ and quasi-stable particles and resonances have a complex value for s, which is often also parameterized by the two real values "mass" m and "width" Γ. These can be combined as $s = (m - i\frac{\Gamma}{2})^2$ among other ways.

The ratio of the resonance characterization parameters (Γ/m) runs over a wide range: from $(\Gamma/m) \sim 10^{-2}$-10^{-1} for hadron resonances and $(\Gamma/m) \sim 3\times 10^{-2}$ for the Z-boson to much smaller values for other electroweakly decaying particles, e.g. $(\Gamma/m)_{\pi^0} \sim 10^{-7}$, $(\Gamma/m)_{\pi^\pm} \sim 10^{-15}$, $(\Gamma/m)_\mu \sim 3 \times 10^{-18}$and $(\Gamma/m)_{K^0} \sim 10^{-14}$. The experimental definition or method of determination of Γ and m is quite different for different magnitudes of (Γ/m).

1. For $\Gamma/m > 10^{-7}$, Γ and m are determined as the lineshape parameters of relativistic Breit-Wigner (BW) amplitudes from the cross sections and asymmetries (e.g. for the Z-boson in $e\bar{e} \to Z \to f\bar{f}$ [6]). In what follows we discuss two different relativistic BW's.

2. For $\Gamma/m \sim 10^{-14}$ and less, the lineshape cannot be resolved and instead of Γ one measures the lifetime τ. The definition of τ is based on the exponential law for the partial decay rates $\dot{P}_\eta(t)$ and/or the decay probability $P(t) = \sum_\eta P_\eta(t)$. One measures τ by a fit to the exponential law [16].

Observationally, Γ and τ are different qualities; Γ comes from the BW energy distribution and τ from the exponential counting rate. Statements that the exponential law is not exact and the use of alternate forms for the relativistic BW have further confused the connection between the width and the lifetime. Connected with the definition of resonance width is the definition of resonance mass. Experimentalists have in the past identified it with the peak of the invariant mass distribution which was sufficient since the data where not accurate enough. S-matrix theoreticians associate a resonance to a pole s_R in the S-matrix and prefer to define the mass as $\mathrm{Re}\sqrt{\mathrm{s_R}} = \mathrm{M_R}$ instead of the peak $\sqrt{\mathrm{Re\, s_R}} = M_\rho$ or the on-shell definition M_Z. Only since the LEP data for the Z-boson has accuracy reached a level where these differences have become of practical importance. Nevertheless, the exact definition of m and Γ and the relation of Γ to τ have always been of fundamental importance. Conventionally, the identification $\hbar/\Gamma \approx \tau$ is justified by some heuristic and approximate arguments [17, 18] based on the Weisskopf-Wigner (WW) methods [19]. So far, no generally accepted theory precisely related $\hbar/\Gamma$ and τ because there did "not exist … a rigorous theory to which these various methods [WW, etc.] can be considered as approximations" [15].

A rigorous description of non-relativistic resonances and decaying states by Gamow kets $\psi^G = |E - i\Gamma/2^-\rangle$ has been provided during the last two decades by the Rigged Hilbert Space (RHS) formulation of quantum mechanics [4, 20, 21]. For these Gamow kets one can derive both an exact Breit-Wigner with width Γ for the lineshape and an exact, exponential time evolution $\psi^G(t) = e^{-iEt}e^{-\frac{\Gamma}{2}t}\psi^G(0)$. From the latter follows the exact Golden Rule for the partial decay rate of the resonance R into the channel η, $\dot{\mathcal{P}}_\eta(t) = \Gamma_\eta e^{-\Gamma t}$. Thus with the properties of the Gamow vectors, the precise relation $\tau = \hbar/\Gamma$ between the differently defined quantities τ and $\hbar/\Gamma$ was established for non-relativistic resonances.

Guided and motivated by recent discussions concerning the ambiguities in the definition of mass and width of relativistic quasi-stable particles, in particular for the Z-boson [5–13,22], we explore the use of relativistic Gamow kets to provide unambiguous definitions of mass and width. The theoretical foundations of the relativistic Gamow vectors are based on the time asymmetric Poincaré transformations of relativistic spacetime and have recently been presented elsewhere [23].

2 Lineshape Parameters and Relativistic Breit-Wigners

The determination of the Z-boson mass and width from the lineshape has been performed with two different definitions of mass and width and two different relativistic Breit-Wigner amplitudes. As a result, two different values for m and Γ have been obtained from the same experimental data [5–7].

The standard approach, followed by practically all experimental analyses of the LEP and SLC data [5], is based on the on-shell definition of mass and width, which we shall denote M_Z and Γ_Z. The Z-boson mass and width are defined in perturbation theory by the (transverse) self-energy $\Pi(s)$ of the Z-boson propagator. The part of the Z-boson propagator proportional to $\eta_{\mu\nu}$ is given by

$$D(s) = \frac{1}{s - M_0^2 - \Pi(s)}. \tag{1}$$

The on-shell renormalization scheme defines the (real) mass M_Z and the width Γ_Z by the renormalization conditions:

$$M_Z^2 = M_0^2 + \mathrm{Re}\Pi(M_Z^2), \quad M_Z\Gamma_Z = \frac{-\mathrm{Im}\Pi(M_Z^2)}{1 - \mathrm{Re}\Pi'(M_Z^2)}, \tag{2a}$$

where the prime indicates differentiation with respect to s. Using these conditions and expanding ReΠ(s) about $s = M_Z^2$, $D(s)$ is written

$$\begin{aligned} D(s) &= \frac{1}{(s - M_Z^2)[1 - \mathrm{Re}\Pi'(M_Z^2)] - i\mathrm{Im}\Pi(\mathrm{s})} \\ &= \frac{1}{1 - \mathrm{Re}\Pi'(M_Z^2)} \times \frac{1}{s - M_Z^2 + i\sqrt{s}\Gamma_Z(s)} \end{aligned} \tag{2b}$$

where the "s-dependent width" $\Gamma_Z(s)$ has been defined by

$$\sqrt{s}\Gamma_Z(s) \equiv \frac{-\mathrm{Im}\Pi(s)}{1 - \mathrm{Re}\Pi'(M_Z^2)} \tag{3}$$

This leads, to the "relativistic Breit-Wigner with energy-dependent width" [e.g. p. 189 of [5]] for the lineshape of the Z-boson

$$a_j^Z(s) = \frac{-\sqrt{s}\sqrt{\Gamma_e(s)\Gamma_f(s)}}{s - M_Z^2 + i\sqrt{s}\Gamma_Z(s)} \tag{4}$$

where the partial widths have the same s dependence: $\Gamma_{e,f}(s) = (\text{const.}) \times \Gamma_Z(s)$. Neglecting the fermion mass, for the Z-boson one calculates near the resonance peak (except for some gauge dependent bosonic contributions)

$$\frac{\mathrm{Im}\Pi(s)}{1 - \mathrm{Re}\Pi'(M_Z^2)} = -\frac{s}{M_Z}\Gamma_Z. \tag{5}$$

Inserting this in (4) (and neglecting the irrelevant s-dependence in the numerator), one obtains the standard expression for the jth partial wave amplitude used in the line shape analysis of all experiments:

$$a_j^Z(s) \approx \frac{-M_Z B_{ef}\Gamma_Z}{s - M_Z^2 + i\frac{s}{M_Z}\Gamma_Z} = \frac{R_Z}{s - M_Z^2 + i\frac{s}{M_Z}\Gamma_Z}. \tag{6}$$

This on-shell renormalization scheme is arbitrary [9–14] and the on-shell mass M_Z is gauge dependent at $\mathrm{O}(g^4)$ and higher and the on-shell width Γ_Z is gauge dependent at $\mathrm{O}(g^6)$ and higher [8,13]. Therefore, another approach has been suggested in which the definition for the mass and width is based on the complex-valued position of the propagator pole:

$$s_R - M_0^2 - \Pi(s_R) = 0. \tag{7}$$

Since s_R is the pole position of the S-matrix, it should be gauge invariant. With the pole position s_R, several alternatives for the mass and width of the Z-boson have been defined [10–13]:

1. (M_ρ, Γ_ρ) (also called (m_2, Γ_2) [13] and $(\bar{M}_Z, \bar{\Gamma}_Z)$ [5,7]):

$$s_R = M_\rho^2 - iM_\rho\Gamma_\rho. \tag{8}$$

2. (m_1, Γ_1) [13] which can be defined in terms of (M_ρ, Γ_ρ):

$$m_1 = \sqrt{M_\rho^2 + \Gamma_\rho^2} \quad \text{and} \quad \Gamma_1 = \frac{m_1}{M_\rho}\Gamma_\rho \tag{9a}$$

or vice versa

$$M_\rho^2 = m_1^2(1 + (\frac{\Gamma_1}{m_1})^2)^{-1} \quad \text{and} \quad \Gamma_\rho^2 = \Gamma_1^2(1 + (\frac{\Gamma_1}{m_1})^2)^{-1}. \tag{9b}$$

3. (M_R, Γ_R) [11] which is suggested (but not dictated) by the analyticity of the S-matrix as a function of the Mandelstam variable s:

$$s_R = (M_R - i\frac{\Gamma_R}{2})^2 \tag{10a}$$

which is related to (M_ρ, Γ_ρ) by

$$M_\rho = M_R(1 - \frac{1}{4}(\frac{\Gamma_R}{M_R})^2)^{\frac{1}{2}} \text{ and } \Gamma_\rho = \Gamma_R(1 - \frac{1}{4}(\frac{\Gamma_R}{M_R})^2)^{-\frac{1}{2}}. \tag{10b}$$

These are only a few of many possible definitions. The association of the unstable particle to the pole position s_R of the S-matrix element does not specify any particular separation of the pole position into mass and width.

In analytic S-matrix theory, a resonance with spin j is defined by a (pair of) first order pole(s) at the position(s) s_R (and s_R^*) on the second sheet of the analytically continued j-th partial S-matrix element [24]. This is not only a model independent definition of s_R (and therefore free of the arbitrariness due to the different choice of renormalization schemes and free of questions concerning gauge dependence), but it also does not rely at all on a perturbation series.

The conventional expressions of ImΠ and therewith of Γ treat the unstable state as an asymptotically free state, i.e. as eigenvectors of the free Hamiltonian $H_0 = H_f + H_{\bar{f}}$ and not the full Hamiltonian $H = H_0 + V$, where V is the decay interaction, cf. (7.49) and (7.63) of [25]. However, scattering resonances are different from their asymptotically free in- or out-states and the Born approximation (as employed in Dirac's Golden Rule) is only useful for $\Gamma/m \ll 1$ and for the initial decay rate $\dot{\mathcal{P}}(t \to 0)$. The problems with the use of asymptotically free states for resonances in the conventional mass renormalization scheme have been mentioned by Passera and Sirlin [26, 27]. These problems have been the starting point for the introduction of the Gamow vectors associated to the S-matrix pole [20, 21, 28].

Using either the S-matrix definition or the pole definition of the propagator (7), the Z-boson pole term of the jth partial amplitude is given by

$$a_j^R(s) = \frac{R_Z}{s - s_R} = \frac{R_Z}{s - M_\rho^2 + iM_\rho\Gamma_\rho} = \frac{R_Z}{s - (M_R - i\frac{\Gamma_R}{2})^2} \tag{11}$$

where (M_ρ, Γ_ρ) (or (M_R, Γ_R), (m_1, Γ_1), etc.) are basic S-matrix parameters, independent of both the center of mass energy squared s and the gauge choice. We call (11) also a "relativistic Breit-Wigner" or the S-matrix pole Breit-Wigner.

The amplitudes $a_j^Z(s)$ and $a_j^R(s)$ are different functions of s; to compare the parameterizations (M_Z, Γ_Z) and (M_ρ, Γ_ρ) we align the positions of the maxima of the two functions $|a_j^Z(s)|^2$ and $|a_j^R(s)|^2$. The maximum of $|a_j^R(s)|^2$ is M_ρ^2 and the maximum of $|a_j^Z(s)|^2$ is $M_Z^2(1+(\Gamma_Z/M_Z)^2)^{-1}$ and setting these equal gives (M_ρ, Γ_ρ) in terms of the value most frequently quoted Z-boson parameters (M_Z, Γ_Z)[1]:

$$M_\rho^2 = M_Z^2(1+(\frac{\Gamma_Z}{M_Z})^2)^{-1} \qquad \Gamma_\rho^2 = \Gamma_Z^2(1+(\frac{\Gamma_Z}{M_Z})^2)^{-1}. \tag{12a}$$

Using (9b) and (10b), the other parameterizations (M_R, Γ_R) and (m_1, Γ_1) are expressed in terms of (M_Z, Γ_Z)

$$M_R^2 = M_Z^2(1-\frac{3}{4}(\frac{\Gamma_Z}{M_Z})^2+\cdots), \Gamma_R^2 = \Gamma_Z^2(1-\frac{5}{4}(\frac{\Gamma_Z}{M_Z})^2+\cdots) \tag{12b}$$

and

$$m_1 = M_Z \qquad \Gamma_1 = \Gamma_Z. \tag{12c}$$

Numerically one obtains

$$M_R = M_Z - 26\text{MeV} \qquad\qquad M_\rho = M_Z - 34\text{MeV}. \tag{13}$$

using the experimental values of (M_Z, Γ_Z).

Both forms of the partial wave amplitude (6) and (11) were fitted to the experimental cross sections and asymmetries using the LEP data [6,7]. These fits confirmed the differences (13) and also yielded $\Gamma_\rho - \Gamma_Z \approx (1-2)$MeV. Since both formulas (6) and (11) give an equally good fit to the experimental data (as discussed below), empirically one cannot distinguish $a_j^Z(s)$ and $a_j^R(s)$ (and probably never will). To select one form over the other, one must invoke a theoretical principle—the requirement of gauge invariance chooses the Breit-Wigner (11) with s-independent width [11–14].

Even after that decision has been taken, the 'correct' choice for the parameterization of (11) in terms of mass and width is unresolved. Which of the definitions (M_ρ, Γ_ρ), (M_R, Γ_R), (m_1, Γ_1) or others should be called the

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[1] Equivalently we could have aligned the maximum of the cross section derived from (6) and the maximum of the cross section derived from (6).

mass and width of the Z-boson? Gauge independence and the S-matrix pole definition do not specify whether the mass is the real part of the square root of the pole position, $\mathrm{Re}\sqrt{s_R} = M_R$, or the square root of the real part, $\sqrt{\mathrm{Re} s_R} = M_\rho$, or some more complicated separation.

The parameters (m_1, Γ_1) defined by (9) are very useful because for the Z-boson they are numerically very close to the conventional standard model values (M_Z, Γ_Z) as seen from (12c) [8, 13, 27] and have for this reason been suggested as a useful parameterization of the Z-boson lineshape. (Also, in the case of the $W^{\pm}$-boson, the m_1 definition is the mass that appears in the conventional calculation of muon and beta decay [26]). We shall in the following section give arguments in favor of the mass and width definition of a resonance by (M_R, Γ_R), i.e. in favor of the separation (10a) of the complex pole position.

Phenomenologically it will not make any difference which parameterization is used to store and report the experimental data as long as the definitions do not contradict generally accepted consistency conditions. With the present experimental data it is totally irrelevant which definition of the Z-boson mass one uses for a global fit of the Standard Model since other Standard Model parameters such as M_W or $1/\alpha_{EM}(M_Z)$ have errors which far exceed the differences (13) [22]. The fit is good ($\chi^2 \approx 12$ for 12 degrees of freedom), confirming to the present level of accuracy the experimental data and also the Standard Model as a theory, including electroweak radiative corrections. The exact definition of the Z-boson mass (and other resonance masses) is not a central issue of the Standard Model and the prevailing opinion is that there are many ways to parameterize the experimental data of electroweak physics [29] and m_1 is as good a value for the Z-boson mass as M_ρ and M_R. However, though not presently needed for the Standard Model fits, the extraordinary accuracy of the Z-lineshape data, obtained with great expense of time and effort, allowed for the first time to discuss the problem of the definition of mass and width for an unstable relativistic particle. For the first time one could go beyond the level of the Weisskopf-Wigner approximation and one-loop effects. This opened up a new tier of inquiries, as witnessed by [8–13] which as a consequence leads to the question: what is a relativistic quasi-stationary particle?

This more universal question asks for a more universal consistency conditions, one of which is the width-lifetime relation $\Gamma = \hbar/\tau$. This relation has not been tested in the Standard Model or for more general settings, though it has been used to determine the width of the muon and fix the value of

G_F. Why should this relation not also hold for other elementary particles of electroweak physics?

As mentioned in [10,13], the separation of the amplitude $a_j(s)$ into resonant and non-resonant parts is crucial for the selection of the desired mass and width definition from among the many possibilities (M_ρ, Γ_ρ), (M_R, Γ_R), (m_1, Γ_1), etc. We want to choose this separation such that the relation $\Gamma = \hbar/\tau$ can be derived from the resonance part $a_j^R(s)$ of the j-th partial amplitude $a_j(s) = a_j^R(s) + B(s)$. In order to prepare this derivation, we need to discuss a little more about the phenomenological aspects of the lineshape analysis, in particular the important role of the background and the separation of the scattering amplitude into a pole term (or a superposition of pole terms) and the background.

The experimental fits of the lineshape data incorporate more than just (11) or (6), e.g. corrections, background and $\gamma - Z$-interference needed to be incorporated into the analysis. The pole term (11) only describes the part of the scattering which goes through the Z resonance

$$e\overline{e} \to Z \to f\overline{f}. \tag{14}$$

Even if there is only one intermediate particle with $j^P = 1^-$ there is *always* a slowly varying background amplitude $B(s)$ so that $a_j(s) = a_j^R(s) + B(s)$. If there are also other intermediate particles with $j^P = 1^-$, for example one other with (complex) mass squared $s = s_{R_2}$, then the partial amplitude contains a sum of two BW's and a background term

$$a_j(s) = \frac{R_Z}{s - s_R} + \frac{R_2}{s - s_{R_2}} + B(s). \tag{15}$$

This form (15), which is often used in phenomenological analysis, does not follow from the analyticity assumptions of S-matrix theory but can be justified under a stronger hypothesis of scattering theory in which the in-states and out-states are strictly distinguished, cf. Appendix A. A justification and qualification of (15) is given in Appendix B.

The fit to the LEP data [6,7] used for the amplitude an expression which contained, in addition to the Z-boson Breit-Wigner (11) or (6), a photon term ("γ-Breit-Wigner") and a slowly varying background amplitude $B(s)$ which is assumed to be constant in the Z-boson energy region [30]:

$$a_j(s) = \frac{R_Z}{s - s_R} + \frac{R_\gamma}{s} + B(s), \text{ with } s_R = (M_R - i\Gamma_R/2)^2. \tag{16}$$

The external line QED corrections are included by a convolution integral with the basic scattering cross sections given by $|a_j(s)|^2$ of (16). The γ-Breit-Wigner is the analog of the one photon exchange graph of the internal line QED corrections. Its inclusion in (16) treats the scattering as a double multichannel resonance process (see [28] sect. XX.2),

$$e\overline{e} \ \rightarrow {}^{Z}_{\gamma} \rightarrow \ f\overline{f}. \tag{17}$$

In the S-matrix approach the amplitude (16) may be considered as a limiting case of the double resonance amplitude (15) for $s_{R_2} \rightarrow 0$, though there is really no S-matrix theory justification for (16) except the analogy to (15). However, the Z-γ interference term from (16) is important for the fits of various asymmetries [6].

The earlier fits to the LEP data use the amplitude with the s-dependent denominator (6) from the on-mass shell definition:

$$a_j(s) = \frac{R_Z}{s - M_Z^2 + is\frac{\Gamma_Z}{M_Z}} + \frac{R_\gamma}{s} + B(s). \tag{18}$$

After the arbitrariness of this form became known, the same kinds of fits were made using the S-matrix definition (16). The fits of the LEP data to (16) and (18) were equally good and they reproduced the expected difference (13) between the differently defined masses and widths, (M_Z, Γ_Z) from (18) and (M_R, Γ_R) from (16). Thus the experimental lineshape data of the Z-boson do not favor either of these two definitions of Z-boson mass and width.

Recently the conventional approach, using a Breit-Wigner with energy dependent width (6), and the S-matrix approach, using the pole definition (11), were compared for hadron resonances. For baryon resonances like the Δ_{33}, recent editions of reference [5] list both the pole position $s_h = (1210 - i\frac{100}{2})^2$MeV in addition to the conventional parameters $M_h = 1232$MeV and $\Gamma(M_h^2) = 120$MeV. The interpretation given to these different values by [31] is that the pole position s_h belongs to the Δ-resonance whereas the conventional parameters $(M_h,\ \Gamma(M_h^2))$ belong to the Δ and the background together. This interpretation is in agreement with our RHS theory of resonance scattering as expressed by the results (15) and (16) and explained in detail in Appendix B.

When both the S-matrix definition (11) and the on-mass shell definition (6) were applied to the ρ-meson data [32], the conclusion was that the S-matrix definition of M_ρ and Γ_ρ is phenomenologically preferred. The reason given was that these fitted parameters remained largely independent of the parameterization of the background term $B(s)$ and the $\rho-\omega$ interference. A similar fit to the S-matrix Breit-Wigner (11) was performed for the experimental data on πp scattering in the Δ resonance region [33]. As with the ρ, the fits to (11) were better than the fits to (6) and they were independent of the background parameterization. Also, the fitted values of M_Δ and Γ_Δ from (11) turn out to be significantly smaller than the conventional values from (6). Therefore, for the hadron resonances the pole definition (11) appears to be slightly preferable phenomenologically. Again, as stated above, the same conclusion does not follow from the Z-boson data [6, 7] where these two fits are equally good and lead just to different values of the Z-boson parameters.

3 Lifetime Parameter and Exponential Decay Rate

Neither the treatment of the resonant propagator based on the complex pole position $s_R = M_0^2 + \Pi(s_R)$ nor the phenomenologically equivalent S-matrix theory definition of a quasi-stable particle by the pole at s_R define the mass and width separately. A wide assortment of two real numbers can be extracted from s_R and identified with the mass and width [34], e.g. one could write

$$s_R = (M_R - i\frac{\Gamma_R}{2})^2 = M_\rho^2 - iM_\rho\Gamma_\rho \tag{19}$$

and call either M_R, Γ_R or M_ρ, Γ_ρ the resonance mass and width. There is no principle in analytic S-matrix theory that tells us how to separate the complex number s_R into the real mass and width. Lineshape by itself, however accurately determined and however precisely the background and corrections are known, does not determine mass and width separately. However, there is a separate physical meaning for the width (and the resonance mass). For this we turn to the second definition of Γ given by the inverse lifetime $\Gamma = \hbar/\tau$, as mentioned in section 1.

The measurement of the lifetime uses the exponential law for the partial decay rates $\dot{\mathcal{P}}_\eta(t)$ of the quasi-stable particle R into any decay products with the channel quantum numbers η. It is done by a fit of the experimental counting rate $\dot{N}_\eta(t)/N = \dot{\mathcal{P}}_\eta(t)/\Gamma$ to the exponential $\exp(-t/\tau)$, where t is the time in the center of mass frame of the decay products η. This cannot be done for the Z-boson because $\hbar/\Gamma_Z$ would be too short to measure, although it has been done for many other decaying elementary particles like $\mu^{\pm}$, $\pi^{\pm}$, $K^{\pm,0}$.

All relativistic quasistable particles should have a width Γ defined by the Breit-Wigner and a lifetime τ defined by the exponential decay law. Although both Γ and τ may not be measured for the same particle due to experimental limitations, we want to require that a complete theoretical description of a resonance should provide an unambiguous, *exact* connection between them: the relation $\Gamma = \hbar/\tau$ should be fulfilled exactly and universally.

To establish the width Γ of (11) as the inverse lifetime we have to first establish

$$\dot{\mathcal{P}}_\eta(t) = \Gamma^{(\eta)} e^{-\Gamma_R t} \tag{20}$$

as a precise result obtained from the second sheet S-matrix pole definition of a resonance. Then, the relationship $\Gamma = \hbar/\tau$ will hold precisely and not just as a result of the WW approximate methods [17, 19]. So we need an intermediary that connects the Breit-Wigner (11) to the exponential law (20). This will be the “state vector” of the resonance state, the relativistic Gamow vector.

Conventionally state vectors representing decaying states and resonances are obtained using heuristic finite dimensional models with an n-dimensional complex “effective Hamiltonian matrix” H^{eff}. For example, in the two dimensional neutral Kaon system [35], the elementary particle state is defined as the eigenvector of the complex Hamiltonian matrix with a complex eigenvalue

$$H^{eff} f^{K_{S,L}} = (m_{S,L} - i\frac{\Gamma_{S,L}}{2}) f^{K_{S,L}}. \tag{21}$$

From this one concludes

$$f^{K_{S,L}}(t) \equiv e^{-iH^{eff}t} f^{K_{S,L}} = e^{-im_{L,S}t} e^{-\frac{\Gamma_{S,L}t}{2}} f^{K_{S,L}} \tag{22}$$

and the general state vector is taken as the superposition

$$\phi = c_s f^{K_S} + c_L f^{K_L} \tag{23a}$$

with the time evolution

$$\begin{aligned} \phi(t) &= e^{-H^{eff}t}\phi = c_S f^{K_S}(t) + c_L f^{K_L}(t) \\ &= c_S e^{-im_S t} e^{-\Gamma_S t/2} f^{K_S} + c_L e^{-im_L t} e^{-\Gamma_L t/2} f^{L_S}. \end{aligned} \tag{23b}$$

The exponential law (20) is then justified in the following way: one inserts (22) for all values of t into Dirac's Golden Rule and obtains[2] a rate $R(t)$ for all values of t. This $R(t)$ one then equates with the time derivative of the decay probability $\dot{\mathcal{P}}_\eta(t)$. This deduction of the exponential law has the following deficiencies:

1. Dirac's Golden Rule is only an approximation for the initial decay rate (i.e. for $t \to 0$) and has only been obtained for the Dirac Lippmann-Schwinger eigenkets of real energy $|E^-\rangle$ and not for vectors like f^{K_S} with complex energies $(E - i\Gamma/2)$. To justify (20) one should start from the fundamental quantum mechanical formula for the probability of decay from R to η

   $$\mathcal{P}_\eta(t) = \mathrm{Tr}(\Lambda_\eta |\psi^{\mathrm{G}}(t)\rangle\langle\psi^{\mathrm{G}}(t)|), \tag{24}$$

   where ψ^G describes the decaying state R and Λ_η is the projection operator on the space of decay products η. Then the probability rate (20) should result as the time derivative of $\mathcal{P}_\eta(t)$ [21].

2. The widths Γ_S are Γ_L defined by (21) as the imaginary part of a complex energy and from (22) one sees this is thus just another name for $\hbar/\tau$. It is unrelated to the width of a lineshape (11) because (21) and its underlying assumptions do not imply a precise relation between Γ_S and any width Γ (either Γ_R, Γ_ρ, Γ_Z or others) of any relativistic Breit-Wigner. To justify $\tau = \hbar/\Gamma$ one must relate Γ_S of (21) to the Γ_R (or Γ_ρ, etc.) of the pole definition of the lineshape (11) or any other definition for the lineshape, e.g. (6).

3. The energy operator H (or in the rest system, $H \equiv P_0 = M = (P_\mu P^\mu)^{1/2}$ for the relativistic case like the K-meson) is a self-adjoint operator in an infinite dimensional space, usually the representation space of the Poincare group of transformations of spacetime. To justify the two dimensional "complex" eigenvector expansion of a general state vector ϕ (23a), one first has to show that the eigenvectors with complex eigenvalue (21) have a meaning for self-adjoint H, that they can be elements of a basis system for the infinite dimensional space and

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[2] Additionally, one inserts (23b) into Dirac's Golden Rule and obtains the decay rate for the neutral Kaon state considered as a superposition of two exponentially decaying states, but this matter is not the subject of the present paper.

that the truncation to the two-dimensional subspace gives (23a) as an approximation.

All these things can be justified, with some qualifications, within the RHS formulation of time asymmetric quantum theory by augmenting the conventional assumptions of scattering theory with the new hypothesis described in the Appendix A.

From this new hypothesis one derives the form of the j-th partial amplitude (15) for two resonance poles of the partial S-matrix and, corresponding to (15) term by term, a complex basis vector decomposition for the prepared in-state ϕ^+, which in case of the $e\bar{e}$ of (17) is given by (cf. Appendix B):

$$\begin{aligned}\phi^+ &= \sum_{R_i} |j, s_{R_i}; b_i{}^-\rangle c_{R_i} + \phi^{bg} \\ &= |j = 1^-, s_R; b_R{}^-\rangle c_R + |j = 1^-, s_{R_2}; b_{R_2}{}^-\rangle c_{R_2} + \phi^{bg}, \qquad (25)\end{aligned}$$

This expression (25) is reminiscent of the heuristic expansion (23a) except that the heuristic expressions do not include the background term ϕ^{bg}. In (25) the quantum numbers b denote a set of degeneracy labels[3]

The terms in the basis vector decomposition (25) correspond to the terms in the multi-resonance partial amplitude (15). To each vector in the sum over the resonances R_i in (25) corresponds a BW in the jth partial amplitude (15). The vector ϕ^{bg} is the component of the (arbitrary) in-state vector ϕ^+ in the infinite dimensional subspace representing the non-resonant continuum and it corresponds to the $B(s)$ term in (15). In the next section we will show how to obtain the decomposition of the in-state into a sum over resonances and a non-resonant part by the analytic continuation of the partial S-matrix. The vector $|j, s_{R_i}; b^-\rangle$ in the sum over R_i in (25) comes from the pole at s_{R_i} in the second sheet of the S-matrix and the vector ϕ^{bg} is represented by a background integral (see (32) below). It is because we can obtain both the BW amplitude (11) and the resonance vector $|j, s_R; b^-\rangle$ from the S-matrix pole that we can exactly and precisely establish the relation $\tau = \hbar/\Gamma_R$.

Before continuing with the details of the derivation in the following section, a few comments are in order about the vectors $|j, s_R; b^-\rangle$, which (except for a normalization factor) are the relativistic Gamow vectors ψ^G.

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[3] For b one could choose the momentum $\mathbf{p}_m$ and j_3 but we suggest the spatial components of the 4-velocity $\hat{\mathbf{p}}_m = \mathbf{p}_m/s$. However this is not important for our discussions here; for a detailed discussion see [23].

For stable relativistic elementary particles the vector space description is given by the irreducible unitary representation spaces of the Poincare group $\mathcal{P}$ [36], from which we can then define the fields. This is not restricted to the asymptotic, interaction free states [37]. The 'out' and 'in' "states" of [37], which we denote by $|j, s; b^-\rangle$ and $|j, s; b^+\rangle$ and which fulfill the Lippmann-Schwinger equation, are basis vectors of an irreducible unitary representation (j, s) of $\mathcal{P}$. They are eigenkets of the (self-adjoint) invariant mass squared operator $P_\mu P^\mu \equiv M^2$ with real eigenvalue s:

$$P_\mu P^\mu |j, s; b^-\rangle = s|j, s; b^-\rangle \tag{26}$$

$$P_0|j, s; b = b_{rest}{}^-\rangle = \sqrt{s}|j, s; b = b_{rest}{}^-\rangle. \tag{27}$$

We want to consider the Z-boson (or any quasistable relativistic particle) as a fundamental particle in the Wigner sense and therefore give its description in terms of a representation space of Poincaré group transformations. The Gamow vectors $|j, s_R; b^-\rangle$ of (25) should therefore also be basis vectors of an irreducible representation (j, s_R) of Poincaré transformations. The Gamow kets $|j, s_R; b^-\rangle$ are thus just generalizations of the well-established 'out-states' $|j, s; b^-\rangle$ and are obtained from them by analytic continuation in s from its "physical value" $\{s|(m_e + m_{\bar{e}})^2 \leq s < \infty\}$ the the S-matrix pole position s_R in the second sheet (or if there are several poles, a Gamow ket $|j, s_{R_i}; b^-\rangle$ is obtained for each pole s_{R_i}). In place of (26) and (27) for the Dirac Lippmann-Schwinger kets $|j, s; b^\pm\rangle$, the Gamow kets should be generalized eigenvectors of the self-adjoint mass-squared operator $P_\mu P^\mu \equiv M^2$ with complex eigenvalue $s_R = (M_R + i\Gamma_R/2)^2$:

$$P_\mu P^\mu |j, s_R; b^-\rangle = s_R|j, s_R; b^-\rangle \tag{28}$$

$$P_0|j, s_R; b = b^-_{rest}\rangle = (M_R - i\frac{\Gamma_R}{2})|j, s_R; b = b^-_{rest}\rangle. \tag{29}$$

These eigenvalue equations are exact and mathematically rigorous if the $|j, s_R; b^-\rangle$ are defined as continuous functionals over the space of observables Φ_+ (cf. Appendix (A1)). More details will be given in the following section, but for further details see [23].

The eigenvalue equation (29) agrees with the effective complex Hamiltonian eigenvalue equation (21) and the decomposition (23a) is the truncation of (25) to the resonance subspace if the background ϕ^{bg} (or equivalently $B(s)$ in (15)) is neglected. Thus the Lee-Oehme-Yang theory [35] of the neutral kaon system summarized by (21-23) and other finite dimensional effective theories with complex Hamiltonians [38] will be established as approximations of the RHS theory after (25, 28, 29) have been justified in the next section.

4 Relating the BW Lineshape to the Exponential Decay Law: Relativistic Gamow Vectors

The mathematical object that makes the connection between the Γ that appears in the lineshape (11) and the Γ that appears in the exponential decay law (20) is the relativistic Gamow vector ψ^G. The vector ψ^G fulfills (28) and (29) precisely and is obtained from the pole term of the S-matrix. The Gamow vector ψ^G will on the one hand lead to the lineshape (11) and on the other hand to a precise form of the exponential law (22). Since both come from the same vector characterized by Γ_R, the lineshape-Γ and the inverse lifetime-Γ are the same and given by Γ_R. Because the Gamow vector distinguishes Γ_R as the characterizing parameter, it implies a preferred separation of the complex value s_R into two real parameters (M_R, Γ_R): $s_R = (M_R - i\Gamma_R/2)^2$.

We start with the S-matrix element

$$\begin{aligned}
(\psi^{out}, S\phi^{in}) &= (\Omega^-\psi^{out}, \Omega^+\phi^{in}) = (\psi^-, \phi^+) \\
&= \sum_{j,j_3} \int \frac{d^3p}{2E(p)}\, ds \sum_{j'j'_3} \int \frac{d^3p'}{2E'(p')}\, ds' \langle\psi^-|\mathbf{p}j_3[sj]^-\rangle \\
&\quad \times \langle\mathbf{p}j_3[sj]|S|\mathbf{p}'j'_3[s'j']\rangle\langle^+\mathbf{p}'j'_3[s'j']|\phi^+\rangle \\
&= \sum_j \int_{(m_e+m_{\bar{e}})^2}^{\infty} ds \sum_b \langle\psi^-|j,s;b^-\rangle S_j(s)\langle^+j,s;b|\phi^+\rangle, \qquad (30)
\end{aligned}$$

where we have made use of the symmetry properties of the S-matrix and abbreviated all additional quantum numbers by b.

Here $\phi^+ \in \Phi_-$ represents the prepared in-state (e.g. e^+e^-) and $\psi^- \in \Phi_+$ represents the detected out-state of the decay products (e.g. $f\bar{f}$). For the case under consideration we are only interested in the term of the sum with $j = 1^-$. To obtain the Gamow vectors one proceeds in exactly the same way as in the non-relativistic case [20]. This is explained in more detail for the case of two resonances in Appendix B. One deforms the contour of integration from the upper rim of the cut along the positive real axis $(m_e+m_{\bar{e}})^2 \leq s < \infty$ through the cut into the second (or higher) sheet past the pole at s_R (or past the poles at s_R,$s_{R_2}, \ldots$ if there is more than one resonance pole). For the contour around each resonance pole s_R, the integral in (30) splits off a pole term which gives the Gamow vector $|j, s=s_R; b^-\rangle$ as an analytic continuation of the Dirac-Lippmann-Schwinger kets $|j, s; b^-\rangle$. The Gamow kets are defined (using the Cauchy formula) by the contour integral around s_R, the resonance pole:

$$\begin{aligned} \langle\psi^-|j, s_R; b^-\rangle &\equiv -\frac{i}{2\pi}\oint ds \langle\psi^-|j, s; b^-\rangle \frac{1}{s-s_R} & \text{(31a)} \\ &= \frac{i}{2\pi}\int_{-\infty_{II}}^{+\infty} ds \langle\psi^-|j, s; b^-\rangle \frac{1}{s-s_R}, & \text{(31b)} \end{aligned}$$

for all $\psi^- \in \Phi_+$. This we also write as a functional equation in the space $\Phi_+^\times$:

$$|j, s_R; b^-\rangle \equiv \frac{i}{2\pi}\int_{-\infty_{II}}^{+\infty} ds \frac{|j, s; b^-\rangle}{s-s_R} \tag{31c}$$

As explained in more details in Appendix B, the integral in (30) becomes a sum of such pole terms (one for each s_{R_i}) plus a background integral

$$\int_{\mathcal{C}} ds \langle\psi^-|j, s; b^-\rangle S_j(s) \langle^+j, s; b|\phi^+\rangle \equiv \langle\psi^-|\phi^{bg}\rangle \tag{32}$$

over a contour $\mathcal{C}$, which we are largely free to choose far away from the resonance poles (e.g. along the negative real axis on the second sheet). In this way one arrives at the representation (15) for the j-th partial amplitude (or similarly for the j-th partial S-matrix $\langle b'|S_j(s)|b\rangle = S_j(s)$). By omitting the arbitrary $\psi^- \in \Phi_+$ in (30) one arrives at the complex basis vector decomposition (25) for the in-state vector ϕ^+ where ϕ^{bg} is given by (32). Summarizing, we have obtained by contour deformation the BW amplitude (15) and the basis vector decomposition (25) from the S-matrix element and established the correspondence between the terms of these equations. From (31b) we see that the variable s in (11), (15), etc. extends from $-\infty_{II}$ to $+\infty$; however for "unphysical" values of $s \leq (m_e + m_{\bar{e}})^2 = 4m_e^2$ these values are in the second sheet.

In order to perform the contour deformation that separates (30) into a sum over resonant terms like (31) plus the background term (32) and in order to derive the Breit-Wigner energy distribution of (31b) from the pole in (31a) some mathematical properties must be fulfilled in addition to the conventional assumptions. This is the new hypothesis of the Appendix A and here means specifically that the energy wave functions $\langle^- j,s;b|\psi^-\rangle$ and $\langle^+ j,s;b|\phi^+\rangle$ must be well-behaved Hardy class functions[4] of the upper and lower half-plane, respectively, in the second sheet of the energy surface of the S-matrix, e.g. $\langle^- s|\psi^-\rangle \in \mathcal{S}\cap\mathcal{H}^2_+$.

The relativistic Gamow vectors $|j,s_R,b^-\rangle$ defined by (31) satisfy the eigenvalue equations

$$\begin{aligned}\langle\psi^-|(M^2)^\times|j,s_R;b^-\rangle &= \frac{i}{2\pi}\int_{-\infty}^{+\infty} ds\, s\langle\psi^-|j,s;b^-\rangle\frac{1}{s-s_R}\\ &= s_R\langle\psi^-|j,s;b^-\rangle \qquad (33)\end{aligned}$$

for every $\psi^-\in\Phi_+\subset\mathcal{H}\subset\Phi_+^\times$. Similarly, one can show that the Gamow vectors in the rest frame $|j,s_R;b^-\rangle\rightarrow|j,s_R;{b_{rest}}^-\rangle$ are generalized eigenvectors of the energy operator $H=P_0$

$$\begin{aligned}\langle\psi^-|H^\times|j,s_R;{b_{rest}}^-\rangle &= \frac{i}{2\pi}\int_{-\infty}^{+\infty} ds\sqrt{s}\langle\psi^-|j,s;{b_{rest}}^-\rangle\frac{1}{s-s_R}\\ &= (M_R-i\frac{\Gamma_R}{2})\langle\psi^-|j,s_R;{b_{rest}}^-\rangle, \qquad (34)\end{aligned}$$

where $\sqrt{s_R}=(M_R-i\frac{\Gamma_R}{2})$. The eigenvalue equations (33) and (34) are precise formulations of (28) and (29). The operators $H^\times$ and $(M^2)^\times$ denote the conjugate operators in $\Phi_+^\times$ of the operators H and $M^2\equiv P_\mu P^\mu$ in the space Φ_+ and are defined by $\langle H\psi^-|F^-\rangle\equiv\langle\psi^-|H^\times|F^-\rangle$ and $\langle M^2\psi^-|F^-\rangle\equiv\langle\psi^-|{M^2}^\times|F^-\rangle$ for all $\psi^-\in\Phi_+$ and $F^-\in\Phi_+^\times$ (see [20] for more details on their definition and use).

The Gamow kets $|j,s_R;b^-\rangle$ are also generalized eigenvectors with generalized eigenvalues $b=\{b_1,b_2,\cdots\}$ in the sense of Dirac kets. Continuous

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[4] The proof of the second equality in (31) and of relations (33) and (34) is a consequence of the following Titchmarsh theorem for Hardy functions $G_-(s)\in\mathcal{H}^2_-$:

For $G_-(s)\in\mathcal{H}_-$ (Hardy class in the lower plane) and $\mathrm{Im}\, z<0$ one has

$$G_-(z) = -\frac{1}{2\pi i}\int_{-\infty}^{+\infty}\frac{G_-(s)}{s-z}ds$$

Choosing $G_-(s)=s^{\frac{n}{2}}\langle\psi^-|s^-\rangle$, which according to hypothesis (A2) of Appendix A are well-behaved Hardy functions, one obtains (31), (33), (34) and more.

linear combinations of the Gamow kets with an arbitrary weight function $\phi_{j_3}(b) \in \mathcal{S}$ (Schwartz space)

$$\psi^G_{js_R} = \sum_{j_3} \int d\mu(b) |j, s_R; b^-\rangle \phi_{j_3}(b), \tag{35}$$

also represent relativistic Gamow states with complex mass $\sqrt{s_R} = (M_R - i\Gamma_R/2)$ and a Breit-Wigner energy distribution $\langle^- j, s; b|\psi_{js_R}\rangle \propto (s - s_R)^{-1}$.

For the time evolution we consider here only the Gamow states at rest, where t is the proper time. We calculate for all $\psi^- \in \Phi_+$

$$\begin{aligned}\langle e^{iHt}\psi^- | s_R; b_{rest}{}^-\rangle &= \langle \psi^- | e^{-iH^\times t} | s_R; b_{rest}{}^-\rangle \\ &= \frac{i}{2\pi}\int ds \frac{\langle \psi^- | e^{-iH^\times t} | s; b_{rest}{}^-\rangle}{s - s_R} \\ &= \frac{i}{2\pi}\int ds \frac{e^{-i\sqrt{s}\,t}\langle \psi^- | s; b_{rest}{}^-\rangle}{s - s_R} \\ &= e^{-i\sqrt{s_R}t}\langle \psi^- | s_R; b_{rest}\text{-}\rangle \end{aligned} \tag{36}$$

The last equality holds iff $\langle\psi^-|s^-\rangle \in \mathcal{H}_-$ *and* $e^{-i\sqrt{s}t}\langle\psi^-|s^-\rangle \in \mathcal{H}_-$ because only then can one also use the Titchmarsh formula of footnote 4 for $G_-(s) = e^{-i\sqrt{s}t}\langle\psi^-|s^-\rangle$ $(\pi < \arg\sqrt{s} \leq 2\pi)$ and obtain $G_-(s_R)$. This is fulfilled only for $t \geq 0$ and not for $t < 0$. Thus, due to the new hypothesis of Appendix A, we have exponential time evolution, but only for $t \geq 0$. The dual operator of e^{iHt}, $(e^{iHt})^\times \equiv e^{-iH^\times t}$, is not defined for $t < 0$ because e^{iHt} is not a continuous operator in Φ_+. Omitting the arbitrary $\psi^- \in \Phi_+$ in (36), we obtain the time evolution of the relativistic Gamow states at rest:

$$e^{-iH^\times t}|j, s_R; b_{rest}{}^-\rangle = e^{-iM_Rt}e^{-\Gamma_Rt/2}|j, s_R; b_{rest}{}^-\rangle, \ t \geq 0. \tag{37}$$

This is a functional equation over the space $\Phi_+ = \{\psi^-\}$, describing the detected decay products. This means it only makes mathematical sense when used to calculate the probabilities $|\langle\psi^-|e^{-iH^\times t}|j, s_R\rangle|^2$ of the decay products ψ^-; the "scalar products" of (37) with ϕ^+ or $\psi^G_{js_R}$ are not defined. This is as far as we can establish the heuristic equation (22) by the mathematically rigorous result (37). That it holds for $t \geq 0$ only does not constitute a limitation for the physics. Just to the contrary, $t \geq 0$ describes the physical situation correctly because the decay products η, described by ψ^-, can only be detected *after* the decaying state R, described by the Gamow vector $|j, s_R^-\rangle$, has been created at $t = 0$ [4].

The time evolution (37) holds for every Gamow vector in the basis vector expansion (25). Therewith the time evolution of the heuristic state vector (23b) as a superposition of two exponentials can also be justified for $t \geq 0$ if ϕ^{bg} in (25) can be neglected. While the time evolution of the resonance terms in (25) depends only upon the parameters (M_{R_i}, Γ_{R_i}) and is exponential, the time evolution of the non-resonant background ϕ^{bg} is non-exponential and depends upon the particular choice of the prepared in-state ϕ^+ as seen from (32). The time evolution of $|\langle\psi^-|\phi^+(t)\rangle|^2$ can be very close to exponential if ϕ^+ is prepared such that $\phi^{bg} \approx 0$ [39] (and there is only one resonance in (25)).

5 Summary and Conclusion

The relativistic Gamow vector has the exact exponential time evolution (37) and the exact S-matrix pole Breit-Wigner energy distribution (11). Since these are the signatures of a quasistable particle and a relativistic resonance, we want to assign the relativistic Gamow vectors as the "state vectors" of quasistable relativistic particles. On the basis of (37) we want M_R and Γ_R to define "mass" and "width" of a quasistable relativistic particle, and from (37) it follows that the lifetime is exactly $\tau = \hbar/\Gamma_R$ and not $\hbar/\Gamma_Z$ or $\hbar/\Gamma_\rho$.

The resonance always occurs with at least some background, described by $B(s)$ of the amplitude (15) and alternatively described by the much lesser known ϕ^{bg} of the prepared state ϕ^+ (much lesser known because it is standardly ignored, e.g. in (23)). Since the perturbation theoretical understanding of resonances fosters the idea that resonances are complicated phenomena, one may doubt the utility of separating out a well-defined Gamow state vector and a well-defined Breit-Wigner amplitude and of associating these with the resonance state. However, although stable elementary particles also never occur in total isolation, they still are considered as separate entities. The accuracy with which the exponential law has been observed in some cases [16] shows that the isolation of a microphysical Gamow state ψ^G from the background ϕ^{bg} can be very good. The popularity of the effective theories with finite dimensional complex Hamiltonian matrices not only in particle physics [35] but also in other areas [38] and the phenomenological success of formulas like (15) or (B18) for interfering resonances testifies to the usefulness of separating exponentially evolving Gamow state vectors and ideal Breit-Wigner amplitudes related to them by the correspondence (B19).

The relativistic Gamow vector is not more nor less of an idealization of reality than Wigner's unitary representations for stable particles. A vector description (or in general a density operator description) is needed because the fundamental probabilities (20) and (24) of quantum mechanics are calculated in terms of operators or vectors. To define a quantum physical entity entirely by the S-matrix pole alone would be incomplete.

If the description of resonances by relativistic Gamow vectors is valid, then the lineshape (11) with mass (and by (37)) resonance energy in the rest frame) given by $M_R = 91.1626 \pm .0031\text{GeV}$ and the width (by (37) the inverse lifetime) given by $\Gamma_R = 2.4934 \pm .0024\text{GeV}$ is its first prediction. This differs from the conventional mass definition by the lineshape (6) of the on-mass shell renormalization scheme $M_Z = 91.1871 \pm .0021\text{GeV}$ and it also differs from the definition by the peak position of the relativistic BW (11), $M_\rho = 91.1541 \pm .0031\text{GeV}$ [40].

This prediction was obtained, like the relativistic Gamow vector, from the definition of the resonance as a pole of the S-matrix at s_R, which by itself is insufficient to fix $M_R = \text{Re}\sqrt{s_R}$ and $\Gamma_R = -2\text{Im}\sqrt{s_R}$. Fixing the mass and width could only be done using the new hypothesis of time asymmetric quantum physics, (A1) and (A2) of the Appendix.

The other results of this theory like the exponential law with the precise $\tau = \frac{\hbar}{\Gamma_R}$, the basis vector expansion (25) or its truncation to the complex "effective" theories like (21-23b), the representation (15) of interfering Breit-Wigners associated to (25) all have been introduced before as separate assumptions and it may be welcome that here they all follow from the same new hypothesis about the boundary conditions.

The only result which may be difficult to accept is the semigroup property of the time evolution (37). In the relativistic theory this means that Gamow vectors can only undergo Poincaré transformations into the forward light cone [23]. These Poincaré transformations form only a semigroup $\mathcal{P}_+ = \{(a, \Lambda)\}$, where Λ is a proper orthochronous Lorentz transformation and $a = (a_0, \mathbf{a})$ is a four vector which fulfills[5] $\hat{p} \cdot a = \sqrt{1 + \hat{\mathbf{p}}^2} a_0 - \hat{\mathbf{p}} \cdot \mathbf{a} \geq 0$ for any $\hat{\mathbf{p}} \in \mathbb{R}^3$.

This semigroup property was a surprising and unintended result when it was first derived for the non-relativistic theory. It expresses a time asymmetry on the microphysical level which is connected with neither the violation of

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[5] One checks by direct calculations that $(a_1, \Lambda_1) \circ (a_2, \Lambda_2) = (a_1 + \Lambda_1 a_2, \Lambda_1 \Lambda_2) \in \mathcal{P}_+$ for $(a_i, \Lambda_i) \in \mathcal{P}_+$ but that not every $(a, \Lambda) \in \mathcal{P}_+$ has an inverse in $\mathcal{P}_+$, i.e. $\mathcal{P}_+$ is a semigroup.

time reversal invariance of the Hamiltonian nor entropy increase (at least not in an obvious way, although for a contrary opinion see [1]). In the meanwhile, the irreversible character of quantum mechanical decay has been mentioned in a few textbooks [2] and more general considerations support the existence of a fundamental time asymmetry in the quantum theory of cosmology [1] and of microsystems [3]. The semigroup time evolution of the Gamow states is just an example of this general time asymmetry.

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Appendix A: Time Asymmetric Quantum Theory of Scattering

In this section we state the new hypothesis by which time asymmetric quantum theory (TAQT) differs from the assumptions of standard scattering theory. The dynamical (differential) equations, the algebras of observables and the physical interpretation (probability) remain the same as in time symmetric quantum theory in the Hilbert space $\mathcal{H}$. The only difference is that TAQT uses time asymmetric boundary conditions, an assumption that is already implicit in the heuristic Lippmann-Schwinger (integral) equations used for the calculation of the transition matrix (T-matrix). Of the two alternate ways of calculating the T-matrix [41], we admit as physically valid only the one which agrees with our intuitive notion of causality. In order to give this a mathematical formulation, we need two Rigged Hilbert Spaces (RHS) with the same Hilbert space $\mathcal{H}$.

The relativistic Gamow vector is a precisely defined mathematical object in the RHS formulation of quantum theory. This new quantum theory is not a drastic departure from the usual Hilbert space (HS) formulation but an extension so that the theory includes Dirac kets $|E\rangle$ and the solutions of the Lippmann-Schwinger equation $|E^-\rangle$ and $|E^+\rangle$ which fulfill outgoing and incoming boundary conditions, respectively. The RHS and the HS theory have the same (time symmetric) dynamical equations but different boundary conditions. The standard HS boundary conditions are time symmetric: the space of in-states $\{\phi^+\}$ and the space of out-states $\{\psi^-\}$ of scattering theory are identified with the Hilbert space $\mathcal{H} = \{\phi^+\} = \{\psi^-\}$ or at least with the same subspace thereof $\{\phi^+\} = \{\psi^-\} \subset \mathcal{H}$.

The RHS theory *distinguishes* meticulously between prepared *states* (in-states) $\{\phi^+\}$ and *observables* (out-states) $\{\psi^-\}$ for which it uses two RHS's of Hardy class

$$\phi^+ \in \Phi_- \subset \mathcal{H} \subset \Phi_-^\times \tag{A1a}$$

$$\psi^- \in \Phi_+ \subset \mathcal{H} \subset \Phi_+^\times . \tag{A1b}$$

The space Φ_- (Φ_+) is of Hardy class type in the lower (upper) half plane; this means mathematically that the energy wave functions $\langle^+E|\phi^+\rangle$ ($\langle^-E|\psi^-\rangle$) are well-behaved Hardy class functions in the lower (upper) half plane, $\mathcal{S} \cap \mathcal{H}^2_-|_{\mathbb{R}_+}$ ($\mathcal{S} \cap \mathcal{H}^2_+|_{\mathbb{R}_+}$):

$$\phi^+ \in \Phi_- \quad \Leftrightarrow \quad \langle^+E|\phi^+\rangle \in \mathcal{S} \cap \mathcal{H}^2_-|_{\mathbb{R}_+} \tag{A2a}$$

$$\psi^- \in \Phi_+ \quad \Leftrightarrow \quad \langle^-E|\psi^-\rangle \in \mathcal{S} \cap \mathcal{H}^2_+|_{\mathbb{R}_+} . \tag{A2b}$$

This condition allow the energy wave functions to be analytically continued into well-behaved, rapidly decreasing analytic functions in the lower or upper half plane of the second sheet of the S-matrix. The notation $|_{\mathbb{R}_+}$ means the restriction to the positive real line, i.e. the physical values of energy, and $\mathcal{S}$ denotes the Schwartz space. The vectors ϕ^+ represent states that are prepared by the accelerator and the vectors ψ^- represent observables or out-states that are defined by the detector. The dual spaces in the RHS's contain, in addition to the Dirac-Lippmann-Schwinger kets $|E^\pm\rangle \in \Phi_\mp^\times$, also Gamow kets $|E_R \pm i\frac{\Gamma}{2}^\pm\rangle \in \Phi_\mp^\times$.

This mathematical assumption of distinct RHS's of Hardy class for states and observables is the new hypothesis of TAQT. At least in the non-relativistic theory [4, 21] one can connect this mathematical assumption to the causality condition that a state must be prepared at a time t_0 before an observable can be measured at time $t > t_0$. It is remarkable that from this single new assumption (A2) of the Hardy class property such a variety of new results follow.

Appendix B: Representing a Superposition of Resonances with Gamow Vectors and with Breit-Wigner Amplitudes

In this appendix we want to derive the expansion of the scattering amplitude (15), the complex eigenvector decomposition (25) and the correspondence between the terms in the sums. These results are consequences of the new hypothesis of Appendix A and the assumption (A2) will be employed extensively in the derivation.

We start with the S-matrix element (30) and consider the case that there are two resonances in the j-th partial wave—although these results are also proven for a countably infinite number of resonances (with qualifications) under the standard mathematical assumptions used for the S-matrix or potential V [42]. We restrict our analysis to only one partial wave, e.g. $j = 1$, and ignore all the other terms in the sum over j on the r.h.s. of (30). We also ignore the additional quantum numbers b and use the nonrelativistic notation, thinking of s as the energy E as we have done in Appendix A. (With a suitable change in the mathematical assumption (A2), this can be immediately carried over to the relativistic case.) Then we have for the j-th partial S-matrix element

$$(\psi^-, \phi^+)_j = \int_0^\infty dE \langle \psi^- | E j^- \rangle S_j(E) \langle {}^+ E j | \phi^+ \rangle \tag{B1}$$

The integration extends over the physical values (spectrum of H) of energy which starts at $E = 0$ and runs along the upper edge of the cut in the Riemann surface of $S_j(E)$. According to our assumption, the analytically continued S-matrix $S_j(z)$ has two (or countably many) resonance poles in the second sheet below the cut at positions $z_{R_i} = E_{R_i} - i\Gamma_{R_i}/2$, $i = 1, 2$. We shall make use of the Hardy class property given by our new hypothesis (causality) (A1) or (A2): ϕ^+ describes the prepared $(e\bar{e})$ in-state and ψ^- describes the observed $(f\bar{f})$ 'out-state' registered by the detector. Therefore, their energy wave functions $\langle\psi^-|Ej^-\rangle = \langle{}^-Ej|\psi^-\rangle^*$ and $\langle{}^+E|\phi^+\rangle$ are not just Lebesgue square integrable as in standard Hilbert space quantum mechanics, but fulfill a stronger condition:

$$\langle\psi^{out}|E\rangle = \langle\psi^-|E^-\rangle \in \mathcal{S}\cap\mathcal{H}^2_-, \quad \langle E|\phi^{in}\rangle = \langle{}^+E|\phi^+\rangle \in \mathcal{S}\cap\mathcal{H}^2_-. \tag{B2}$$

In the above relation, the eigenvectors $|E\rangle$ of the free Hamiltonian H_0 are connected to the eigenvectors $|E^\pm\rangle$ of the full Hamiltonian $H = H_0 + V$ by the Lippmann-Schwinger equation or the Møller wave operators:

$$|E^\pm\rangle = |E\rangle + \frac{1}{E - H \pm i0} V|E\rangle = \Omega^\pm|E\rangle. \tag{B3}$$

With the assumption (B2), the integrand in (B1) can be analytically continued into the lower half plane of the second sheet and the values of the functions are completely determined by their values on the positive real axis $\mathbb{R}_+$, i.e. by their physical values $\langle E|\phi^{in}\rangle$ and $\langle E|\psi^{out}\rangle$, where $|\langle E|\phi^{in}\rangle|^2$ describes the energy distribution in the beam and the energy resolution of the detector is given by $|\langle E|\psi^{out}\rangle|^2$. Now one can deform the contour of integration from the positive real line through the cut into the lower half plane as shown in Fig. 1.

The r.h.s. of (B1) becomes (dropping the j notation)

$$\begin{aligned}(\psi^-,\phi^+) \;=\; & \int_0^{-\infty} dE\,\langle\psi^-|E^-\rangle S^{bg}_{II}(E)\langle{}^+E|\phi^+\rangle \\ & + \oint_{\mathcal{C}_1} d\omega\,\langle\psi^-|\omega^-\rangle S_{II}(\omega)\langle{}^+\omega|\phi^+\rangle \\ & + \oint_{\mathcal{C}_2} d\omega\,\langle\psi^-|\omega^-\rangle S_{II}(\omega)\langle{}^+\omega|\phi^+\rangle\end{aligned} \tag{B4}$$

The first term has nothing to do with the resonances and we call it the background term

$$\int_0^{-\infty} dE\,\langle\psi^-|E^-\rangle S^{bg}_{II}(E)\langle{}^+E|\phi^+\rangle \equiv \langle\psi^-|\phi^{bg}\rangle; \tag{B5}$$

it is the same as (32).

We now use the expansion around the pole z_{R_i}

$$S(\omega) = \frac{s^{(i)}_{-1}}{\omega - z_{R_i}} + s_0 + s_1(\omega - z_{R_i}) + \cdots \tag{B6}$$

for each of the two (or countably many) integrals *separately.* The integrals around the poles we call the pole terms; they are given by

$$
\begin{aligned}
(\psi^-,\phi^+)_{\mathrm{poleterm}_i} &= \oint_{\mathcal{C}_i} d\omega\,\langle\psi^-|\omega^-\rangle S_{II}(\omega)\langle^+\omega|\phi^+\rangle && \text{(B7)}\\
&= \oint_{\mathcal{C}_i} d\omega\,\langle\psi^-|\omega^-\rangle \frac{s_{-1}^{(i)}}{\omega - z_{R_i}}\langle^+\omega|\phi^+\rangle && \text{(B8)}\\
&= -2\pi i s_{-1}^{(i)}\langle\psi^-|z_{R_i}{}^-\rangle\langle^+z_{R_i}|\phi^+\rangle && \text{(B9)}\\
&= \int_{-\infty_{II}}^{+\infty} dE\,\langle\psi^-|E^-\rangle\langle^+E|\phi^+\rangle\frac{s_{-1}^{(i)}}{E - z_{R_i}}. && \text{(B10)}
\end{aligned}
$$

It is worthwhile to explain what is used in each step of this calculation. To get from (B8) to (B9), the Cauchy theorem has been applied; to get from (B8) to (B10), the contour $\mathcal{C}_i$ of each integral separately has been deformed into the integral along the real axis (second sheet) from $-\infty_{II} < E < +\infty$ and along the infinite semicircle (which vanishes because of the Hardy class property). The equality of (B9) and (B10) is the Titchmarsh theorem for Hardy class functions (cf. footnote 4). More details are given in earlier publications, see for example the review [20], Section 5, where also the case of higher order S-matrix poles is considered.

The integral in (B10) extends from $E = -\infty_{II}$ in the second sheet along the real axis to $E = 0$ and then from $E = 0$ to $E = +\infty$ in either sheet. It does not matter whether we take the second part of the integral over the physical values of E, $0 \le E < \infty$, along the lower edge of the real axis in the second sheet or in the first sheet immediately above the real axis. The major contribution to the integral comes from the physical values $0 \le E < \infty$ if z_{R_i} is not too far from the real axis. Equation (B10) is a Breit-Wigner amplitude

$$
\frac{s_{-1}^{(i)}}{E - z_{R_i}} = \frac{s_{-1}^{(i)}}{E - (E_{R_i} - i\Gamma_{R_i}/2)} \text{ with } -\infty_{II} < E < +\infty_I. \tag{B11}
$$

integrated over along with the smooth and well-behaved apparatus functions $\langle^-E|\psi^-\rangle^*\langle^+E|\phi^+\rangle \in \mathcal{H}_-^2$. But unlike the conventional BW which (if one worries about these mathematical details) one takes only $0 \le E < +\infty$, it is an idealized or exact BW whose domain extends to $-\infty_{II}$ in the second (unphysical) sheet. This does not violate the condition of semi-boundedness of the Hamiltonian operator H. It is this BW that we mean by the "S-matrix Breit-Wigner" (11).

Each of the BW integrals in (B10) define a Gamow ket $|z_{R_i}{}^-\rangle$ corresponding the exact BW. From (B9) and (B10) one obtains the representation

$$\langle\psi^-|z_{R_i}{}^-\rangle = \frac{i}{2\pi}\int_{-\infty_{II}}^{+\infty} dE\, \frac{\langle\psi^-|E^-\rangle}{E-(E_{R_i}-i\Gamma_{R_i}/2)} s_{-1}^{(i)} \frac{\langle^+E|\phi^+\rangle}{\langle^+z_{R_i}|\phi^+\rangle}. \tag{B12}$$

This is equivalent to the definition (31).

We shall now write the j-th partial S-matrix element (ψ^-,ϕ^+) in two different ways. On the one hand, using (B9) (or (B12)) and (B5) in (B4) we obtain:

$$(\psi^-,\phi^+) = \langle\psi^-|\phi^{bg}\rangle + \sum_i \langle\psi^-|z_{R_i}{}^-\rangle(2\pi/i)\langle^+z_{R_i}|\phi^+\rangle. \tag{B13}$$

On the other hand, using (B1) and (B10) in (B4) we obtain:

$$\int_0^\infty dE\langle\psi^-|Ej^-\rangle S_j(E)\langle^+Ej|\phi^+\rangle = -\int_{-\infty}^0 dE\,\langle\psi^-|Ej^-\rangle S_j^{bg}(E)\langle^+Ej|\phi^+\rangle$$

$$+\sum_i \int_{-\infty_{II}}^{+\infty} dE\,\langle\psi^-|Ej^-\rangle\langle^+Ej|\phi^+\rangle \frac{s_{-1}^{(i)}}{E-z_{R_i}}. \tag{B14}$$

According to the van Winter theorem [43], a Hardy class function on the negative real axis (and therefore, by the Titchmarsh theorem, on the whole half plane) is uniquely determined by its values on the positive real axis, cf. Appendix A2 of [20]. Therefore we can use the Mellin transform to rewrite the first integral on the r.h.s. of (B14) and obtain [44]

$$\int_0^\infty dE\langle\psi^-|Ej^-\rangle S_j(E)\langle^+Ej|\phi^+\rangle = \int_0^\infty dE\,\langle\psi^-|Ej^-\rangle b_j(E)\langle^+Ej|\phi^+\rangle$$

$$+\sum_i \int_{-\infty_{II}}^{+\infty} dE\,\langle\psi^-|Ej^-\rangle\langle^+Ej|\phi^+\rangle \frac{s_{-1}^{(i)}}{E-z_{R_i}}. \tag{B15}$$

Without more specific information about the S-matrix, we cannot be certain about the energy dependence of the background $b_j(E)$. If there are no further poles or other singularities besides those included in the sum, then $b_j(E)$ is likely to be a slowly varying function of the energy. However, as long as we are only interested in the two resonances in some energy range under investigation, we can consider $b_j(E)$ as an irrelevant background.

The equations (B13) and (B15) carry the same information and each term in (B15) has a corresponding term in (B13). The first term on the r.h.s. represents the non-resonant background and the terms in the sum represent the resonant terms.

We first discuss (B13). Omitting the arbitrary $\psi^- \in \Phi_+$ (the observable) one writes this as a functional equation in the space $\Phi_+^\times$ and obtains the complex basis vector expansion (25) of the prepared in-state $\phi^+ \in \Phi_-$:

$$\phi^+ = \phi^{bg} + \sum_i |z_{R_i}{}^-\rangle c_{R_i}, \text{ where } c_{R_i} = (2\pi/i)\langle^+ z_{R_i}|\phi^+\rangle. \tag{B16}$$

The in-state ϕ^+ can be decomposed into a vector representing the non-resonant part ϕ^{bg} and a sum over the resonance states described by the Gamow kets. The complex spectral resolution (B16) is a generalized eigenvector expansion like Dirac's eigenvector expansion,

$$\phi^+ = \int_0^\infty dE\, |E^+\rangle\langle^+ E|\phi^+\rangle. \tag{B17}$$

While (B17) expresses the in-state ϕ^+ in terms of scattering states described by the eigenkets $|E^+\rangle$ of the Hamiltonian H with real generalized eigenvalue E, (B16) is an expansion in terms of eigenkets $|z_{R_i}{}^-\rangle$ of the same *self-adjoint* Hamiltonian H with complex generalized eigenvalue $z_{R_i} = E_{R_i} - i\Gamma_{R_i}/2$.

We now discuss (B15). This equality holds for the whole space of functions $\langle\psi^-|E^-\rangle\langle^+E|\phi^+\rangle \in \mathcal{S} \cap \mathcal{H}^2_-$. Therefore we can omit these arbitrary energy wave functions

$$\langle\psi^-|E^-\rangle\langle^+E|\phi^+\rangle = \langle\psi^{out}|E\rangle\langle E|\phi^{in}\rangle \in \mathcal{S} \cap \mathcal{H}^2_-$$

which describe the resolution of the preparation apparatus and of the registration apparatus and write equation (B15) as an equation between distributions over the function space $\mathcal{S} \cap \mathcal{H}^2_-$:

$$S_j(E) = b_j(E) + \sum_i \frac{s_{-1}^{(i)}}{E - z_{R_i}}. \tag{B18}$$

Each term in (B16) has a corresponding term in (B18). The vector ϕ^{bg} describing the non-resonant part of the state ϕ^+ (which was prepared in the distant past as the non-interacting prepared in-state ϕ^{in}) corresponds to the background part of the S-matrix element S^{bg} (or in the amplitude (15) to the background amplitude $B(s) = (b_j(E) - 1)/2ip$). Note that there must always be such a background term $B(s)$ or ϕ^{bg} phenomenologically *and* theoretically, according to (B16) and (B18), but in many cases it may not be relevant for the physical problem and can be ignored. To the superposition of resonance state vectors described by the Gamow kets $|z_{R_i}{}^{-}\rangle$ in (B16) corresponds in (B18) the "superposition" of resonance amplitudes described by the exact (with $-\infty_{II} < E < +\infty$) Breit-Wigner amplitudes. Thus we have the correspondence:

$$\left\{\begin{array}{l}\text{Gamow vector } |j, s_r; b_{rest}{}^{-}\rangle \\ \text{of (37) with exact exponen-} \\ \text{tial decay with lifetime } \tau = \\ \hbar/\Gamma_R\end{array}\right\} \Leftrightarrow \left\{\begin{array}{l}\text{Breit-Wigner amplitude } a_j^R(s) \\ \text{of (11), where } s \text{ extends over} \\ -\infty_{II} < s < +\infty \text{ with width} \\ \Gamma_R = \hbar/\tau\end{array}\right\} \tag{B19}$$

The precise meaning of the superposition (B18) is the full equation (B15). The representation (15) of the j-th partial amplitude is the analogue of (B18) and must therefore be understood as a convolution integral using the apparatus wave functions with the property (B2). Expressions like (B18) are not a consequence of the usual assumptions of analyticity but follow from the new hypothesis of the RHS formulation of quantum mechanics stated in Appendix A. The interference of resonances as predicted by (B18) has been tested experimentally for the non-relativistic case in nuclear physics [45] and for the relativistic case in the $\rho - \omega$ interference which arises as a consequence of isospin symmetry breaking [32]. This phenomenological success of formulas like (15) or (B18) for two interfering resonances is another argument in favor of the new hypotheses (A1) and (A2) for quantum theory.

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Group and the SLD Heavy Flavour and Electroweak Groups, CERN-preprint 99-15. However, the values quoted there are $\bar{M}_Z + 34.1$MeV and $\bar{\Gamma}_Z + 0.9$MeV. We calculate M_R and Γ_R from the values of M_ρ and Γ_ρ using the algebraic relation (10b) since (M_R, Γ_R) and (M_ρ, Γ_ρ) are just alternate parameterizations of the same BW amplitude with s-independent width.

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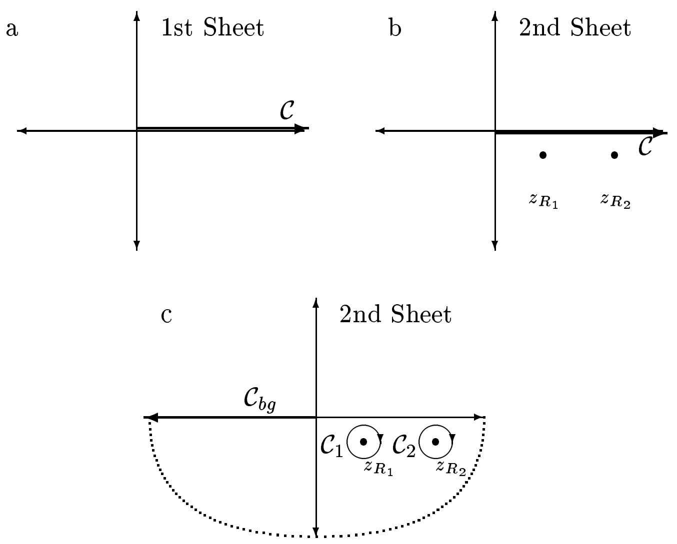

Figure 1: In this figure, the cut lies along the positive real axis. We begin with (a)—the integral is taken $i\epsilon$ above the cut in the first sheet. This integral can equivalently be taken $-i\epsilon$ below the cut in the second sheet (b). This contour in (b) can be deformed as in (c) and is equivalent to a sum of contours $\mathcal{C}_{bg}$, $\mathcal{C}_1$ and $\mathcal{C}_2$ since the integral along infinite semicircle in the lower half plane vanishes (because of the Hardy class properties of (B2)).