Four-Fermion Working Group Report

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DRAFT

Contents

1. The CC03 cross section

In order to extract the WW signal from the full set of $\mathrm{e^+e^-} \to 4f$ processes, the CC03 cross section was introduced and discussed in [4]. In lowest order, this cross section is simply based on the three WW signal diagrams with the full four-particle kinematics with off-shell W bosons. Compared to the full set of diagrams, the CC03 subset depends only trivially on the final state and allows to combine all channels easily. However, since the CC03 cross section is based on a subset of diagrams, it is gauge-dependent and usually defined in the 't Hooft–Feynman gauge. While the CC03 cross section is not an observable, it is nevertheless a useful quantity at LEP 2 energies where it can be classified as a pseudo-observable. It contains the interesting physics, such as the non-abelian couplings and the sensitivity of the total cross section to M_W near the W-pair threshold. The goal of this common definition is to be able to combine the different final state measurements from different experiments so that the new theoretical calculations can be checked with data at a level better than 1%. Note, however, that the CC03 cross section will become very problematic at linear-collider energies, where the background diagrams and the gauge dependences are much larger.

The double-pole approximation (DPA) of the lowest-order cross section emerges from the CC03 diagrams upon projecting the W-boson momenta in the matrix element to their on-shell values. This means that the DPA is based on the residue of the double resonance, which is a gauge-invariant quantity, because it is directly related to the sub-processes of on-shell W-pair production and on-shell W decay. In contrast to the CC03 cross section, the DPA is theoretically well-defined. The price to be paid for this is the exclusion of the threshold region, where the DPA is not valid. On the other hand, the DPA provides a convenient framework for the inclusion of radiative corrections.

It is worth summarizing the status of the WW cross-section prior to the 2000 Winter Conferences. Nominally, any $\mathrm{e^+e^-} \to \mathrm{WW} \to 4f$ is a tree level calculation and one could try the standard procedure of including, in a reasonable way, as much as possible of the universal corrections by constructing an improved Born approximation (hereafter IBA). This is the way the data have been analyzed so far, mostly with the help of `GENTLE`. Experimenters have compared `GENTLE` and `grc4f` for CC03. When one puts the same input parameters, renormalization scheme, etc, a technical agreement at the 0.1% level is found. This is not enough, we wish the theoretical uncertainty to be below 1% (0.5% seems possible) with justification.

Indeed, we have clear indications that non-universal electroweak corrections for WW(CC03) cross-section are not small and even larger than the experimental LEP accuracy. `GENTLE` will produce a CC03 cross-section, typically in the G_F-scheme, with universal ISR QED and non-universal ISR/FSR QED corrections, implemented with the so-called current-splitting technique. The corresponding curve has been used for the definition of the Standard Model prediction with a $\pm 2\%$ systematic error assigned to it. Note that, in `GENTLE`, the non-universal ISR correction with current-splitting technique reads as $+0.4\%$ effect at LEP 2 energies.

Recently, a new electroweak $\mathcal{O}(\alpha)$ CC03 cross-section has become available, in the framework of DPA, showing a result that is 2–2.5% smaller than the CC03 cross section from `GENTLE`. This is a **new numbers** big effect since the combined experimental accuracy of LEP experiments is of the same magnitude. It is, therefore, of the upmost importance to understand the structure of a DPA-corrected CC03 cross-section.

2. Dealing with unstable particles

The main purpose of this report is to qualify as a reference handbook for precision experiments at LEP 2 energies. As such, the report is essentially devoted to describe the state-of-the-art in the predictions of the MC adopted by the experimental Collaborations, with particular emphasis on reaching a reliable estimate of the Standard Model background to searches.

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Comment Sasha and Wim: Agreed ←

Nevertheless, most of our *technical* problems originate from the complications naturally pertaining to the gauge structure of the theory and to the presence of unstable particles. As an interlude, we would like to summarize the nominal essence of the theoretical basis of all generators. In this respect one should remember that several, new, theoretical ideas were fully developed also as a consequence of a previous workshop on WW-physics and, in turn, many generators have profited from the most recent theoretical development. Furthermore, this Section will be a natural place where to add some consideration about the fine points in the DPA-procedure.

Four-fermion production processes, with or without radiative corrections, all involve fermions in the initial and final state and unstable gauge bosons as intermediate particles. Sometimes a photon is also present in the final state. If complete sets of graphs contributing to a given process are taken into account, the associated matrix elements are in principle gauge-invariant, i.e. they are independent of gauge fixing and respect Ward identities. This is, however, not guaranteed for incomplete sets of graphs like the ones corresponding to the off-shell W-pair production process (CC03). Indeed this process has been found to violate the $SU(2)$ Ward identities [25].

In addition, the unstable gauge bosons that appear as intermediate particles can give rise to poles $1/(p^2 - M^2)$ in physical observables if they are treated as stable particles. In view of the high precision of the LEP 2 experiments, the proper treatment of these unstable particles has become a demanding exercise, since on-shell approximations are simply not good enough anymore. A proper treatment of unstable particles requires the re-summation of the corresponding self-energies to all orders. In this way the singularities originating from the poles in the on-shell propagators are regularized by the imaginary parts contained in the self-energies, which are closely related to the decay widths (Γ) of the unstable particles. The perturbative re-summation itself involves a simple geometric series and is therefore easy to perform. However, this simple procedure harbours the serious risk of breaking gauge invariance. Gauge invariance is guaranteed order by order in perturbation theory. Unfortunately one takes into account only part of the higher-order terms by re-summing the self-energies. This results in a mixing of different orders of perturbation theory and thereby jeopardizes gauge invariance, even if the self-energies themselves are extracted in a gauge-invariant way. Apart from being theoretically unacceptable, gauge-breaking effects can also lead to large errors in the MC predictions. At LEP 2 energies this problem occurs in the reactions $e^+e^- \to e^-\bar{\nu}_e u\bar{d}$, $e^+\nu_e\bar{u}d$ for forward-scattered beam particles [26].

Based on this observation, it is clear that a gauge-invariant scheme is required for the treatment of unstable particles. It should be stressed, however, that any such scheme is arbitrary to a greater or lesser extent: since the Dyson summation must necessarily be taken to all orders of perturbation theory, and we are not able to compute the complete set of *all* Feynman diagrams to *all* orders, the various schemes differ even if they lead to formally gauge-invariant results. Bearing this in mind, we need besides gauge invariance some physical motivation for choosing a particular scheme. In this context two options can be mentioned. Either one can try to *subtract* gauge-violating terms or one can try to *add* gauge-restoring terms to the calculation.

The first option is the so-called *pole scheme* [27]. In this scheme one decomposes the complete amplitude by expanding around the poles. As the physically observable residues of the poles are gauge-invariant, gauge invariance is not broken if the finite width is taken into account in the pole terms $\propto 1/(p^2 - M^2)$. In reactions with multiple unstable-particle resonances it is rather awkward to perform the complete pole-scheme expansion with all its subtleties in the treatment of the off-shell phase space [28, 17]. Therefore one usually approximates the expansion by retaining only the terms with the highest degree of resonance. This approximation is called the leading-pole approximation and is closely related to on-shell production and decay of the unstable particles. The accuracy of the approximation is typically $\mathcal{O}(\Gamma/M)$, making it a suitable tool for calculating *radiative corrections*, since in that case the errors are further suppressed by powers of the coupling constant [28, 17]. Since diagrams with a lower degree of resonance do not feature in the leading-pole approximation, it is not an adequate approach for describing lowest-order reactions. So, for lowest-order reactions one needs an alternative approach.

The second option is based on the fundamentally different philosophy of trying to determine and include the minimal set of Feynman diagrams that is necessary for compensating the gauge violation caused by the self-energy graphs. This is obviously the theoretically most satisfying solution, but it may cause an increase in the complexity of the matrix elements and consequently a slowing down of the numerical calculations. Two methods have been developed along these lines. First of all, for the gauge bosons we are guided by the observation that the *lowest-order* decay widths are exclusively given by the imaginary parts of the fermion loops in the one-loop self-energies. It is therefore natural to perform a Dyson summation of these fermionic one-loop self-energies and to include the other possible one-particle-irreducible fermionic one-loop corrections (*fermion-loop scheme*) [26, 29, 30, 31]. For the *lowest-order* LEP 2 process $e^+e^- \to 4\mathrm{f}$ this amounts to adding the fermionic corrections to the triple gauge-boson vertex. The complete set of fermionic contributions forms a manifestly gauge-invariant subset, since it involves the closed subset of all $\mathcal{O}([N_c^f\alpha/\pi]^n)$ contributions (with N_c^f denoting the colour degeneracy of fermion f). Moreover, it obeys all Ward identities exactly, even with re-summed propagators, as shown in Ref. [30] for two- and four-fermion production (see also Ref. [180] for a more general discussion of the interplay between Ward identities and Dyson resummation). The limitation of the fermion-loop scheme is due to the fact that it does not apply to particles with bosonic decay modes and that on resonance one perturbative order is lost. This in turn disqualifies it as a candidate for handling radiative corrections. Moreover, the inclusion of a full-fledged set of one-loop corrections in a lowest-order amplitude tends to over-complicate things for reactions like $e^+e^- \to 4\mathrm{f}\gamma$.

Comment Sasha and Wim: Let's stick to the fermion-loop scheme as implemented in the MC. If we start to mention the background-field method we will have to discuss it completely (concept + problems like the gauge-dependence in the bosonic sector) AND we will have to mention the pinch technique. That would really open a can of worms.

A very simple, *ad hoc* way to compensate width effects of propagators by vertex modifications is to analytically continue the corresponding mass parameters in all Feynman rules consistently, leading to complex couplings. This *complex-mass scheme* preserves all Ward identities and works for lowest-order predictions. It should be noted, however, that the so-obtained complex couplings and masses do not follow from perturbation theory. For space-like gauge-boson momenta, for instance, there should be no imaginary contribution to the propagator. As such the *complex-mass scheme* harbours the potential risk of changing the CP-structure of the theoretical predictions. In [6] this procedure was applied to $e^+e^- \to 4\mathrm{f}$ and $e^+e^- \to 4\mathrm{f}\gamma$.

Recently a novel non-diagrammatic technique has been proposed for arbitrary *tree-level* reactions, involving all possible unstable particles and an unspecified amount of stable external particles [32]. By using gauge-invariant non-local effective Lagrangians, it is possible to generate the self-energy effects in the propagators as well as the required gauge-restoring terms in the multi-particle (3-point, 4-point, etc.) interactions. In this way the full set of Ward identities can be solved, while keeping the gauge-restoring terms to a minimum. At present the first MC implementation of this new method in an existing program, HELAC [33], is in progress. Hence, its full potential is still under scrutiny.

3. The leading-pole approximation

As mentioned above, the pole scheme consists in decomposing the complete amplitude by expanding around the poles of the unstable particles. The residues in this expansion are physically observable and therefore gauge-invariant. The pole-scheme expansion can be viewed as a gauge-invariant prescription for performing an expansion in powers of Γ/M. It should be noted that there is no unique definition of the residues. Their calculation involves a mapping of off-shell matrix elements with off-shell kinematics on on-resonance matrix elements with restricted kinematics. This mapping, however, is not unambiguously fixed. After all, it involves more than just the invariant masses of the unstable particles and one thus has to specify the variables that have to be kept fixed in the mapping. The resulting implementation dependence manifests itself in differences of sub-leading nature, e.g. $\mathcal{O}(\Gamma/M)$ suppressed deviations in the leading pole-scheme residue. In special regions of phase space, where the matrix elements vary rapidly, the implementation dependence can take noticeable proportions. This happens in particular near phase-space boundaries, like thresholds.

In order to make these statements a bit more transparent, we sketch the pole-scheme method for a single unstable particle. In this case the Dyson re-summed lowest-order matrix element is given by

$$\begin{aligned}\mathcal{M}^{\infty} &= \frac{W(p^2,\omega)}{p^2-\tilde{M}^2}\sum_{n=0}^{\infty}\left(\frac{-\tilde{\Sigma}(p^2)}{p^2-\tilde{M}^2}\right)^n = \frac{W(p^2,\omega)}{p^2-\tilde{M}^2+\tilde{\Sigma}(p^2)}\\ &= \frac{W(M^2,\omega)}{p^2-M^2}\,\frac{1}{Z(M^2)}+\left[\frac{W(p^2,\omega)}{p^2-\tilde{M}^2+\tilde{\Sigma}(p^2)}-\frac{W(M^2,\omega)}{p^2-M^2}\,\frac{1}{Z(M^2)}\right],\end{aligned} \tag{1}$$

where $\tilde{\Sigma}(p^2)$ is the unrenormalized self-energy of the unstable particle with momentum p and unrenormalized mass $\tilde{M}$. The renormalized quantity M^2 is the pole in the complex p^2 plane, whereas $Z(M^2)$ denotes the wave-function factor:

$$M^2-\tilde{M}^2+\tilde{\Sigma}(M^2)=0, \qquad Z(M^2)=1+\tilde{\Sigma}'(M^2). \tag{2}$$

The first term in the last expression of Eq. (1) represents the single-pole residue, which is closely related to on-shell production and decay of the unstable particle. The second term between the square brackets has no pole and can be expanded in powers of p^2-M^2. The argument ω denotes the dependence on the other variables, i.e. the implementation dependence. After all, the unstable particle is always accompanied by other particles in the production and decay stages. For instance, consider the LEP1 reaction $e^+e^-\to\bar{f}f$. In the mapping $p_Z^2\to M^2$ one can either keep $t=(p_{e^-}-p_f)^2=-p_Z^2(1-\cos\theta)/2$ fixed or $\cos\theta$. In the former mapping $\cos\theta_{\rm pole}$ is obtained from the on-shell relation $\cos\theta_{\rm pole}=1+2t/M^2$, whereas in the latter mapping $t_{\rm pole}=-M^2(1-\cos\theta)/2$. It may be that a particular mapping leads to an unphysical point in the on-shell phase space. In the present example $t_{\rm pole}$ will always be physical when $\cos\theta$ is kept fixed in the mapping. However, since $|\cos\theta_{\rm pole}|>1$ for $t<-{\rm Re}M^2$, it is clear that mappings with fixed Mandelstam variables harbour the potential risk of producing such unphysical phase-space points.[1] This can have repercussions on the convergence of the pole-scheme expansion. Therefore it is recommended to use implementations that are free of unphysical on-shell phase-space points.

The at present only workable approach for evaluating the radiative corrections to resonance-pair-production processes, like W-pair production, involves the so-called leading-pole approximation (LPA). This approximation restricts the complete pole-scheme expansion to the term with the highest degree of resonance. In the case of W-pair production only the double-pole residues are hence considered. This is usually referred to as the double-pole approximation (DPA). The intrinsic error associated with this procedure is $\alpha\Gamma_W/(\pi M_W)\times\log(\ldots)\lesssim 0.5\%$, $\leftarrow$

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except far off resonance, where the pole-scheme expansion cannot be viewed as an effective expansion in powers of Γ/M, and close to phase-space boundaries, where the DPA cannot be trusted to produce the dominant contributions. In the latter situations also the implementation dependence of the double-pole residues can lead to enhanced errors. Close to the nominal (on-shell) W-pair threshold, for instance, the intrinsic error is effectively enhanced by a factor $M_W/(\sqrt{s}-2M_W)\equiv M_W/\Delta E$. In view of this it is wise to apply the DPA only if the energy is several Γ_W above the threshold.

In the DPA one can identify two types of contributions. One type comprises all diagrams that are strictly reducible at both unstable W-boson lines (see Fig. 1). These corrections are therefore called

[1] In the resonance region, $|p_Z^2-M^2|\ll|M^2|$, the unphysical on-shell phase-space points occur near the edge of the

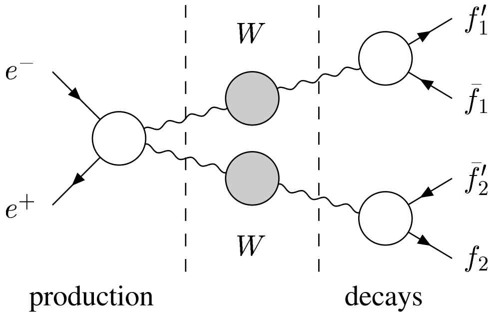


Fig. 1: The generic structure of the factorizable W-pair contributions. The shaded circles indicate the Breit–Wigner resonances, whereas the open circles denote the Green functions for the production and decay sub-processes up to $\mathcal{O}(\alpha)$ precision.

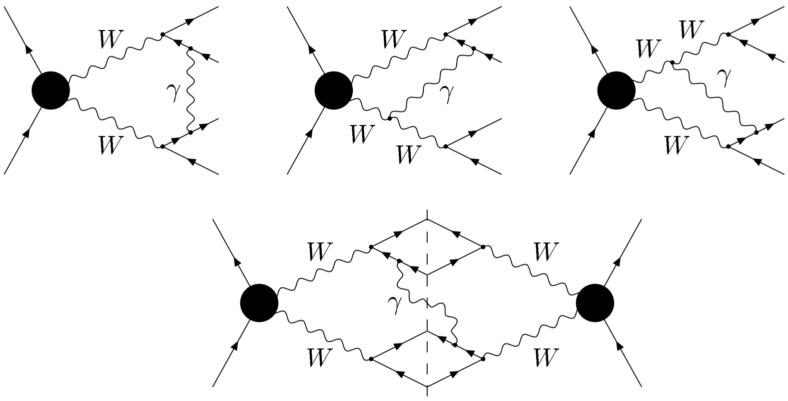


Fig. 2: Examples for virtual (top) and real (bottom) non-factorizable corrections to W-pair production. The black circles denote the lowest-order Green functions for the production of the virtual W-boson pair.

factorizable and can be attributed unambiguously either to the production of the W-boson pair or to one of the subsequent decays. The second type consists of all diagrams in which the production and/or decay sub-processes are not independent and which therefore do not seem to have two overall W propagators as factors (see Fig. 2). We refer to these effects as non-factorizable corrections.[2] In the DPA the non-factorizable corrections arise exclusively from the exchange or emission of photons with $E_\gamma \lesssim \Gamma_W$ [34]. Hard photons as well as massive-particle exchanges do not lead to double-resonant contributions. The physical picture behind all of this is that in the DPA the W-pair process can be viewed as consisting of several sub-processes: the production of the W-boson pair, the propagation of the W bosons, and the subsequent decay of the unstable W bosons. The production and decay are hard sub-processes, which occur on a relatively short time interval, $\mathcal{O}(1/M_W)$. They are in general distinguishable as they are well separated by a relatively big propagation interval, $\mathcal{O}(1/\Gamma_W)$. Consequently, the corresponding amplitudes have certain factorization properties. The same holds for the radiative corrections to the sub-processes. The only way the various stages can be interconnected is via the radiation of soft photons with energy of $\mathcal{O}(\Gamma_W)$. These photons induce relatively long range interactions and thereby allow the various sub-processes to communicate with each other.

As is clear from the above-given discussion of the DPA, a specific prescription has to be given for

off-shell phase space, since $t < -\mathrm{Re}M^2$ requires $\cos\theta \approx -1$.

[2]It should be noted that the exact split-up between factorizable and non-factorizable radiative corrections requires a precise (gauge-invariant) definition. We will come back to this point.

the calculation of the DPA residues. Or, in other words, we have to fix the implementation of the mapping of the full off-shell phase space on the kinematically restricted (on-resonance) one. Two strategies have been adopted in the literature [28, 176]. One can opt to always extract pure double-pole residues [28]. This means in particular that after the integration over decay kinematics and invariant masses has been performed the on-shell cross-section should be recovered. Alternatively, one can decide to exclude the off-shell phase space from the mapping and apply the residue only to the matrix elements [176]. We will come back to the conceptual and numerical differences between these two implementation strategies in the detailed discussion of the DPA programs. At this point we merely note that the numerical differences can serve as an estimate of the theoretical uncertainty of the DPA procedure.

In the rest of this section we will explain those aspects of the DPA procedure that are common to both implementation methods. To this end we focus on the lowest-order reaction

$$e^+(q_1)\,e^-(q_2) \to W^+(p_1)\,W^-(p_2) \to \bar{f}_1(k_1)f_1'(k_1')\,f_2(k_2)\bar{f}_2'(k_2'), \tag{3}$$

involving only those diagrams that contain as factors the Breit–Wigner propagators for the W^+ and W^- bosons. Here $\bar{f}_1$ and f_1' are the decay products of the W^+ boson, and f_2 and $\bar{f}_2'$ those of the W^- boson. It should be noted that a large part of the radiative corrections in DPA to this reaction can be treated in a way similar to the lowest-order case, which is therefore a good starting point. The amplitude for process (3) takes the form

$$\mathcal{M} = \sum_{\lambda_1,\lambda_2} \Pi_{\lambda_1\lambda_2}(M_1,M_2)\,\frac{\Delta^{(+)}_{\lambda_1}(M_1)}{D_1}\,\frac{\Delta^{(-)}_{\lambda_2}(M_2)}{D_2}\,, \tag{4}$$

where any dependence on the helicities of the initial- and final-state fermions has been suppressed, and

$$D_i = M_i^2 - M_W^2 + iM_W\Gamma_W, \qquad M_i^2 = (k_i + k_i')^2. \tag{5}$$

The quantities $\Delta^{(+)}_{\lambda_1}(M_1)$ and $\Delta^{(-)}_{\lambda_2}(M_2)$ are the off-shell W-decay amplitudes for specific spin-polarization states λ_1 (for the W^+) and λ_2 (for the W^-), with $\lambda_i = (-1,0,+1)$. The off-shell W-pair production amplitude $\Pi_{\lambda_1\lambda_2}(M_1,M_2)$ depends on the invariant fermion-pair masses M_i and on the polarizations λ_i of the virtual W bosons. In the limit $M_i \to M_W$ the amplitudes Π and $\Delta^{(\pm)}$ go over into the on-shell production and decay amplitudes.

In the cross-section the above factorization leads to

$$\sum_{\text{fermion helicities}} |\mathcal{M}|^2 \;=\; \sum_{\lambda_1,\lambda_2,\lambda_1',\lambda_2'} \mathcal{P}_{[\lambda_1\lambda_2][\lambda_1'\lambda_2']}(M_1,M_2)\,\frac{\mathcal{D}_{\lambda_1\lambda_1'}(M_1)}{|D_1|^2}\,\frac{\mathcal{D}_{\lambda_2\lambda_2'}(M_2)}{|D_2|^2}. \tag{6}$$

In Eq. (6) the production part is given by a 9×9 density matrix

$$\mathcal{P}_{[\lambda_1\lambda_2][\lambda_1'\lambda_2']}(M_1,M_2) \;=\; \sum_{e^\pm\ \text{helicities}} \Pi_{\lambda_1\lambda_2}(M_1,M_2)\,\Pi^*_{\lambda_1'\lambda_2'}(M_1,M_2). \tag{7}$$

Similarly the decay part is governed by 3×3 density matrices

$$\mathcal{D}_{\lambda_i\lambda_i'}(M_i) \;=\; \sum_{\text{fermion helicities}} \Delta_{\lambda_i}(M_i)\,\Delta^*_{\lambda_i'}(M_i), \tag{8}$$

where the summation is performed over the helicities of the final-state fermions.

It is clear that Eq. (7) is closely related to the absolute square of the matrix element for stable unpolarized W-pair production. In that case the cross-section contains the trace of the above density matrix

$$\mathbf{Tr}\,\mathcal{P}(M_W,M_W) = \sum_{\lambda_1,\lambda_2} \mathcal{P}_{[\lambda_1\lambda_2][\lambda_1\lambda_2]}(M_W,M_W) \;=\; \sum_{\text{all polarizations}} |\Pi_{\lambda_1\lambda_2}(M_W,M_W)|^2. \tag{9}$$

The decay of an unpolarized on-shell W boson is determined by

$$\mathbf{Tr}\,\mathcal{D}(M_W) = \sum_{\lambda_i} \mathcal{D}_{\lambda_i\lambda_i}(M_W) \;=\; \sum_{\text{all polarizations}} |\Delta_{\lambda_i}(M_W)|^2. \tag{10}$$

Note, however, that also the off-diagonal elements of $\mathcal{P}(M_W, M_W)$ and $\mathcal{D}(M_W)$ are required for determining Eq. (6) in the limit $M_i \to M_W$.

As a next step it is useful to describe the kinematics of process (3) in a factorized way, i.e. using the invariant masses M_1 and M_2 of the fermion pairs. The differential cross-section takes the form

$$d\sigma = \frac{1}{2s}\sum |\mathcal{M}|^2\, d\Gamma_{4\text{f}} = \frac{1}{2s}\sum |\mathcal{M}|^2\, d\Gamma_{\text{pr}} \cdot d\Gamma^{+}_{\text{dec}} \cdot d\Gamma^{-}_{\text{dec}} \cdot \frac{dM_1^2}{2\pi} \cdot \frac{dM_2^2}{2\pi}, \tag{11}$$

where $d\Gamma_{4\text{f}}$ indicates the complete four-fermion phase-space factor and $s = (q_1+q_2)^2$ the centre-of-mass energy squared. The phase-space factors for the production and decay sub-processes, $d\Gamma_{\text{pr}}$ and $d\Gamma^{\pm}_{\text{dec}}$, read

$$\begin{aligned}
d\Gamma_{\text{pr}} &= \frac{1}{(2\pi)^2}\,\delta(q_1+q_2-p_1-p_2)\,\frac{d\vec{p}_1}{2p_{10}}\,\frac{d\vec{p}_2}{2p_{20}}, \\
d\Gamma^{+}_{\text{dec}} &= \frac{1}{(2\pi)^2}\,\delta(p_1-k_1-k_1')\,\frac{d\vec{k}_1}{2k_{10}}\,\frac{d\vec{k}_1'}{2k_{10}'}, \\
d\Gamma^{-}_{\text{dec}} &= \frac{1}{(2\pi)^2}\,\delta(p_2-k_2-k_2')\,\frac{d\vec{k}_2}{2k_{20}}\,\frac{d\vec{k}_2'}{2k_{20}'}.
\end{aligned} \tag{12}$$

When the factorized form for $\sum |\mathcal{M}|^2$ is inserted one obtains

$$\begin{aligned}
d\sigma &= \frac{1}{2s}\sum_{\lambda_1,\lambda_2,\lambda_1',\lambda_2'} \mathcal{P}_{[\lambda_1\lambda_2][\lambda_1'\lambda_2']}(M_1,M_2)\,d\Gamma_{\text{pr}} \times \mathcal{D}_{\lambda_1\lambda_1'}(M_1)\,d\Gamma^{+}_{\text{dec}} \times \mathcal{D}_{\lambda_2\lambda_2'}(M_2)\,d\Gamma^{-}_{\text{dec}} \times \\
&\quad \times \frac{1}{2\pi}\,\frac{dM_1^2}{|D_1|^2} \times \frac{1}{2\pi}\,\frac{dM_2^2}{|D_2|^2},
\end{aligned} \tag{13}$$

which is the common starting point for any of the DPA implementations.

4. Radiative corrections in double-pole approximation

A full calculation of the complete electroweak $\mathcal{O}(\alpha)$ corrections to $\mathrm{e}^+\mathrm{e}^- \to 4f(+\gamma)$ for all four-fermion final states is beyond present possibilities. While the real bremsstrahlung corrections induced by $\mathrm{e}^+\mathrm{e}^- \to 4f\gamma$ are known exactly [5, 6, 7], there are severe technical and conceptual problems with the virtual corrections to four-fermion production. Fortunately, the full account of the $\mathcal{O}(\alpha)$ corrections is not needed at the level of accuracy demanded by LEP 2. For W-pair-mediated processes, $\mathrm{e}^+\mathrm{e}^- \to \mathrm{WW} \to 4f$, the required accuracy of predictions is of the order of 0.5% for integrated quantities. At this level, the corrections to W-pair production can be treated in the DPA. In regions of phase space where two resonant W bosons do not dominate the cross sections, such as in the WW-threshold region or in the single-W domain, the DPA is, of course, not valid and one should resort to other approximations as the Weizsäcker-Williams for single-W.

Since only diagrams with two nearly resonant W bosons are relevant for the DPA, the number of graphs is reduced considerably, and a generic treatment of all four-fermion final states is possible. Obviously all diagrams that appear for the pair production and the decay of on-shell W bosons are also relevant for the pole expansion in the DPA. Since such contributions involve a product of two independent Breit–Wigner factors for the W resonances, they are called *factorizable* corrections. However, there exist also doubly-resonant corrections in which the production and decay sub-processes do not proceed independently. Power counting reveals that such corrections are only doubly-resonant if the particle that is exchanged by the sub-processes is a low-energetic photon. Owing to the complicated off-shell behaviour of these corrections, they are called *non-factorizable*.

While the definition of the DPA is straightforward for the virtual corrections, it is problematic for the real corrections. The problem is due to the momentum carried away by photon radiation. The invariant masses of the W bosons in contributions in which the photon is emitted in the W-pair production subprocess differ from those where the photon is emitted in the W-decay sub-processes. The corresponding Breit–Wigner resonances overlap if the energy of the emitted photon is of the order of Γ_{W}. It is not obvious how to define the DPA for such photons. Therefore, the results based on a DPA for the real corrections have to be treated with some caution.

According to the above classification, there are four categories of contributions to $\mathcal{O}(\alpha)$ corrections in DPA: factorizable and non-factorizable ones both for virtual and real corrections. In the following the salient features of those four parts are described.

Comment Sasha and Wim: we have moved the last but one paragraph to the next subsection and we have removed the last paragraph in order to avoid repetitions

4.01 Virtual corrections

As a first step we discuss how to separate the virtual corrections into a sum of factorizable and non-factorizable virtual corrections. The diagrammatic split-up according to reducible and irreducible W-boson lines is an illustrative way of understanding the different nature of the two classes of corrections, but since the double-resonant diagrams are not gauge-invariant by themselves the precise split-up needs to be defined properly.

We can make use of the fact that there are effectively two scales in the problem: M_W and Γ_W. Let us now consider virtual corrections coming from photons with different energies:

- soft photons, $E_\gamma \ll \Gamma_W$,
- semi-soft photons, $E_\gamma = \mathcal{O}(\Gamma_W)$,
- hard photons, $\Gamma_W \ll E_\gamma = \mathcal{O}(M_W)$.

Only soft and semi-soft photons contribute to both factorizable and non-factorizable corrections. The latter being defined to describe interactions between different stages of the off-shell process. The reason for this is that only these photons can induce relatively long-range interactions and thereby allow the various sub-processes, which are separated by a propagation interval of $\mathcal{O}(1/\Gamma_W)$, to communicate with each other. Virtual corrections involving the exchange of hard photons or of massive particles contribute exclusively to the factorizable corrections. In view of the short range of the interactions induced by these particles, their contribution to the non-factorizable corrections are suppressed by at least $\mathcal{O}(\Gamma_W/M_W)$.

As hard photons contribute to the factorizable corrections only, we merely need to define a split-up for soft and semi-soft photons. It is impossible to do this in a consistent gauge-invariant way on the basis of diagrams. In Refs. [28, 17] it was shown that only part of particular diagrams should be attributed to the non-factorizable corrections, the rest being of factorizable nature. The complete set of non-factorizable corrections was obtained by collecting all terms that contain the ratios $D_i/[D_i \pm 2kp_i]$, where k denotes the momentum of the (semi-)soft photon. The so-defined non-factorizable corrections read [28]

$$\mathcal{M}_{\mathrm{nf}}^{\mathrm{virt}} = i\mathcal{M}_{\mathrm{DPA}}^{0} \int \frac{d^4k}{(2\pi)^4[k^2+io]} \Big[(\mathcal{J}_0^\mu + \mathcal{J}_\oplus^\mu)\mathcal{J}_{+,\mu} + (\mathcal{J}_0^\mu + \mathcal{J}_\ominus^\mu)\mathcal{J}_{-,\mu} + \mathcal{J}_+^\mu \mathcal{J}_{-,\mu} \Big], \qquad (14)$$

which contains the gauge-invariant currents

$$\mathcal{J}_0^\mu = e\left[\frac{p_1^\mu}{kp_1+io} + \frac{p_2^\mu}{-kp_2+io} \right],$$

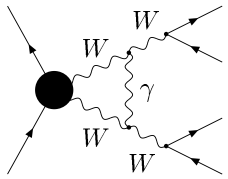


Fig. 3: The Coulomb graph, contributing to both factorizable and non-factorizable corrections.

$$\mathcal{J}^{\mu}_{\oplus} = -\,e\left[\frac{q_1^{\mu}}{kq_1+io} - \frac{q_2^{\mu}}{kq_2+io}\right], \quad \mathcal{J}^{\mu}_{\ominus} = +\,e\left[\frac{q_1^{\mu}}{-kq_1+io} - \frac{q_2^{\mu}}{-kq_2+io}\right] \tag{15}$$

for photon emission from the production stage of the process, and

$$\begin{aligned} \mathcal{J}^{\mu}_{+} &= -\,e\left[\frac{p_1^{\mu}}{kp_1+io} + Q_{f_1}\frac{k_1^{\mu}}{kk_1+io} - Q_{f_1'}\frac{k_1'^{\,\mu}}{kk_1'+io}\right]\frac{D_1}{D_1+2kp_1}, \\ \mathcal{J}^{\mu}_{-} &= -\,e\left[\frac{p_2^{\mu}}{-kp_2+io} + Q_{f_2}\frac{k_2^{\mu}}{-kk_2+io} - Q_{f_2'}\frac{k_2'^{\,\mu}}{-kk_2'+io}\right]\frac{D_2}{D_2-2kp_2} \end{aligned} \tag{16}$$

for photon emission from the decay stages of the process. Here $\mathcal{M}^0_{\rm DPA}$ is the lowest-order matrix element in DPA and Q_f stands for the charge of fermion f in units of e. Since Eq. (14) contains (at least) all contributions from diagrams with irreducible W-boson lines, it can be viewed as a gauge-invariant extension of the set of W-irreducible diagrams. In general one has to calculate all of the integrals appearing in the above expressions (see Ref. [39]b for a complete set). However, if one is interested in the sum of virtual corrections and real-photon radiation, then some simplifications occur depending on the treatment of the photon.

If the radiated (real) photon is treated inclusively, then many of the terms in Eq. (14) cancel. In this context the difference in the signs of the io parts appearing in the currents $\mathcal{J}_{\ominus}$ and $\mathcal{J}_{\oplus}$ are crucial. These signs actually determine which interference terms give rise to a non-vanishing non-factorizable contribution after virtual and real-photon corrections have been added. As a result of such considerations only a very limited subset of 'final-state' interferences survives for inclusive photons: the virtual corrections corresponding to Figs. 2 and 3 as well as the associated real-photon corrections. The sum of virtual and real non-factorizable corrections has been calculated in Refs. [105, 16, 17][3]. It has been shown in Ref. [19] that this sum vanishes if the invariant masses of both W bosons are integrated over, i.e. in particular that the full non-factorizable correction to the total cross section is zero in DPA.

Comment Sasha and Wim: we have removed the comments about the numerical results. They should go to the numerical discussion.

The virtual factorizable corrections consist of all hard contributions and the left-over part of the semi-soft ones. The so-defined factorizable corrections have the nice feature that they can be expressed in terms of corrections to on-shell sub-processes, i.e. the production of two on-shell W bosons and their subsequent on-shell decays. The corresponding matrix element can be expressed in the same way as described at lowest-order:

$$\mathcal{M}^{\rm virt}_{\rm fact} = \sum_{\lambda_1,\lambda_2} \Pi_{\lambda_1\lambda_2}(M_1,M_2)\,\frac{\Delta^{(+)}_{\lambda_1}(M_1)}{D_1}\,\frac{\Delta^{(-)}_{\lambda_2}(M_2)}{D_2}. \tag{17}$$

Here two of the amplitudes are taken at lowest order, whereas the remaining one contains all possible one-loop contributions, including the W wave-function factors that appear in Eq. (1). In this way the

[3]The original result of the older calculation [105] does not agree with the two more recent results [16, 17], which are in mutual agreement. As known from the authors of Ref. [105], their corrected results also agree with the ones of Refs. [16, 17].

well-known on-shell radiative corrections to the production and decay of pairs of W bosons [36]–[37] appear as basic building blocks of the factorizable corrections.[4] In the semi-soft limit the photonic virtual factorizable corrections to the production stage, contained in Π, cancel against the corresponding real-photon corrections. Non-vanishing contributions from Π occur as soon as the k^2 terms in the propagators cannot be neglected anymore. An example of this is the factorizable correction from the Coulomb graph Fig. 3. For the on-shell (factorizable) part of the Coulomb effect photons with momenta $k_0 = \mathcal{O}(\Delta E)$ and $|\vec{k}| = \mathcal{O}(\sqrt{M_W \Delta E})$ are important [40], i.e. k^2 cannot be neglected in the propagators of the unstable particles. Since we stay well away from the W-pair threshold ($\Delta E \equiv \sqrt{s} - 2M_W \gg \Gamma_W$), this situation occurs outside the realm of the semi-soft photons. This fits nicely into the picture of the production stage being a hard subprocess, governed by relatively short time scales as compared with the much longer time scales required for the non-factorizable corrections, which interconnect the different sub-processes.

footnote changed ←

4.02 Real-photon radiation

In this subsection we discuss the aspects of real-photon radiation in the DPA as used in [28]. To this end we consider the process ←

$$e^+(q_1)\, e^-(q_2) \to W^+(p_1)\, W^-(p_2)\, [\gamma(k)] \to \bar{f}_1(k_1) f_1'(k_1')\, f_2(k_2) \bar{f}_2'(k_2')\, \gamma(k), \tag{18}$$

where in the intermediate state there may or may not be a photon. We will show how to extract the gauge-invariant double-pole residues in different situations. The exact cross-section for process (18) can be written in the following form

$$d\sigma = \frac{1}{2s}|\mathcal{M}_\gamma|^2 d\Gamma_{4f\gamma} = \frac{1}{2s}\left[2\mathrm{Re}\Big(\mathcal{M}_0\mathcal{M}_+^* + \mathcal{M}_0\mathcal{M}_-^* + \mathcal{M}_+\mathcal{M}_-^*\Big) + |\mathcal{M}_0|^2 + |\mathcal{M}_+|^2 + |\mathcal{M}_-|^2\right] d\Gamma_{4f\gamma}, \tag{19}$$

where $d\Gamma_{4f\gamma}$ indicates the complete five-particle phase-space factor, and the matrix elements $\mathcal{M}_0$ and $\mathcal{M}_\pm$ correspond to the diagrams where the photon is attached to the production or decay stage of the three W-pair diagrams, respectively. This split-up can be achieved with the help of the partial-fraction decomposition [13] ←

$$\frac{1}{D_i(D_i + 2pk)} = \frac{1}{2pk}\left(\frac{1}{D_i} - \frac{1}{D_i + 2pk}\right). \tag{20}$$

Each contribution to the cross-section can be written in terms of polarization density matrices, which originate from the amplitudes

$$\mathcal{M}_0 = \Pi_\gamma(M_1, M_2)\, \frac{\Delta^{(+)}(M_1)}{D_1}\, \frac{\Delta^{(-)}(M_2)}{D_2}\,, \tag{21}$$

$$\mathcal{M}_+ = \Pi(M_{1\gamma}, M_2)\, \frac{\Delta_\gamma^{(+)}(M_{1\gamma})}{D_{1\gamma}}\, \frac{\Delta^{(-)}(M_2)}{D_2}\,, \tag{22}$$

$$\mathcal{M}_- = \Pi(M_1, M_{2\gamma})\, \frac{\Delta^{(+)}(M_1)}{D_1}\, \frac{\Delta_\gamma^{(-)}(M_{2\gamma})}{D_{2\gamma}}\,, \tag{23}$$

where all polarization indices for the W bosons and the photon have been suppressed, and

$$D_{i\gamma} = D_i + 2kk_i + 2kk_i', \quad M_{i\gamma}^2 = M_i^2 + 2kk_i + 2kk_i', \quad M_i^2 = (k_i + k_i')^2. \tag{24}$$

[4] Note that the complete density matrix is required in this case, in contrast to the pure on-shell calculation which involves the diagonal elements of the density matrix only.

The matrix elements Π_γ and $\Delta_\gamma^{(\pm)}$ describe the production and decay of the W bosons accompanied by the radiation of a photon. The matrix elements without subscript γ have been introduced in Eq. (4).

In the calculation of the Born matrix element and virtual corrections only two poles could be identified in the amplitudes, originating from the Breit–Wigner propagators $1/D_i$. The pole-scheme expansion was performed around these two poles. In contrast, the bremsstrahlung matrix element has four in general different poles, originating from the four Breit–Wigner propagators $1/D_i$ and $1/D_{i\gamma}$. As mentioned above, the matrix element can be rewritten as a sum of three matrix elements $(\mathcal{M}_0, \mathcal{M}_+, \mathcal{M}_-)$, each of which only contain two Breit–Wigner propagators. For these three individual matrix elements the pole-scheme expansion is fixed, as before, to an expansion around the corresponding two poles. However, when calculating cross-sections [see Eq. (19)] the mapping of the five-particle phase space introduces a new type of ambiguity. The interference terms in Eq. (19) involve two different double-pole expansions simultaneously. One might think this will pose a problem, since there is no natural choice for the phase-space mapping in those cases. As we will see later, however, only photons with $E_\gamma \lesssim \Gamma_W \ll M_W$ give noticeable contributions to these interference terms. This means that one can apply a soft-photon-like approximation (see below). In Ref. [28] it was argued that the resulting ambiguity in the phase-space mapping will not have significant repercussions on the quality of the DPA calculation, in the same way as stable-particle calculations are not significantly affected by the photon momentum in the soft-photon regime. We note, however, that there is still some controversy on this issue.

Comment Ansgar: The proof in Ref. [28] holds for instance not for the photon spectrum in the semi-soft photon region

Comment Sasha and Wim: we think it does hold, just like in any soft approximation. This is the above-mentioned controversy.

Let us return now to the three earlier-defined regimes for the photon energy:

- for hard photons $[E_\gamma \gg \Gamma_W]$ the Breit–Wigner poles of the W-boson resonances before and after photon radiation are well separated in phase space (see $M_{i\gamma}^2$ and M_i^2 defined above). As a result, the interference terms in Eq. (19) can be neglected. This leads to three *distinct* regions of on-shell contributions, where the photon can be assigned unambiguously to the W-pair-production subprocess or to one of the two decays. This assignment is determined by the pair of invariant masses (out of M_i^2 and $M_{i\gamma}^2$) that is in the M_W^2 region. Therefore, the double-pole residue can be expressed as the sum of the three on-shell contributions without increasing the intrinsic error of the DPA. Note that in the same way it is also possible to assign the photon to one of the sub-processes, since misassignment errors are suppressed, assuming for convenience that all final-state momenta can ideally be measured.
- for semi-soft photons $[E_\gamma = \mathcal{O}(\Gamma_W)]$ the Breit–Wigner poles are relatively close together in phase space, resulting in a substantial overlap of the line shapes. The assignment of the photon is now subject to larger errors. Moreover, since the interference terms in Eq. (19) cannot be neglected, a proper prescription for calculating the DPA residues (i.e. the phase-space mapping) is required [28].
- for soft photons $[E_\gamma \ll \Gamma_W]$ the Breit–Wigner poles are on top of each other, resulting in a pole-scheme expansion that is identical to the one without the photon.

Let us first consider the hard-photon regime in more detail. Due to the fact that the poles are well separated in the hard-photon regime, it is clear that the interference terms are suppressed and can be neglected:

$$d\sigma = \frac{1}{2s}\left[|\mathcal{M}_0|^2 + |\mathcal{M}_+|^2 + |\mathcal{M}_-|^2\right] d\Gamma_{4f\gamma}. \tag{25}$$

Note that each of the three terms has two poles, originating from two resonant propagators. However, these poles are different for different terms. The phase-space factor can be rewritten in three equivalent ways. The first is

$$d\Gamma_{4f\gamma} = d\Gamma_0^\gamma = d\Gamma_{\rm pr}^\gamma \cdot d\Gamma_{\rm dec}^+ \cdot d\Gamma_{\rm dec}^- \cdot \frac{dM_1^2}{2\pi} \cdot \frac{dM_2^2}{2\pi}, \tag{26}$$

with

$$d\Gamma_{\rm pr}^\gamma = \frac{1}{(2\pi)^2}\,\delta(q_1+q_2-p_1-p_2-k)\,\frac{d\vec{p}_1}{2p_{10}}\,\frac{d\vec{p}_2}{2p_{20}}\,\frac{d\vec{k}}{(2\pi)^3 2k_0}. \tag{27}$$

The two others are

$$d\Gamma_{4f\gamma} = d\Gamma_+^\gamma = d\Gamma_{\rm pr} \cdot d\Gamma_{\rm dec}^{+\gamma} \cdot d\Gamma_{\rm dec}^- \cdot \frac{dM_{1\gamma}^2}{2\pi} \cdot \frac{dM_2^2}{2\pi}, \tag{28}$$

with

$$d\Gamma_{\rm dec}^{+\gamma} = \frac{1}{(2\pi)^2}\,\delta(p_1-k_1-k_1'-k)\,\frac{d\vec{k}_1}{2k_{10}}\,\frac{d\vec{k}_1'}{2k_{10}'}\,\frac{d\vec{k}}{(2\pi)^3 2k_0}, \tag{29}$$

and a similar expression for $d\Gamma_-^\gamma$. The phase-space factors $d\Gamma_{\rm pr}$ and $d\Gamma_{\rm dec}^\pm$ are just the lowest-order ones. The cross-section can then be written in the following equivalent form

$$d\sigma = \frac{1}{2s}\Bigl[|\mathcal{M}_0|^2\,d\Gamma_0^\gamma + |\mathcal{M}_+|^2\,d\Gamma_+^\gamma + |\mathcal{M}_-|^2\,d\Gamma_-^\gamma\Bigr]. \tag{30}$$

In order to extract gauge-invariant quantities, the DPA limit should be taken. This amounts to taking the limit $p_{1,2}^2 \to M_W^2$, using a particular prescription for mapping the full off-shell phase space on the kinematically restricted on-resonance one. Note however that $p_{1,2}$ can be different according to the δ-functions in the decay parts of the different phase-space factors. To be specific, the production term $|\mathcal{M}_0|^2$ has poles at $p_i^2 = M_i^2 = M_W^2$, $|\mathcal{M}_+|^2$ has poles at $p_1^2 = M_{1\gamma}^2 = M_W^2$ and $p_2^2 = M_2^2 = M_W^2$, and $|\mathcal{M}_-|^2$ has poles at $p_1^2 = M_1^2 = M_W^2$ and $p_2^2 = M_{2\gamma}^2 = M_W^2$.

We complete our survey of the different photon-energy regimes by considering semi-soft and soft photons. The split-up of factorizable and non-factorizable real-photon corrections proceeds in the same way as described in the previous subsection for virtual corrections. The result reads in semi-soft approximation

$$d\sigma = \frac{1}{2s}|\mathcal{M}_\gamma|^2 d\Gamma_{4f\gamma} \approx -\,d\sigma^0_{\rm DPA}\frac{d\vec{k}}{(2\pi)^3 2k_0}\left[2{\rm Re}\Bigl(\mathcal{I}_0^\mu\mathcal{I}_{+,\mu}^* + \mathcal{I}_0^\mu\mathcal{I}_{-,\mu}^* + \mathcal{I}_+^\mu\mathcal{I}_{-,\mu}^*\Bigr) + |\mathcal{I}_0^2| + |\mathcal{I}_+^2| + |\mathcal{I}_-^2|\right]. \tag{31}$$

The gauge-invariant currents $\mathcal{I}_0$ and $\mathcal{I}_\pm$ are given by

$$\begin{aligned}
\mathcal{I}_0^\mu &= e\left[\frac{p_1^\mu}{kp_1} - \frac{p_2^\mu}{kp_2} - \frac{q_1^\mu}{kq_1} + \frac{q_2^\mu}{kq_2}\right], \\
\mathcal{I}_+^\mu &= -\,e\left[\frac{p_1^\mu}{kp_1} + Q_{f_1}\frac{k_1^\mu}{kk_1} - Q_{f_1'}\frac{k_1'^{\,\mu}}{kk_1'}\right]\frac{D_1}{D_1+2kp_1}, \\
\mathcal{I}_-^\mu &= +\,e\left[\frac{p_2^\mu}{kp_2} + Q_{f_2}\frac{k_2^\mu}{kk_2} - Q_{f_2'}\frac{k_2'^{\,\mu}}{kk_2'}\right]\frac{D_2}{D_2+2kp_2}.
\end{aligned} \tag{32}$$

The first three interference terms in Eq. (31) correspond to the real non-factorizable corrections. The last three squared terms in Eq. (31) belong to the factorizable real-photon corrections. They constitute the semi-soft limit of Eq. (30). ↓

4.1 A hybrid scheme – virtual corrections in DPA and real corrections from full matrix elements

The reliability of the error estimate of $(\alpha/\pi) \times (\Gamma_{\rm W}/M_W) \times \log(\cdots) \lesssim 0.5\%$ for the accuracy of the DPA can, of course, only be controlled by a comparison to calculations that are based on the full matrix elements. While for the virtual corrections such results do not exist yet, the situation for the real corrections is much better, since full matrix-element calculations for the processes $\mathrm{e}^+\mathrm{e}^- \to 4f\gamma$ are available [5, 6, 111]. The latter results seem to be of particular importance, because the above error estimate for *real* corrections in DPA is subject of some controversy.

Although it deserves some care, it is possible to combine the virtual $\mathcal{O}(\alpha)$ corrections in DPA with real corrections from the full $\mathrm{e}^+\mathrm{e}^- \to 4f\gamma$ lowest-order matrix elements. The non-trivial point in this combination lies in the relations of IR and mass singularities in virtual and real corrections. The singularities have the form of a universal radiator function multiplied or convoluted with the respective lowest-order matrix element $\mathcal{M}_0$ of the non-radiative process. Since $\mathcal{M}_0$ appears in DPA for the virtual correction ($\mathcal{M}_0^{\mathrm{DPA}}$), but as full matrix element for the real ones, a simple summation of virtual and real corrections would lead to a mismatch in the singularity structure and eventually to totally wrong results. A solution of this problem is to extract those singular parts from the real photon contribution which exactly match the singular parts of the virtual photon contribution, then to replace the full $|M^0|^2$ by $|M_0^DPA|^2$ in these terms and finally to add this modified part to the virtual corrections. This modification is allowed in the range of validity of the DPA and leads to a proper matching of all IR and mass singularities. The described approach for such a hybrid DPA scheme is followed in the RACOONWW program [176, 18]. More details of this approach can also be found in Section **??**. **complete ref.**

A particular advantage of this method is due to the fact that the leading ISR logarithms, which are part of the extracted singularities of the real corrections, can be easily kept with the full matrix element $\mathcal{M}_0$ (see [18] for details). In this way, the logarithmic enhancement factor $\log(\ldots)$ does not involve large contributions from the electron mass.

Comment Sasha and Wim: we have removed the last paragraph in order to avoid too many repetitions ↑

4.2 Intrinsic ambiguities and reliability of the double-pole approximation

A naive estimate for the intrinsic uncertainty of the DPA is given by $(\alpha/\pi) \times (\Gamma_{\mathrm{W}}/M_W) \times \log(\cdots) \lesssim 0.5\%$, which accounts for the relative size of the neglected corrections to background diagrams, including possible logarithmic enhancements. However, this estimate can fail whenever small scales play an essential role in the cross section. As mentioned above, this is definitely the case near the W-pair threshold and in the single-W region.

Whether the naive estimate is reliable for the emission of semi-soft photons, where $E_\gamma \sim \Gamma_{\mathrm{W}}$, is not clear a priori. Only a detailed numerical comparison between the DPA and the exact calculation of $4\mathrm{f} + \gamma$ production can answer this question. Note that this issue is also closely related to the impact of non-factorizable corrections, since the definition of real non-factorizable corrections implicitly assumes a treatment of $4\mathrm{f}+\gamma$ production in DPA. Thus, the question about the size of non-factorizable corrections should always be discussed in connection with the more general question about the quality of treating photon radiation within DPA.

An improved assessment of the theoretical uncertainty can be obtained by varying predictions within the intrinsic freedom of the followed approach for the DPA. For instance, any kind of DPA makes use of an on-shell projection of the off-shell four-fermion phase space to the phase space with on-shell W bosons. The difference between different on-shell projections is part of the theoretical uncertainty of the DPA approach and should be considered in predictions.

Besides the matrix element, the DPA can also be applied to the phase space. While this further approximation is not needed in a generator, it is useful in a semi-analytic approach. Moreover, it can help to assess the intrinsic ambiguity of the DPA.

It is a fact of life that questions of principle are sometimes of scarce practical relevance. CC03 contains gauge-invariance-breaking terms but what is their numerical impact at LEP 2 energies? It is quite a known fact that, when computed in the 't Hooft-Feynman gauge, they are unimportant. At least they are for the WW total cross-section – the signal – and we can verify this statement by comparing the gauge-dependent CC03 with the full gauge-invariant cross-section (CC11 for instance) including background diagrams. There is a general agreement, dating from the '95 workshop that the difference is less than 0.2% at LEP 2 energies.

It is bizarre that one can render the Born CC03 diagrams gauge-invariant at the prize of large numerical variations; it is enough to project the kinematics in the matrix elements onto the on-shell phase space, while keeping the off-shellness in the Breit-Wigner propagators. However, this changes the cross-section by several per cent! At the Born level, therefore, one can accept a non-gauge-invariant CC03 cross-section (at least in the 't Hooft-Feynman gauge) as a reasonable quantity at LEP 2 energies. For higher energies one should be more careful.

The same phenomenon will occur when we include radiative corrections and we would like to add some comment on the DPA procedure, in particular on the arbitrariness introduced by projecting the kinematics.

For high enough energies, any process $e^+e^- \to VV$ will be a dominant source of fermion pairs due to the double resonant enhancement and hence CC03(NC02) will be a good approximation to the total cross-section for four-fermion production.

Comment Ansgar: This is only correct for appropriate cuts (which exclude for instance single W production)

Thus, for example, to calculate the cross-section $e^+e^- \to VV$ one proceeds as described above; one calculates the matrix element for $e^+e^- \to VV \to 4\mathrm{f}$ and extracts the part resonant in the invariant masses of the pairs, k_+^2, k_-^2. The general matrix element takes the form

$$\mathcal{M}\left(\dots, k_+, k_-, \dots\right) = \sum_i M_i\left(\dots, k_+, k_-, \dots\right)\, A_i\left(\dots, k_+^2, k_-^2\right), \tag{33}$$

where the M_i contain the spinor and Lorentz tensor structure of the matrix element, e.g. they have the external fermionic wave-functions attached. The A_i are Lorentz scalars that depends on the invariants of the problem and become non-trivial and difficult to compute when higher order corrections are included. One way of looking at the DPA-procedure is to say that the resonant part is extracted from the A_i, by Laurent expansion. The external particle wave functions should not be affected by the process hence the kinematics of the problem should be left unchanged because the final state integrations involve only the fermions, stable on-shell particles. The gauge nature of the theory is intimately connected with the A_i not with kinematics. This is hardly a surprise, even in Born CC03 two calculations performed in two different gauges differ by single resonant terms.

Comment Ansgar: I would omit the preceeding paragraph. None of the actual implementations of the DPA is done in this way. Moreover, I can give examples where the splitting into M_i and A_i with a pole approximation for the A_i can introduce unitarity violating terms.

Comment Sasha and Wim: we agree on Ansgar's comment, in fact we propose to remove the whole subsection. The first four paragraphs are yet again a repetition. Moreover, the use of DPA at Born level (CC03) is numerically not recommendable. One should instead use the pole-scheme to its full extent, i.e. the full expansion.

4.21 Remarks on DPA corrections to distributions inclusive w.r.t. photons

The DPA corrections to distributions that are inclusive w.r.t. photons depend in a very sensitive way on how the four-particle phase space is parametrized, or, in other words, on the way the distributions are defined after the photon has been integrated out. This statement sounds obvious, but nevertheless deserves some special attention.

In particular the invariant-mass distributions (W line shapes) are affected. In reactions with two resonances the invariant masses have to be defined from the decay products. Depending on the precise

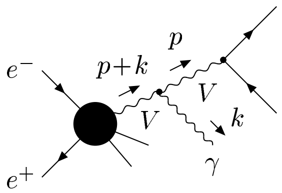


Fig. 4: Photon radiation from an unstable particle V. Virtualities: $s_V = p^2$ and $s'_V = (p+k)^2$.

definition of the invariant masses different sources of large Breit–Wigner distortions can be identified [41, 42, 176], in contrast to the situation at LEP1 where only initial-state radiation (ISR) can cause such distortions.

In Ref. [41] it has been shown that also final-state radiation (FSR) can induce distortions. This is a general property of resonance-pair reactions, irrespective of the adopted scheme for implementing the finite-width effects. The only decisive factor for the distortion to take place is whether the virtuality of the unstable particle is defined with (s'_V) or without (s_V) the radiated photon (see Fig. 4). Upon integration over the photon momentum, the former definition (cf $M^2_{i\gamma}$ defined in Sec. 4.02) is free of large FSR effects from the V-decay system. It can only receive large corrections from the other (production or *decay*) stages of the process. The latter definition (cf M^2_i defined in Sec. 4.02), however, does give rise to large FSR effects from the *V-decay system.* In contrast to the LEP1 case, where the ISR-corrected line shape receives contributions from effectively *lower* Z-boson virtualities, the s_V line shape receives contributions from effectively *higher* virtualities s'_V of the unstable particle. As was argued above, only sufficiently hard photons ($E_\gamma \gg \Gamma_V$) can be properly assigned to one of the on-shell production or decay stages of the process in the DPA. For semi-soft photons [$E_\gamma = \mathcal{O}(\Gamma_V)$], however, the assignment lacks a solid motivation and it may happen, e.g. for final-state muons, that the invariant-mass definition in terms of the decay products without the photon is more natural. As was pointed out in Ref. [41], exactly these semi-soft photons are responsible for the FSR-induced distortion effects. The hard FSR photons move the virtuality s'_V of the unstable particle far off resonance for near-resonance s_V values, resulting in a suppressed contribution to the s_V line shape. This picture fits in nicely with the negligible overlap of the three on-shell double-pole contributions for hard photons, discussed above. The reason why the FSR distortions can be rather large lies in the fact that the final-state collinear singularities [$\propto \frac{\alpha}{\pi} Q_f^2 \log(m_f^2/M_V^2) \log(\Gamma_V/M_V)$] do not vanish, even not for fully inclusive photons. After all, a fixed value of s_V makes it impossible to sum over all degenerate final states by a mere integration over the photon momentum. So the KLN theorem does not apply in this case. These FSR distortion effects result in shifts in the measurement of the W-boson mass of the order of 20 MeV, as has been qualitatively confirmed in Ref. [42].

This situation changes for event-selection procedures in which not all photons can be separated from the charged fermions (as is the case for final-state $e^\pm$). If photon recombination has to be taken into account, i.e. if photons within a finite cone around the charged fermions have to be combined with the corresponding fermion into a single pseudo particle, the mentioned mass singularities connected to final-state fermions disappear. The KLN theorem applies and the large fermion-mass logarithms are effectively replaced by logarithms depending on the cone size [41]. In Ref. [42] this expectation has been confirmed numerically, showing that the large negative shifts in the peak position of the W invariant-mass distribution obtained without photon recombination are reduced. In Ref. [176] it has been shown that the effect of photon recombination can even overcompensate the momentum loss from FSR if the recombination is very inclusive. This is due to the recombination of photons that are radiated off the initial state or off particles belonging to the other decaying W boson. The resulting positive peak shifts can amount to several 10 MeV. Explicit numerical results on W invariant-mass distributions can also be found in Section **??**. **complete ref.**

When considering pure angular inclusive-photon distributions, one should integrate over the photon phase space *and* the invariant masses M_i^2. After integrating out both invariant masses the non-factorizable corrections will vanish, which is a typical feature of the non-factorizable interference effects [34].

4.3 Double-pole approximations in practice

For LEP 2 energies three different groups[5] have formulated versions of a DPA for $\mathrm{e^+e^-} \to \mathrm{WW} \to 4f(+\gamma)$. While Beenakker, Berends and Chapovsky [22], called BBC in the following, formulated a semi-analytic DPA, the other two groups implemented variants of the DPA in the event generators `YFSWW` [23, 24] and `RacoonWW` [18, 176]. The basic features of these different implementations are summarized in the following.

4.31 The `YFSWW` approach

`YFSWW`: $\mathcal{O}(\alpha)$ correction to $\mathrm{e^+e^-} \to \mathrm{W^+W^-}$, leading-log corrections to W decays via PHOTOS and branching ratios, YFS exponentiation for ISR and photon emission from W-bosons, off-shell Coulomb singularity, no non-factorizable corrections, no W spin correlations

4.32 The BBC approach

BBC: semi-analytical calculation of complete $\mathcal{O}(\alpha)$ corrections in DPA (with both factorizable and non-factorizable corrections and W spin correlations), no background. ←

Comment Sasha and Wim: the Coulomb singularity is automatically included as part of the virtual corrections. It is not on-shell. Since the DPA is only valid well above threshold, the off-shell part of the Coulomb singularity is contained in the non-factorizable corrections. ←

4.33 The `RacoonWW` approach

`RacoonWW` treats the virtual $\mathcal{O}(\alpha)$ corrections to $\mathrm{e^+e^-} \to \mathrm{WW} \to 4f$ in DPA. No further approximations beyond the pole expansion of the matrix element are made, i.e. non-factorizable corrections are included, and W-spin correlations are respected. The off-shell Coulomb singularity is part of the virtual corrections. The real $\mathcal{O}(\alpha)$ corrections are based on the full $4\mathrm{f}+\gamma$ matrix element (of the CC11 class), so that the full kinematics is supported also for photon radiation. All matrix elements are based on massless fermions, and fermion masses are introduced only for collinear photon emission that is inclusive within a (small) finite cone for each fermion. Thus, a photon collinear to an outgoing fermion has to be recombined with the corresponding fermion, and a photon close to the beams has to be considered as invisible. Initial-state radiation beyond $\mathcal{O}(\alpha)$ is treated in the structure-function approach, including soft-photon exponentiation and leading-log contributions in $\mathcal{O}(\alpha^2)$.

5. The fermion-loop and non-local approaches

As was mentioned above, the alternative to *subtracting* sub-leading gauge-violating terms is to *add* gauge-restoring terms to the calculation. In order to do this, one has to add to the amplitude those terms that are needed for satisfying the Ward identities. This is not easy to do in general. The following observation helps. The very fact that the perturbative amplitudes require re-summation of the self-energies indicates that either the perturbative expansion parameter (coupling constant) is not the proper one, or alternatively that the quantity that is expanded (i.e. the lowest-order Lagrangian of the Standard Model) is not the best choice. This observation leads one to consider first the one-loop corrected effective potential of the Standard Model before doing Born calculations, in order to avoid Dyson re-summation of the self-energies.

[5] Another DPA has been discussed in Ref. [21] for linear-collider energies.

For the discussion of the fermion-loop and non-local approaches it is therefore worthwhile to first have a closer look at the origin of the gauge-invariance problem associated with the re-summation of self-energies. To this end we consider the simple example of an unbroken non-abelian $SU(N)$ gauge theory with fermions and subsequently integrate out these fermions.

First we fix the notations and introduce some conventions. The $SU(N)$ generators in the fundamental representation are denoted by T^a with $a = 1, \cdots, N^2-1$. They are normalized according to $\mathrm{Tr}\,(\mathrm{T}^a\mathrm{T}^b) = \delta^{ab}/2$ and obey the commutation relation $\Big[\mathrm{T}^a, \mathrm{T}^b\Big] = i f^{abc}\,\mathrm{T}^c$. In the adjoint representation the generators F^a are given by $(\mathrm{F}^a)^{bc} = -i f^{abc}$. The Lagrangian of the unbroken $SU(N)$ gauge theory with fermions can be written as

$$\mathcal{L}(x) = -\frac{1}{2}\,\mathrm{Tr}\Big[\boldsymbol{F}_{\mu\nu}(x)\,\boldsymbol{F}^{\mu\nu}(x)\Big] + \bar{\psi}(x)\,(i\,\not{D} - m)\,\psi(x), \tag{34}$$

with

$$\boldsymbol{F}_{\mu\nu} \equiv \mathrm{T}^a F^a_{\mu\nu} = \frac{i}{g}\,[D_\mu, D_\nu], \qquad D_\mu = \partial_\mu - ig\,\mathrm{T}^a A^a_\mu \equiv \partial_\mu - ig\boldsymbol{A}_\mu. \tag{35}$$

Here ψ is a fermionic N-plet in the fundamental representation of $SU(N)$ and A^a_μ are the (N^2-1) non-abelian $SU(N)$ gauge fields, which form a multiplet in the adjoint representation. The Lagrangian (34) is invariant under the $SU(N)$ gauge transformations

$$\begin{aligned} \psi(x) &\to \psi'(x) = \mathrm{G}(x)\,\psi(x), \\ \boldsymbol{A}_\mu(x) &\to \boldsymbol{A}'_\mu(x) = \mathrm{G}(x)\,\boldsymbol{A}_\mu(x)\,\mathrm{G}^{-1}(x) + \frac{i}{g}\,\mathrm{G}(x)\Big[\partial_\mu \mathrm{G}^{-1}(x)\Big], \end{aligned} \tag{36}$$

with the $SU(N)$ group element defined as $\mathrm{G}(x) = \exp[ig\,\mathrm{T}^a\theta^a(x)]$. The covariant derivative D_μ and field strength $\boldsymbol{F}_{\mu\nu}$ both transform in the adjoint representation

$$D_\mu \to \mathrm{G}(x)\,D_\mu\,\mathrm{G}^{-1}(x), \qquad \boldsymbol{F}_{\mu\nu}(x) \to \mathrm{G}(x)\,\boldsymbol{F}_{\mu\nu}(x)\,\mathrm{G}^{-1}(x). \tag{37}$$

Since the Lagrangian is quadratic in the fermion fields, one can integrate them out exactly in the functional integral. The resulting effective action is then given by

$$i\,S_{\text{eff}}[J] = i\int d^4x\,\Big\{-\frac{1}{2}\,\mathrm{Tr}\Big[\boldsymbol{F}_{\mu\nu}(x)\,\boldsymbol{F}^{\mu\nu}(x)\Big] + J^a_\mu(x)\,A^{a,\mu}(x)\Big\} + \mathrm{Tr}\Big[\ln(-\not{D} - i\,m)\Big], \tag{38}$$

with $J^a_\mu(x)$ denoting the gauge-field sources. The trace on the right-hand side has to be taken in group, spinor, and coordinate space. As a next step one can expand the effective action in terms of the coupling constant

$$\begin{aligned} \mathrm{Tr}\Big[\ln(-\not{D} - i\,m)\Big] &= \mathrm{Tr}\Big[\ln(-\not{\partial} - i\,m)\Big] + \mathrm{Tr}\Big[\ln\Big(1 + \frac{g}{i\,\not{\partial} - m}\,\not{\boldsymbol{A}}\Big)\Big] \\ &= \mathrm{Tr}\Big[\ln(-\not{\partial} - i\,m)\Big] + \sum_{n=1}^{\infty}\frac{(-1)^{n-1}}{n}\,\mathrm{Tr}\Big[\Big(\frac{g}{i\,\not{\partial} - m}\,\not{\boldsymbol{A}}\Big)^n\Big]. \end{aligned} \tag{39}$$

Note that the left-hand side of Eq. (39) is gauge-invariant as a result of the trace-log operation. In contrast, the separate terms of the expansion on the right-hand side are not gauge-invariant. This is due to the fact that, unlike in the abelian case, the non-abelian gauge transformation (36) mixes different powers of the gauge field A_μ in Eq. (39). Thus, if one truncates the series on the right-hand side of Eq. (39) one will in general break gauge invariance. From Eq. (39) it is also clear that the fermionic part of the effective action induces higher-order interactions between the gauge bosons.

What are these higher-order interactions? Let us consider the quadratic gauge-field contribution

$$-\frac{1}{2}\,\mathrm{Tr}\left[\left(\frac{g}{i\,\not\partial - m}\,\not{\!\!A}\right)^2\right] = -\,\frac{1}{2}\int d^4x\,d^4y\,\mathrm{Tr}\Big[O(x,y)\,O(y,x)\Big], \tag{40}$$

where

$$O(x,y) = g\,S_{\mathrm{F}}^{(0)}(x-y)\,\not{\!\!A}(y) \tag{41}$$

and $i\,S_{\mathrm{F}}^{(0)}(x-y) = <0\,|\,T(\psi(x)\,\bar{\psi}(y))\,|\,0>_{\text{free}}$ is the free fermion propagator. The trace on the right-hand side of Eq. (40) has to be taken in group and spinor space. A quick glance at this quadratic gauge-field contribution reveals that it is just the one-loop self-energy of the gauge boson induced by a fermion loop. In the same way, the higher-order terms $\sim g^n A^n$ in Eq. (39) are just the fermion-loop contributions to the n-point gauge-boson vertices.

One can truncate the expansion in Eq. (39) at $n=2$, thus taking into account only the gauge-boson self-energy term and neglecting the fermion-loop contributions to the higher-point gauge-boson vertices. This is evidently the simplest procedure for performing the Dyson re-summation of the fermion-loop self-energies. However, as was pointed out above, truncation of Eq. (39) at any finite order in g in general breaks gauge invariance. This leads to the important observation that, *although the re-summed fermion-loop self-energies are gauge-independent by themselves, the re-summation is nevertheless responsible for gauge-breaking effects in the higher-point gauge-boson interactions through its inherent mixed-order nature*. Another way of understanding this is provided by the gauge-boson Ward identities. Since the once-contracted n-point gauge-boson vertex can be expressed in terms of $(n-1)$-point vertices, it is clear that gauge invariance is violated if the self-energies are re-summed without adding the necessary compensating terms to the higher-point vertices.

An alternative is to keep all the terms in Eq. (39). Then the matrix elements derived from the effective action will be gauge-invariant. Keeping all the terms means that we will have to take into account not only the fermion-loop self-energy in the propagator, but also all the possible fermion-loop contributions to the higher-point gauge-boson vertices. This is exactly the prescription of the fermion-loop scheme (FLS) [26, 29, 30, 31]. Although the FLS guarantees gauge invariance of the matrix elements, it has disadvantages as well. Its general applicability is limited to those situations where non-fermionic particles can effectively be discarded in the self-energies, as is for instance the case for Γ_W and Γ_Z at lowest order. Another disadvantage is that in the FLS one is forced to do the loop calculations, even when calculating lowest-order quantities. For example, the calculation of the tree-level matrix element for the process $e^+e^- \to 4\mathrm{f}\gamma$ involves a four-point gauge-boson interaction, which has to be corrected by fermion loops in the FLS. This overcomplicates an otherwise lowest-order calculation.

It is clear that the FLS provides more than we actually need. It does not only provide gauge invariance for the Dyson re-summed matrix elements at a given order in the coupling constant, but it also takes into account all fermion-loop corrections at that given order. In the vicinity of unstable particle resonances the imaginary parts of the fermion-loop self-energies are effectively enhanced by $\mathcal{O}(1/g^2)$ with respect to the other fermion-loop corrections. Therefore, what is really needed is only a minimal subset of the non-enhanced contributions such that gauge invariance is restored. In a sense one is looking for a minimal solution of a system of Ward identities. The FLS provides a solution, but this solution is far from minimal and is only practical for particles that decay exclusively into fermions. Since the decay of unstable particles is a physical phenomenon, it seems likely that there exists a simpler and more natural method for constructing a solution to a system of Ward identities, without an explicit reference to fermions. This is precisely the philosophy behind the non-local approach [32]. This approach consists in using gauge-invariant non-local effective Lagrangians for generating both the self-energy effects in the propagators and the required gauge-restoring terms in the higher-point interactions. In this way the full set of Ward identities can be solved, while keeping the gauge-restoring terms to a minimum.

5.1 The fermion-loop scheme

The Fermion-Loop scheme developed in [26] and refined in [30] makes the approximation of neglecting all masses for the incoming and outgoing fermions in the processes $e^+e^- \to n$ fermions.

It is possible, however, to give the construction of a complete, massive-massive, Fermion-Loop scheme (MFL), i.e., a scheme for

Comment Ansgar: What means "massive-massive"? $\leftarrow$

incorporating the finite-width effects in the theoretical predictions for tree-level, LEP 2 and beyond, processes [43].

One can work in the 't Hooft-Feynman gauge and create all relevant building blocks, namely the vector-vector, vector-scalar and scalar-scalar transitions of the theory, all of them one-loop re-summed. The loops, entering the scheme, contain fermions and, as done before in [30], we allow for a non-zero top quark mass inside loops. There is a very simple relation between re-summed transitions and running parameters, since Dyson re-summation is most easily expressed in terms of running couplings and running mixing angles. $\leftarrow$

$\leftarrow$ In the MFL generalization, it is particularly convenient to introduce additional running quantities. They are the running masses of the vector bosons, $M_0^2(p^2) = M^2(p^2)/c^2(p^2)$, formally connected to the location of the W and Z complex poles. After introducing these running masses, it is straightforward to prove that all $\mathcal{S}$-matrix elements of the theory assume a very simple structure. Coupling constants, mixing angles and masses are promoted to running quantities and the $\mathcal{S}$-matrix elements retain their $\leftarrow$ Born-like structure, with running parameters instead of bare ones, and vector-scalar or scalar-scalar transitions disappear if we employ unitary-gauge–like vector boson propagators where the masses appearing in the denominator of propagators are the running ones.

The renormalization of ultraviolet divergences can be easily extended to the MFL-scheme by showing that all ultraviolet divergent parts of the one-loop vertices, $\gamma WW, \gamma W\phi, \gamma\phi W$ and $\gamma\phi\phi$ for instance, are proportional to the lowest order part. Therefore, the only combinations that appear are of the form $1/g^2 + VVV$ vertex or $M^2/g^2 + VV\phi$ vertex etc. All of them are, by construction, ultraviolet finite.

Equipped with this generalization of the Fermion-Loop scheme, one can prove the fully-massive U(1) Ward identity which is required for a correct treatment of the single-W processes. As a by-product $\leftarrow$ of the method, the cross-section for single-W production automatically evaluates all channels at the right scale, without having to use ad hoc re-scalings and avoiding the approximation of a unique scale for all terms contributing to the cross-section.

The generalization of the Fermion-Loop scheme goes beyond its, most obvious, application to single-W processes and allows for a gauge invariant treatment of all $e^+e^- \to n$ fermion processes with a correct evaluation of the relevant scales. Therefore, the MFL-scheme can be applied to several other processes like $e^+e^- \to Z\gamma^*$ and, in general to all $e^+e^- \to 6$ fermion processes, with the inclusion of top quarks.

Comment Sasha and Wim: the MFL does not apply to reactions involving the physical Higgs boson, right? $\leftarrow$

5.2 The non-local approach

The main idea of the non-local approach is to rearrange the series on the right-hand side of Eq. (39) in such a way that each term becomes gauge-invariant by itself. Subsequent truncation of the series at a given term is then allowed. It is possible to approximate Eq. (39) by means of an effective Lagrangian in such a way that the resulting effective action has the following properties:

- it generates the Dyson re-summed transverse gauge-boson self-energy in the propagator. This means that it contributes to the gauge-boson two-point function. Hence, the effective lagrangian should depend at least on two gauge-boson fields.

- the Dyson re-summed self-energy is in general not a constant, but rather a function dependent on the interaction between the gauge-bosons and the fermions. This means that the effective Lagrangian should in general be non-local (bi-local) in the gauge fields. Thus the gauge fields should be taken at two different space–time points.
- it is gauge-invariant. As such the effective Lagrangian should have the form of an infinite tower of gauge fields.

For the gauging procedure of the non-local Lagrangians we will need a special ingredient, the *path-ordered exponential*, which is defined as

$$U(x,y) = U^{\dagger}(y,x) \;=\; \mathrm{P}\exp\left[-ig\int\limits_{x}^{y} \boldsymbol{A}_{\mu}(\omega)\, d\omega^{\mu}\right] \tag{42}$$

Here $d\omega^{\mu}$ is the element of integration along some path $\Omega(x,y)$ that connects the points x and y [6]. The so-defined path-ordered exponential transforms as

$$U(x,y) \to \mathrm{G}(x)\, U(x,y)\, \mathrm{G}^{-1}(y) \tag{43}$$

under the $SU(N)$ gauge transformations. It hence carries the gauge transformation from one space-time point to the other.

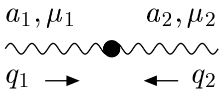
For a $SU(N)$ Yang–Mills theory the non-local action with the above-described properties takes the form

$$\mathcal{S}_{\mathrm{NL}} = -\frac{1}{2}\int d^4x\, d^4y\, \Sigma_{\mathrm{NL}}(x-y)\, \mathrm{Tr}\left[U(y,x)\, \boldsymbol{F}_{\mu\nu}(x)\, U(x,y)\, \boldsymbol{F}^{\mu\nu}(y)\right] \equiv \int d^4x\, d^4y\, \mathcal{L}_{\mathrm{NL}}(x,y), \tag{44}$$

with $\mathcal{L}_{\mathrm{NL}}(x,y)$ the non-local effective Lagrangian. As required, the action contains bilinear gauge-boson interactions. The induced infinite tower of higher-point gauge-boson interactions, which are also of progressively higher order in the coupling constant g, is needed for restoring gauge invariance.

It is important to stress at this point that this term in the effective action should not be understood as a new fundamental interaction. It is generated by radiative corrections. From the point of view of general properties of non-local Lagrangians, the non-local coefficient $\Sigma_{\mathrm{NL}}(x-y)$ is arbitrary. In practice, however, it is fixed by the explicit interaction between the gauge-bosons and the fermions in the underlying fundamental theory. In our simple example this connection is given by Eq. (39).

Let us now derive the two-point function as an example of the Feynman rules generated by Eq. (44):



$$: \; i\,\Sigma^{a_1 a_2,\,\mu_1\mu_2}(x_1,x_2) = \left.\frac{i\,\delta^2(\mathcal{S}_{\mathrm{L}}+\mathcal{S}_{\mathrm{NL}})}{\delta A^{a_1}_{\mu_1}(x_1)\,\delta A^{a_2}_{\mu_2}(x_2)}\right|_{A=0}, \tag{45}$$

where the local action $\mathcal{S}_{\mathrm{L}}$ follows from the gauge-boson term in Eq. (34). The Fourier transform of this two-point function can be calculated in a straightforward way, since the path-ordered exponentials are effectively unity. The result reads

$$i\,\tilde{\Sigma}^{a_1 a_2,\,\mu_1\mu_2}(q_1,q_2) = i\,\delta^{a_1 a_2}\left(q_1^{\mu}q_1^{\nu} - q_1^2 g^{\mu\nu}\right)\left[1+\tilde{\Sigma}_{\mathrm{NL}}(q_1^2)\right](2\pi)^4\,\delta^{(4)}(q_1+q_2). \tag{46}$$

[6] In principle we are free to choose this particular path. This freedom is just one out of the many freedoms that characterize the treatment of unstable particles (as mentioned earlier). It just reflects the fact that in a perturbative expansion one is free to pick up additional higher-order contributions, since the answer at any given (truncated) order will not be changed by such additional terms

Note that this two-point interaction is transverse, as it should be for an unbroken theory. The non-local coefficient acts as a (dimensionless) correction to the transverse free gauge-boson propagator. So, exactly what is needed for the Dyson re-summation of the gauge-boson self-energies. The infinite tower of gauge-restoring higher-point gauge-boson interactions are provided by the gauge-boson fields present in both $\boldsymbol{F}_{\mu\nu}$ and the path-ordered exponential occurring in Eq. (44). For explicit Feynman rules we refer to Ref. [32].

Although the above-described non-local procedure provides a gauge-invariant framework for performing the Dyson re-summation of the gauge-boson self-energies, we want to stress that it is not unique. We have seen above that the FLS provides a different solution of the system of gauge-boson Ward identities. In the context of non-local effective Lagrangians it is always possible to add additional towers of gauge-boson interactions that start with three-point interactions and therefore do not influence the Dyson re-summation of the gauge-boson self-energies.

In the light of the discussion presented in Sect. 5., we rearrange the series on the right-hand side of Eq. (39) according to gauge-invariant towers of gauge-boson interactions labelled by the minimum number of gauge bosons that are involved in the non-local interaction. Effectively this constitutes an expansion in powers of the coupling constant g, since a higher minimum number of particles in the interaction is equivalent to a higher minimum order in g. In order to achieve minimality we have truncated this series at the lowest effective order. This should not be viewed as some *ad hoc* recipe, but rather as a systematic expansion of the effective potential.

Up to now we have seen how the non-local effective Lagrangian method works for unstable gauge bosons in a simple $SU(N)$ gauge theory with fermions. In the Standard Model there are different types of unstable particles: the top-quark, the massive gauge-bosons, the Higgs boson. In Ref. [32] it was shown how to extend the above-described method in such a way that it allows the description of all the unstable particles in terms of bi-local effective Lagrangians.

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