The Standard Model

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Contents

1 QED Lagrangian, $U(1)$ invariance and Feynman rules

We will consider the simplest case of a gauge theory, quantum electrodynamics (QED), describing the interaction of spin-$\frac{1}{2}$ particles with photons. A gauge-fixed Lagrangian for QED is given by

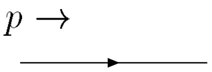
$$\mathcal{L}_{\text{QED}} = -\frac{1}{4} F_{\mu\nu} F_{\mu\nu} - \frac{1}{2} \left(\mathcal{C}_{_A}\right)^2 - \sum_f \overline{\psi}_f \left(\not\partial - ieQ_f \not{A} + m_f\right) \psi_f , \qquad (1)$$

where

$$F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu , \qquad \mathcal{C}_{_A} = -\frac{1}{\xi} \partial_\mu A_\mu , \qquad (2)$$

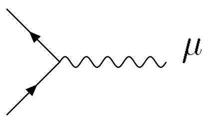
and where the sum runs over the fermion fields, f. Each with charge eQ_f, e, being the charge of the positron, and mass m_f. We have leptons $f = l = e, \mu, \tau$, with $Q_l = -1$, quarks $f = u, c$ and t with $Q_f = \frac{2}{3}$, and quarks $f = d, s$ and b with $Q_f = -\frac{1}{3}$.

The Feynman rules of QED are particularly simple:



$$\frac{1}{(2\pi)^4 i} \frac{-i\not{p} + m_f}{p^2 + m_f^2 - i\epsilon},$$

$$\frac{1}{(2\pi)^4 i} \frac{1}{p^2 + i\epsilon} \left[\delta_{\mu\nu} + \left(\xi^2 - 1\right) \frac{p_\mu p_\nu}{p^2}\right], \qquad (3)$$



$$(2\pi)^4 i \; ieQ_f \gamma_\mu .$$

We will confine the calculations to a special gauge, the renormalizable or Feynman gauge where $\xi = 1$. It is well known that the ξ-dependence cancels in the $\mathcal{S}$-matrix for a given physical process, although this is not necessarily true for Green functions.

2 The processes $e^+e^- \rightarrow \mu^+\mu^-, e^+e^-$

A process of special interest in QED is the annihilation of an e^+e^- pair with the creation of a pair of different fermions, for example, $\mu^+\mu^-$, as in Fig. 1

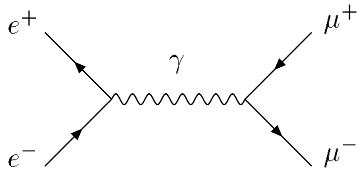


Figure 1: The diagram for $e^+e^- \to \mu^+\mu^-$ annihilation.

In this case, only the annihilation diagram contributes to the cross-section at the lowest order. If, instead, the flavors of the incoming and outgoing fermions are the same, then we have two diagrams and, for $f = e$ (Bhabha scattering).

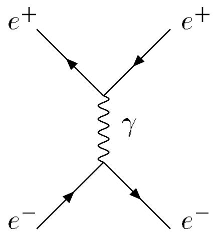


Figure 2: The t-channel diagram for Bhabha scattering, $e^+e^- \to e^+e^-$.

Beyond the lowest order the situation is more complicated due to the presence of radiative corrections. We will face multi-loop diagrams with all the complications inherent to the renormalization procedure but also another class of divergences will appear: infrared and collinear.

We can now compute the unpolarized cross-section in the Born approximation.

$$e^-(p_-)\, e^+(p_+) \to f(q_-)\, \overline{f}(q_+)\,. \tag{4}$$

The total cross-section for such a reaction is given by

$$\sigma_{f\overline{f}} = \frac{(2\pi)^4}{2\sqrt{\lambda\left(s, m_e^2, m_e^2\right)}} \int d\Gamma_2 \overline{\sum_{\text{spins}}} \mid \mathcal{M} \mid^2, \tag{5}$$

where $\mathcal{M}$ is the amplitude of the process. The Eq.(5) defines *the normalization* of the amplitude; symbol $\overline{\sum}_{\text{spins}}$ stands for summation over initial and final spin degrees of freedom. If the initial pair is assumed to be unpolarized, then we should also average over the initial spin states, with an additional factor 1/4 in Eq.(5) and below. The $d\Gamma_2$ is an element of $2 \to 2$ phase space

$$d\Gamma_2 = \frac{dq_+}{(2\pi)^3}\frac{dq_-}{(2\pi)^3}\delta\left(q_+^2+m_f^2\right)\delta\left(q_-^2+m_f^2\right)\theta\left(q_{+0}\right)\theta\left(q_{-0}\right)\delta\left(p_+ + p_- - q_+ - q_-\right). \tag{6}$$

The variable s is the square of the total energy in the centre-of-mass system (c.m.s.) and the Kállen λ-function is defined by

$$\lambda\left(x,y,z\right) = x^2+y^2+z^2-2\left(xy+xz+yz\right). \tag{7}$$

We now introduce the usual Mandelstam invariants s, t and u to replace the non-covariant quantities, for example, energies and scattering angles:

$$\begin{aligned} s &= -\left(p_+ + p_-\right)^2 = -\left(q_+ + q_-\right)^2 > 0, \\ t &= -\left(p_+ - q_+\right)^2 = -\left(p_- - q_-\right)^2 \leq 0, \\ u &= -\left(p_+ - q_-\right)^2 = -\left(p_- - q_+\right)^2 \leq 0. \end{aligned} \tag{8}$$

If we introduce the scattering angle θ in the centre-of-mass system as the angle between the incoming e^+ and the outgoing $\overline{f}$, then

$$\begin{aligned} t &= m_e^2+m_f^2+2\left(\mid \vec{p_+}\mid\mid \vec{q_+}\mid\cos\theta - E_{e^+}E_{\overline{f}}\right) \\ &= -\frac{1}{2}[s-2\left(m_e^2+m_f^2\right)-s\,\beta_e\beta_f\cos\theta], \\ u &= -\frac{1}{2}[s-2\left(m_e^2+m_f^2\right)+s\,\beta_e\beta_f\cos\theta], \end{aligned} \tag{9}$$

where

$$\beta_f^2 = \frac{\lambda\left(s,m_f^2,m_f^2\right)}{s^2} = 1-\frac{4m_f^2}{s}, \tag{10}$$

with β_f being the relativistic velocity $\mid\vec{p}\mid/E$. The quantities s, t and u are not independent but fulfil the identity

$$s+t+u = 2\left(m_e^2+m_f^2\right). \tag{11}$$

In this way we obtain to the following expressions for the differential phase space and cross-section:

$$d\Gamma_2 = \frac{1}{4\left(2\pi\right)^5}\frac{1}{s\beta_e}dt, \qquad \frac{d\sigma_{f\overline{f}}}{dt} = \frac{1}{16\,\pi\,s^2\beta_e^2}\sum_{\text{spins}}\mid\mathcal{M}^{\text{Born}}\mid^2, \tag{12}$$

with conditions defining the physical portion of the phase space, i.e.

$$s > 4m_e^2\,, \qquad X > 0\,, \tag{13}$$

where the complete expression for X is

$$\begin{aligned} X &= -\frac{1}{(16s)^2}\lambda\left(\lambda\left(s,m_e^2,m_e^2\right),\lambda\left(s,m_f^2,m_f^2\right),\lambda\left(-(P+q)^2,-P^2,-q^2\right)\right), \\ P &= p_+ + p_- = q_+ + q_-\,, \qquad q = p_+ - q_+ = q_- - p_-\,. \end{aligned} \tag{14}$$

The kinematic interpretation of the condition $X > 0$ is that the cosine of the scattering angle in the centre-of-mass system must lie between -1 and $+1$. The differential cross-section, Eq.(12), can be related to the differential cross-section for the scattering angle in the centre-of-mass system

$$\frac{d\sigma_{f\bar{f}}}{d\Omega_{\text{c.m.s}}} = \frac{d\sigma_{f\bar{f}}}{dt}\frac{s}{4\,\pi}\beta_e\beta_f = \frac{1}{64\,\pi^2 s}\frac{\beta_f}{\beta_e}\overline{\sum_{\text{spins}}}\mid \mathcal{M}^{\text{Born}}\mid^2. \tag{15}$$

2.1 The Born cross-sections

In this section we shall give the complete Born cross-section for two of the relevant processes: Bhabha scattering and annihilation into fermion pairs.

2.2 Bhabha scattering.

Consider now the amplitude of electron–positron scattering at $\mathcal{O}\left(e^2\right)$, $e^-\left(p_-\right)e^+\left(p_+\right) \to e^-\left(q_-\right)e^+\left(q_+\right)$:

$$\begin{aligned} \mathcal{M}^{\text{Born}} = &- \; i\,e^2\bar{v}\left(p_+\right)\gamma_\mu u\left(p_-\right)\;\bar{u}\left(q_-\right)\gamma_\mu v\left(q_+\right)\;\frac{1}{\left(p_+ + p_-\right)^2} \\ &+ \; i\,e^2\bar{u}\left(q_-\right)\gamma_\mu u\left(p_-\right)\;\bar{v}\left(p_+\right)\gamma_\mu v\left(q_+\right)\;\frac{1}{\left(p_+ - q_+\right)^2}. \end{aligned} \tag{16}$$

The full expression for the unpolarized cross-section is therefore

$$\begin{aligned} \frac{d\sigma_{e^+e^-}}{dt} = & \frac{2\,\pi\alpha^2}{s\left(s-4\,m_e^2\right)}\Big\{\frac{1}{s^2}\left[\left(u-2\,m_e^2\right)^2+\left(t-2\,m_e^2\right)^2+4\,m_e^2 s\right] \\ &+\frac{2}{st}[(s+t)^2-4\,m_e^4]+\frac{1}{t^2}\left[\left(u-2\,m_e^2\right)^2+\left(s-2\,m_e^2\right)^2+4\,m_e^2 t\right]\Big\}, \end{aligned} \tag{17}$$

with the standard notation for the fine-structure constant:

$$\alpha = \frac{e^2}{4\,\pi}. \tag{18}$$

The amplitude Eq.(16) is the sum of two terms: s- and t-channel photon exchange. Of particular interest is the behaviour of the cross-section for two limiting cases:

Case 1 $m_e^2/s \to 0$. Here we have

$$\frac{d\sigma_{e^+e^-}}{d\Omega}_{\mathrm{c.m.s}} = \frac{\alpha^2}{2\,s}\left[\sin^4(\theta/2) + \cos^4(\theta/2) - 2\,\frac{\cos^4(\theta/2)}{\sin^2(\theta/2)} + \frac{1+\cos^4(\theta/2)}{\sin^4(\theta/2)}\right], \tag{19}$$

where the first two terms are $s-s$, the third is the $s-t$ interference and the last is $t-t$.

Case 2 $4m_e^2/s \to 1$. Here we have:

$$\frac{d\sigma_{e^+e^-}}{d\Omega}_{\mathrm{c.m.s}} = \frac{\alpha^2 m_e^2}{16\,\mid \vec{p_+} \mid^4 \sin^4(\theta/2)}, \tag{20}$$

where only the $t-t$ contribution survives.

2.3 The s-channel annihilation process.

The cross-section for the annihilation $e^+e^- \to f\overline{f}\,(f \neq e)$ can be easily derived from the previous one by omitting out the t-channel diagram and by scaling the charges of the flavors involved. In the high-energy limit we obtain

$$\frac{d\sigma_{f\overline{f}}}{d\Omega}_{\mathrm{c.m.s}} = Q_f^2\,\frac{\alpha^2}{4\,s}\left(1+\cos^2\theta\right). \tag{21}$$

Note that the total cross-section for the annihilation process is finite, which is not the case for Bhabha scattering. The corresponding expression in Bhabha scattering cannot be integrated over all angles because the integral diverges at $\theta = 0$. This divergence is connected with the physically unrealizable requirement that the two fermions scatter without the emission of photons. Very low-energy photon emission cannot be ignored when the momentum transfer becomes very small ($\theta \to 0$). Therefore, it is one more manifestation of the infrared problem in QED.

$$\sigma_{\mathrm{T}}\left(e^+e^- \to f\overline{f}\right) = \frac{4\,\pi\alpha^2}{3\,s}Q_f^2. \tag{22}$$

The differential cross-section is an even function of the scattering angle, which results in the vanishing of the forward–backward asymmetry, defined by

$$A_{\text{FB}} = \frac{\displaystyle\int_0^1 d\cos\theta \, \frac{d\sigma}{d\cos\theta} - \int_{-1}^0 d\cos\theta \, \frac{d\sigma}{d\cos\theta}}{\displaystyle\int_0^1 d\cos\theta \, \frac{d\sigma}{d\cos\theta} + \int_{-1}^0 d\cos\theta \, \frac{d\sigma}{d\cos\theta}}. \tag{23}$$

This property is peculiar to the QED Lagrangian, which conserves parity. It is modified in higher orders, since they may induce charge asymmetric effects. Charge conjugation invariance can be invoked to show that only the interference terms between the lowest order graph and the two-photon (box) diagrams contribute to the forward–backward asymmetry to order α^3. Similarly, for bremsstrahlung contributions, $e^+e^- \to f\overline{f}\gamma$, only the interference between the C-odd initial state radiation diagrams and the C-even final state radiation diagrams has to be considered for the asymmetry.

Modifications are also expected by the inclusion of initial and final state helicities for the fermions, which induce P-odd effects. Finally, the inclusion of resonances with both vector and axial couplings to fermions will produce P-odd effects, already in the lowest order.

For Bhabha scattering, however, the lowest order cross-section shows forward–backward asymmetry, in contrast with the annihilation cross-section. This is due to the presence of the t-dependent scattering diagram containing the propagator $1/t$ in the differential cross-section Eq.(17). This term is $\cos\theta$-odd and it causes a non-zero A_{FB}. This particular example is telling us that there are many different reasons why the forward–backward asymmetry may arise: from P, C-non-invariance to a trivial kinematical origin.

3 Electroweak Lagrangian and Feynman rules

3.1 Lagrangian building

In this section we give the explicit form of the Standard Model (SM) Lagrangian in the R_ξ gauge. We assume the simplest (minimal) scalar sector.

Within the SM Lagrangian there is a triplet of vector bosons B^a_μ, a singlet B^0_μ, a complex scalar field K, fermion families, and Faddeev–Popov ghost-fields (hereafter FP) $X^\pm, Y^Z, Y^A$. The physical fields Z and A are related to B^3_μ and B^0_μ by

$$\begin{pmatrix} Z \\ A \end{pmatrix} = \begin{pmatrix} c_\theta & -s_\theta \\ s_\theta & c_\theta \end{pmatrix} \begin{pmatrix} B_3 \\ B_0 \end{pmatrix}, \tag{24}$$

where $s_\theta (c_\theta)$ denote as usual the sine and cosine of the weak mixing angle. The scalar field in the minimal realization of the SM is

$$K = \frac{1}{\sqrt{2}} \begin{pmatrix} \chi \\ \\ \sqrt{2} i \phi^- \end{pmatrix}, \qquad \chi = H + 2\frac{M}{g} + i\phi^0, \tag{25}$$

where by H we denote the physical Higgs boson and moreover M and g are Lagrangian parameters corresponding to the bare W mass and to the $SU(2)$ bare coupling constant. The total Lagrangian will be the sum of various pieces. The first is $\mathcal{L}_{\rm YM} + \mathcal{L}_{\rm S}$, with the standard Yang–Mills Lagrangian given by

$$\mathcal{L}_{\rm YM} = -\frac{1}{4} F^a_{\mu\nu} F^a_{\mu\nu} - \frac{1}{4} F^0_{\mu\nu} F^0_{\mu\nu}, \tag{26}$$

and the *minimal* Higgs sector by

$$\mathcal{L}_{\rm S} = -\left(D_\mu K\right)^+ D_\mu K - \mu^2 K^+ K - \frac{1}{2}\lambda \left(K^+ K\right)^2, \tag{27}$$

where $\lambda > 0$ and symmetry breaking requires $\mu^2 < 0$. Moreover, we use standard definitions for

$$\begin{aligned} F^a_{\mu\nu} &= \partial_\mu B^a_\nu - \partial_\nu B^a_\mu + g \varepsilon_{abc} B^b_\mu B^c_\nu, \\ F^0_{\mu\nu} &= \partial_\mu B^0_\nu - \partial_\nu B^0_\mu, \end{aligned} \tag{28}$$

and the covariant derivative for the scalar field assumes the following form

$$D_\mu K = \left(\partial_\mu - \frac{i}{2} g B_\mu^a \tau^a - \frac{i}{2} g g_1 B_\mu^0\right) K, \tag{29}$$

with the standard Pauli matrices τ^a and $g_1 = -s_\theta / c_\theta$. They follow from the fact that K, as defined in Eq.(25), belongs to a doublet representation of the symmetry group. The scalar field can be rewritten as

$$K = \frac{1}{\sqrt{2}} \left(H + 2\frac{M}{g} + i\phi^a \tau^a\right) \begin{pmatrix} 1 \\ 0 \end{pmatrix}, \tag{30}$$

so that its covariant derivative becomes

$$\begin{aligned} D_\mu K &= \frac{1}{\sqrt{2}} \left(\partial_\mu - \frac{i}{2} g B_\mu^a \tau^a - \frac{i}{2} g g_1 B_\mu^0\right) \left(H + 2\frac{M}{g} + i\phi^a \tau^a\right) \begin{pmatrix} 1 \\ 0 \end{pmatrix} \\ &= \frac{1}{\sqrt{2}} \Big\{ \partial_\mu H - \frac{i}{2} g g_1 B_\mu^0 \left(H + 2\frac{M}{g}\right) + \frac{1}{2} g B_\mu^a \phi^a \\ &\qquad + i \Big[\partial_\mu \phi^a - \frac{1}{2} g B_\mu^a \left(H + 2\frac{M}{g}\right) - \frac{i}{2} g g_1 B_\mu^0 \phi^a + \frac{1}{2} g \varepsilon_{cba} B_\mu^c \phi^b \Big] \tau^a \Big\} \begin{pmatrix} 1 \\ 0 \end{pmatrix}. \end{aligned} \tag{31}$$

We split the Lagrangian into $\mathcal{L}_{\rm YM} - (D_\mu K)^+ D_\mu K$ and $\mathcal{L}_{\rm S}^{\rm I}$, the latter containing the interactions of the scalar sector and write

$$\mathcal{L}_{\rm YM} - (D_\mu K)^+ D_\mu K = \mathcal{L}_0 + M \left(\frac{1}{c_\theta} Z_\mu \partial_\mu \phi^0 + W_\mu^+ \partial_\mu \phi^- + W_\mu^- \partial_\mu \phi^+\right), \tag{32}$$

where the charged fields have been introduced as

$$W_\mu^\pm = \frac{1}{\sqrt{2}} \left(B_\mu^1 \mp i B_\mu^2\right), \qquad \phi^\pm = \frac{1}{\sqrt{2}} \left(\phi^1 \mp i\phi^2\right), \qquad \phi^0 \equiv \phi^3. \tag{33}$$

This part of the Lagrangian contains $Z - \phi^0, W^\pm - \phi^\mp$ mixing terms; they are of the zeroth order in the coupling constant and their contribution must be summed up if we want to develop perturbation theory. There we discover the singularity of the Lagrangian. The construction of the SM continues as follows.

First we add a *gauge-fixing* piece to the Lagrangian (called $\mathcal{L}_{\rm gf}$ in the following) that cancels these mixing terms. However, it breaks the gauge invariance and successively we must introduce the so-called Faddeev–Popov ghost fields to compensate for this breaking. We now specify a non-singular gauge; in fact, a set of gauges R_ξ depending on a single parameter ξ. We have a renormalizable gauge for finite ξ and the physical (unitary) gauge is obtained for $\xi \to \infty$. The gauge-fixing piece is

$$\mathcal{L}_{\rm gf} \;=\; -\frac{1}{2}\mathcal{C}^a\mathcal{C}^a - \frac{1}{2}\left(\mathcal{C}^0\right)^2 \;=\; -\mathcal{C}^+\mathcal{C}^- - \frac{1}{2}\left[\left(\mathcal{C}^3\right)^2 + \left(\mathcal{C}^0\right)^2\right], \tag{34}$$

where we can write

$$\mathcal{C}^a = -\frac{1}{\xi}\partial_\mu B^a_\mu + \xi M \phi^a. \tag{35}$$

The various components are given in the following equations: first

$$\mathcal{C}^{\pm} = -\frac{1}{\xi}\partial_\mu W^{\pm}_\mu + \xi M \phi^{\pm}, \qquad \mathcal{C}^0 = -\frac{1}{\xi}\partial_\mu B^0_\mu + \xi\frac{s_\theta}{c_\theta}M\phi^0. \tag{36}$$

Then we write

$$-\frac{1}{2}\left[\left(\mathcal{C}^3\right)^2 + \left(\mathcal{C}^0\right)^2\right] = -\frac{1}{2}\mathcal{C}^2_{\scriptscriptstyle Z} - \frac{1}{2}\mathcal{C}^2_{\scriptscriptstyle A}, \tag{37}$$

and derive the gauge-fixing term in the $Z - A$ basis,

$$\mathcal{C}_{\scriptscriptstyle A} = -\frac{1}{\xi}\partial_\mu A_\mu, \qquad \mathcal{C}_{\scriptscriptstyle Z} = -\frac{1}{\xi}\partial_\mu Z_\mu + \xi\frac{M}{c_\theta}\phi^0. \tag{38}$$

In the R_ξ gauge we have that

$$\mathcal{L}_{\scriptscriptstyle \rm YM} - \left(D_\mu K\right)^+ D_\mu K - \mathcal{C}^+\mathcal{C}^- - \frac{1}{2}\mathcal{C}^2_{\scriptscriptstyle Z} - \frac{1}{2}\mathcal{C}^2_{\scriptscriptstyle A} = \mathcal{L}_{\rm prop} + \mathcal{L}^{\rm bos,I}. \tag{39}$$

The quadratic part of the Lagrangian, $\mathcal{L}_{\rm prop}$, now reads

$$\begin{aligned}
\mathcal{L}_{\rm prop} \;=\; & -\partial_\mu W^+_\nu \partial_\mu W^-_\nu + \left(1 - \frac{1}{\xi^2}\right)\partial_\mu W^+_\mu \partial_\nu W^-_\nu \\
& -\frac{1}{2}\partial_\mu Z_\nu \partial_\mu Z_\nu + \frac{1}{2}\left(1 - \frac{1}{\xi^2}\right)\left(\partial_\mu Z_\mu\right)^2 \\
& -\frac{1}{2}\partial_\mu A_\nu \partial_\mu A_\nu + \frac{1}{2}\left(1 - \frac{1}{\xi^2}\right)\left(\partial_\mu A_\mu\right)^2 \\
& -\frac{1}{2}\partial_\mu H \partial_\mu H - \partial_\mu \phi^+ \partial_\mu \phi^- - \frac{1}{2}\partial_\mu \phi^0 \partial_\mu \phi^0 \\
& -M^2 W^+_\mu W^-_\mu - \frac{1}{2}\frac{M^2}{c^2_\theta}Z_\mu Z_\mu \\
& -\xi^2 M^2 \phi^+\phi^- - \frac{1}{2}\xi^2\frac{M^2}{c^2_\theta}\phi^0\phi^0 - \frac{1}{2}M_{\scriptscriptstyle H} H^2.
\end{aligned} \tag{40}$$

The quadratic part of the Lagrangian allows us to derive propagators. Those for the gauge fields are as follows:

$$
\begin{aligned}
\mathcal{L}_{\rm prop} \;\rightarrow\; W^{\pm} \qquad & \frac{1}{p^2+M^2}\left\{\delta_{\mu\nu} + \left(\xi^2-1\right)\frac{p_\mu p_\nu}{p^2+\xi^2 M^2}\right\} \\
= \;\; & \frac{1}{p^2+M^2}\left(\delta_{\mu\nu} + \frac{p_\mu p_\nu}{M^2}\right) - \frac{p_\mu p_\nu}{M^2\left(p^2+\xi^2 M^2\right)} \\
= \;\; & \frac{1}{p^2+M^2}\left(\delta_{\mu\nu} - \frac{p_\mu p_\nu}{p^2}\right) + \frac{\xi^2}{p^2+\xi^2 M^2}\frac{p_\mu p_\nu}{p^2}, \qquad (41) \\
Z \qquad & \text{is obtained from } W^{\pm} \text{ by replacing } \;\; M \rightarrow \frac{M}{c_\theta}, \\
A \qquad & \frac{1}{p^2}\left\{\delta_{\mu\nu} + \left(\xi^2-1\right)\frac{p_\mu p_\nu}{p^2}\right\}.
\end{aligned}
$$

The scalar field propagators are given by

$\phi^{\pm}$ $\qquad \dfrac{1}{p^2+\xi^2 M^2}, \qquad$

ϕ^{0} $\qquad \dfrac{1}{p^2+\xi^2 \dfrac{M^2}{c_\theta^2}}. \qquad (42)$

3.2 Interactions.

Having fixed the propagators we can spell out the weak Lagrangian, describing the vector bosons and their interactions including interactions with the scalar system:

$$
\begin{aligned}
\mathcal{L}^{\rm bos,I} = \; & -igc_\theta\left\{\partial_\nu Z_\mu W_\mu^{[+}W_\nu^{-]} - Z_\nu W_\mu^{[+}\partial_\nu W_\mu^{-]} + Z_\mu W_\nu^{[+}\partial_\nu W_\mu^{-]}\right\} \qquad (43) \\
& -igs_\theta\left\{\partial_\nu A_\mu W_\mu^{[+}W_\nu^{-]} - A_\nu W_\mu^{[+}\partial_\nu W_\mu^{-]} + A_\mu W_\nu^{[+}\partial_\nu W_\mu^{-]}\right\} \\
& +\frac{1}{2}g^2\left\{\left(W_\mu^+ W_\nu^-\right)^2 - \left(W_\mu^+ W_\mu^-\right)^2\right\} \\
& +g^2 c_\theta^2\left\{Z_\mu Z_\nu W_\mu^+ W_\nu^- - Z_\mu Z_\mu W_\nu^+ W_\nu^-\right\} \\
& +g^2 s_\theta^2\left\{A_\mu A_\nu W_\mu^+ W_\nu^- - A_\mu A_\mu W_\nu^+ W_\nu^-\right\} \\
& +g^2 s_\theta c_\theta\left\{A_\mu Z_\nu W_\mu^{[+}W_\nu^{-]} - 2A_\mu Z_\mu W_\nu^+ W_\nu^-\right\} \\
& -gMH\left\{W_\mu^+ W_\nu^- + \frac{1}{2c_\theta^2}Z_\mu Z_\mu\right\} \\
& -\frac{i}{2}g\left\{W_\mu^+\left(\phi^0\partial_\mu\phi^- - \phi^-\partial_\mu\phi^0\right) - W_\mu^-\left(\phi^0\partial_\mu\phi^+ - \phi^+\partial_\mu\phi^0\right)\right\}
\end{aligned}
$$

$$
\begin{aligned}
&+\frac{1}{2}g\left\{W_\mu^+\left(H\partial_\mu\phi^- - \phi^-\partial_\mu H\right) - W_\mu^-\left(H\partial_\mu\phi^+ - \phi^+\partial_\mu H\right)\right\}\\
&+\frac{1}{2}\frac{g}{c_\theta}Z_\mu\left(H\partial_\mu\phi^0 - \phi^0\partial_\mu H\right)\\
&+ig\left(s_\theta A_\mu - \frac{s_\theta^2}{c_\theta}Z_\mu\right)MW_\mu^{[+}\phi^{-]}\\
&+ig\left(s_\theta A_\mu + \frac{c_\theta^2 - s_\theta^2}{c_\theta}Z_\mu\right)\left(\phi^+\partial_\mu\phi^- - \phi^-\partial_\mu\phi^+\right)\\
&-\frac{1}{4}g^2W_\mu^+W_\mu^-\left(HH + \phi^0\phi^0 + 2\phi^+\phi^-\right)\\
&-\frac{1}{8}\frac{g^2}{c_\theta^2}Z_\mu Z_\mu\left\{HH + \phi^0\phi^0 + 2\left(c_\theta^2 - s_\theta^2\right)^2\phi^+\phi^-\right\}\\
&-\frac{1}{2}g^2\frac{s_\theta^2}{c_\theta}Z_\mu\phi^0W_\mu^{[+}\phi^{-]} - \frac{i}{2}g^2\frac{s_\theta^2}{c_\theta}Z_\mu HW_\mu^{[+}\phi^{-]} + \frac{1}{2}g^2s_\theta A_\mu\phi^0W_\mu^{[+}\phi^{-]}\\
&+\frac{i}{2}g^2s_\theta A_\mu HW_\mu^{[+}\phi^{-]} - g^2\frac{s_\theta}{c_\theta}\left(c_\theta^2 - s_\theta^2\right)Z_\mu A_\mu\phi^+\phi^- - g^2s_\theta^2A_\mu A_\mu\phi^+\phi^-,
\end{aligned}
$$

where we have introduced the antisymmetrized combination:

$$
A^{[+}B^{-]} = A^+B^- - A^-B^+. \tag{44}
$$

The interactions in the scalar sector will be given by the scalar potential written as

$$
\mathcal{L}_{\mathrm{s}}^{\mathrm{I}} = -\mu^2K^+K - \frac{1}{2}\lambda\left(K^+K\right)^2, \tag{45}
$$

where it is particularly convenient to introduce new parameters,

$$
M_{_H}^2 = 4\frac{\lambda}{g^2}M^2, \qquad \beta_{_H} = \mu^2 + 2\frac{\lambda}{g^2}M^2, \qquad \alpha_{_H} = \frac{1}{4}\frac{M_{_H}^2}{M^2}. \tag{46}
$$

In terms of these we arrive at the following expression for the interaction Lagrangian:

$$
\begin{aligned}
\mathcal{L}_{\mathrm{s}}^{\mathrm{I}} \;=\; & -\beta_{_H}\left\{2\frac{M}{g}H + \frac{1}{2}\left[H^2 + \left(\phi^0\right)^2 + 2\phi^+\phi^-\right]\right\}\\
& -g\alpha_{_H}M\left[H^3 + H\left(\phi^0\right)^2 + 2H\phi^+\phi^-\right]\\
& -\frac{1}{8}g^2\alpha_{_H}\Big[H^4 + \left(\phi^0\right)^4 + 2H^2\left(\phi^0\right)^2\\
& +4H^2\phi^+\phi^- + 4\left(\phi^0\right)^2\phi^+\phi^- + 4\left(\phi^+\phi^-\right)^2\Big].
\end{aligned} \tag{47}
$$

3.3 Faddeev–Popov ghosts.

In order to define the FP ghost Lagrangian we must subject $\mathcal{C}^a$ to a gauge transformation. In what follows we list the $SU(2) \otimes U(1)$ transformation laws of the various fields:

$$\begin{aligned}
B^a_\mu &\to B^a_\mu + g\varepsilon_{abc}\Lambda^b B^c_\mu - \partial_\mu \Lambda^a, \qquad B^0_\mu \to B^0_\mu - \partial_\mu\Lambda^0, \\
K &\to \left(1 - \frac{i}{2} g\Lambda^a \tau^a - \frac{i}{2} g g_1 \Lambda^0\right) K, \qquad \text{with} \quad g_1 = -\frac{s_\theta}{c_\theta}, \\
H + i\phi^0 &\to H + i\phi^0 - \frac{i}{2} g \left[\left(\Lambda^3 + g_1\Lambda^0\right)\left(H + 2\frac{M}{g} + i\phi^0\right) + 2i\Lambda^+\phi^-\right], \\
\phi^0 &\to \phi^0 - \frac{1}{2} g \left(\Lambda^3 + g_1\Lambda^0\right)\left(H + 2\frac{M}{g}\right) + \frac{i}{2} g \left(\Lambda^-\phi^+ - \Lambda^+\phi^-\right), \\
\phi^- &\to \phi^- - \frac{1}{2} g\Lambda^- \left(H + 2\frac{M}{g} + i\phi^0\right) - \frac{i}{2} g \left(-\Lambda^3 + g_1\Lambda^0\right)\phi^-, \qquad (48)
\end{aligned}$$

where the appropriate combinations of gauge parameters are

$$\begin{aligned}
\Lambda^1 &= \frac{1}{\sqrt{2}}\left(\Lambda^+ + \Lambda^-\right), & \Lambda^2 &= \frac{i}{\sqrt{2}}\left(\Lambda^+ - \Lambda^-\right), \\
\Lambda^3 &= c_\theta \Lambda^Z + s_\theta \Lambda^A, & \Lambda^0 &= -s_\theta\Lambda^Z + c_\theta\Lambda^A, \\
\Lambda^3 + g_1\Lambda^0 &= \frac{1}{c_\theta}\Lambda^Z, & -\Lambda^3 + g_1\Lambda^0 &= -\frac{c_\theta^2 - s_\theta^2}{c_\theta}\Lambda^Z - 2s_\theta\Lambda^A.
\end{aligned} \qquad (49)$$

So that we may write

$$\begin{aligned}
\phi^0 &\to \phi^0 - \frac{1}{2} g \frac{\Lambda^Z}{c_\theta}\left(H + 2\frac{M}{g}\right) + \frac{i}{2} g \left(\Lambda^-\phi^+ - \Lambda^+\phi^-\right), \\
\phi^- &\to \phi^- - \frac{1}{2} g\Lambda^- \left(H + 2\frac{M}{g} + i\phi^0\right) + \frac{i}{2} g \left(\frac{c_\theta^2 - s_\theta^2}{c_\theta}\Lambda^Z + 2s_\theta\Lambda^A\right)\phi^-, \\
W^-_\mu &\to W^-_\mu - ig\Lambda^-\left(c_\theta Z_\mu + s_\theta A_\mu\right) + ig\left(c_\theta\Lambda^Z + s_\theta\Lambda^A\right)W^-_\mu - \partial_\mu\Lambda^-, \\
A_\mu &\to A_\mu + igs_\theta\left(\Lambda^- W^+_\mu - \Lambda^+ W^-_\mu\right) - \partial_\mu\Lambda^A, \\
Z_\mu &\to Z_\mu + igc_\theta\left(\Lambda^- W^+_\mu - \Lambda^+ W^-_\mu\right) - \partial_\mu\Lambda^Z. \qquad (50)
\end{aligned}$$

The gauge transformations can be summarized in terms of the following equation

$$\mathcal{C}^i \to \mathcal{C}^i + \left(\mathrm{M}^{ij} + g\mathrm{L}^{ij}\right)\Lambda^j. \qquad (51)$$

We can see that M^{ij} has an inverse and we thus have a permissible gauge. In the charged sector we obtain

$$
\begin{aligned}
\mathcal{C}^- &= -\frac{1}{\xi}\partial_\mu W_\mu^- + \xi M \phi^- \\
&\to \mathcal{C}^- - \frac{1}{\xi}\partial_\mu\{-ig\Lambda^-\left(c_\theta Z_\mu + s_\theta A_\mu\right) + ig\left(c_\theta \Lambda^Z + s_\theta \Lambda^A\right) W_\mu^- - \partial_\mu \Lambda^-\} \\
&\quad + g\xi M\left\{-\frac{1}{2}\Lambda^-\left(H + 2\frac{M}{g} + i\phi^0\right) + \frac{i}{2}\frac{c_\theta^2 - s_\theta^2}{c_\theta}\Lambda^Z\phi^- + i s_\theta \Lambda^A \phi^-\right\} \\
&= \mathcal{C}^- + \frac{1}{\xi}\Box\Lambda^- - \xi M^2\Lambda^- + \frac{i}{\xi}g\partial_\mu\{\Lambda^-\left(c_\theta Z_\mu + s_\theta A_\mu\right)\} \\
&\quad - \frac{i}{\xi}g\partial_\mu\{\left(c_\theta\Lambda^Z + s_\theta\Lambda^A\right)W_\mu^-\} - \frac{1}{2}\xi g M\left(H + i\phi^0\right)\Lambda^- \\
&\quad + \frac{i}{2}\xi g M\frac{c_\theta^2 - s_\theta^2}{c_\theta}\Lambda^Z\phi^- + i\xi g s_\theta M \Lambda^A\phi^-.
\end{aligned}
\tag{52}
$$

Since we have

$$
\mathcal{C}^- \;\to\; \mathcal{C}^- + \left(\mathrm{M}^{-i} + g\mathrm{L}^{-i}\right)\Lambda^i, \qquad i = \pm, Z, A, \tag{53}
$$

the corresponding propagator is

$$
\frac{1}{\xi}\Box - \xi M^2 \qquad \xrightarrow[X^\pm]{} \qquad \frac{\xi}{p^2 + \xi^2 M^2}. \tag{54}
$$

For the transformation of $\mathcal{C}_A$ we obtain:

$$
\begin{aligned}
\mathcal{C}_A &= -\frac{1}{\xi}\partial_\mu A_\mu \to \mathcal{C}_A - \frac{1}{\xi}\partial_\mu[igs_\theta\left(\Lambda^- W_\mu^+ - \Lambda^+ W_\mu^-\right) - \partial_\mu\Lambda^A] \\
&= \mathcal{C}_A + \frac{1}{\xi}\Box\Lambda^A - \frac{i}{\xi}g s_\theta \partial_\mu\left(\Lambda^- W_\mu^+ - \Lambda^+ W_\mu^-\right),
\end{aligned}
\tag{55}
$$

giving the propagator of Y^A

$$
\frac{1}{\xi}\Box \qquad \xrightarrow[Y^A]{} \qquad \frac{\xi}{p^2}. \tag{56}
$$

For the transformation of $\mathcal{C}_Z$ we find:

$$
\begin{aligned}
\mathcal{C}_Z &= -\frac{1}{\xi}\partial_\mu Z_\mu + \xi\frac{M}{c_\theta}\phi^0 \\
&\to \mathcal{C}_Z - \frac{1}{\xi}\partial_\mu\{igc_\theta\left(\Lambda^- W_\mu^+ - \Lambda^+ W_\mu^-\right) - \partial_\mu\Lambda^Z\}
\end{aligned}
$$

$$
\begin{aligned}
& +\xi\frac{M}{c_\theta}\left\{-\frac{M}{c_\theta}\Lambda^z - \frac{1}{2}g\frac{\Lambda^z}{c_\theta}H + \frac{i}{2}g\left(\Lambda^-\phi^+ - \Lambda^+\phi^-\right)\right\} \\
= \; & \mathcal{C}_z\frac{1}{\xi}\Box\Lambda^z - \xi\frac{M^2}{c_\theta^2}\Lambda^z - \frac{i}{\xi}gc_\theta\partial_\mu\left(\Lambda^- W_\mu^+ - \Lambda^+ W_\mu^-\right) \\
& -\frac{1}{2}\xi g\frac{M}{c_\theta^2}\Lambda^z H + i\xi g\frac{M}{c_\theta}\left(\Lambda^-\phi^+ - \Lambda^+\phi^-\right), \qquad (57)
\end{aligned}
$$

giving the propagator of Y^Z as follows:

$$
\frac{1}{\xi}\Box - \xi\frac{M^2}{c_\theta^2} \qquad \underset{Y^Z}{\cdots\cdots\blacktriangleright\cdots\cdots} \qquad \frac{\xi}{p^2 + \xi^2\dfrac{M^2}{c_\theta^2}}. \qquad (58)
$$

The interaction is derived from Eq.(51) and is given by $g\overline{X}^i \mathrm{L}^{ij} X^j$, where $X^i = X^+, X^-, Y^Z, Y^A$ and

$$
\begin{aligned}
X^1 &= \frac{1}{\sqrt{2}}\left(X^+ + X^-\right), & X^2 &= \frac{i}{\sqrt{2}}\left(X^+ - X^-\right), \\
X^3 &= c_\theta Y^Z + s_\theta Y^A, & X^0 &= -s_\theta Y^Z + c_\theta Y^A,
\end{aligned}
\qquad (59)
$$

where $X^\pm$ and Y^Z are the FP ghosts associated with the three vector bosons of weak interaction. Y^A is the FP ghost associated with the photon. The interaction Lagrangian in the FP sector may be cast in the following form

$$
\begin{aligned}
\mathcal{L}_{\mathrm{gf}}^{\mathrm{I}} = \; & igc_\theta\frac{1}{\xi}W_\mu^+\left(\partial_\mu\overline{Y}^z X^- - \partial_\mu\overline{X}^+ Y^z\right) + igc_\theta\frac{1}{\xi}W_\mu^-\left(\partial_\mu\overline{X}^- Y^z - \partial_\mu\overline{Y}^z X^+\right) \\
& + igs_\theta\frac{1}{\xi}W_\mu^+\left(\partial_\mu\overline{Y}^A X^- - \partial_\mu\overline{X}^+ Y^A\right) + igs_\theta\frac{1}{\xi}W_\mu^-\left(\partial_\mu\overline{X}^- Y^A - \partial_\mu\overline{Y}^A X^+\right) \\
& + igc_\theta\frac{1}{\xi}Z_\mu\left(\partial_\mu\overline{X}^+ X^+ - \partial_\mu\overline{X}^- X^-\right) + igs_\theta\frac{1}{\xi}A_\mu\left(\partial_\mu\overline{X}^+ X^+ - \partial_\mu\overline{X}^- X^-\right) \\
& -\frac{1}{2}g\xi MH\left(\overline{X}^+ X^+ + \overline{X}^- X^- + \frac{1}{c_\theta^2}\overline{Y}^z Y^z\right) \\
& -ig\xi M\frac{c_\theta^2 - s_\theta^2}{c_\theta}\left(\overline{X}^+ Y^z\phi^+ - \overline{X}^- Y^z\phi^-\right) \\
& +\frac{i}{2}g\xi M\frac{1}{c_\theta}\left(\overline{Y}^z X^-\phi^+ - \overline{Y}^z X^+\phi^-\right) \\
& +igs_\theta\xi M\left(\overline{X}^- Y^A\phi^- - \overline{X}^+ Y^A\phi^+\right) \\
& +\frac{i}{2}g\xi M\left(\overline{X}^+ X^+\phi^0 - \overline{X}^- X^-\phi^0\right). \qquad (60)
\end{aligned}
$$

Although most of the calculations are usually done in the R_ξ gauge described above (and in its limit $\xi \to 1$) there is the possibility of introducing a three-parameter gauge-fixing term.

3.4 Interactions with fermions.

Having derived the first part of the Lagrangian we now switch to discussing the coupling of vector bosons with fermions. A generic fermion-isodoublet will be denoted by

$$\psi = \begin{pmatrix} u \\ d \end{pmatrix}, \qquad \psi_{_{L,R}} = \frac{1}{2}\left(1 \pm \gamma_5\right)\psi, \tag{61}$$

with a covariant derivative for the L-fields which we write as

$$D_\mu \psi_{_L} \;=\; \left(\partial_\mu + g B^i_\mu T^i\right)\psi_{_L}, \qquad i = 0, ..., 3 \tag{62}$$

and which is written in terms of the following generators of $SU(2) \otimes U(1)$:

$$T^a \;=\; -\frac{i}{2}\tau^a, \qquad T^0 \;=\; -\frac{i}{2} g_2 I. \tag{63}$$

For the R-fields we have instead

$$D_\mu \psi_{_R} \;=\; \left(\partial_\mu + g B^i_\mu t^i\right)\psi_{_R}, \qquad i = 0, ..., 3, \tag{64}$$

$$t^a \;=\; 0, \qquad t^0 = -\frac{i}{2}\begin{pmatrix} g_3 & 0 \\ 0 & g_4 \end{pmatrix}. \tag{65}$$

This part of the Lagrangian can be written as

$$\mathcal{L}^{\mathrm{fer,I}}_{_V} = -\overline{\psi}_{_L} \not{D} \psi_{_L} - \overline{\psi}_{_R} \not{D} \psi_{_R}, \qquad g_i = -\frac{s_\theta}{c_\theta}\lambda_i. \tag{66}$$

The parameters g_2, g_3 and g_4 are arbitrary constants. Actually, there is a Ward identity, ensuring the relation $g_3 = g_1 + g_2$. In other words, these constants are not completely free if we want to generate fermion masses with the help of the Higgs system.

Thus, $\psi_{_L}$ transforms as a doublet under $SU(2)$ and the $\psi_{_R}$ as a singlet. The parameters λ_i are then fixed by the requirement that the e.m. current has the conventional structure, $iQ_f e\overline{f}\gamma_\mu f$. We put $e = gs_\theta$ and derive the solution as

$$\lambda_2 = 1 - 2Q_u \;=\; -1 - 2Q_d\,, \qquad \lambda_3 = -2Q_u\,, \qquad \lambda_4 = -2Q_d\,, \tag{67}$$

where the charge is

$$Q_f = 2I_f^{(3)}|Q_f|. \tag{68}$$

$W^\pm$ always couples to a $V+A$ current and $\mathcal{L}_V^{\rm fer,I}$ reads

$$\begin{aligned}
\mathcal{L}_V^{\rm fer,I} \;&=\; \sum_f \left[igs_\theta Q_f A_\mu \overline{f}\gamma_\mu f + i\frac{g}{2c_\theta}Z_\mu \overline{f}\gamma_\mu\left(I_f^{(3)} - 2Q_f s_\theta^2 + I_f^{(3)}\gamma_5\right)f\right] \\
&+ \sum_{\rm d}\left[i\frac{g}{2\sqrt{2}}W_\mu^+\overline{u}\gamma_\mu\left(1+\gamma_5\right)d + i\frac{g}{2\sqrt{2}}W_\mu^-\overline{d}\gamma_\mu\left(1+\gamma_5\right)u\right],
\end{aligned} \tag{69}$$

where the first sum runs over all fermions, f, and the second over all doublets, d, of the SM.

For the Higgs-fermion sector, in the presence of quarks, we need not only the field K but its conjugate K^c too; that is, we need both K and K^c in order to give mass to the up- and down-partner of the fermionic isodoublet. The K^c is

$$K^c = -\frac{1}{\sqrt{2}}\begin{pmatrix} \sqrt{2}i\phi^+ \\ \\ \chi^* \end{pmatrix}, \tag{70}$$

with the corresponding part of the Lagrangian:

$$\mathcal{L}_S^{\rm fer} = -\alpha_f \overline{\psi}_L K u_R - \beta_f \overline{\psi}_L K^c d_R + h.c. \tag{71}$$

The solution for the Yukawa couplings gives

$$\alpha_f = \frac{1}{\sqrt{2}}g\frac{m_u}{M}, \qquad \beta_f = -\frac{1}{\sqrt{2}}g\frac{m_d}{M}. \tag{72}$$

The last part of the Lagrangian is now

$$\mathcal{L}_S^{\rm fer} = -\sum_f m_f\overline{f}f + \mathcal{L}_S^{\rm fer,I}, \tag{73}$$

with an interaction Lagrangian given by

$$\begin{aligned}
\mathcal{L}_S^{\rm fer,I} \;&=\; \sum_{\rm d}\left\{i\frac{g}{2\sqrt{2}}\phi^+\left[\frac{m_u}{M}\overline{u}\left(1+\gamma_5\right)d - \frac{m_d}{M}\overline{u}\left(1-\gamma_5\right)d\right]\right. \\
&\left. + i\frac{g}{2\sqrt{2}}\phi^-\left[\frac{m_d}{M}\overline{d}\left(1+\gamma_5\right)u - \frac{m_u}{M}\overline{d}\left(1-\gamma_5\right)u\right]\right\} \\
&+ \sum_f\left(-\frac{1}{2}gH\frac{m_f}{M}\overline{f}f + igI_f^{(3)}\phi^0\frac{m_f}{M}\overline{f}\gamma_5 f\right),
\end{aligned} \tag{74}$$

which completes the construction of the SM Lagrangian.

3.5 Tadpoles

In the SM the role of tadpoles is particularly delicate.

- In the Lagrangian, a tadpole constant should appear that is zero in the lowest order, and must be adjusted in such a way that the vacuum expectation value of the H field remains zero order by order in perturbation theory.

Here we will adopt a strategy different from that used in Eqs.(25) and (46). Instead of trading μ^2 for a new parameter $\beta_{_H}$, as carried out in Eq.(46), we *renormalize* the vacuum expectation value itself as follows:

$$K = \frac{1}{\sqrt{2}} \begin{pmatrix} \chi \\ \sqrt{2} i\phi^- \end{pmatrix}, \qquad \chi = H + 2\frac{M}{g}\left(1 + g^2\beta_t\right) + i\phi^0, \tag{75}$$

- Now we set $\mu^2 + 2\left(\lambda/g^2\right) M^2 = 0$ and, in turn, it is β_t that one fixes by the requirement of a zero vacuum expectation value of the H field. The $\mathcal{L}_{\rm S}^{\rm I}$ part of the Lagrangian now reads:

$$\begin{aligned}
\mathcal{L}_{\rm s}^{\rm I} \;=\; & -2\, gMM_{_H}^2\beta_t\, H - \frac{1}{2}M_{_H}^2\left(1 + 3\, g^2\beta_t\right) H^2 \\
& -\frac{1}{2}\, g^2 M_{_H}^2\beta_t\left[\left(\phi^0\right)^2 + 2\phi^+\phi^-\right] - g\alpha_{_H} M\left[H^3 + H\left(\phi^0\right)^2 + 2H\phi^+\phi^-\right] \\
& -\frac{1}{8}g^2\alpha_{_H}\Big[H^4 + \left(\phi^0\right)^4 + 2H^2\left(\phi^0\right)^2 \\
& +4H^2\phi^+\phi^- + 4\left(\phi^0\right)^2\phi^+\phi^- + 4\left(\phi^+\phi^-\right)^2\Big].
\end{aligned} \tag{76}$$

- In the two different procedures two different parameters are introduced, $\beta_{_H}$ and β_t, to be fixed by the requirement of cancelling the one-loop contribution to the vacuum expectation value. These two parameters are related by

$$\beta_t = \frac{\beta_{_H}}{g^2 M_{_H}^2}. \tag{77}$$

Note that the only practical difference (cf. Eq.(47)) is related, so far, to the H^2 term and it will be shown that this difference is irrelevant insofar that it can be renormalized away.

- From the *renormalized* shift of the H field, we are automatically led to the addition of tadpoles in the $W-W$ and $Z-Z$ self-energies and in the corresponding vector–scalar transitions. It can be seen from the following terms:

$$-g^2\beta_t\,\left(M_{_0}^2\,Z_\mu Z_\mu + 2\,M^2 W_\mu^+ W_\mu^-\right)$$
$$-g^2\,M\beta_t\,\left(\frac{1}{c_\theta}\,\phi^0\partial_\mu Z_\mu + \phi^+\partial_\mu W_\mu^- + \phi^-\partial_\mu W_\mu^+\right). \qquad (78)$$

- These tadpoles are usually not added to the various bare self-energies, since they do not contribute to the renormalized ones. However, they are essential for proving that the same self-energies are ξ-independent when put on their bare mass shell; that is $p^2 = -M^2$ and $p^2 = -M_{_0}^2$, respectively. At the same time, the $\beta_t\,H^2$ terms will be crucial for showing ξ-independence of the $H-H$ self-energy at $p^2 = -M_{_H}^2$.

3.6 The QCD Lagrangian

For the QCD Lagrangian there are eight 3×3 Hermitian matrices λ^a, a direct generalization of the 2×2 Pauli matrices, which satisfy

$$\mathrm{Tr}\lambda^a = 0, \qquad \mathrm{Tr}\lambda^a\lambda^b = 2\,\delta_{ab}\,,$$
$$\left[T^a, T^b\right] = f^{abc}\,T^c, \qquad \left\{T^a, T^b\right\} = -i\,d^{abc}\,T^c - \frac{1}{3}\,\delta_{ab}\,, \qquad (79)$$

with $T^a = -i\lambda^a/2$. The structure constants f are antisymmetric in all three indices and satisfy the Jacobi identity, while the d are symmetric in all indices. The QCD Lagrangian contains three pieces:

- the colour gluon Lagrangian, $\mathcal{L}_c$;
- the colour fermion Lagrangian, $\mathcal{L}_c^{\rm fer}$;

- the colour Faddeev–Popov Lagrangian, $\mathcal{L}_c^{\mathrm{FP}}$.

All indices $a, b, \ldots$ take the values $1, \ldots, 8$ corresponding to the eight gluon vector fields, G_μ^a. The indices $i, j, \ldots$ take the values $1, \ldots, 3$, corresponding to three colours. An index σ designates the flavors: u, d, c, s, t, b, of quark fields, q_i^σ. Furthermore, we limit the presentation of the QCD Lagrangian to the Feynman (covariant) gauge. There are other sets of standard choices, for example, the non-covariant axial (or physical) gauges. The first two pieces read:

$$\begin{aligned}
\mathcal{L}_c &= -\frac{1}{2}\,\partial_\nu G_\mu^a \partial_\nu G_\mu^a - g_s\, f^{abc}\,\partial_\mu G_\nu^a G_\mu^b G_\nu^c - \frac{1}{4}\,g_s^2\, f^{abc} f^{ade}\, G_\mu^b G_\nu^c G_\mu^d G_\nu^e, \\
\mathcal{L}_c^{\mathrm{fer}} &= \frac{1}{2} i g_s \left(\overline{q}_i^\sigma \gamma^\mu \lambda_{ij}^a q_j^\sigma\right) G_\mu^a . \qquad (80)
\end{aligned}$$

The FP ghost Lagrangian of QCD is written in terms of a ghost color field κ^a:

$$\mathcal{L}_c^{\mathrm{FP}} = \overline{\kappa}^a \partial^2 \kappa^a + g_s\, f^{abc}\,\partial_\mu \overline{\kappa}^a \kappa^b G_\mu^c . \qquad (81)$$

Here, g_S is the strong coupling constant. We will use also

$$\alpha_s = \frac{g_s^2}{4\,\pi}, \qquad a = \frac{\alpha_s}{\pi}. \qquad (82)$$

Finally, note that even for an arbitrary gauge the Lagrangian $\mathcal{L}_c^{\mathrm{FP}}$ would not change; that is, in QCD the coupling of the ghosts to vectors is independent of the gauge-fixing parameter.

4 Appendix: Feynman rules for vertices

In this appendix, we shall present all the vertices in the electroweak sector of the minimal SM and in the R_ξ gauge. There are a few conventions deserving a comment:

1. there are three gauge parameters, denoted by ξ, ξ_Z and ξ_A.
2. $s_\theta(c_\theta)$ denotes the sine (cosine) of the weak mixing angle.
3. $Q_f, I_f^{(3)}$ denote the electric charge (in units of e) and the third component of isospin of a genuine fermion.
4. we will show the particle symbol next to the line.

   To summarize:

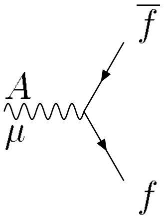
$A, Z, W^\pm$	for vector bosons;
$\phi^0, \phi^\pm$	for the unphysical components of the scalar field, Eq.(25);
H	for the physical Higgs boson;
$X^\pm, Y^A, Y^Z$ etc.	for FP ghosts;
$u(d)$	for a generic $I_f^{(3)} = \frac{1}{2}(-\frac{1}{2})$ fermionic field.

   We should keep in mind that $\overline{X}^-$ is not equal to X^+.

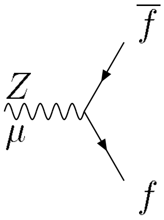
The arrow convention is as follows:

1. The arrows occurring in lines are denoting fermion lines, or the flow of the electric charge or the flow of the FP ghost number. An incoming W^+ will, therefore, be denoted by an incoming arrow.
2. An arrow pointing inwards implies a positive charge flowing into the vertex. For a negatively charged FP field the flow of the charge is opposite to the direction of the arrow; for a positively charged FP field it is in the direction of the arrow.
3. In vertices all momenta are taken to be ingoing.

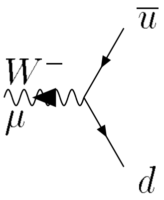
First the *fermionic* Feynman rules:



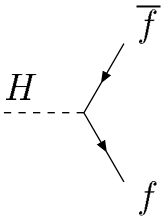
$ieQ_f\gamma_\mu$



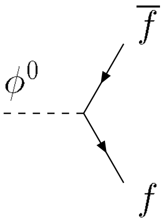
$i\frac{g}{2c_\theta}\gamma_\mu\left(v_f + a_f\gamma_5\right)$



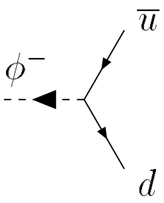
$i\frac{g}{2\sqrt{2}}\gamma_\mu\gamma_+$



$-\frac{gm_f}{2M}$



$igI_f^{(3)}\frac{m_f}{M}\gamma_5$



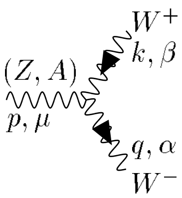
$i\frac{g}{2\sqrt{2}}\left(\frac{m_d}{M}\gamma_+ - \frac{m_u}{M}\gamma_-\right)$

Here

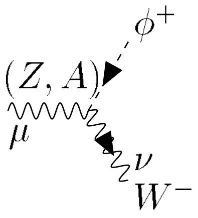
$$\gamma_\pm = 1 \pm \gamma_5\,, \qquad v_f = I_f^{(3)} - 2\,Q_f s_\theta^2, \qquad a_f = I_f^{(3)}. \tag{83}$$

Secondly the *bosonic* Feynman rules:

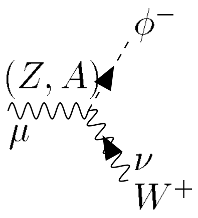
- Trilinear vertices



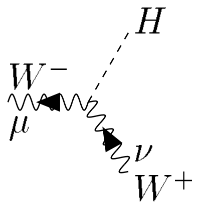
$g\left(c_\theta, s_\theta\right)\left\{\delta_{\mu\alpha}\left(p-q\right)_\beta + \delta_{\alpha\beta}\left(q-k\right)_\mu + \delta_{\mu\beta}\left(k-p\right)_\alpha\right\}$



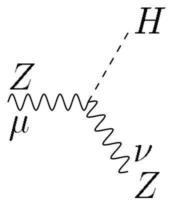
$$ig\left(\frac{s_\theta^2}{c_\theta}, -s_\theta\right) M\delta_{\mu\nu}$$



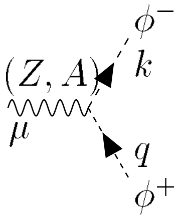
$$ig\left(-\frac{s_\theta^2}{c_\theta}, s_\theta\right) M\delta_{\mu\nu}$$



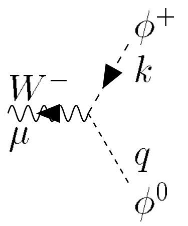
$$-gM\delta_{\mu\nu}$$



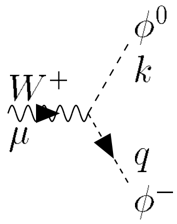
$$-g\frac{M}{c_\theta^2}\delta_{\mu\nu}$$



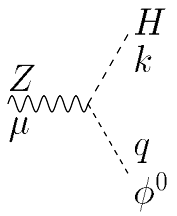
$$g\left(\frac{c_\theta^2 - s_\theta^2}{2c_\theta}, s_\theta\right)(q-k)_\mu$$



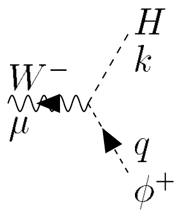
$$\frac{1}{2}g\,(q-k)_\mu$$



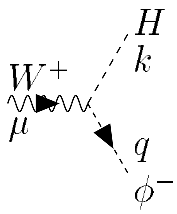
$$\frac{1}{2}g\,(q-k)_\mu$$



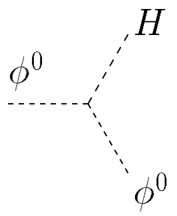
$$\frac{i}{2}\frac{g}{c_\theta}(q-k)_\mu$$



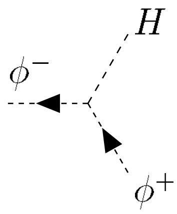
$$\frac{i}{2}g\,(q-k)_\mu$$



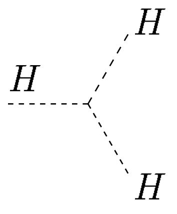
$$\frac{i}{2}g\,(q-k)_\mu$$



$$-\frac{1}{2}g\frac{M_H^2}{M}$$

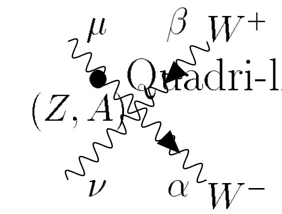


$$-\frac{1}{2}g\frac{M_H^2}{M}$$

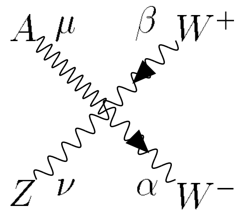


$$-\frac{3}{2}g\frac{M_H^2}{M}$$

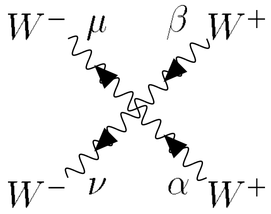
- Quadri-linear vertices



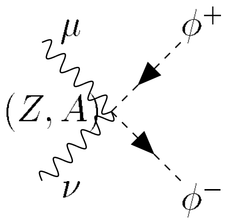
$$-g^2 \left(c_\theta^2, s_\theta^2\right) \left\{2\delta_{\mu\nu}\delta_{\alpha\beta} - \delta_{\mu\alpha}\delta_{\nu\beta} - \delta_{\mu\beta}\delta_{\nu\alpha}\right\}$$



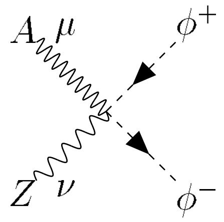
$$-g^2 s_\theta c_\theta \left\{2\delta_{\mu\nu}\delta_{\alpha\beta} - \delta_{\mu\alpha}\delta_{\nu\beta} - \delta_{\mu\beta}\delta_{\nu\alpha}\right\}$$



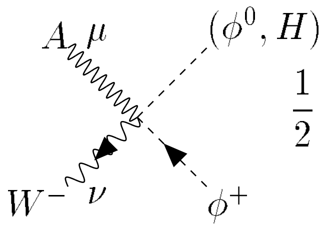
$$g^2 \left\{2\delta_{\mu\nu}\delta_{\alpha\beta} - \delta_{\mu\alpha}\delta_{\nu\beta} - \delta_{\mu\beta}\delta_{\nu\alpha}\right\}$$



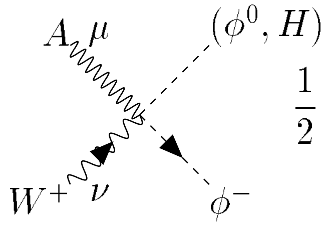
$$g^2 \left(-\frac{\left(c_\theta^2 - s_\theta^2\right)^2}{2c_\theta^2}, -2s_\theta^2\right) \delta_{\mu\nu}$$



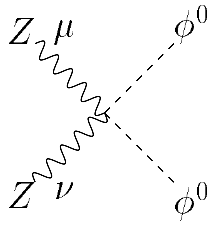
$$-g^2 \frac{s_\theta}{c_\theta} \left(c_\theta^2 - s_\theta^2\right) \delta_{\mu\nu}$$



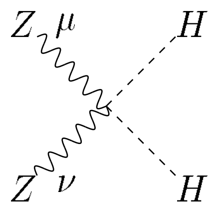
$$\frac{1}{2} \left(1, -i\right) g^2 s_\theta \delta_{\mu\nu}$$



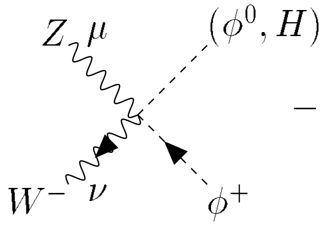
$$\frac{1}{2}\,(1,i)\;g^2 s_\theta \delta_{\mu\nu}$$



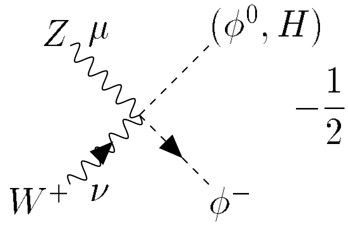
$$-\frac{1}{2}\frac{g^2}{c_\theta^2}\delta_{\mu\nu}$$



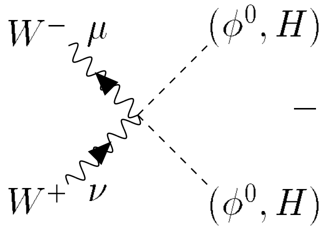
$$-\frac{1}{2}\frac{g^2}{c_\theta^2}\delta_{\mu\nu}$$



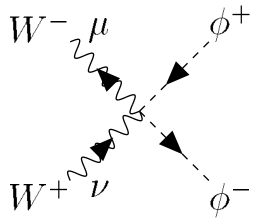
$$-\frac{1}{2}\,(1,-i)\;g^2 \frac{s_\theta^2}{c_\theta}\delta_{\mu\nu}$$



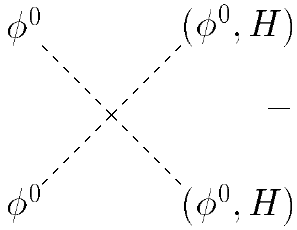
$$-\frac{1}{2}\,(1,i)\;g^2 \frac{s_\theta^2}{c_\theta}\delta_{\mu\nu}$$



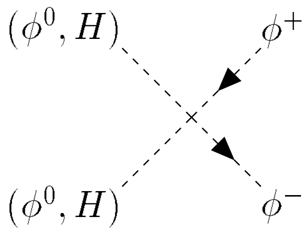
$$-\frac{1}{2}\,(1,1)\;g^2 \delta_{\mu\nu}$$



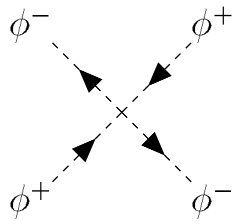
$$-\frac{1}{2} g^2 \delta_{\mu\nu}$$



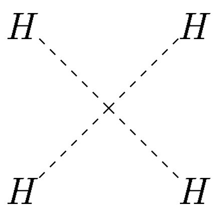
$$-\frac{1}{4}\,(3,1)\;g^2 \frac{M_H^2}{M^2}$$



$$-\frac{1}{4}\,(1,1)\;g^2 \frac{M_H^2}{M^2}$$

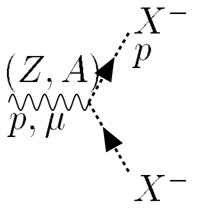


$$-\frac{1}{2} g^2 \frac{M_H^2}{M^2}$$

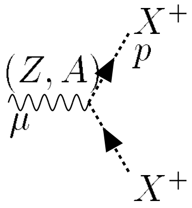


$$-\frac{3}{4} g^2 \frac{M_H^2}{M^2}$$

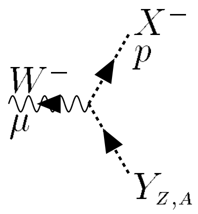
- Trilinear vertices involving FP ghosts



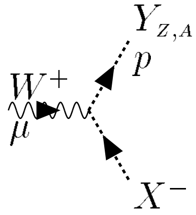
$$g\frac{1}{\xi}\left(c_\theta, s_\theta\right) p_\mu$$



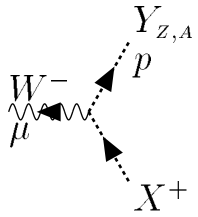
$$-g\frac{1}{\xi}\left(c_\theta, s_\theta\right) p_\mu$$



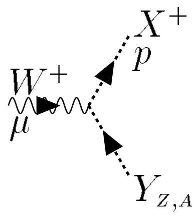
$$-g\frac{1}{\xi}\left(c_\theta, s_\theta\right) p_\mu$$



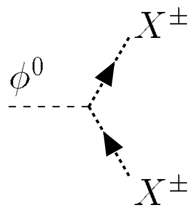
$$-g\left(\frac{c_\theta}{\xi_Z}, \frac{s_\theta}{\xi_A}\right) p_\mu$$



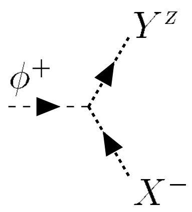
$$g\left(\frac{c_\theta}{\xi_Z}, \frac{s_\theta}{\xi_A}\right) p_\mu$$



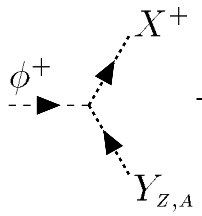
$$g\frac{1}{\xi}\left(c_\theta, s_\theta\right) p_\mu$$



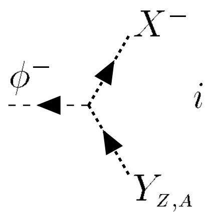
$$\pm\frac{i}{2}g\xi M$$



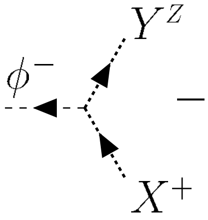
$$\frac{i}{2}\frac{g}{c_\theta}\xi_Z M$$



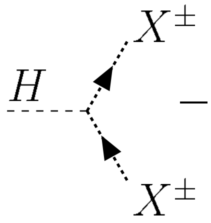
$$-ig\left(\frac{c_\theta^2 - s_\theta^2}{2c_\theta}, s_\theta\right)\xi M$$



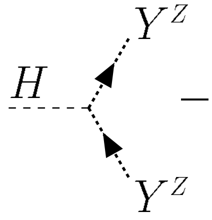
$$ig\left(\frac{c_\theta^2 - s_\theta^2}{2c_\theta}, s_\theta\right)\xi M$$



$$-\frac{i}{2}\frac{g}{c_\theta}\xi_Z M$$



$$-\frac{1}{2}g\xi M$$



$$-\frac{1}{2}\frac{g}{c_\theta^2}\xi_Z M$$

5 A list of QED one-loop diagrams

The one-loop corrections in QED are given by the photon self-energy, the electron self-energy, the $e^+e^-\gamma$ vertex and by the $\gamma - \gamma$ boxes. The first three diagrams will enter into any renormalization scheme. QED boxes, however, are free from ultraviolet divergences and therefore irrelevant from the point of view of renormalization.

Before going on, we should emphasize that the quantities of interest in QED have two sources of infinities. Correspondingly, we must introduce two regulators.

- The first corresponds to the ultraviolet singularities where we use dimensional regularization. The corresponding regulator has been denoted by ε and we have to consider a number of space dimensions, $n = 4 - \varepsilon < 4$.
- For infrared divergences we could use massive regulators or regulate the mass singularities again in the dimensional scheme, $n = 4 + \varepsilon' > 4$. This will be referred to as the $\varepsilon \to -\varepsilon'$ correspondence and it implies that the theory is not simultaneously ultraviolet-regular and mass-singularity-free for an arbitrary number of dimensions. This leads us to the following prescription:

The general prescription is to first renormalize the theory dimensionally and, after the counter-terms are included, to continue to $n = 4 + \varepsilon'$.

- In summary, we have to introduce two *epsilon*-parameters, ε and ε', defined by $n = 4 - \varepsilon$ and $n = 4 + \varepsilon'$. They are both positive and allow us to perform the integration in the complex n-plane. Correspondingly, we will use two regulators:

ultraviolet $\bar{\varepsilon}$ and *infrared* $\hat{\varepsilon}$:

$$\frac{1}{\bar{\varepsilon}} = \frac{2}{\varepsilon} - \gamma - \ln\pi, \qquad \frac{1}{\hat{\varepsilon}} = \frac{2}{\varepsilon'} + \gamma + \ln\pi, \tag{84}$$

where $\gamma = 0.577216$ is the Euler constant.

- The regulators satisfy the following relevant identity:

$$\frac{1}{\bar{\varepsilon}} + \frac{1}{\hat{\varepsilon}} = 0. \tag{85}$$

5.1 Photonic self-energy

The photon self-energy in QED consists of a single Feynman diagram with an internal fermion loop of a given flavor f and is described by a tensor, $\Pi_{\mu\nu}$, as in Fig. 3

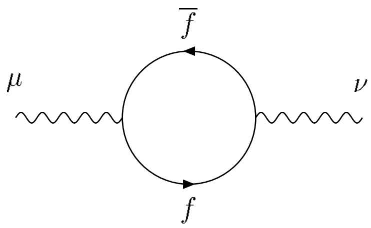


Figure 3: Photonic self-energy in QED.

Note that $\Pi_{\mu\nu}$ is *transverse*, a consequence of the QED $U(1)$ gauge invariance. We obtain the following expression for $\Pi_{\mu\nu}$, written in terms of the scalar function $\Pi\left(p^2\right)$

$$\begin{aligned} \Pi_{\mu\nu} &= 4\, i\pi^2 e^2 \left(p^2\delta_{\mu\nu} - p_\mu p_\nu\right) \Pi\left(p^2\right), \\ \Pi\left(p^2\right) &= 2[B_{21}\left(p^2; m, m\right) + B_1\left(p^2; m, m\right)]. \end{aligned} \tag{86}$$

For QED things are relatively easy and we obtain

$$\Pi(p^2) - \Pi(0) = \frac{1}{9} + \frac{1}{3}\left(1 - 2\frac{m^2}{p^2}\right)\int_0^1 dx \ln\left[1 + \frac{p^2}{m^2}x\left(1-x\right)\right]. \tag{87}$$

5.2 Fermionic self-energy

Fermionic self-energy is correspondingly given by a 4×4 matrix:

$$\Sigma\left(\not{p}\right) = \left(2\pi\right)^4 i\frac{e^2}{16\pi^2}\{[2B_1\left(p^2; m, 0\right) + 1]i\not{p} + [-4B_0\left(p^2; m, 0\right) + 2]m\}, \tag{88}$$

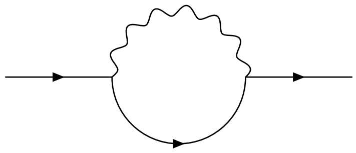


Figure 4: Fermionic self-energy in QED.

originating from the diagram of Fig. 4

Again, in QED things are easy. Direct calculation results in

$$\Sigma\,(\not p) = i\pi^2 e^2 \{ -\left[\left(\frac{n}{2} - 1 \right)\, i\not p + nm \right] \frac{1}{\bar{\varepsilon}} + 2 \int_0^1 dx\ (x\, i\not p + 2\,m) \ln \chi \}, \quad (89)$$

with $\chi = -p^2 x^2 + (p^2 - m^2)x + m^2$.

It should be stressed that for the electron the corresponding self-energy diagram has a well-defined value in the mass shell limit but not its derivative, which shows a singularity due to the zero mass of the photon.

- This is the first example of an infrared divergence and it raises the question of the interplay between ultraviolet and infrared singularities. Also, the QED vertex function gives rise to an infrared divergence.

- However the renormalization of the e.m. coupling in QED through the definition of the fine-structure constant introduces no infrared divergences in the perturbation series. In summary, we have a theorem stating that

In QED, on-shell renormalization is possible, because the vertex correction at zero momentum transfer cancels the electron wave-function renormalization exactly, and because the photon self-energy is infrared finite.

After ultraviolet renormalization, we are left with the resolution of the infrared problem in QED; that is, of the momentum-dependent infrared divergences that requires the introduction of 'real' (as opposite to virtual) radiative corrections. At the present stage the theory must be understood as regularized in the infrared regime by means of dimensional regularization.

5.3 QED vertex

The one-loop QED $e^+e^-\gamma$ vertex corresponds to the diagram in Fig. 5.

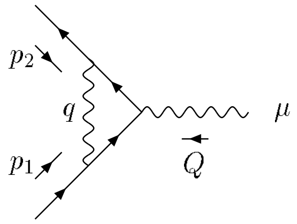


Figure 5: QED vertex diagram.

With both $e^\pm$ on their mass shell the QED vertex is reducible to the following structure: $(2\pi)^4\, i\, ie\gamma_\mu \to (2\pi)^4\, i\, ie\gamma_\mu + \Lambda_\mu$, where

$$\Lambda_\mu = -\,(2\pi)^4\, i\,\frac{ie^3}{16\pi^2}[\gamma_\mu V_1\left(Q^2; m, m\right) + \sigma_{\mu\nu}\left(p_1+p_2\right)_\nu V_2\left(Q^2; m, m\right)]. \quad (90)$$

- The V_1 part is the Dirac electric form factor, containing ultraviolet and infrared divergences.
- The V_2 part, giving the anomalous magnetic moment of the electron, is ultraviolet finite.

For the on-shell vertex we can use the relations

$$\bar{v}\left(p_2\right) \not{p}_2 = -i\, m\bar{v}\left(p_2\right), \qquad \not{p}_1\, u\left(p_1\right) = i\, mu\left(p_1\right). \quad (91)$$

With $p_1^2 = p_2^2 = -m^2$ and $Q^2 = \left(p_1+p_2\right)^2$ and μ as *the mass scale* we have

$$\begin{aligned}
\Lambda_\mu &= -ie^3\mu^{4-n}\int d^nq\frac{1}{d_0d_1d_2}N_\mu\,, \\
N_\mu &= -4p_1\cdot p_2\gamma_\mu + 2\left(\not{p}_1\gamma_\alpha\gamma_\mu - \gamma_\mu\gamma_\alpha\not{p}_2\right)q_\alpha + (2-n)\,\gamma_\alpha\gamma_\mu\gamma_\beta q_\alpha q_\beta\,, \quad (92)
\end{aligned}$$

with propagators given by

$$d_0 = q^2, \qquad d_1 = \left(q+p_1\right)^2 + m^2, \qquad d_2 = \left(q-p_2\right)^2 + m^2. \quad (93)$$

Introducing the auxiliary vector $k_x = xp_2 - (1-x)\,p_1$, and performing the standard Feynman parameterization, we obtain

$$\Lambda_\mu \;=\; -ie^3\Gamma\left(3\right)\int_0^1 dx \int_0^1 dy y\, \mu^{4-n}\int d^n q \frac{1}{\left(q^2 - 2\,y\,q\cdot k_x\right)^3} N_\mu\,. \qquad (94)$$

For the scalar integral we use the *infrared* regulator ε':

$$\frac{\mu^{-\varepsilon'}}{i\pi^2}\int d^n q \frac{1}{\left(q^2 - 2yq\cdot k_x\right)^3} = \pi^{\varepsilon'/2}\frac{\Gamma\left(1-\varepsilon'/2\right)}{\Gamma\left(3\right)} y^{-1+\varepsilon'}\frac{1}{\chi\left(Q^2,x\right)}\Big[\frac{\chi\left(Q^2,x\right)}{\mu^2}\Big]^{\varepsilon'/2}. \qquad (95)$$

Here and below the following quadratic form has been introduced,

$$\chi\left(Q^2,x\right) = Q^2 x\left(1-x\right) + m^2. \qquad (96)$$

For the vector and tensor integrals we use instead the *ultraviolet* regulator ε and calculate integrals with the aid of the following equations:

$$\begin{aligned}
\frac{\mu^{\varepsilon}}{i\pi^2}\int d^n q \frac{q_\alpha}{\left(q^2 - 2yq\cdot k_x\right)^3} &= k_{x,\alpha}\pi^{-\varepsilon/2}\frac{\Gamma\left(1+\varepsilon/2\right)}{\Gamma\left(3\right)} y^{-\varepsilon}\frac{1}{\chi\left(Q^2,x\right)}\Big[\frac{\chi\left(Q^2,x\right)}{\mu^2}\Big]^{-\varepsilon/2},\\
\frac{\mu^{\varepsilon}}{i\pi^2}\int d^n q \frac{q_\alpha q_\beta}{\left(q^2 - 2yq\cdot k_x\right)^3} &= \frac{1}{2}[\delta_{\alpha\beta}\chi\left(Q^2,x\right) + \varepsilon\, k_{x,\alpha}\, k_{x,\beta}]\pi^{-\varepsilon/2}\frac{\Gamma\left(\varepsilon/2\right)}{\Gamma\left(3\right)} y^{1-\varepsilon}\\
&\quad\times\frac{1}{\chi\left(Q^2,x\right)}\Big[\frac{\chi\left(Q^2,x\right)}{\mu^2}\Big]^{-\varepsilon/2}. \qquad (97)
\end{aligned}$$

The y-integration can be performed for all values of n, leading to

$$\int_0^1 dy y^{-k-\varepsilon} = \frac{1}{1-k-\varepsilon}, \qquad k = 1,2,3\,. \qquad (98)$$

Substituting all the integral Eqs.(95)–(98) into Eq.(94) and expanding around $\varepsilon = 0$ and $\varepsilon' = 0$, we arrive at some expression for Λ_μ:

$$\Lambda_\mu\left(Q^2;m,m\right) = -\left(2\pi\right)^4 i\,\frac{ie^3}{16\pi^2}\Big[\gamma_\mu V_1\left(Q^2;m,m\right) + im\left(p_1-p_2\right)_\mu \int_0^1 dx \frac{1}{\chi\left(Q^2,x\right)}\Big], \qquad (99)$$

where the scalar part in charge renormalization may be written in the compact form

$$\begin{aligned}
V_1\left(Q^2;m,m\right) \;&=\; -\left(Q^2+2m^2\right)\int_0^1 dx \frac{1}{\chi\left(Q^2,x\right)}\left[\frac{1}{\hat{\varepsilon}} + \ln\frac{\chi\left(Q^2,x\right)}{\mu^2}\right] \qquad (100)\\
&\quad + \frac{1}{\bar{\varepsilon}} - \int_0^1 dx \ln\frac{\chi\left(Q^2,x\right)}{\mu^2} + 2\left(Q^2+4m^2\right)\int_0^1 dx \frac{1}{\chi\left(Q^2,x\right)} - 2.
\end{aligned}$$

After applying the Gordon identity, and using the usual C and B-functions instead of one-fold integrals, we arrive at the final representation for $V_{1,2}\left(Q^2;m,m\right)$ where no approximation has been made, not even ignoring the electron mass:

$$\begin{aligned} V_1\left(Q^2;m,m\right) &= -2\left(Q^2+2m^2\right)C_0\left(-m^2,-m^2,Q^2;m,0,m\right) \\ &\quad +B_0\left(Q^2;m,m\right)-4B_{ff}\left(Q^2;m,m\right)-2, \end{aligned} \tag{101}$$

$$V_2\left(Q^2;m,m\right) = \frac{2}{Q^2+4m^2}B_{ff}\left(Q^2;m,m\right), \tag{102}$$

where B_{ff} is a peculiar combinations of B_0-functions, namely

$$\begin{aligned} B_{ff}\left(Q^2;m,m\right) &= B_0\left(Q^2;m,m\right)-B_0\left(-m^2;m,0\right) \\ &= -\frac{1}{2}\left(Q^2+4m^2\right)\int_0^1 dx\frac{1}{\chi\left(Q^2,x\right)}. \end{aligned} \tag{103}$$

There are two limiting cases of special interest, $s=-Q^2\gg m^2$ and $Q^2=0$. Here, we shall content ourselves with the large s limit, where we derive

$$sC_0\left(-m^2,-m^2,-s;m,0,m\right)\approx-\left(\frac{1}{\hat{\varepsilon}}+\ln\frac{m^2}{\mu^2}\right)\ln\frac{-s-i\epsilon}{m^2}-\frac{1}{2}\ln^2\frac{-s-i\epsilon}{m^2}+\frac{1}{6}\pi^2, \tag{104}$$

where $m\to 0$, and also,

$$B_0\left(-s;0,0\right)\approx\frac{1}{\bar{\varepsilon}}-\ln\frac{-s-i\epsilon}{\mu^2}+2, \qquad B_0\left(-m^2;m,0\right)=\frac{1}{\bar{\varepsilon}}-\ln\frac{m^2}{\mu^2}+2. \tag{105}$$

Collecting the various terms we obtain for $m\to 0$:

$$\begin{aligned} V_1\left(-s;m,m\right) &= \frac{1}{\bar{\varepsilon}}-\ln\frac{m^2}{\mu^2}-2\left(\frac{1}{\hat{\varepsilon}}+\ln\frac{m^2}{\mu^2}\right)\ln\frac{-s-i\epsilon}{m^2} \\ &\quad -\ln^2\frac{-s-i\epsilon}{m^2}+\frac{1}{3}\pi^2+3\ln\frac{-s-i\epsilon}{m^2}, \end{aligned} \tag{106}$$

For $Q^2=0$ we obtain instead

$$V_1\left(0;m,m\right) = \frac{1}{\bar{\varepsilon}}-\frac{2}{\hat{\varepsilon}}-3\,\ln\frac{m^2}{\mu^2}+4. \tag{107}$$

In the above results we have kept an explicit distinction between the ultraviolet and the infrared poles. The quantity of physical interest is always V_1 subtracted at zero momentum, which reads therefore as

$$V_1^{\rm sub} = -2\left(\frac{1}{\hat{\varepsilon}} + \ln\frac{m^2}{\mu^2}\right)\left(\ln\frac{-s-i\epsilon}{m^2} - 1\right) - \ln^2\frac{-s-i\epsilon}{m^2} + \frac{1}{3}\pi^2 + 3\ln\frac{-s-i\epsilon}{m^2} - 4. \tag{108}$$

If needed, the exact expression for V_1 is also very simple:

$$\begin{aligned}\frac{1}{2}V_1^{\rm sub} &= \left(\frac{1}{\hat{\varepsilon}} + \ln\frac{m^2}{\mu^2}\right)\left(1 + \frac{1+\beta^2}{2\,\beta}\ln\eta\right) - \frac{3}{2}\beta\,\ln\eta - 2 \\ &+ \frac{1+\beta^2}{\beta}\left[\mathrm{Li}_2\left(\eta\right) + \frac{1}{3}\pi^2 - \frac{1}{4}\ln^2\eta + \ln\eta\ln\left(1-\eta\right) - \frac{i\pi}{4}\ln\frac{1-\beta^2}{4\beta^2}\right],\end{aligned} \tag{109}$$

where we have introduced

$$\beta^2 = 1 - 4\,\frac{m^2}{s}, \qquad \eta = \frac{1-\beta}{1+\beta}. \tag{110}$$

5.4 QED box diagrams

QED represents some special case of the full electroweak theory with its distinctive simplicity and, for this reason, we discuss here the specific example of QED $\gamma - \gamma$ boxes. Let us consider the annihilation $e^+e^- \to f\overline{f}$. There are two QED box diagrams, the direct one and the crossed one (see Fig. 6)

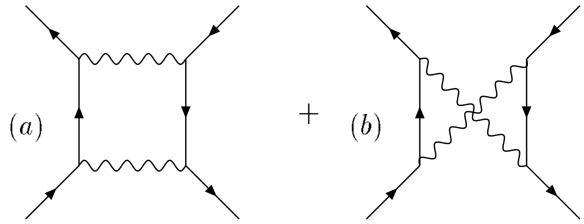


Figure 6: Two QED box diagrams: (a) direct; (b) crossed.

These are easily related and if the final expression is given in terms of t, u variables, this is tantamount to exchanging $t \leftrightarrow u$ with an additional overall minus sign in the cross-section. We can define two different distributions, $D_\pm\left(\theta\right)$:

$$D_\pm\left(\theta\right) = \frac{d\sigma\left(\theta\right)}{d\Omega} \pm \frac{d\sigma\left(\pi-\theta\right)}{d\Omega}. \tag{111}$$

- D_+ is the relevant angular distribution when the charges of the final states are not detected, while
- D_-—the asymmetry function—is available when one measures the differential cross-section with charge detection.

Note that charge conjugation invariance implies that only the interference terms between the lowest order amplitude and the box diagrams contribute to D_- to order α^3, as far as virtual radiative corrections are concerned. The lowest order amplitude squared and summed over polarization is

$$\mathcal{A}_{0} = \frac{1}{4}\overline{\sum_{\text{spins}}} \mid \mathcal{M}_{0} \mid^2 = 2\, e^4 Q_e^2\, Q_f^2\, \frac{t^2+u^2}{s^2}. \tag{112}$$

The corresponding contribution from the interference of the direct box diagram can be written as

$$\mathcal{A}_{\text{int}}^{\text{dr}} = -\frac{e^6}{2\,\pi^2}\, Q_e^3 Q_f^3\, \frac{1}{s}\delta_{\gamma\gamma}^{\text{box}}\left(s,t,u\right), \tag{113}$$

where

$$\delta_{\gamma\gamma}^{\text{box}}\left(s,t,u\right) = u^2\mathcal{D}_{\gamma\gamma}^{+}\left(s,t,u\right) + t^2\mathcal{D}_{\gamma\gamma}^{-}\left(s,t,u\right). \tag{114}$$

Similarly, the crossed box is obtained with the replacement $t \leftrightarrow u$,

$$\mathcal{A}_{\text{int}}^{\text{cr}} = \frac{e^6}{2\,\pi^2}\, Q_e^3 Q_f^3\, \frac{1}{s}\delta_{\gamma\gamma}^{\text{box}}\left(s,u,t\right). \tag{115}$$

Thus, only two functions $\mathcal{D}_{\gamma\gamma}^{\pm}\left(s,t,u\right)$ are needed to describe boxes and they are given by

$$\begin{aligned} t^2\mathcal{D}_{\gamma\gamma}^{-}\left(s,t,u\right) &= \frac{t^2}{s}\left[d_0\left(s,t\right) + c_0\left(s;0,m_e,0\right) + c_0\left(s;0,m_f,0\right)\right], \\ u^2\mathcal{D}_{\gamma\gamma}^{+}\left(s,t,u\right) &= \frac{t^2+u^2}{2\,s}\left[d_0\left(s,t\right) + c_0\left(s;0,m_e,0\right) + c_0\left(s;0,m_f,0\right)\right] \\ &\quad + \left(u-t\right)\; c_0\left(t;m_e,0,m_f\right) + u\left[B_0\left(-s;0,0\right) - B_0\left(-t;m_e,m_f\right)\right], \end{aligned} \tag{116}$$

where we have introduced scaled functions:

$$\begin{aligned} d_0\left(s,t\right) &= st\, D_0\left(-m_e^2,-m_e^2,-m_f^2,-m_f^2,-s,-t;0,m_e,0,m_f\right), \\ c_0\left(s;0,m_e,0\right) &= s\, C_0\left(-m_e^2,-m_e^2,-s;0,m_e,0\right), \\ c_0\left(t;m_e,0,m_f\right) &= t\, C_0\left(-m_e^2,-m_f^2,-t;m_e,0,m_f\right). \end{aligned} \tag{117}$$

The infrared-divergent scalar function d_0 is split into an infrared divergent c_0-function plus a finite remainder; namely

$$d_0\,(s,t) = t\,\bar{J}_{\gamma\gamma}\,(-s,-t;m_e,m_f) - 2\,c_0\,(t;m_e,0,m_f)\,. \tag{118}$$

The functions B_0 (C_0, D_0) are the scalar two- (three-, four-) point integrals (see Section 6). From this result it is immediately obvious that the infrared divergences in the direct (and also crossed) diagram do factorize into the lowest order. Indeed, the infrared-divergent part is fully specified in terms of

$$\frac{u^2}{s}\mathcal{D}^{+}_{\gamma\gamma}\,(s,t,u) + \frac{t^2}{s}\mathcal{D}^{-}_{\gamma\gamma}\,(s,t,u)\Big|_{\mathrm{IR}} = -2\,\frac{t^2+u^2}{s^2}\,c_0\,(t;m_e,0,m_f)\,, \tag{119}$$

the remainder being infrared finite. For completeness of presentation we write again all the ingredients entering the final results for the interference of box diagrams with the lowest order:

$$\bar{J}_{\gamma\gamma}\,(-s,-t;m_e,m_f) = \frac{1}{t}\Big[\ln\frac{m_e^2 m_f^2}{t^2}\ln\frac{-t}{s} + \frac{1}{2}\ln^2\frac{m_e^2}{-t} + \frac{1}{2}\ln^2\frac{m_f^2}{-t} + \frac{1}{3}\pi^2\Big], \tag{120}$$

$$C_0\left(-m_e^2,-m_e^2,-s;0,m_e,0\right) = -\frac{1}{s}\left(\frac{1}{2}\ln^2\frac{m_e^2}{s} + \frac{1}{6}\pi^2 + i\,\pi\ln\frac{m_e^2}{s}\right), \tag{121}$$

$$C_0\left(-m_e^2,-m_f^2,-t;m_e,0,m_f\right) = \frac{1}{2t}\Big[\ln\frac{m_e^2 m_f^2}{t^2}\left(\frac{1}{\hat{\varepsilon}} + \ln\frac{-t}{\mu^2}\right) + \frac{1}{2}\ln^2\frac{m_e^2}{-t} + \frac{1}{2}\ln^2\frac{m_f^2}{-t} + \frac{1}{3}\pi^2\Big], \tag{122}$$

$$B_0\,(-s;0,0) - B_0\,(-t;m_e,m_f) = -\ln\frac{s}{-t} + i\,\pi. \tag{123}$$

These relations hold for $m_e^2,\ m_f^2 \ll -t$ and $m_e^2 \ll s$. For the total interference terms, lowest order $\times$ box diagrams we have therefore

$$\mathcal{A}^{\mathrm{box}}_{\mathrm{int}} = -\frac{e^6}{2\,\pi^2}\,Q_e^3 Q_f^3\,f^{\mathrm{box}}_{\gamma\gamma}\,(s,t,u)\,, \tag{124}$$

$$f^{\mathrm{box}}_{\gamma\gamma}\,(s,t,u) = \frac{1}{s}[\delta^{\mathrm{box}}_{\gamma\gamma}\,(s,t,u) - \delta^{\mathrm{box}}_{\gamma\gamma}\,(s,u,t)], \tag{125}$$

$$\mathrm{Re}\ f^{\mathrm{box}}_{\gamma\gamma}\,(s,t,u) = 2\,\frac{t^2+u^2}{s^2}\left(\frac{1}{\hat{\varepsilon}} + \ln\frac{s}{\mu^2}\right)\ln\frac{t}{u} + \frac{t}{s}\ln\left(-\frac{s}{u}\right) - \frac{u}{s}\ln\left(-\frac{s}{t}\right) + \frac{t-u}{s}\left[\ln^2\left(-\frac{s}{t}\right) + \ln^2\left(-\frac{s}{u}\right)\right]. \tag{126}$$

As expected, there are no collinear divergences and the limit of zero fermion masses can be taken.

6 Scalar integrals, vectorial and tensorial reduction

To cope with the complications of the SM, we must derive a complete set of formulas valid for arbitrary internal and external masses. We will deal with expressions for scalar diagrams with one, two, three and four external lines. Besides scalar functions we also need tensor integrals with up to four external legs and as many powers of momentum as allowed in a renormalizable theory. These tensor structures can be reduced to linear combinations of scalar functions.

6.1 One-point integrals, A-functions

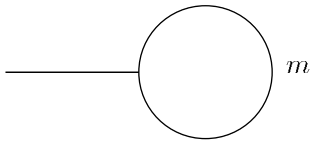


Figure 7: The one-point Green function.

The one-point function is given in Fig. 7 and the corresponding expression will be discussed below.

6.2 The scalar one-point integral.

We start by introducing the one-point scalar integrals which are needed for tadpole diagrams and in the reduction of higher-order functions:

$$i\pi^2 A_0(m) = \mu^{4-n} \int d^n q \, \frac{1}{q^2 + m^2 - i\epsilon}. \tag{127}$$

This integral can be easily evaluated in terms of the Euler Γ-function giving

$$A_0(m) = \pi^{n/2-2} \Gamma\left(1 - \frac{n}{2}\right) m^2 \left(\frac{m^2}{\mu^2}\right)^{n/2-2}. \tag{128}$$

If we introduce $\varepsilon = 4 - n$ and expand around $n = 4$, then the following expression is derived:

$$A_0\left(m\right) = m^2\left(-\frac{2}{\varepsilon} + \gamma + \ln\pi - 1 + \ln\frac{m^2}{\mu^2}\right) + \mathcal{O}\left(\varepsilon\right). \qquad (129)$$

where $\gamma = 0.577216$ is the Euler constant.

It is customary to define a quantity $1/\bar{\varepsilon}$ by

$$\frac{1}{\bar{\varepsilon}} = \frac{2}{\varepsilon} - \gamma - \ln\pi, \qquad (130)$$

and to write

$$A_0\left(m\right) = m^2\left(-\frac{1}{\bar{\varepsilon}} - 1 + \ln\frac{m^2}{\mu^2}\right) + \mathcal{O}\left(\varepsilon\right). \qquad (131)$$

6.3 Two-point integrals, B-functions

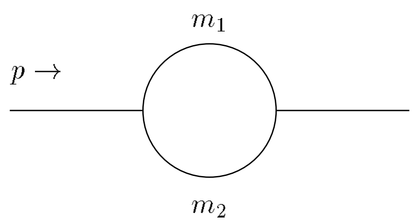


Figure 8: The two-point Green function.

The family of two-point functions is given in Fig. 8 and it is discussed below.

6.4 The scalar two-point integral.

Consider the scalar two-point function which is met in the calculation of self-energy diagrams containing two propagators, d_0 and d_1:

$$\begin{aligned} &i\pi^2 B_0\left(p^2; m_1, m_2\right) = \mu^{4-n}\int d^n q \frac{1}{d_0 d_1}, \\ &d_0 = q^2 + m_1^2 - i\epsilon, \qquad d_1 = \left(q+p\right)^2 + m_2^2 - i\epsilon. \end{aligned} \qquad (132)$$

It is convenient to introduce the general expression for propagators:

$$d_i = \left(q + p_1 + \cdots + p_i\right)^2 + m_{i+1}^2 - i\epsilon. \qquad (133)$$

A_0, considered above, involves the simplest, external momentum independent propagator d_0. For arbitrary internal masses the B_0 function becomes

$$B_0\left(p^2; m_1, m_2\right) = \frac{1}{\bar{\varepsilon}} - \ln\frac{m_1 m_2}{\mu^2} - R + \frac{m_1^2 - m_2^2}{2p^2}\ln\frac{m_1^2}{m_2^2} + 2, \tag{134}$$

where

$$R = -\frac{\Lambda}{p^2}\ln\frac{p^2 - i\epsilon + m_1^2 + m_2^2 - \Lambda}{2\,m_1 m_2}. \tag{135}$$

and where we have introduced $\Lambda^2 = \lambda\left(-p^2, m_1^2, m_2^2\right)$.

There are simplifications for special values of the arguments. For instance, if $m_1 = m_2 = m$, then we find

$$B_0\left(p^2; m, m\right) = \frac{1}{\bar{\varepsilon}} - \ln\frac{m^2}{\mu^2} + 2 - \beta\,\ln\frac{\beta+1}{\beta-1}, \tag{136}$$

where $\beta^2 = 1 + 4\,m^2/(p^2 - i\epsilon)$. Similarly, if one of the internal masses is zero, then we have

$$B_0\left(p^2; 0, m\right) = \frac{1}{\bar{\varepsilon}} - \ln\frac{m^2}{\mu^2} + 2 - \left(1 + \frac{m^2}{p^2}\right)\ln\left(1 + \frac{p^2 - i\epsilon}{m^2}\right). \tag{137}$$

Finally, for massless internal lines we obtain

$$B_0\left(p^2; 0, 0\right) = \frac{1}{\bar{\varepsilon}} - \ln\frac{p^2 - i\epsilon}{\mu^2} + 2. \tag{138}$$

From all these functions we can easily extract the corresponding imaginary parts. With $s = -p^2$ we write:

$$\begin{aligned}
\mathrm{Im}B_0\left(p^2; 0, 0\right) &= \pi\theta\left(s\right), \quad \mathrm{Im}B_0\left(p^2; 0, m\right) = \pi\left(1 - \frac{m^2}{s}\right)\theta\left(s - m^2\right), \\
\mathrm{Im}B_0\left(p^2; m, m\right) &= \pi\left(1 - 4\frac{m^2}{s}\right)^{1/2}\theta\left(s - 4m^2\right), \\
\mathrm{Im}B_0\left(p^2; m_1, m_2\right) &= \pi\frac{\sqrt{\lambda\left(s, m_1^2, m_2^2\right)}}{s}\,\theta\left(s - (m_1 + m_2)^2\right).
\end{aligned} \tag{139}$$

6.5 Tensor two-point integrals.

Tensor two-point integrals can be reduced to linear combinations of scalar functions. We start with

$$i\pi^2 B_\mu\left(p^2; m_1, m_2\right) = \mu^{4-n}\int d^n q\frac{q_\mu}{d_0 d_1} = i\pi^2 B_1\left(p^2; m_1, m_2\right)\,p_\mu. \tag{140}$$

Using the relation $q^2 = d_0 - m_1^2$, with

$$q \cdot p = \frac{1}{2}\left(d_1 - d_0 + f_1^b\right), \qquad f_1^b = -p^2 + m_1^2 - m_2^2, \tag{141}$$

we derive the following identity:

$$p^2 B_1\left(p^2; m_1, m_2\right) = \frac{1}{2}\left[A_0\left(m_1\right) - A_0\left(m_2\right) + f_1^b B_0\left(p^2; m_1, m_2\right)\right]. \tag{142}$$

The function B_1 obeys the symmetry

$$B_1\left(p^2; m_2, m_1\right) = -B_1\left(p^2; m_1, m_2\right) - B_0\left(p^2; m_1, m_2\right). \tag{143}$$

The rank two tensor integral can be reduced as follows:

$$\begin{aligned} i\pi^2 B_{\mu\nu}\left(p^2; m_1, m_2\right) &= \mu^{4-n} \int d^n q \frac{q_\mu q_\nu}{d_0 d_1} \\ &= i\pi^2\left[B_{21}\left(p^2; m_1, m_2\right) p_\mu p_\nu + B_{22}\left(p^2; m_1, m_2\right) \delta_{\mu\nu}\right]. \end{aligned} \tag{144}$$

The last relation can be multiplied by $\delta_{\mu\nu}$ and by p_ν to give

$$\begin{aligned} p^2 B_{21}\left(p^2; m_1, m_2\right) + n B_{22}\left(p^2; m_1, m_2\right) &= A_0\left(m_2\right) - m_1^2 B_0\left(p^2; m_1, m_2\right), \\ p^2 B_{21}\left(p^2; m_1, m_2\right) + B_{22}\left(p^2; m_1, m_2\right) &= \frac{1}{2}\left[A_0\left(m_2\right) + f_1^b B_1\left(p^2; m_1, m_2\right)\right]. \end{aligned} \tag{145}$$

In order to solve this system of equations we have to compute the singular parts of the scalar one- and two-point functions in terms of the quantity $\bar{\varepsilon}$ defined by Eq.(130). First we define a function χ as

$$\chi(x) = -p^2 x^2 + \left(p^2 + m_2^2 - m_1^2\right) x + m_1^2 - i\epsilon. \tag{146}$$

A simple calculation shows that

$$\begin{aligned} B_0\left(p^2; m_1, m_2\right) &= \frac{1}{\bar{\varepsilon}} - \int_0^1 dx \ln\left(\frac{\chi}{\mu^2}\right) \xrightarrow{\text{sing}} \frac{1}{\bar{\varepsilon}}, \\ B_1\left(p^2; m_1, m_2\right) &= -\frac{1}{2}\frac{1}{\bar{\varepsilon}} + \int_0^1 dx x \ln\left(\frac{\chi}{\mu^2}\right) \xrightarrow{\text{sing}} -\frac{1}{2}\frac{1}{\bar{\varepsilon}}, \\ B_{21}\left(p^2; m_1, m_2\right) &= \frac{1}{3}\frac{1}{\bar{\varepsilon}} - \int_0^1 dx x^2 \ln\left(\frac{\chi}{\mu^2}\right) \xrightarrow{\text{sing}} \frac{1}{3}\frac{1}{\bar{\varepsilon}}, \\ B_{22}\left(p^2; m_1, m_2\right) &= -\frac{1}{2}\left(\frac{1}{\bar{\varepsilon}} + 1\right) \int_0^1 dx \chi + \frac{1}{2}\int_0^1 dx \chi \ln\left(\frac{\chi}{\mu^2}\right) \\ &\xrightarrow{\text{sing}} -\frac{1}{4}\left(m_1^2 + m_2^2 + \frac{1}{3}p^2\right)\frac{1}{\bar{\varepsilon}}. \end{aligned} \tag{147}$$

By using these relations we arrive at a system of equations, Eq.(145), with

$$n\, B_{22}\left(p^2; m_1, m_2\right) = 4\, B_{22}\left(p^2; m_1, m_2\right) + \frac{K^2}{6}, \quad K^2 = p^2 + 3\left(m_1^2 + m_2^2\right). \tag{148}$$

At this point we introduce an X_2-matrix

$$X_2 = \begin{pmatrix} p^2 & 4 \\ p^2 & 1 \end{pmatrix} \tag{149}$$

and the vector b whose components are

$$\begin{aligned} b_1 &= A_0\left(m_2\right) - m_1^2 B_0\left(p^2; m_1, m_2\right) - \frac{K^2}{6}, \\ b_2 &= \frac{1}{2}[A_0\left(m_2\right) + f_1^b B_1\left(p^2; m_1, m_2\right)]. \end{aligned} \tag{150}$$

The $B_{2i}\left(p^2; m_1, m_2\right)$ form factors can, therefore, be obtained by using the inverse matrix of Eq.(149)

$$B_{2i}\left(p^2; m_1, m_2\right) = [X_2]^{-1}_{ij}\; b_j\,. \tag{151}$$

We explicitly list the final results:

$$\begin{aligned} B_1\left(p^2; m_1, m_2\right) &= \frac{1}{2p^2}[A_0\left(m_1\right) - A_0\left(m_2\right) + \left(\Delta m^2 - p^2\right) B_0\left(p^2; m_1, m_2\right)], \\ B_{21}\left(p^2; m_1, m_2\right) &= \frac{3\left(m_1^2 + m_2^2\right) + p^2}{18p^2} + \frac{\Delta m^2 - p^2}{3p^4} A_0\left(m_1\right) - \frac{\Delta m^2 - 2p^2}{3p^4} A_0\left(m_2\right) \\ &\quad + \frac{\lambda\left(-p^2, m_1^2, m_2^2\right) - 3p^2 m_1^2}{3p^4} B_0\left(p^2; m_1, m_2\right), \\ B_{22}\left(p^2; m_1, m_2\right) &= -\frac{3\left(m_1^2 + m_2^2\right) + p^2}{18} - \frac{\Delta m^2 - p^2}{12\, p^2} A_0\left(m_1\right) + \frac{\Delta m^2 + p^2}{12\, p^2} A_0\left(m_2\right) \\ &\quad - \frac{\lambda\left(-p^2, m_1^2, m_2^2\right)}{12p^2} B_0\left(p^2; m_1, m_2\right), \quad \Delta m^2 = m_1^2 - m_2^2. \end{aligned} \tag{152}$$

6.6 Derivatives of B-functions

In the actual evaluation of one-loop radiative corrections we will also need derivatives of the B-functions. They will appear in renormalization factors associated with external lines which are derived by the corresponding two-point Green functions and are given by the following results:

$$
\begin{aligned}
\frac{\partial B_{\{0;1;21\}}}{\partial p^2} &= -\int_0^1 dx \frac{\{x; -x^2; x^3\}\,(1-x)}{\chi}, \\
\frac{\partial B_{22}}{\partial p^2} &= -\frac{1}{12}\frac{1}{\bar{\varepsilon}} + \frac{1}{2}\int_0^1 dx x\,(1-x)\ln\left(\frac{\chi}{\mu^2}\right).
\end{aligned} \tag{153}
$$

For the QED corrections some of the previous derivatives are infrared divergent and must be regulated. For instance, with $\chi(x) = (1-x)\left(p^2 x + m^2\right)$ we have for the scalar integral

$$
B_0\left(p^2; m, 0\right) = \pi^{n/2-2}\,\Gamma\left(2-\frac{n}{2}\right)\int_0^1 dx \left(\frac{\chi}{\mu^2}\right)^{n/2-2}. \tag{154}
$$

With $n = 4 + \varepsilon'$ we, therefore, derive

$$
\frac{\partial}{\partial p^2} B_0\left(p^2; m, 0\right) = -\pi^{\varepsilon'/2}\Gamma\left(1-\frac{\varepsilon'}{2}\right)\int_0^1 dx \frac{x\,(1-x)}{\chi\,(x)}\left(\frac{\chi\,(x)}{\mu^2}\right)^{\varepsilon'/2}, \tag{155}
$$

which in turn gives

$$
\frac{\partial}{\partial p^2} B_0\left(p^2; m, 0\right)|_{p^2=-m^2} = -\pi^{\varepsilon'/2}\Gamma\left(1-\frac{\varepsilon'}{2}\right)\frac{1}{m^2}\left(\frac{m^2}{\mu^2}\right)^{\varepsilon'/2}\left(\frac{1}{\varepsilon'} - \frac{1}{1+\varepsilon'}\right). \tag{156}
$$

Expanding the various terms in ε' we derive the Laurant series

$$
\frac{\partial}{\partial p^2} B_0\left(p^2; m, 0\right)|_{p^2=-m^2} = -\frac{1}{2m^2}\left(\frac{1}{\hat{\varepsilon}} - 2 + \ln\frac{m^2}{\mu^2}\right), \tag{157}
$$

where we may use

$$
\frac{1}{\hat{\varepsilon}} = \frac{2}{\varepsilon'} + \gamma + \ln\pi = \frac{2}{n-4} + \gamma + \ln\pi = -\frac{1}{\bar{\varepsilon}}. \tag{158}
$$

Similarly, for the derivative of B_1 we obtain

$$
\begin{aligned}
\frac{\partial}{\partial p^2} B_1\left(p^2; m, 0\right)|_{p^2=-m^2} &= \pi^{\varepsilon'/2}\Gamma\left(1-\frac{\varepsilon'}{2}\right)\left(m^2\right)^{-1+\varepsilon'/2}\left(\mu^2\right)^{-\varepsilon'/2} \\
&\times\left(\frac{1}{\varepsilon'} - \frac{2}{1+\varepsilon'} + \frac{1}{2+\varepsilon'}\right),
\end{aligned} \tag{159}
$$

giving the following result:

$$
\frac{\partial}{\partial p^2} B_1\left(p^2; m, 0\right)|_{p^2=-m^2} = \frac{1}{2m^2}\left(\frac{1}{\hat{\varepsilon}} - 3 + \ln\frac{m^2}{\mu^2}\right). \tag{160}
$$

6.7 Three-point integrals, C-functions

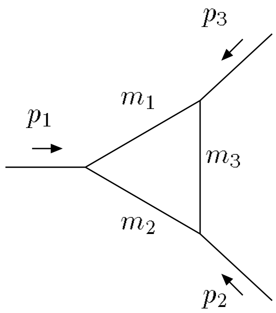


Figure 9: The three-point Green function.

The scalar three-point function, (Fig. 9) associated with vertex corrections is more involved and will require some additional work.

6.8 Basic definition.

First we define of the scalar three-point function,

$$i\pi^2\, C_0\left(p_1^2, p_2^2, Q^2; m_1, m_2, m_3\right) = \mu^{4-n} \int d^n q \frac{1}{d_0 d_1 d_2}, \tag{161}$$

with d_i given by Eq.(133), which in this case are

$$d_0 = q^2 + m_1^2 - i\epsilon, \quad d_1 = (q + p_1)^2 + m_2^2 - i\epsilon, \quad d_2 = (q + Q)^2 + m_3^2 - i\epsilon, \tag{162}$$

where $Q = p_1 + p_2$ and $Q^2 = (p_1 + p_2)^2$ denotes one of the Mandelstam variables, $Q^2 = -s, t$ or u, for an arbitrary $2 \to 2$ amplitude. Two Feynman parameters are enough for the three-point function, and in terms of a particular choice of Feynman parameters C_0 becomes

$$\begin{aligned} C_0\left(p_1^2, p_2^2, Q^2; m_1, m_2, m_3\right) &= \int_0^1 dx \int_0^x dy \left(ax^2 + by^2 + cxy + dx + ey + f\right)^{-1}, \\ a &= -p_2^2, \qquad b = -p_1^2, \qquad c = p_1^2 + p_2^2 - Q^2, \qquad d = p_2^2 + m_2^2 - m_3^2, \\ e &= -p_2^2 + Q^2 + m_1^2 - m_2^2, \qquad f = m_3^2 - i\epsilon. \end{aligned} \tag{163}$$

6.9 Some particular cases of C_0-functions.

Before deriving the general result we consider a few special cases. First, we select

$$p_{1,2}^2 = 0, \qquad (p_1 + p_2)^2 = Q^2, \qquad m_1 = m_3 = 0, \qquad m_2 = M. \tag{164}$$

In this case

$$C_0\left(0,0,Q^2;0,M,0\right) = \int_0^1 dx \int_0^x dy[-Q^2xy + M^2x + \left(Q^2 - M^2\right)y - i\epsilon]^{-1}$$
$$= -\int_0^1 dx \frac{1}{-Q^2x + M^2} \ln\left(\frac{Q^2}{M^2}x\right) = \frac{1}{Q^2}\left[\mathrm{Li}_2\left(1\right) - \mathrm{Li}_2\left(1 - \frac{Q^2 - i\epsilon}{M^2}\right)\right]. \quad (165)$$

Actually, there is only one (generic) three-point scalar integral that occurs in the calculation of two-fermion production when we use the approximation where all fermionic masses, with the exception of the top-quark mass, are ignored. It corresponds to the following choice:

$$p_{1,2}^2 = 0, \quad \left(p_1 + p_2\right)^2 = Q^2, \quad m_1 = M_1\,, \quad m_2 = M_2\,, \quad m_3 = M_3\,. \quad (166)$$

Then the coefficients, in quadratic form, become

$$\begin{aligned} a &= 0, & b &= 0, & c &= -Q^2, \\ d &= M_2^2 - M_3^2, & e &= Q^2 + M_1^2 - M_2^2, & f &= M_3^2 - i\epsilon, \end{aligned} \quad (167)$$

and the result for C_0 reads

$$C_0\left(0,0,Q^2;M_1,M_2,M_3\right) = \int_0^1 dx \int_0^x dy \frac{1}{\chi(x,y)}, \quad (168)$$

where the function χ is a quadratic form in x and y,

$$\chi(x,y) = Q^2 y\left(1-x\right) + M_1^2 y + M_2^2\left(x - y\right) + M_3^2\left(1-x\right). \quad (169)$$

In this particular case we obtain

$$C_0 = \frac{1}{Q^2}\sum_{i=1}^{3}\left(-1\right)^{\delta_{i3}}\left[\mathrm{Li}_2\left(\frac{x_0 - 1}{x_0 - x_i}\right) - \mathrm{Li}_2\left(\frac{x_0}{x_0 - x_i}\right)\right], \quad (170)$$

with four different roots

$$\begin{aligned} x_0 &= 1 + \frac{M_1^2 - M_2^2}{Q^2}, \qquad x_3 = \frac{M_3^2}{M_3^2 - M_2^2}, \\ x_{1,2} &= \frac{Q^2 + M_1^2 - M_3^2 \mp \sqrt{\lambda\left(-Q^2, M_1^2, M_3^2\right)}}{2Q^2}. \end{aligned} \quad (171)$$

All masses squared are understood to have equal infinitesimal imaginary parts: $M_i^2 \to M_i^2 - i\epsilon$, necessary to properly define the analytic continuation at $Q^2 \to -s$.

The following special cases are also met in any realistic calculation:

$$C_0\left(0,0,Q^2;M_1,0,M_3\right) = \frac{1}{Q^2}\ln\frac{x_2}{x_2-1}\ln\frac{x_1-1}{x_1}, \tag{172}$$

$$C_0\left(0,0,Q^2;M_1,0,M_1\right) = \frac{1}{Q^2}\ln^2\frac{\beta_Q+1}{\beta_Q-1}, \qquad \beta_Q = \sqrt{1+\frac{4M_1^2}{Q^2}}, \tag{173}$$

$$\begin{aligned} C_0\left(0,0,Q^2;M_1,M_2,0\right) &= C_0\left(0,0,Q^2;0,M_2,M_1\right) \\ &= \frac{1}{Q^2}\left[\mathrm{Li}_2\left(1-\frac{M_1^2}{M_2^2}\right)-\mathrm{Li}_2\left(1-\frac{Q^2+M_1^2}{M_2^2}\right)\right] \end{aligned} \tag{174}$$

One more interesting case is

$$\begin{aligned} &C_0\left(-m^2,-m^2,Q^2;0,m,0\right) = \frac{1}{m^2\left(y_1-y_2\right)}\Big[2\mathrm{Li}_2\left(\frac{1}{y_1}\right)-2\mathrm{Li}_2\left(\frac{1}{y_2}\right) \\ &+\mathrm{Li}_2\left(y_1\right)-\mathrm{Li}_2\left(y_2\right)\Big], \quad \text{with} \quad y_{1,2} = -\frac{Q^2}{2m^2}\left(1\pm\sqrt{1+\frac{4m^2}{Q^2}}\right). \end{aligned} \tag{175}$$

Also of some relevance is the scalar integral with all internal masses set to zero:

$$\begin{aligned} C_0\left(p_1^2,p_2^2,Q^2;0,0,0\right) &= \frac{1}{Q^2\left(a_+-a_-\right)}\left[\ln\left(a_+a_-\right)\ln\frac{a_+^{(1)}}{a_-^{(1)}}+2\mathrm{Li}_2\left(a_+\right)-2\mathrm{Li}_2\left(a_-\right)\right], \\ a_\pm^{(1)} &= 1-a_\pm, \quad a_\pm = \frac{Q^2+p_1^2-p_2^2\pm\sqrt{\lambda\left(Q^2,p_1^2,p_2^2\right)}}{2\,Q^2}. \end{aligned} \tag{176}$$

The result (176) is valid in the Euclidean region.

Another simple case is given by a scalar integral with one very small mass and two external momenta on-mass-shell, $p_1^2 = -m_1^2$, $p_2^2 = -m_3^2$ and $m_2 = \lambda$, with λ small with respect to all other quantities. Although we are dealing with the infrared singularities within the dimensional regularization approach, this example is a useful bridge to the mass-regularization method. Holding $m_3 \neq m_1$ will allow us to discuss QED corrections to the decay $W^+ \to u\overline{d}$. By using an appropriate implementation of the Feynman parameters we can write

$$C_0\left(-m_1^2,-m_3^2,Q^2;m_1,\lambda,m_3\right) = \int_0^1 dy\int_0^1 dx\frac{x}{\chi(x,y)}, \tag{177}$$

with the integrand

$$\chi(x,y) = x^2\chi(y)+\lambda^2\,(1-x)-i\epsilon, \quad \chi(y) = m_1^2\,(1-y)+m_3^2 y+Q^2 y\,(1-y)\,. \tag{178}$$

Using the fact that

$$\int_0^1 dx \frac{x}{\chi(y)x^2+\lambda^2\,(1-x)} = \frac{1}{2\chi(y)}\ln\left(\frac{\chi(y)}{\lambda^2}\right)+\mathcal{O}\Big(\frac{\lambda}{\sqrt{\chi(y)}}\Big), \tag{179}$$

we obtain the following decomposition:

$$\begin{aligned}
C_0 &= F_1 \ln\left(\frac{\mu}{\lambda}\right)+\frac{1}{2}F_2\,, && (180)\\
F_1 &= \int_0^1 dy\,\frac{1}{\chi\,(y)} = \frac{1}{Q^2\,(y_1-y_2)}\left[\ln\left(\frac{y_2-1}{y_2}\right)-\ln\left(\frac{y_1-1}{y_1}\right)\right], && (181)\\
F_2 &= \int_0^1 dy\,\frac{1}{\chi\,(y)}\ln\frac{\chi\,(y)}{\mu^2} = F_1\,\ln\left(\frac{Q^2-i\epsilon}{\mu^2}\right)+\frac{f\,(y_1,y_2)-f\,(y_2,y_1)}{Q^2\,(y_1-y_2)} && (182)
\end{aligned}$$

with $f\,(y_1,y_2)$ given by:

$$\begin{aligned}
f\,(y_1,y_2) &= \frac{1}{2}\ln\left(\frac{1-y_2}{-y_2}\right)\ln\left[y_2\,(y_2-1)\,(y_1-y_2)^2\right]\\
&\quad -\mathrm{Li}_2\left(\frac{1-y_2}{y_1-y_2}\right)+\mathrm{Li}_2\left(\frac{-y_2}{y_1-y_2}\right). && (183)
\end{aligned}$$

In this equation $y_{1,2}$ are the roots of the equation $\chi\,(y)=0$, i.e.

$$y_{1,2} = \frac{Q^2+m_3^2-m_1^2\pm\sqrt{\lambda\,(-Q^2,m_1^2,m_3^2)}}{2Q^2}. \tag{184}$$

Note that μ is artificially introduced in (180) in order to show the correspondence with the method of dimensional regularization, i.e.

$$\ln\left(\frac{\mu}{\lambda}\right)^2 \leftrightarrow \frac{1}{\hat{\varepsilon}}. \tag{185}$$

Expressions (180)–(183) greatly simplify if $m_1 = m_3 = m$, since in this case we have $y_1+y_2 = 1$. Two integrals are useful for this case. With $\beta_m^2 = 1+4m^2/Q^2$ we have:

$$\begin{aligned}
F_1 &= \int_0^1 dy\,\left[m^2+Q^2 y\,(1-y)\right]^{-1} = \frac{2}{Q^2\beta_m}\ln\frac{\beta_m+1}{\beta_m-1}, && (186)\\
F_2 &= \int_0^1 dy\,\frac{1}{m^2+Q^2 y\,(1-y)}\ln\left[\frac{m^2+Q^2 y\,(1-y)}{\mu^2}\right] = F_1\,\ln\left(\frac{Q^2-i\varepsilon}{\mu^2}\right)\\
&\quad +\frac{1}{Q^2\beta_m}\left[\ln\frac{\beta_m+1}{\beta_m-1}\ln\frac{m^2\beta_m^2}{Q^2}-2\mathrm{Li}_2\left(\frac{\beta_m+1}{2\beta_m}\right)+2\mathrm{Li}_2\left(\frac{\beta_m-1}{2\beta_m}\right)\right]. && (187)
\end{aligned}$$

6.10 Reduction of the vector three-point integral.

The rank-1 tensor associated with the three-point function is given by

$$i\pi^2 C_\mu\left(p_1^2, p_2^2, Q^2; m_1, m_2, m_3\right) = \mu^{4-n} \int d^n q \, \frac{q_\mu}{d_0 d_1 d_2}, \tag{188}$$

which leads to the following decomposition:

$$i\pi^2 [C_{11}\left(p_1^2, p_2^2, Q^2; m_1, m_2, m_3\right) p_{1\mu} + C_{12}\left(p_1^2, p_2^2, Q^2; m_1, m_2, m_3\right) p_{2\mu}]. \tag{189}$$

The corresponding reduction is based on the following relations:

$$\begin{aligned} p_1 \cdot q &= \frac{1}{2}\left(d_1 - d_0 + f_1^c\right), \qquad p_2 \cdot q = \frac{1}{2}\left(d_2 - d_1 + f_2^c\right), \\ f_1^c &= -p_1^2 + m_1^2 - m_2^2, \qquad f_2^c = -Q^2 + p_1^2 + m_2^2 - m_3^2. \end{aligned} \tag{190}$$

For the final result three additional pinches are needed:

$$\begin{aligned} C_k^{(0)} &= B_k(1,2) = B_k\left(p_2^2; m_2, m_3\right), \\ C_k^{(1)} &= B_k(0,2) = B_k\left(Q^2; m_1, m_3\right), \\ C_k^{(2)} &= B_k(0,1) = B_k\left(p_1^2; m_1, m_2\right), \end{aligned} \tag{191}$$

where k runs over all possible indices of the B_k-functions, i.e. $0, 1, 21$ and 22. For instance,

$$\begin{aligned} & i\pi^2 B_0(i,j) = \mu^{4-n} \int d^n q \frac{1}{d'_i d'_j}, \\ & d'_i = d_i, \qquad d'_j = d_j, \qquad \text{for} \quad i = 0, \\ & d'_1 = q^2 + m_2^2, \qquad d'_2 = (q + p_2)^2 + m_3^2. \end{aligned} \tag{192}$$

As we did for the two-point integrals, we introduce a matrix

$$X_{3,ij} = p_i \cdot p_j, \tag{193}$$

which satisfies $\det X_3 = -\Delta_3$, and also the vector $R_{12}^{(1)}$

$$R_{12}^{(1)} = \frac{1}{2}\begin{pmatrix} C_0^{(1)} - C_0^{(0)} + f_1^c C_0 \\ C_0^{(2)} - C_0^{(1)} + f_2^c C_0 \end{pmatrix}. \tag{194}$$

With their help we derive

$$C_{1i}\left(p_1^2, p_2^2, Q^2; m_1, m_2, m_3\right) = \left(X_3^{-1}\right)_{ij} R_{12\,j}^{(1)}. \tag{195}$$

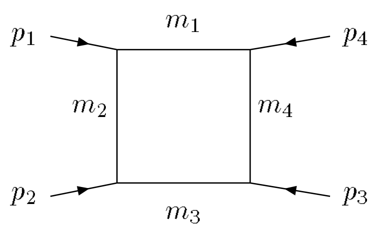


Figure 10: The four-point Green function.

6.11 Four-point integrals, D-functions

The four-point functions, are again much more complicated than the previous ones, including the three-point functions.

6.12 The scalar four-point integral, D_0-function.

We start with the definition,

$$\begin{aligned} i\pi^2 D_0 &= i\pi^2 D_0\left(p_1^2, p_2^2, p_3^2, p_4^2, (p_1+p_2)^2, (p_2+p_3)^2; m_1, m_2, m_3, m_4\right) \\ &= \mu^{4-n}\int d^n q \frac{1}{d_0 d_1 d_2 d_3}, \end{aligned} \tag{196}$$

with d_i as in Eq.(133), which in this case are written down as

$$\begin{array}{ll} d_0 = q^2 + m_1^2 - i\epsilon, & d_1 = (q+p_1)^2 + m_2^2 - i\epsilon, \\ d_2 = (q+p_1+p_2)^2 + m_3^2 - i\epsilon, & d_3 = (q+p_1+p_2+p_3)^2 + m_4^2 - i\epsilon, \end{array} \tag{197}$$

with all four-momenta flowing inwards (as shown in Fig. 10), so that $p_1+p_2+p_3+p_4 = 0$. After making use of an alternative Feynman parameterization, we arrive at the following representation:

$$D_0 = \int d^4 u \delta\left(\sum_i u_i - 1\right)\prod_i \theta(u_i)\, I(\{u\}), \tag{198}$$

where the integrand may be written in compact form as

$$I(\{u\}) = \left[\sum_j m_{j+1}^2 u_{j+1} + \sum_{i<j} p_{ij}^2 u_{i+1} u_{j+1}\right]^{-2}. \tag{199}$$

By explicit evaluation we can show that the latter is exactly the one-loop four-point function that is needed. Introducing the variables x, y and z this may be cast in the form

$$D_0 = \int_0^1 dx \int_0^x dy \int_0^y dz (ax^2+by^2+gz^2+cxy+hxz+jyz+dx+ey+kz+f)^{-2}. \tag{200}$$

The coefficients of the quadratic form are:

$$
\begin{array}{lll}
a = -p_{23}^2 = -p_3^2\,, & b = -p_{12}^2 = -p_2^2\,, & g = -p_{01}^2 = -p_1^2\,, \\
c = -p_{13}^2 + p_{12}^2 + p_{23}^2\,, & h = -p_{03}^2 - p_{12}^2 + p_{02}^2 + p_{13}^2\,, & j = -p_{02}^2 + p_{01}^2 + p_{12}^2\,, \\
d = m_3^2 - m_4^2 + p_{23}^2\,, & e = m_2^2 - m_3^2 + p_{13}^2 - p_{23}^2\,, & k = m_1^2 - m_2^2 + p_{03}^2 - p_{13}^2\,,
\end{array}
$$

$f = m_4^2 - i\epsilon\,.$

6.13 Some particular cases of D_0-functions.

For four fermion processes in the approximation when all external fermionic masses are ignored, we may derive rather compact expressions for D_0-functions. We consider two different cases where

1. there are no virtual photons in a box diagram;
2. box diagrams contain one or two virtual photons.

The treatment of D_0 in these two cases is substantially different. In the first case the D_0-function is infrared finite and we have no particular problem in computing it. In the second case, however, an infrared singularity will show up and it is more convenient to isolate the singular part first by performing a splitting of the basic integral.

Infrared-divergent boxes are always split into a combination of infrared singular three-point functions plus an additional integral which is finite and for which a direct calculation is more convenient as compared with a standard scalar reduction.

Case 1. The most general expression we encounter in considering ZZ and WW boxes corresponds to the following choice:

$$
\begin{array}{rcl}
p_i^2 &=& 0\,, \qquad (p_1+p_2)^2 = Q^2, \qquad (p_2+p_3)^2 = P^2, \\
m_1 &=& M_1\,, \qquad m_2 = 0\,, \qquad m_3 = M_1\,, \qquad m_4 = M_2\,.
\end{array}
\qquad (201)
$$

With an appropriate choice of Feynman parameters it may be presented and calculated as follows:

$$
D_0\left(0,0,0,0,Q^2,P^2;M_1,0,M_1,M_2\right) = \int_0^1 dz \int_0^1 ydy \int_0^1 dx
$$

$$\times \frac{1}{\left[M_1^2 y + M_2^2(1-y) + P^2(1-y)(1-z) + Q^2 z y^2 x(1-x)\right]^2}$$

$$= \frac{1}{Q^2(P^2+M_2^2)\sqrt{d_4}} \sum_{i=1}^{4}\sum_{j=1}^{2} (-1)^{\delta_{i3}+\delta_{j2}} \left[\mathrm{Li}_2\left(\frac{\bar{x}_j}{\bar{x}_j - x_i}\right) - \mathrm{Li}_2\left(\frac{\bar{x}_j - 1}{\bar{x}_j - x_i}\right)\right],$$

with the six roots given by

$$\begin{aligned} x_{1,2} &= \frac{1}{2}\left(1 \mp \sqrt{1+\frac{4M_1^2}{Q^2}}\right), \qquad & \bar{x}_{1,2} = \frac{x_4}{2}\left(1 \mp \sqrt{d_4}\right), \\ x_3 &= \frac{M_2^2}{M_2^2 - M_1^2}, \qquad & x_4 = \frac{P^2+M_2^2}{P^2+M_2^2-M_1^2}. \end{aligned} \tag{202}$$

and with

$$d_4 = 1 + \frac{4M_1^2 P^2\left(P^2+M_2^2-M_1^2\right)}{Q^2\left(P^2+M_2^2\right)^2}. \tag{203}$$

For $M_2 = 0$, which in practical applications means $m_t = 0$, it simplifies to

$$D_0\left(0,0,0,0,Q^2,P^2;M_1,0,M_1,0\right) = \frac{2}{Q^2 P^2 \sqrt{d_4^{(0)}}} \sum_{ij=1}^{2} (-1)^{i+1}\, \mathrm{Li}_2\left(\frac{\tilde{x}_i}{\tilde{x}_i - x_j}\right), \tag{204}$$

where the roots now read as follows:

$$\tilde{x}_{1,2} = \frac{x_4}{2}\left(1 \mp \sqrt{d_4^{(0)}}\right), \qquad d_4^{(0)} = 1 + \frac{4M_1^2\left(P^2-M_1^2\right)}{Q^2 P^2}. \tag{205}$$

Case 2. We encounter this case when considering ZA and AA boxes where we introduce three auxiliary integrals:

$$\begin{aligned} i\pi^2 \bar{J}_{\gamma\gamma}\left(Q^2,P^2;m_e,m_f\right) &= \mu^{4-n}\int d^n q \frac{2q\cdot(q+Q)}{d_0(0)\, d_1(m_e)\, d_2(0)\, d_3(m_f)}, \qquad (206)\\ i\pi^2 \bar{J}_{\gamma Z}\left(Q^2,P^2;m_e,m_f\right) &= \mu^{4-n}\int d^n q \frac{2q\cdot Q}{d_0(0)\, d_1(m_e)\, d_2(M_Z)\, d_3(m_f)}, \\ i\pi^2 \bar{J}_{Z\gamma}\left(Q^2,P^2;m_e,m_f\right) &= \mu^{4-n}\int d^n q \frac{2Q\cdot(q+Q)}{d_0(M_Z)\, d_1(m_e)\, d_2(0)\, d_3(m_f)}, \end{aligned}$$

which are simple to calculate.

Performing the standard reduction, we express the corresponding D_0 functions in terms of these integrals:

$$D_0\left(-m_e^2,-m_e^2,-m_f^2,-m_f^2,Q^2,P^2;0,m_e,0,m_f\right) \tag{207}$$

$$= \frac{1}{Q^2}[-\bar{J}_{\gamma\gamma}\left(Q^2,P^2;m_e,m_f\right)$$
$$+C_0\left(-m_e^2,-m_f^2,P^2;m_e,0,m_f\right)+C_0\left(-m_f^2,-m_e^2,P^2;m_f,0,m_e\right)],$$
$$D_0\left(-m_e^2,-m_e^2,-m_f^2,-m_f^2,Q^2,P^2;0,m_e,M_z,m_f\right) \tag{208}$$
$$= \frac{1}{Q^2+M_z^2}[-\bar{J}_{\gamma z}\left(Q^2,P^2;m_e,m_f\right)$$
$$-C_0\left(-m_e^2,-m_f^2,P^2;m_e,M_z,m_f\right)+C_0\left(-m_f^2,-m_e^2,P^2;m_f,0,m_e\right)],$$
$$D_0\left(-m_e^2,-m_e^2,-m_f^2,-m_f^2,Q^2,P^2;M_z,m_e,0,m_f\right) \tag{209}$$
$$= \frac{1}{Q^2+M_z^2}[\bar{J}_{z\gamma}\left(Q^2,P^2;m_e,m_f\right)$$
$$+C_0\left(-m_e^2,-m_f^2,P^2;m_e,0,m_f\right)-C_0\left(-m_f^2,-m_e^2,P^2;m_f,M_z,m_e\right)].$$

To conclude, we present the answers for the auxiliary integrals in terms of one-fold integrals. An explicit form is also given:

$$\begin{aligned}
\bar{J}_{\gamma\gamma}\left(Q^2,P^2;m_e,m_f\right) &= \int_0^1 dx \frac{1}{\chi\left(P^2;m_e,m_f\right)} \ln\frac{\chi\left(P^2;m_e,m_f\right)}{Q^2} \\
&= F_2|_{m_1\to m_e,\, m_3\to m_f,\, Q^2\to P^2,\, \mu^2\to Q^2}\,, \qquad (210)\\
\bar{J}_{\gamma z}\left(Q^2,P^2;m_e,m_f\right) &= -\bar{J}_{z\gamma}\left(Q^2,P^2;m_e,m_f\right) = \ln\frac{M_z^2+Q^2}{M_z^2} J_0\left(P^2;m_e,m_f\right).
\end{aligned}$$

Here $\chi\left(P^2;m_e,m_f\right) = P^2x\left(1-x\right)+m_e^2\left(1-x\right)+m_f^2x$ is the usual quadratic form and F_2 is given by Eq.(183).

7 Renormalization in QED

7.1 The basic approach to renormalization

Before entering the details of the renormalization in QED at the one-loop level we briefly summarize the main procedure for dealing with infinities. To discuss renormalization we assume QED to be the theory of photon and electrons, therefore everywhere we put $Q_f = Q_e = -1$.

In computing one- or multi-loop diagrams we face the problem of having to deal with ultraviolet infinities. In any theory the first step will be to define its regularized version and only afterwards address the procedure for infinity subtraction. Regularization is simply the replacement of a theory by a slightly different one, using some cut-off. As it happens, there is now a general consensus on what regularization scheme to use, i.e. dimensional regularization.

- Any Lagrangian contains two types of objects: fields and parameters – masses and other than mass parameters; for example, the coupling constant e in QED. We may replace the bare parameters of the Lagrangian, $\{p_0\}$, by renormalized ones by multiplicative renormalization. For each bare parameter p_0 we write

$$p_0 = Z_p\, p = p + \delta p, \qquad \delta p = e^2 \delta p^{(1)} + \cdots, \tag{211}$$

  with renormalization constants Z_p different from 1 by loop corrections.

- An example is the electron mass m, The quantity $m + e^2 \delta m$ is called the bare mass, m_0, and m itself the experimental mass. This notion also reflects an intuition about the physical meaning of the bare mass: if the interactions could be switched off ($e = 0$) that is what we would see.

  The renormalization constants are, in general, infinite and fixed by a finite set of renormalization conditions.

- The decomposition in Eq.(211) is to a large extent arbitrary. Only the divergent parts are determined directly by the structure of the divergences of the one-loop amplitudes. The finite parts depend on the choice of the explicit renormalization conditions which, in turn, define the renormalization scheme.

- The choice of a renormalization scheme—a rather technical subject—is mostly dictated by practical considerations, but where physical observables are concerned, all renormalization schemes (RS) have been made equal. This, of course, applies as long as they respect gauge invariance and do not involve ad hoc treatments of leading and sub-leading higher-order corrections.

Before actually discussing the options that we have in working with specific RS, let us briefly summarize the main ingredients that enter into the calculation.

1. From any unrenormalized Lagrangian and from the corresponding Feynman rules we compute the Green function of the theory; say, at one-loop. These Green functions are controlled by Ward identities, which reflect the gauge invariance of the theory and after subtracting the infinities we will again need these identities in order to see that the renormalization does not spoil gauge invariance.

2. In dealing with Ward identities for Green functions there is no need to confine ourselves to external lines that are transversal (photon sources satisfying the condition $\partial_\mu J_\mu = 0$) or on their mass shell. Physical observables are obtained when we move from Green functions to $\mathcal{S}$-matrix elements. In any renormalization scheme this is a crucial step.

The main object to discuss, in summarizing the steps leading from a Green function to the corresponding $\mathcal{S}$-matrix elements, is the two-point Green function, which will, in general, have a pole. In any theory this pole becomes a property of the $\mathcal{S}$-matrix and therefore in any gauge theory the pole is gauge-invariant by construction.

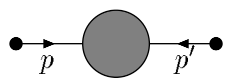


Figure 11: The two-point Green function.

Whenever massive and unstable vector bosons—as well as any other unstable particle—are present in the theory, these poles must be examined with due care since one can be shown that they lay in the complex plane, actually on the second Riemann sheet.

- In QED, however, the only mass that we care about is the electron mass, a stable particle. For many applications in QED it is most natural to use our knowledge of the electron mass, and all RS will use m as an input parameter to be related to the pole of the $\mathcal{S}$-matrix.

- For QCD, however, many calculations have to be independent of quark masses and in this case it will be natural to choose RS where we do not need to use a renormalization condition related to the poles of the $\mathcal{S}$-matrix.

- At the physical mass pole the two-point Green function can be cast in the following form

$$G_{ij}\left(p, p'\right) = \left(2\pi\right)^4 i\, \delta^4\left(p + p'\right) \frac{K_{ij}\left(p\right)}{p^2 + m^2}, \qquad p^2 \to -m^2, \tag{212}$$

  where the index i stands for a spinor, Lorentz, etc. index. Next, wave-functions J_i are defined, for each non-zero eigenvalue of K, and they must be normalized.

As an example, we consider fermions in QED. Here, we compute one-loop electron self-energy,

$$\Sigma\left(\not p\right)=\left(2\pi\right)^{4}i\frac{e^{2}}{16\pi^{2}}\{[2B_{1}\left(p^{2};m,0\right)+1]i\not p+[-4B_{0}\left(p^{2};m,0\right)+2]m\},\quad(213)$$

then the Dyson re-summed (or complete) propagator

$$S=\frac{1}{\left(2\pi\right)^{4}i}\left[i\not p+m+e^{2}\delta m-\frac{\Sigma\left(\not p\right)}{\left(2\pi\right)^{4}i}\right]^{-1},\qquad(214)$$

where we introduce the mass renormalization counter-term

$$e^{2}\delta m=\frac{\Sigma\left(im\right)}{\left(2\pi\right)^{4}i}.\qquad(215)$$

Expanding $\Sigma\left(\not p\right)$ in a Taylor series around the physical electron mass $i\not p=-m$ (the so-called *subtraction point*), we obtain:

$$\Sigma\left(\not p\right)=\Sigma\left(im\right)+\left(i\not p+m\right)\Sigma_{\mathrm{WF}}+\mathcal{O}\left(\left(i\not p+m\right)^{2}\right),\qquad(216)$$

where the coefficient of the linear term is the so-called wave-function factor,

$$\Sigma_{\mathrm{WF}}\;=\;\frac{\partial\Sigma\left(\not p\right)}{\partial\left(i\not p\right)}|_{i\not p=-m}.\qquad(217)$$

After mass renormalization, Eq.(215), and wave-function renormalization, Eq.(217), we arrive at the following residual matrix:

$$K\left(p\right)=\frac{1}{Z}\left(-i\not p+m\right).\qquad(218)$$

Here, the factor

$$Z=1-\frac{\Sigma_{\mathrm{WF}}}{\left(2\pi\right)^{4}i},\qquad(219)$$

is, by definition, the fermion wave-function renormalization constant, which is infinite, since Σ_{WF} is ultraviolet divergent. Thus, we must take for the fermion wave-function

$$J=u\left(p\right)\frac{Z^{1/2}}{2\,m}\sqrt{2\,E},\qquad(220)$$

in order to preserve wave-function normalization to 1 (the Dirac spinors are assumed to be also normalized to 1). A similar procedure will apply to the normalization of v spinors.

Finally, we consider any arbitrary Green function with an external fermion, multiply it by $p^2 + m^2$ and put the momentum of the external line on its mass shell. The net effect in passing to the $\mathcal{S}$-matrix elements is to multiply each external fermion line (actually, every external line) by a factor $Z^{-1/2}$, which contains infinities.

Different RS may have different procedures at any intermediate step but all of them will give the same answer for the $\mathcal{S}$-matrix, as long as we respect the proper treatment of the external lines.

In QED there is more than mass renormalization and we also have charge renormalization. Here, the situation is again simplified because of the basic properties of the Lagrangian.

- At zero momentum transfer the vertex corrections in QED cancel the electron wave-function factors exactly and, moreover, the photon self-energy is infrared finite.

Thus, in QED we can define a perturbative coupling for on-shell scattering.

The parameter renormalization, Eq.(211), is sufficient to obtain finite $\mathcal{S}$-matrix elements if, in addition, wave-function renormalization factors for external on-shell particles are included.

Off-shell Green functions, however, are not finite by themselves. If we choose a procedure where also vertices and self-energies are to be made finite, then, besides parameters (coupling and masses), the bare fields have to be redefined in terms of renormalized fields by another set of multiplicative renormalizations

$$\phi_0 = Z_\phi \, \phi. \tag{221}$$

Expanding the renormalization constants

$$Z_i = 1 + e^2 \delta Z_i \tag{222}$$

gives

$$\mathcal{L}\left(\phi_0; p_0\right) = \mathcal{L}\left(\phi; p\right) + \mathcal{L}_{\rm ct}\left(\phi, \delta Z_\phi; p, \delta p\right), \tag{223}$$

where with $\{p\}$ we denote the set of parameters e, m etc. and $\mathcal{L}_{\rm ct}$ denotes the counter-term Lagrangian. Before actually discussing the various options for choosing a renormalization scheme we will describe in detail the notion of a counter-term.

- Consider some theory, described by a Lagrangian, depending on certain fields and parameters. At the tree level there is no ambiguity and theoretical predictions from this Lagrangian can be compared with the experiment. One data point, i.e. one measurement, is needed to fix one p. After that any other comparison is a test of the theory.

- Now suppose that we want to go beyond the tree approximation. Then radiative corrections must be calculated. The relation between the parameters $\{p\}$ and the experimental data becomes much more complicated but it remains precisely true that one measurement is needed to fix one free parameter p, the rest is a test. Of course, the values of $\{p\}$ as determined using only the tree approximation will be different from the values determined taking into account radiative corrections. As it happens, this difference is usually infinitely large because the radiative corrections contain infinities. Such infinities are well-defined and understood.

- Because of the awkward situation that the corrected $\{p\}$ and the tree $\{p\}$ are so different one introduces the notion of a counter-term. As we have explained before, in the Lagrangian we write $\{p + \delta p\}$ instead of $\{p\}$, and $\{\delta p\}$ is chosen in some well-defined manner such that now $\{p\}$ remain in the neighbourhood of the tree $\{p\}$. It is, however, purely a matter of convenience; the only thing that ever emerges in the confrontation with the data is $\{p + \delta p\}$.

- This is why all RS are indeed equivalent. Of course different theoretical predictions for some observable quantity may refer to a different choice of experimental data points needed for the renormalization conditions or to the same data points taken at two different scales. This fact alone should not be related, under any circumstance, to a difference in the renormalization scheme and we prefer, therefore, to introduce the notion of input parameter set (IPS).

Before we can make predictions from a theory described by n independent parameters we must specify an IPS, i.e. a choice for n experimental data points to be used as input.

Two predictions for the same observable will inevitably differ by an amount proportional to the missing higher orders if they refer to different IPS, even if they are performed within the same renormalization scheme. In turn, the use of different scales as a subtraction point is closely connected to the scale behaviour of the theory that is controlled by the renormalization-group equation, to which we will return towards the end of this section.

In order to define a consistent procedure, it is necessary, when talking about $\{p\}$, to specify what $\{\delta p\}$ are used. Stating our conventions on this matter is what is usually termed the renormalization scheme. Two essentially different approaches may be distinguished:

1. prescribe $\{p\}$ precisely;

2. prescribe $\{\delta p\}$ precisely.

Again, only the combinations $\{p+\delta p\}$ appear in the confrontation with the data, and we are discussing here a matter of convention. As a matter of terminology, we will call quantities such as $\{\delta p\}$ counter-terms.

- In the early days of QED, method 1 was used. The convention was to prescribe $\{p\}$, and to use for that some very well-defined experimental quantities. The quantities $\{\delta p\}$ were then obtained from the data, including radiative corrections.

- In QED, the mass and the charge of the electron are very well known, and the scheme is well understood. Again, the situation will be different in QCD where, on the contrary, we would like to avoid any reference to quark masses.

- An example of approach 1 is the *on-shell* renormalization scheme, defined as the procedure of parameters and field redefinition by Eq.(211), Eq.(221) when the renormalization factors are fixed for external on-shell particles such that the meaning of these parameters is preserved to be the same as in the tree approximation.

- Convention 1 has many advantages but sometimes there is no clear precisely known experimental quantity that can play the role of defining $\{p\}$. Such is the case for QCD with respect to the coupling constant g_S of that theory. This g_S, at least as seen experimentally, is a function of the scale, and cannot be measured at a low scale due to confinement. Consequently, theorists, after considerable wrangling, have carefully considered method 2.

- The quantities $\{\delta p\}$ are prescribed and $\{p\}$ are determined from some experiments depending on $\{p\}$. This method is consequently realized within the *minimal subtraction* $\overline{MS}$ renormalization scheme.

Perhaps the best way to illustrate how RS work in practice is to consider mass renormalization in QED. Coming back to Eq.(214), we expand $\Sigma\,(\not p)$ around a finite *intermediate* mass m_R. Then the complete propagator will become,

$$\begin{aligned} S &= \frac{1}{(2\pi)^4\, i}\Big(i\not p + m_R + e^2\delta m + \frac{e^2}{16\,\pi^2}\,\Delta S\Big)^{-1}, \\ \Delta S &= 3\, m_R\Big(\frac{1}{\bar\varepsilon} - \ln\frac{m_R^2}{\mu^2} + \frac{4}{3}\Big) + \Sigma_{\rm rest}\,(p)\,, \end{aligned} \tag{224}$$

where $\Sigma_{\rm rest}\,(p)$ contains additional ultraviolet divergences. Therefore, we have a freedom in fixing the mass counter-term. This is done with a condition

$$\delta m = -\frac{1}{16\,\pi^2}\, m_R\left[3\left(\frac{1}{\bar\varepsilon} - \ln\frac{m_R^2}{\mu^2} + \frac{4}{3}\right) + C\right], \tag{225}$$

containing an arbitrary finite constant C.

The Renormalization scheme (RS): *Any explicit definition of the constant C in Eq.(225) is a definition of the RS.* Clearly, μ and C are arbitrary parameters and once we have specified our IPS; say, m and the fine-structure constant α, then different choices of μ and of C will correspond to bare Lagrangians with different bare parameters m_0 and e_0. *The choice $C = 0$ defines the familiar $\overline{MS}$ scheme. Leaving C as well as μ and ε arbitrary and including wave function renormalization factors defines the* **generalized minimal subtraction** *scheme, or GMS.*

The parameter m_R is fixed in terms of the physical electron mass,

$$m_R = m_R\,(\alpha, m, \mu, C)\,, \tag{226}$$

and this relation is uniquely governed by the requirement that the physical mass be in the position of the single-particle pole in the two-point Green function.

In conclusion, different RS may or may not indulge in presenting different intermediate parameters, like m_R, but all of them will agree in any prediction for physical observables—at a fixed order in perturbation theory—once the IPS is uniquely chosen. Stated differently, as long as the theory is renormalizable and the scale at which we perform the subtraction is the same (if we can find an on-shell $\mathcal{S}$-matrix element that corresponds to some well-measured quantity) then any procedure for cancelling the infinities will predict $\mathcal{S}$-matrix elements that are finite, μ-independent and scheme independent.

Renormalization procedure: *A renormalization procedure comprises the specification of the gauge fixing term including the corresponding ghost Lagrangian, the choice of the regularization scheme—nowadays dimensional regularization—, the prescription for the RS and a choice for the IPS.*

A typical example of what we call the IPS dependence of radiative corrections is the following. Suppose that in a theory like QED with two parameters we have already made use of the definition of the physical mass of the electron. Suppose, in addition, we assume that some $\mathcal{S}$-matrix element has some value S_a. Then up to one-loop we will have

$$\mathcal{S}_a = e^2_{_{Ra}}\, \mathcal{S}^{(1)}\left(m_{_{Ra}}, \mu\right). \tag{227}$$

where we have explicitly indicated the number of loops. Then we can solve for $m_{_{Ra}}$ and $e_{_{Ra}}$ in terms of m and $\mathcal{S}_a$. Suppose we compute the second matrix element $\mathcal{S}_b$, giving the second solution $m_{_{Rb}}$ and $e_{_{Rb}}$ in terms of m and $\mathcal{S}_b$. In predicting any matrix element $\mathcal{M}$ we will have options

$$\mathcal{M}_{a,b} = \mathcal{M}_0 \alpha^n_{a,b}\left(1 + \mathcal{M}_1 \alpha_{a,b}\right), \tag{228}$$

with $\alpha_{a,b} = e^2_{_{Ra,b}}$ and where to one-loop order $\alpha_b = \alpha_a\left(1 + \gamma\alpha_a\right)$.

Then we obtain $\mathcal{M}_b = \mathcal{M}_a + \delta\mathcal{M}$, with

$$\delta\mathcal{M} = \mathcal{M}_0 \alpha_a^n \left\{ \left[\frac{n(n-1)}{2}\gamma^2 + (n+1)\gamma\mathcal{M}_1 \right] \alpha_a^2 + \cdots + \gamma^{n+1}\mathcal{M}_1 \alpha_a^{n+2} \right\}. \tag{229}$$

Unless higher orders are computed we always consider $\delta\mathcal{M}$ to be the uncertainty associated with the two IPS.

7.2 On-shell versus $\overline{MS}$ renormalization in QED

The main motivation of this subsection will consist in carrying out the one-loop renormalization programmes in QED within two schemes, on-shell and $\overline{MS}$, and in illustrating how the physical result is RS independent.

The QED Lagrangian in the Feynman gauge can be derived from Eq.(1), setting $\xi = 1$ and $Q_e = -1$. It is unambiguous at the tree level. Moving to higher orders, we assume that it is made of bare fields and parameters labelled with sup- or sub-indices 0 and specifies the renormalization constants for the two fields—A_μ and ψ—and the two QED parameters—the electron mass m and the charge e:

$$\begin{aligned} A_\mu^0 &= Z_A^{1/2} A_\mu, \qquad \psi^0 = Z_\psi^{1/2}\psi, \\ e_0 &= Z_e\, e, \qquad m_0 = Z_m\, m = m + e^2\delta m + \mathcal{O}\left(e^4\right), \\ Z_i &= 1 + e^2\delta Z_i + \mathcal{O}\left(e^4\right). \end{aligned} \tag{230}$$

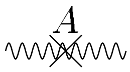
The Lagrangian can now be rewritten, up to terms $\mathcal{O}\left(e^2\right)$, as

$$\mathcal{L}_{\text{QED}}^{\text{R}} = \mathcal{L}_{\text{QED}} + \mathcal{L}_{\text{ct}}, \tag{231}$$

with a counter-term Lagrangian:

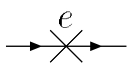
$$\begin{aligned} \mathcal{L}_{\text{ct}} &= e^2\,\mathcal{L}_{\text{ct}}^{(2)} + \mathcal{O}\left(e^4\right), \\ \mathcal{L}_{\text{ct}}^{(2)} &= -\frac{1}{4}\delta Z_A\, F_{\mu\nu}F_{\mu\nu} - \frac{1}{2}\delta Z_A\, \left(\partial_\mu A_\mu\right)^2 - \delta Z_\psi \overline{\psi}\partial\!\!\!/\psi \\ &\quad - \left(\delta Z_\psi m + \delta m\right)\overline{\psi}\psi - i\left(\delta Z_e + \delta Z_\psi + \frac{1}{2}\delta Z_A\right) eA_\mu\overline{\psi}\gamma_\mu\psi. \end{aligned} \tag{232}$$

The counter-term part of the Lagrangian is made of three terms: the first is bilinear in the photon fields, the second is bilinear in the fermion field and the third is a three-linear QED-like interaction. We may say that it generates a new set of QED Feynman rules to be denoted by a cross. First, the δZ_A counter-term:



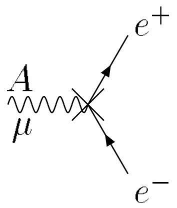
$$\rightarrow \qquad -e^2 \delta Z_{A}\,. \tag{233}$$

Then the δZ_ψ and δm counter-terms:



$$\rightarrow \qquad -e^2 \left(\delta Z_\psi i \not{p} + \delta Z_\psi m + \delta m\right). \tag{234}$$

And finally, the remaining combinations:



$$\rightarrow \qquad -ie\gamma_\mu e^3 \left(\delta Z_e + \delta Z_\psi + \frac{1}{2}\delta Z_{A}\right). \tag{235}$$

Equipped with these additional Feynman rules and using the results for the one-loop QED diagrams we may write down answers generated by both pieces of the Lagrangian Eq.(231). This part of the presentation is absolutely general and common to all approaches.

We begin with the photon self-energy. The electron-loop diagram gives

$$\Pi_{\mu\nu} = i\pi^2 e^2 \left(p^2 \delta_{\mu\nu} - p_\mu p_\nu\right) 4\Pi\left(p^2\right). \tag{236}$$

Here we introduce

$$\Pi\left(p^2\right) = 2[B_{21}\left(p^2; m, m\right) + B_1\left(p^2; m, m\right)], \tag{237}$$

where the limit $p^2 \rightarrow 0$ gives,

$$\Pi\left(0\right) = \frac{1}{3}\left(-\frac{1}{\bar{\varepsilon}} + \ln\frac{m^2}{\mu^2}\right). \tag{238}$$

The $p_\mu p_\nu$ part does not contribute whenever we consider $\Pi_{\mu\nu}$ as coupled to conserved fermionic currents. Thus, for $\Pi_{\mu\nu}$ we may use

$$\Pi_{\mu\nu} = \Pi_0\, p^2 \delta_{\mu\nu}\,, \tag{239}$$

with a scalar coefficient defined by

$$\Pi_0 = (2\pi)^4\, i\, \frac{e^2}{4\pi^2}\Pi\left(p^2\right). \tag{240}$$

We may now compute the Dyson re-summed (sometimes called complete or dressed) photon propagator

$$D_{\mu\nu} = \frac{1}{(2\pi)^4\, i}\, \frac{\delta_{\mu\nu}}{p^2}\, \frac{1}{1 + e^2\delta Z_{A} - \dfrac{e^2}{4\pi^2}\Pi\left(p^2\right)}. \tag{241}$$

Similarly, for the dressed electron propagator, the following is obtained:

$$\begin{aligned} S \;=\; & \frac{1}{(2\pi)^4\, i}\Big\{\left(1 + e^2\delta Z_\psi\right)(i\not{p} + m) + e^2\delta m \\ & - \frac{1}{(2\pi)^4\, i}[\Sigma\,(im) + (i\not{p} + m)\,\Sigma_{\rm WF} + \mathcal{O}\left((i\not{p} + m)^2\right)]\Big\}^{-1}, \end{aligned} \tag{242}$$

We shall also need a few other ingredients. From the fermion self-energy, Eq.(213), we derive the first two terms in the Taylor expansion

$$\Sigma\,(im) = i\pi^2 e^2 m\left(-\frac{3}{\bar{\varepsilon}} + 3\ln\frac{m^2}{\mu^2} - 4\right), \tag{243}$$

and the wave-function coefficient,

$$\begin{aligned} \Sigma_{\rm WF} &= i\pi^2 e^2\left\{2B_1\left(-m^2; m, 0\right) + 1 - 4\, m^2[B_{1p}\left(-m^2; m, 0\right) + 2\, B_{0p}\left(-m^2; m, 0\right)]\right\} \\ &= i\pi^2 e^2\left(-\frac{1}{\bar{\varepsilon}} + \frac{2}{\hat{\varepsilon}} + 3\ln\frac{m^2}{\mu^2} - 4\right). \end{aligned} \tag{244}$$

In terms of $V_1\left(Q^2; m, m\right)$, the γ_μ-part of the one-loop $e^+e^-\gamma$ vertex becomes

$$-\,(2\pi)^4\, i\, ie\left\{1 + e^2\left[\delta Z_e + \frac{1}{2}\delta Z_{A} + \delta Z_\psi + \frac{1}{16\pi^2}V_1\left(Q^2; m, m\right)\right]\right\}\gamma_\mu. \tag{245}$$

Having at our disposal Eq.(241), Eq.(242), and Eq.(245), we can easily illustrate the practical implementation of different RS.

7.3 The on-mass-shell renormalization scheme.

The essence of the on-mass-shell (hereafter OMS) RS is to preserve the meaning of the original parameters of the Lagrangian. We begin with a discussion of the dressed photonic propagator Eq.(241), requiring that its residue should be unchanged at the photonic mass shell, $p^2 = 0$, i.e.

$$e^2 \delta Z_{A} = \frac{e^2}{4\pi^2} \Pi(0) \,. \tag{246}$$

This requirement guarantees that the wave-function for external photonic lines does not change due to one-loop radiative corrections. Using Eq.(238), this requirement fixes one of the counter-terms to be

$$\delta Z_{A} = \frac{1}{12\pi^2} \Big(-\frac{1}{\bar{\varepsilon}} + \ln \frac{m^2}{\mu^2} \Big). \tag{247}$$

Now we consider the dressed electron propagator that we require to be of the form

$$S = \frac{1}{(2\pi)^4 \, i \, (i\not{p} + m)} \tag{248}$$

at the electron-mass-shell, $i\not{p} = -m$. This requirement preserves the external line electron wave-function from being renormalized by one-loop radiative corrections. It allows us to fix two other counter-terms from the condition

$$\begin{aligned} &\frac{1}{i\not{p} + m} \Big\{ e^2 \delta Z_\psi \, (i\not{p} + m) + e^2 \delta m \\ &\qquad - \frac{1}{(2\pi)^4 \, i} \left[\Sigma(im) + (i\not{p} + m) \, \Sigma_{\text{WF}} + \mathcal{O}\left((i\not{p} + m)^2 \right) \right] \Big\} u(p) = 0, \end{aligned} \tag{249}$$

yielding the following two equations:

$$e^2 \delta m = \frac{\Sigma(im)}{(2\pi)^4 \, i}, \qquad e^2 \delta Z_\psi = \frac{\Sigma_{\text{WF}}}{(2\pi)^4 \, i}, \tag{250}$$

which allow us to write

$$\begin{aligned} \delta m &= \frac{m}{16\pi^2} \left(-\frac{3}{\bar{\varepsilon}} + 3 \ln \frac{m^2}{\mu^2} - 4 \right), \\ \delta Z_\psi &= \frac{1}{16\pi^2} \left(-\frac{1}{\bar{\varepsilon}} + \frac{2}{\hat{\varepsilon}} + 3 \ln \frac{m^2}{\mu^2} - 4 \right). \end{aligned} \tag{251}$$

Finally, we move to the one-loop corrected vertex Eq.(245) and require it to be

$$-(2\pi)^4 \, i \, ie\gamma_\mu \,, \tag{252}$$

at $Q^2 = 0$, which preserves the Thomson limit of the electric charge from being renormalized by one-loop radiative corrections and which leads to the condition:

$$\delta Z_e + \frac{1}{2}\delta Z_A + \delta Z_\psi + \frac{1}{16\pi^2} V_1\left(0; m, m\right) = 0. \tag{253}$$

Substituting the counter-term δZ_ψ, which is already fixed and the derived expression for $V_1\left(0; m, m\right)$, Eq.(107), we observe the well-known QED Ward identity:

$$\delta Z_\psi + \frac{1}{16\pi^2} V_1\left(0; m, m\right) \equiv 0, \tag{254}$$

which allows us to fix the last counter-term:

$$\delta Z_e \equiv -\frac{1}{2}\delta Z_A. \tag{255}$$

Now all the counter-terms in the Lagrangian Eq.(232) are fixed and we may calculate any QED process at the one-loop level with the Lagrangian Eq.(231), that is, accounting for diagrams generated by the renormalized part and by the counter-terms.

The QED coupling constant becomes the $e^+e^-\gamma$ coupling in the Thomson limit of Compton scattering. Then, the theorem is telling us that α—free of infrared singularities—has a value independent of the order of perturbation theory, only determined by the accuracy of the experiment.

In full generality, the one-loop and counter-term contributions for any external on-shell line compensate each other identically (this is known as the principle of non-renormalizability for external lines). For any $2 \to 2$ fermion process, at the one-loop level, we encounter only two building blocks, Eqs.(241) and (245), while Eq.(242) has only played an auxiliary role in the counter-term fixation. These two building blocks become ultraviolet finite once we substitute the counter-terms as dictated by the renormalization procedure. They may be described in terms of two quantities—the effective (running) electric charge, $e^2\left(p^2\right)$, and the renormalized vertex, $V_1^{\text{ren}}\left(Q^2; m, m\right)$.

1. The photon propagation is now described by
$$e^2 D_{\mu\nu} = \frac{e^2\left(p^2\right)}{(2\pi)^4 i}\,\frac{\delta_{\mu\nu}}{p^2}\,. \tag{256}$$
The following point is important: in Eq.(256) we observe the presence of the running parameter
$$e^2\left(p^2\right) = \frac{e^2}{1 - \dfrac{e^2}{4\pi^2}\Pi^{\rm ren}\left(p^2\right)}, \tag{257}$$
where the evolution is governed by the renormalized quantity:
$$\Pi^{\rm ren}\left(p^2\right) = \Pi\left(p^2\right) - \Pi\left(0\right). \tag{258}$$

2. The electromagnetic interaction
$$\Lambda_\mu = (2\pi)^4 i\,\frac{ie^3}{16\pi^2}[\gamma_\mu V_1^{\rm ren}\left(Q^2; m, m\right) + \sigma_{\mu\nu}\left(p_1+p_2\right)_\nu V_2\left(Q^2; m, m\right)], \tag{259}$$
is expressed in terms of the renormalized vertex
$$V_1^{\rm ren}\left(Q^2; m, m\right) = V_1\left(Q^2; m, m\right) - V_1\left(0; m, m\right). \tag{260}$$

To understand the quantitative behaviour of the running of α we start from
$$\Pi^{\rm ren}\left(p^2\right) = \frac{1}{9} + \frac{1}{3}\left(1 - 2\,\frac{m^2}{p^2}\right)\int_0^1 dx \ln\frac{\chi\left(p^2, x\right)}{m^2}, \tag{261}$$
with $\chi\left(p^2, x\right)$ determined by Eq.(96), and derive its behaviour, for both low and high p^2. For instance
$$\Pi^{\rm ren}\left(p^2\right) = \frac{p^2}{15\,m^2}, \qquad \text{for} \quad p^2 \to 0, \tag{262}$$
where we find the well-known contribution to the Uehling effect; that is, the modification of Coulomb's law due to vacuum polarization. Alternatively, for large $s = -p^2$ we have
$$\Pi^{\rm ren}\left(p^2\right) = \frac{1}{3}\left(\ln\frac{s}{m^2} - i\,\pi\right), \qquad \text{for} \quad s = -p^2 \to \infty. \tag{263}$$

It is perhaps worth mentioning that the re-summation in Eq.(257) will remain valid also when QED is embedded into the Standard

Model, but only as long as we limit ourselves to the inclusion of fermion loops. There are problems, however, with boson loops that are not gauge-invariant by themselves.

We also present the $V_1^{\text{ren}}\left(Q^2; m, m\right)$ once more in an integral form:

$$V_1^{\text{ren}}\left(Q^2; m, m\right) = 2\left(\frac{1}{\hat{\varepsilon}} + \ln\frac{m^2}{\mu^2}\right)\left[1 - \left(Q^2 + 2m^2\right)\int_0^1 dx \frac{1}{\chi\left(Q^2, x\right)}\right] \quad (264)$$
$$-2\left(Q^2 + 2m^2\right)\int_0^1 dx \frac{1}{\chi\left(Q^2, x\right)} \ln\frac{\chi\left(Q^2, x\right)}{m^2}$$
$$-\int_0^1 dx \ln\frac{\chi\left(Q^2, x\right)}{m^2} + 2\left(Q^2 + 3m^2\right)\int_0^1 dx \frac{1}{\chi\left(Q^2, x\right)} - 6,$$

in order to emphasize that there remains a pole and a scale-dependent factor:

$$\frac{1}{\hat{\varepsilon}} + \ln\frac{m^2}{\mu^2}, \quad (265)$$

which has an infrared origin and which will be compensated for in any realistic calculation by the contribution of the real *soft photons* emission and also by the *box* diagrams, which are ultraviolet finite by themselves.

7.4 The $\overline{MS}$ renormalization scheme.

The main motivation of this subsection will consist in carrying out the one-loop renormalization programme by prescribing precisely what the counter-terms are, after which the parameters of the Lagrangian are determined from some set of experiments.

We will make contact with the $\overline{MS}$ renormalization scheme where we start by computing the ultraviolet singularities of the one-loop diagrams for defining the counter-terms. These will include self-energy diagrams as well as vertices, since boxes in QED are free from ultraviolet poles. The residues of the pole at $n = 4$ are listed in the following where we adopt the general strategy of the $\overline{MS}$ scheme, where not only the pole but also the various factors containing γ and $\ln\pi$, i.e. $\bar{\varepsilon}$, are renormalized away. Thus, in $\overline{MS}$ the singular parts are subtracted and the parameters are defined at an arbitrary scale. This scheme has its natural habitat in QCD where, because of confinement, there is no special mass scale in the renormalization procedure.

Actually our main emphasis in this section will rather be on the fact that, in principle, any value could be assigned to the constant C in Eq.(225) since physical observables will not depend on any particular choice of C, assuming that gauge invariance is preserved, or, in other words, that no ad hoc procedure spoils the underlying cancellations in the theory.

So, the residues of the pole at $n = 4$ are

- $\Pi_{\mu\nu}$, the photon self-energy:

$$\mathrm{PP}\left[\Pi_{\mu\nu}\right] = (2\pi)^4\, i\, \frac{e^2}{12\,\pi^2}\, \left(p_\mu p_\nu - p^2 \delta_{\mu\nu}\right)\, \frac{1}{\bar{\varepsilon}}. \tag{266}$$

- Σ, the electron self-energy:

$$\mathrm{PP}\left[\Sigma\right] = -\,(2\pi)^4\, i\, \frac{e^2}{16\,\pi^2}\, \left(i\not{p} + 4\,m\right)\, \frac{1}{\bar{\varepsilon}}. \tag{267}$$

- Λ_μ, the $e^+e^-\gamma$ vertex:

$$\mathrm{PP}\left[\Lambda_\mu\right] = -\,(2\pi)^4\, i\, (ie)\, \frac{e^2}{16\,\pi^2} \gamma_\mu\, \frac{1}{\bar{\varepsilon}}. \tag{268}$$

The structure of the divergences fixes the renormalization constants up to a choice of C that we fix according to the $\overline{MS}$ renormalization scheme.

First, we deal with the photon propagator. From Eq.(240):

$$\mathrm{PP}\left[\Pi_0\right] = -\,(2\pi)^4\, i\, \frac{e^2}{12\,\pi^2}\, \frac{1}{\bar{\varepsilon}}. \tag{269}$$

The renormalization in this case amounts to the requirement that

$$\delta Z_{A} = -\frac{1}{12\,\pi^2}\, \frac{1}{\bar{\varepsilon}}. \tag{270}$$

We now consider the dressed electron propagator, Eq.(242), with

$$\mathrm{PP}\left[\frac{\Sigma}{(2\pi)^4\, i}\right] = -\frac{e^2}{16\,\pi^2}\left[3\,m + (i\not{p} + m)\right]\frac{1}{\bar{\varepsilon}}. \qquad (271)$$

The denominator of $\mathcal{S}$ becomes, for $i\not{p} \to -m$

$$\begin{aligned}\left[(2\pi)^4\, i\, S\right]^{-1} &= i\not{p} + m + e^2\delta S,\\ \delta S &= \delta Z_\psi\ (i\not{p} + m) + \delta m + \frac{3}{16\,\pi^2} m\frac{1}{\bar{\varepsilon}} + \frac{1}{16\,\pi^2}\,(i\not{p} + m)\ \frac{1}{\bar{\varepsilon}},\end{aligned}$$

which we require to be $-i\not{p} + m$. This gives

$$\delta m = -\frac{3}{16\,\pi^2} m\,\frac{1}{\bar{\varepsilon}}, \qquad \delta Z_\psi = -\frac{1}{16\,\pi^2}\,\frac{1}{\bar{\varepsilon}}. \qquad (272)$$

From the $e^+e^-\gamma$ vertex, Eq.(245), which we require to be $-\,(2\pi)^4\, i\, ie\gamma_\mu$, we obtain another counter-term:

$$\delta Z_e = -\frac{1}{2}\,\delta Z_{A} - \delta Z_\psi - \frac{1}{16\,\pi^2}\,\frac{1}{\bar{\varepsilon}} = \frac{1}{24\,\pi^2}\,\frac{1}{\bar{\varepsilon}}. \qquad (273)$$

As a result, the $\overline{MS}$ renormalized QED Lagrangian becomes

$$\begin{aligned}\mathcal{L}^{\mathrm{R}}_{\mathrm{QED}} &= -\frac{1}{4}\left(1 - \frac{e^2}{12\pi^2}\frac{1}{\bar{\varepsilon}}\right) F_{\mu\nu}F_{\mu\nu} - \frac{1}{2}\left(1 - \frac{e^2}{12\pi^2}\frac{1}{\bar{\varepsilon}}\right)(\partial_\mu A_\mu)^2 \qquad (274)\\ &\quad - \left(1 - \frac{e^2}{16\pi^2}\frac{1}{\bar{\varepsilon}}\right)\overline{\psi}\not{\partial}\psi - m\left(1 - \frac{e^2}{4\pi^2}\frac{1}{\bar{\varepsilon}}\right)\overline{\psi}\psi - ie\left(1 - \frac{e^2}{16\pi^2}\frac{1}{\bar{\varepsilon}}\right)A_\mu\overline{\psi}\gamma_\mu\psi.\end{aligned}$$

In this way, all one-loop Green functions of QED in the $\xi = 1$ gauge have been made finite and the renormalized parameters are subsequently fixed by comparing with some set of experimental data points, noticeably the electron mass and the fine structure constant.

For instance we start from the Lagrangian Eq.(274) and compute the residue of the pole at $p^2 = 0$ in $e^2 D_{\mu\nu}$. The electric charge is defined through the coefficient of the pole at zero momentum transfer of the scattering between two charged particles. From the definition of the fine structure constant, $e^2 = 4\pi\alpha$, we obtain

$$e^2 = \frac{4\,\pi\alpha}{1 + \dfrac{\alpha}{3\,\pi}\,\ln\dfrac{m^2}{\mu^2}}. \qquad (275)$$

Then $e^2 D_{\mu\nu}$ for arbitrary p^2 is considered, where according to Eq.(270) and Eq.(237)

$$\Pi\left(p^2\right) + 4\pi^2 \delta Z_{_A} = \frac{1}{9} + \frac{1}{3}\,\left(1 - 2\,\frac{m^2}{p^2}\right) \int_0^1 \,dx\,\ln\frac{\chi\left(p^2, x\right)}{m^2}. \tag{276}$$

The physically relevant object is the effective electric charge, Eq.(257), for which we need

$$\Pi\left(p^2\right) - \frac{1}{3}\,\ln\frac{m^2}{\mu^2}. \tag{277}$$

In this way, it becomes clear why the physical building blocks are identical to those in the OMS scheme. The main reason is that in computing Feynman integrals we always have the combination

$$\frac{1}{\overline{\varepsilon}} - \ln\frac{\text{scale}^2}{\mu^2}, \tag{278}$$

and in any observable—where ultraviolet infinities cancel—evaluated at a given scale the renormalization condition replaces the μ-dependence with a physical scale. In the effective electric charge this replacement is $\mu \to m$. In short, while $\Pi\left(p^2\right)$ is μ-dependent in the $\overline{MS}$ scheme the whole scale dependence disappears in $\Pi^{\text{ren}}\left(p^2\right)$.

The same will remain true for the renormalized vertex, Eq.(260). Indeed, from the definition of the QED vertex, Eq.(100) we see that the ultraviolet pole is Q^2-independent; thus, in its renormalization the μ- (ultraviolet) dependence will drop out.

7.5 Parameter renormalization and the $\mathcal{S}$-matrix: the GMS framework

To continue our discussion of the renormalization procedures within QED we turn to parameter renormalization, Eq.(211), with wave-function renormalization factors for external on-shell particles. This procedure, as we have already stressed, will be enough for dealing with finite $\mathcal{S}$-matrix elements.

The starting point will be the following Lagrangian:

$$\begin{aligned}\mathcal{L}^{\text{R}}_{\text{QED}} &= -\frac{1}{4}\,F_{\mu\nu}F_{\mu\nu} - \frac{1}{2}\left(\partial_\mu A_\mu\right)^2 - \overline{\psi}\left(\partial\!\!\!/ + m\right)\psi - e^2\delta m\overline{\psi}\psi \\ &\quad -ieA_\mu\overline{\psi}\gamma_\mu\psi - i\delta Z_e e^3 A_\mu\overline{\psi}\gamma_\mu\psi.\end{aligned} \tag{279}$$

In this subsection the finite parts of all diagrams will always be included. For the electron propagator we have

$$S = \frac{1}{(2\pi)^4\, i} \left[i\not{p} + m + e^2\delta m - \frac{\Sigma\left(\not{p}\right)}{(2\pi)^4\, i} \right]^{-1}. \tag{280}$$

We can rewrite it as a Taylor expansion around $i\not{p} = -m$

$$\begin{aligned} \left[(2\pi)^4\, iS\right]^{-1} &= i\not{p} + m + e^2\delta m - \delta S, \\ \delta S &= \frac{1}{(2\pi)^4\, i} \left[\Sigma\left(im\right) + \left(i\not{p} + m\right)\, \Sigma_{\text{WF}} + \mathcal{O}\left(\left(i\not{p} + m\right)^2\right) \right]. \end{aligned}$$

S will show a pole at $i\not{p} = -m$, where m is the physical electron mass. Thus, mass renormalization should be as follows:

$$e^2\delta m = \frac{1}{(2\pi)^4\, i}\, \Sigma\left(im\right), \tag{281}$$

and the electron propagator S becomes

$$S = \frac{1}{(2\pi)^4\, i} \left\{ \left(1 - \frac{1}{(2\pi)^4\, i} \left[\Sigma_{\text{WF}} + \mathcal{O}\left(i\not{p} + m\right) \right] \right) \left(i\not{p} + m\right) \right\}^{-1}. \tag{282}$$

The proper renormalization of the electron wave function requires us to consider the introduction of Dirac spinors and a limit for on-shell electrons. It gives

$$\bar{u} S u \;=\; \frac{1}{(2\pi)^4\, i}\, \bar{u}\, Z^{-1/2}\, \frac{1}{i\not{p} + m} Z^{-1/2}\, u, \qquad \text{for} \quad p^2 \to -m^2. \tag{283}$$

As before, we have defined a Z factor as

$$Z = 1 - \frac{\Sigma_{\text{WF}}}{(2\pi)^4\, i}, \tag{284}$$

Consider now any amplitude $\mathcal{M}$ with an external electron line, then the corresponding $\mathcal{S}$-matrix element becomes

$$Z^{1/2}\, \bar{u}\, \frac{i\not{p} + m}{Z\ \left(i\not{p} + m\right) + \mathcal{O}\left(\left(i\not{p} + m\right)^2\right)}\, \mathcal{M} \to Z^{-1/2} \bar{u}\, \mathcal{M} = \bar{u} \left[1 + \frac{1}{2}\, \frac{1}{(2\pi)^4\, i} \Sigma_{\text{WF}} \right] \mathcal{M}. \tag{285}$$

Substituting the corresponding expressions for the B functions, we obtain

$$\frac{\Sigma_{\text{WF}}}{(2\pi)^4\, i} = \frac{e^2}{16\, \pi^2} \left(-\frac{1}{\bar{\varepsilon}} + \ln\frac{m^2}{\mu^2} - 4 + \Sigma_{\text{IR}} \right). \tag{286}$$

There are several possible realizations for the infrared part, according to the adopted regularization, but here we just use dimensional regularization:

$$\Sigma_{\rm IR} = \frac{2}{\hat{\varepsilon}} + 2\,\ln\frac{m^2}{\mu^2}. \tag{287}$$

It is instructive to compare $\Sigma_{\rm WF}$, Eqs.(286) and (287), with δZ_ψ derived in Eq.(251).

For the $e^+e^-\gamma$ vertex we use again the definition, Eq.(90). The V_2 part does not contribute to renormalization and for V_1 we could use some general result but, given the inherent simplicity of QED, we simply refer to Eq.(101). For $Q^2 = 0$ we obtain

$$\begin{aligned}\Lambda\left(Q^2\right) &= (2\pi)^4\,i\,\frac{i\,e^3}{16\,\pi^2}\,V_1\left(Q^2;m,m\right),\\ \Lambda\left(0\right) &= (2\pi)^4\,i\,\frac{i\,e^3}{16\,\pi^2}\left(-\frac{1}{\bar{\varepsilon}} + \ln\frac{m^2}{\mu^2} - 4 + 2\,\Sigma_{\rm IR}\right).\end{aligned} \tag{288}$$

Now vertex corrections and fermion wave-function factors are combined and we are naturally led to consider everything in the limit of zero momentum transfer where the residue of the pole in the scattering of two charged particles defines the fine structure constant. We have again verified, by explicit calculation, the Ward identity

$$\Lambda\left(0\right) - ie\,\Sigma_{\rm WF} \equiv 0. \tag{289}$$

Therefore, only the photon self-energy contributes to the electric charge renormalization when we impose the renormalization condition at zero momentum transfer. Note that both contributions, vertex and fermion wave-function factor, are separately infrared-divergent. The dressed photon propagator, which happens to be infrared finite, becomes

$$e^2\,D_{\mu\nu} = \frac{1}{(2\pi)^4\,i}\,\frac{\delta_{\mu\nu}}{p^2}\,e^2\left[1 - \frac{e^2}{4\,\pi^2}\Pi\left(p^2\right)\right]^{-1}, \tag{290}$$

where $\Pi\left(p^2\right)$ is defined by Eq.(237). For our choice of the renormalization condition it follows that

$$4\,\pi\alpha\left(0\right) = e^2\left[1 - \frac{e^2}{4\,\pi^2}\,\Pi\left(0\right)\right]^{-1}. \tag{291}$$

Substituting back in the photon propagator we find the well-known phenomenon of the evolution of coupling constants in field theory. We actually find more, the μ dependence cancels and we find exactly the same result as in any renormalization scheme:

$$e^2 D_{\mu\nu} = \frac{1}{(2\pi)^4 i}\frac{\delta_{\mu\nu}}{p^2}\,4\,\pi\alpha\left(p^2\right), \quad \alpha\left(p^2\right) = \alpha\left(0\right)\left[1 - \frac{\alpha\left(0\right)}{\pi}\,\Pi^{\rm ren}\left(p^2\right)\right]^{-1}, \tag{292}$$

where $\Pi^{\rm ren}\left(p^2\right)$ is given by Eq.(258).

This is exactly Eq.(257). Clearly, all divergences and scales drop out in the difference since—this is really the crucial point—they do not depend on p^2.

Recall now Eq.(238). In QED, as in any other renormalizable theory, the infinities cancel after renormalization in any physical observable. Therefore, we can re-formulate the theory by setting everywhere $1/\bar{\varepsilon}$ to zero and by promoting the bare parameters to $\overline{MS}$ parameters. In other words: *defining an $\overline{MS}$ parameter is equivalent to adopting the heuristic rule (valid at one-loop)*

$$\frac{1}{\bar{\varepsilon}} + \ln\mu^2 \to \ln\mu^2_{\overline{MS}}, \tag{293}$$

in the relation expressing the bare parameters in terms of the renormalized ones. Thus,

$$e^2_{\overline{MS}}\left(\mu^2\right) = 4\,\pi\alpha\left(0\right)\left[1 - \frac{\alpha\left(0\right)}{3\,\pi}\ln\frac{\mu^2_{\overline{MS}}}{m^2}\right]^{-1} \approx 4\,\pi\alpha\left(0\right)\left[1 + \frac{\alpha\left(0\right)}{3\,\pi}\ln\frac{\mu^2_{\overline{MS}}}{m^2}\right]. \tag{294}$$

Obviously, $\alpha\left(p^2\right)$ and $e^2_{\overline{MS}}$ are different objects and only the former has a physical interpretation, while the latter is nothing more than a convenient way of expressing the bare parameters of a renormalizable theory, since it is universal; that is, process independent, and takes into account the universal large effects from fermion loops.

- Note that we could start from a QED Lagrangian without counter-terms and relate the bare parameters directly to the experimental data. Indeed, nowhere from Eq.(290) to Eq.(292) is the notion of counter-term actually needed. Certainly, the bare parameters have no physical meaning but, on the other hand, relations between measurable physical quantities, where the parameters drop out, are finite and it is therefore possible to perform tests of the theory in terms of such relations by eliminating the bare parameters.

By virtue of the renormalizability of the theory, all divergences drop out in the final answer. What is left, for practical convenience, is the introduction of intermediate parameters like $e^2_{\overline{MS}}(\mu^2)$. The only meaningful quantity will be the running coupling constant, Eqs.(257) and (292).

7.6 Gauge dependence and renormalization

It is, of course, important to verify the gauge independence of the $\mathcal{S}$-matrix and for that we reconsider some of the building blocks in a general R_ξ QED-like gauge where the only modification occurs for the photon propagator

$$\frac{1}{p^2}\left[\delta_{\mu\nu} + \left(\xi_{_A}^2 - 1\right)\,\frac{p_\mu p_\nu}{p^2}\right]. \tag{295}$$

At one-loop the photon self-energy remains unchanged, since the diagram of Fig. 3 is manifestly gauge-invariant, while the fermion self-energy receives an additional contribution

$$\Sigma = \Sigma^{(1)} + \left(\xi_{_A}^2 - 1\right)\,\Delta\Sigma, \tag{296}$$

The extra piece is given by

$$\Delta\Sigma = -\,(2\pi)^4\, i\,\frac{e^2}{16\,\pi^2}[(B_0 - B_1)\, i\,\not{p} + m\,B_0 - 2\,\left(p^2 b_{21} + b_{22}\right)\, i\,\not{p}]. \tag{297}$$

Making use of the relations among the B_{ij}-functions and the b_{ij}-functions we arrive at

$$\Delta\Sigma = -\,(2\pi)^4\, i\,\frac{e^2}{16\,\pi^2}\,(i\,\not{p} + m)\,[B_0\,(p;0,m) + (-i\,\not{p} + m)\,\, i\,\not{p}\, b_1\,(p;m)]. \tag{298}$$

From Eq.(298) it is immediately seen that the additional contribution vanishes on-mass-shell, therefore the mass counter-term will be independent of ξ_A. The same is not true for the wave-function renormalization factor or, which is equivalent, for Z_ψ. Indeed, we find that

$$\Delta\Sigma_{\rm WF} = -\,(2\pi)^4\, i\,\frac{e^2}{16\,\pi^2}\,\left[B_0\left(-m^2;0,m\right) - 2\,m^2 b_1\left(-m^2;m\right)\right]. \tag{299}$$

The term in brackets is easily evaluated and gives

$$B_0\left(-m^2;0,m\right) - 2\,m^2 b_1\left(-m^2;m\right) = \frac{1}{\bar{\varepsilon}} + \frac{1}{\hat{\varepsilon}}, \tag{300}$$

where we have used the relation $b_1\left(-m^2;m\right) = -B_{0p}\left(-m^2;0,m\right)$. Thus, the gauge-dependent addition to the wave-function renormalization factor, or Z_ψ, is neither infrared finite nor gauge independent. For the vertex corrections we find that

$$\Lambda\left(Q^2\right) = \Lambda^{(1)}\left(Q^2\right) + (2\pi)^4\, i\,\frac{i\,e^3}{16\,\pi^2}\left(\xi_A^2 - 1\right)\Delta\Lambda\left(Q^2\right), \tag{301}$$

with an extra factor

$$\Delta\Lambda\left(Q^2\right) = -n\,b_{22}\left(-m^2;m\right) + m^2\left[b_{21}\left(-m^2;m^2\right) + 2\,b_1\left(-m^2;m\right)\right]. \tag{302}$$

After using again the properties of the b_{ij} functions we end up with

$$\Delta\Lambda\left(Q^2\right) = -B_0\left(-m^2;0,m\right) + 2\,m^2 b_1\left(-m^2;m\right). \tag{303}$$

In this way we can prove that the usual Ward identity

$$\Lambda\,(0) - ie\,\Sigma_{\rm WF} \equiv 0 \tag{304}$$

is satisfied for arbitrary ξ_A and that, in turn, the Z_e factor is gauge independent.

Therefore, one obtains that the physical parameters e and m in QED are renormalized in a gauge-independent way.

8 The Standard Model vector-boson self-energies.

For the $\delta_{\mu\nu}$ part of the vector–vector transitions all the results can be cast in the following form:

$$S_{\gamma\gamma} \;=\; (2\pi)^4\, i\,\frac{g^2 s_\theta^2}{16\,\pi^2}\,\Pi_{\gamma\gamma}(p^2)\,p^2, \quad S_{ZZ} = (2\pi)^4\, i\,\frac{g^2}{16\pi^2 c_\theta^2}\,\Sigma_{ZZ}(p^2),$$

$$
\begin{aligned}
S_{Z\gamma} &= (2\pi)^4\, i\, \frac{g^2 s_\theta}{16\pi^2 c_\theta}\, \Sigma_{ZA}(p^2), \\
S_{WW} &= (2\pi)^4\, i\, \frac{g^2}{16\pi^2}\, \Sigma_{WW}(p^2), \qquad S_{HH} = (2\pi)^4\, i\, \frac{g^2}{16\pi^2}\, \Sigma_{HH}(p^2), \quad (305)
\end{aligned}
$$

where we have transformed from the (A, Z) basis to the $(3, Q)$ basis, defined by:

$$
\begin{aligned}
\Sigma_{ZZ}(p^2) &= \Sigma_{33}(p^2) - 2s_\theta^2 \Sigma_{3Q}(p^2) + s_\theta^4 \Pi_{\gamma\gamma}(p^2)\, p^2, \\
\Sigma_{ZA}(p^2) &= \Sigma_{3Q}(p^2) - s_\theta^2 \Pi_{\gamma\gamma}(p^2)\, p^2. \qquad (306)
\end{aligned}
$$

Within the SM and in the $\xi = 1$ gauge we also have the following identities, expressing the results for total transitions:

$$
\begin{aligned}
\Pi_{\gamma\gamma}(p^2) &= \frac{2}{3} - 12 B_{21}\left(p^2; M, M\right) + 7 B_0\left(p^2; M, M\right) + 4 \sum_f c_f Q_f^2 B_f\left(p^2; m_f, m_f\right), \\
\Sigma_{3Q}(p^2) &= p^2 \left[\frac{2}{3} - 10 B_{21}\left(p^2; M, M\right) + \frac{13}{2} B_0\left(p^2; M, M\right)\right] \\
&\quad -2M^2 B_0\left(p^2; M, M\right) + p^2 \sum_f c_f |Q_f| B_f\left(p^2; m_f, m_f\right). \qquad (307)
\end{aligned}
$$

The most important observation here is that $\Sigma_{3Q}(0)$ is not zero and so it will be the Z–γ transition that is most important for the electric charge renormalization. Next we split the self-energies into a p^2-proportional part and the rest, according to the following definition:

$$
\Sigma_{33}(p^2) = \Pi_{33}(p^2)\, p^2 + \Sigma_{33}^r(p^2), \quad \Sigma_{WW}(p^2) = \Pi_{WW}(p^2)\, p^2 + \Sigma_{WW}^r(p^2). \quad (308)
$$

The various components are given by:

$$
\begin{aligned}
\Pi_{33}(p^2) &= \frac{2}{3} - 9 B_{21}\left(p^2; M, M\right) + \frac{25}{4} B_0\left(p^2; M, M\right) - B_{21}\left(p^2; M_0, M_H\right) \quad (309) \\
&\quad -B_1\left(p^2; M_0, M_H\right) - \frac{1}{4} B_0\left(p^2; M_0, M_H\right) + \frac{1}{2} \sum_f c_f B_f\left(p^2; m_f, m_f\right), \\
\Sigma_{33}^r(p^2) &= -2M^2 B_0\left(p^2; M, M\right) + \frac{1}{2}\left(M_0^2 - M_H^2\right) B_1\left(p^2; M_0, M_H\right) \quad (310) \\
&\quad +\frac{1}{4}\left(5M_0^2 - M_H^2\right) B_0\left(p^2; M_0, M_H\right) - \frac{1}{2} \sum_f c_f m_f^2 B_0\left(p^2; m_f, m_f\right), \\
\Pi_{WW}(p^2) &= \frac{2}{3} - 9 B_{21}\left(p^2; M_0, M\right) - 9 B_1\left(p^2; M_0, M\right) + \frac{7}{4} B_0\left(p^2; M_0, M\right) \\
&\quad -B_{21}\left(p^2; M, M_H\right) - B_1\left(p^2; M, M_H\right) - \frac{1}{4} B_0\left(p^2; M, M_H\right)
\end{aligned}
$$

$$
\begin{aligned}
&+s_\theta^2[8B_{21}\left(p^2;M_{_0},M\right)-2B_0\left(p^2;M_{_0},M\right)+8B_1\left(p^2;M_{_0},M\right)\\
&-8B_{21}\left(p^2;0,M\right)-8B_1\left(p^2;0,M\right)+2B_0\left(p^2;0,M\right)]\\
&+\sum_{f=d}c_f B_f\left(p^2;m_{f'},m_f\right), \qquad (311)
\end{aligned}
$$

$$
\begin{aligned}
\Sigma^r_{_{WW}}(p^2) \;=\;& \frac{9}{2}\left(M_{_0}^2-M^2\right)B_1\left(p^2;M_{_0},M\right)+\frac{1}{4}\left(13M_{_0}^2-21M^2\right)B_0\left(p^2;M_{_0},M\right)\\
&+\frac{1}{2}\left(M^2-M_{_H}^2\right)B_1\left(p^2;M,M_{_H}\right)+\frac{1}{4}\left(5M^2-M_{_H}^2\right)B_0\left(p^2;M,M_{_H}\right)\\
&+s_\theta^2\{2\left(M^2-M_{_0}^2\right)[2B_1\left(p^2;M_{_0},M\right)+B_0\left(p^2;M_{_0},M\right)] \qquad (312)\\
&-2M^2[2B_1\left(p^2;0,M\right)+B_0\left(p^2;0,M\right)]\}+\sum_f c_f m_f^2 B_1\left(p^2;m_{f'},m_f\right).
\end{aligned}
$$

It is worth presenting here the fermionic component of the Higgs boson self-energy,

$$
\Sigma_{_{HH}}(p^2)=\sum_f c_f\frac{m_f^2}{M_{_W}^2}\Big[A_0\left(m_f\right)-\frac{p^2+4m_f^2}{2}B_0\left(p^2;m_f,m_f\right)\Big]. \qquad (313)
$$

Note that the fermionic component of the Higgs–vector boson transition vanishes identically as it should, since it is proportional to

$$
[B_0\left(p^2;m_f,m_f\right)+2B_1\left(p^2;m_f,m_f\right)]p_\mu . \qquad (314)
$$

In the above expressions we have not yet included tadpoles for the W–W and for Z–Z transitions. To understand that no real problem is hidden in tadpoles we will say that in all physical observables we encounter combinations like $\Sigma_{WW}(p^2)-\Sigma_{WW}(q^2)$, where tadpoles drop out, or like $\Sigma_{WW}(M_W^2)-\Sigma_{ZZ}(M_Z^2)$ where again tadpoles drop out and where the combination is gauge-parameter independent. If we do not add tadpoles, then the combination is still gauge independent but the two pieces are not separately independent.

9 Fermion wave-function renormalization

The fermionic self-energies are the building blocks used for the evaluation of wave-function renormalization factors. After writing the parameterization

$$
\Sigma(i\not{p})=(2\pi)^4\,i\,[a_1+a_2\,\gamma_5+(a_3-a_4\,\gamma_5)\,i\not{p}] \qquad (315)
$$

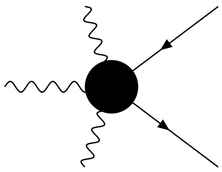


Figure 12: Treatment of external fermion line in an $\mathcal{S}$-matrix element.

and using derivatives, $a'_i = \partial a_i/\partial p^2|_{p^2=-m^2}$, we obtain the wave-function factors shown in Table 1, with

$$W_V = \frac{1}{2}a_3 - m^2 a'_3 + m a'_1\,, \qquad W_A = \frac{1}{2}a_4\,. \tag{316}$$

In the SM we always have $a_2 = 0$. In going from a Green function to the corresponding $\mathcal{S}$-matrix element the wave-function renormalization factors enter multiplicatively.

9.1 Wave-function renormalization factors in the $\xi = 1$ gauge.

In what follows we present the explicit expressions for the quantities W_V^1 and W_A^1, in the $\xi = 1$ gauge. The superscript (1) will be dropped since no confusion is created. It is convenient to split W_V into a QED and a non-QED part, $W_V = W_V^{\rm e.m.} + W_V^{\rm w}$; that is,

Table 1: Wave-function renormalization factors for fermionic lines

$1 + W_V + W_A\gamma_5$	incoming particle
$1 + W_V - W_A^*\gamma_5$	outgoing particle
$1 + W_V + W_A\gamma_5$	outgoing anti-particle
$1 + W_V - W_A^*\gamma_5$	incoming anti-particle

to separate out photonic (em) and purely weak (w) components.

We derive for the electromagnetic part:

$$W_{_V}^{\rm e.m.} = \frac{g^2 s_\theta^2 Q_f^2}{16\pi^2}\Big\{B_1\left(m_f,0\right) - 2m_f^2 B_{1p}\left(m_f,0\right) - 4m_f^2 B_{0p}\left(m_f,0\right) + \frac{1}{2}\Big\}, \quad (317)$$

and for the remaining (weak) factors:

$$\begin{aligned} W_{_V}^{\rm w} &= \frac{g^2}{64\pi^2}\Big\{\frac{m_f^2}{2M^2}\sum_{M=M_{_0},M_{_H}}\left[B_1\left(m_f,M\right) - 2m_f^2 B_{1p}\left(m_f,M\right)\right] \qquad (318)\\ &+\frac{1}{c_\theta^2}\sigma_f^{(2)}\left[B_1\left(m_f,M_{_0}\right) - 2m_f^2 B_{1p}\left(m_f,M_{_0}\right)\right]\\ &+\Big(\frac{m_+^2}{2M^2}+1\Big)\left[B_1\left(m_{f'},M\right) - 2m_f^2 B_{1p}\left(m_{f'},M\right)\right]\\ &-2\frac{m_f^2 m_{f'}^2}{M^2}B_{0p}\left(m_{f'},M\right) - \frac{4}{c_\theta^2}\delta_f^{(2)} m_f^2 B_{0p}\left(m_f,M_{_0}\right) + \frac{1}{2c_\theta^2}\sigma_f^{(2)} + \frac{1}{2}\Big\},\\ W_{_A} &= \frac{g^2}{64\pi^2}\Big\{\frac{v_f a_f}{c_\theta^2}\left[2B_1\left(m_f,M_{_0}\right)+1\right] + B_1\left(m_{f'},M\right) + \frac{1}{2} - \frac{m_-^2}{2M^2}B_1\left(m_{f'},M\right)\Big\}, \end{aligned}$$

with $p^2 = -m_f^2$.

10 The Standard Model $Vf\overline{f}$ vertices

One of the essential ingredients of any calculation of radiative corrections in the SM is given by the three-point vertex functions. With different external and internal lines they will enter into the calculation of decay rates like

$$Z, H \to f\overline{f}, \qquad W \to \overline{u}d(u\overline{d}), \qquad (319)$$

and of processes that share a special relevance for the renormalization procedure, namely

$$e^-\mu^- \to e^-\mu^-, \qquad (\text{at} \quad Q^2 = 0), \qquad \text{or} \quad \mu \to e\overline{\nu}_e\nu_\mu. \qquad (320)$$

Finally, there are distributions for various processes:

$$e^+e^- \to e^+e^-, f\overline{f}, W^+W^-, \gamma\gamma, Z\gamma, ZZ, HZ. \qquad (321)$$

In this section we examine a particular class of vertices defined by having the structure $V(S) \to f\overline{f}$, i.e. vector (scalar) into fermion pairs. We do it first in the 't Hooft–Feynman $\xi = 1$ gauge, and extend the results to arbitrary ξ including the unitary gauge.

Every diagram is expressible as the sum of an appropriate number of Lorentz structures $\otimes$ scalar form factors bearing some sub- and superscripts.

10.1 $Vf\overline{f}$ vertices in the 't Hooft–Feynman gauge and in the massless limit

For the $\xi = 1$ gauge and in the limit where we ignore fermion masses, all diagrams can be classified according to their internal lines in a unique way and only two types will occur, the F_1VF_2 abelian type and the V_1FV_2 non-abelian type, where different F or V internal fields are only required in W-decay. If the top quark occurs in the final state, then H-lines or ϕ-lines will be present and we will also have FHF, etc. structures.

The most general vector boson–fermion–antifermion coupling,

$$V(Q) \to f(-p_1)\overline{f}(-p_2), \tag{322}$$

can be reduced to a combination of six form factors (for vector-like structures):

$$\begin{aligned} V_\mu\left(Q^2\right) &= (2\pi)^4\, i\frac{ig^3}{16\pi^2}[F_{_V}\gamma_\mu + F_{_A}\gamma_\mu\gamma_5 + F_{_M}\sigma_{\mu\nu}Q_\nu \\ &\quad +F_{_S}Q_\mu + F_{_P}\gamma_5 Q_\mu + F_{_E}\gamma_5\left(p_1-p_2\right)_\mu], \end{aligned} \tag{323}$$

with $Q + p_1 + p_2 = 0$ and $p_{+\mu} = (p_1 - p_2)_\mu\,,\ \ p_{-\mu} = (p_1 - p_2)_\mu$.

For the neutral current sector there are 14 diagrams of this kind and only 10 for $\gamma f\overline{f}$, while we have 18 of them for charged currents and 15 for the $Hf\overline{f}$ vertex.

In this subsection we are mostly interested in the limit of small fermion masses where only $F_{_V}$ and $F_{_A}$ contribute. They can be computed starting from the scalar three-point functions. Another way of representing the vertex corrections, always in the massless fermion limit, is

$$\begin{aligned} V_\mu\left(Q^2\right) &= (2\pi)^4\, i\frac{ig^3}{16\pi^2}[F_{_Q}\gamma_\mu + F_{_L}\gamma_\mu\left(1+\gamma_5\right)], \\ F_{_Q} &= F_{_V} - F_{_A}, \qquad F_{_L} = F_{_A}. \end{aligned} \tag{324}$$

We consider first the neutral current (hereafter NC) case, i.e. $\gamma, Z \to f\overline{f}$. Two vertices survive in the massless limit: in our terminology they are of the FVF or VFV type, and are shown in Fig. 13.

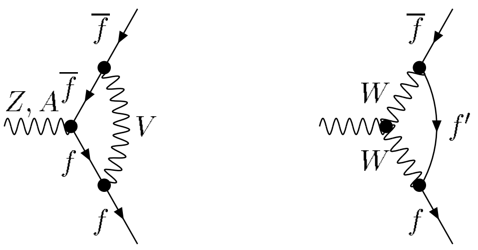


Figure 13: FVF and VFV vertices

The result depends on the V/A ratios for the vertices and these two different structures can be written as

$$\begin{aligned} V_\mu^{FVF}\left(Q^2\right) &= -\int \frac{d^n q}{d_0 d_1 d_2} \gamma_\alpha \left(\lambda_1 + \lambda_2\gamma_5\right) \not{q} \gamma_\mu \\ &\quad \times \left(\lambda_3 + \lambda_4\gamma_5\right) \left(\not{q} + \not{p}_1 + \not{p}_2\right) \gamma_\alpha \left(\lambda_1 + \lambda_2\gamma_5\right), \qquad (325) \\ V_\mu^{VFV}\left(Q^2\right) &= -i \int d^n q \gamma_\alpha \left(\lambda_1 + \lambda_2\gamma_5\right) \left(\not{q} + \not{p}_1\right) \gamma_\beta \left(\lambda_1 + \lambda_2\gamma_5\right) \frac{v_{\mu\alpha\beta}}{d_0 d_1 d_2}, \qquad (326) \end{aligned}$$

where $v_{\mu\alpha\beta}$ is the corresponding three-vector boson vertex and

$$d_0 = q^2 + m_1^2, \qquad d_1 = \left(q + p_1\right)^2 + m_2^2, \qquad d_2 = \left(q - Q\right)^2 + m_3^2. \quad (327)$$

FVF Configuration. For the first diagram we can obtain the final result by using

$$\begin{aligned} p_1^2 &= -m_f^2, \qquad p_2^2 = -m_f^2, \qquad \left(p_1 + p_2\right)^2 = Q^2, \\ m_1 &= m_f, \qquad m_2 = m, \qquad m_3 = m_f, \qquad \text{where} \quad m = 0, M_{_0}, \\ m_1 &= m_{f'}, \qquad m_2 = M, \qquad m_3 = m_{f'}, \qquad (328) \end{aligned}$$

where f' is the isospin partner of the f-fermion. We obtain for the abelian (a) case

$$F_{_{V,Ba}} = \kappa f_{_V} F_{_{Ba}}, \qquad F_{_{A,Ba}} = \kappa g_{_A} F_{_{Ba}}, \qquad (329)$$

where $B = (A, Z)$ and where κ is a coefficient coming from the internal vertices. Furthermore, we have introduced

$$f_{_V} = \left(\lambda_1^2 + \lambda_2^2\right) \lambda_3 + 2\,\lambda_1\lambda_2\lambda_4, \qquad g_{_A} = \left(\lambda_1^2 + \lambda_2^2\right) \lambda_4 + 2\,\lambda_1\lambda_2\lambda_3. \quad (330)$$

Ignoring again terms proportional to the fermion masses we find

$$F_{_{Ba}}\left(Q^2\right) = 4\,(1+n-4)\,C_{24} + 2\,Q^2\,(C_{11}+C_{23})\,. \qquad (331)$$

The pole part of C_{24} is easily computed and gives $\dfrac{1}{4}\dfrac{1}{\bar{\varepsilon}}$. Therefore, we have

$$F_{_{Ba}}\left(Q^2\right) = 4\left(C_{24} - \frac{1}{2}\right) + 2Q^2\,(C_{11}+C_{23})\,. \qquad (332)$$

Within the SM we have six possible diagrams of FVF abelian-type; the corresponding constants are listed in Table 2.

The higher-order C-functions, which appear in Eq.(332), can be reduced to purely scalar functions by using the results of the previous sections. However, this reduction simplifies considerably for massless fermions and in what follows we indicate the results. First, we introduce a subtracted B_0-function:

$$b_{ff}\left(Q^2\right) = B_0\left(Q^2;0,0\right) - B_0\,(0;0,M) = -\ln\left(\frac{|Q^2|}{M^2}\right) + 1 + i\,\pi\theta\left(-Q^2\right). \qquad (333)$$

In terms of this function and of the scalar C_0,

$$C_{_M}(Q^2) = C_0\left(0,0,Q^2;0,M,0\right), \qquad (334)$$

we obtain the complete list of integrals $C_{ij}\,(0,0,Q^2;0,M,0)$:

$$\begin{aligned}
C_{11} &= -\left(1+r^2\right)C_{_M}(Q^2) - \frac{1}{Q^2}b_{ff}\left(Q^2\right),\\
C_{23} &= -2r^2\left(1+r^2\right)C_{_M}(Q^2) - \frac{1+2r^2}{Q^2}b_{ff}\left(Q^2\right) - \frac{1}{2Q^2},\\
C_{24} &= \frac{1}{4}\Delta\,(M) + \frac{1}{2} + \frac{1}{2}r^2\left(1+r^2\right)Q^2C_{_M}(Q^2) + \frac{1}{4}\left(1+2r^2\right)b_{ff}\left(Q^2\right) \qquad (335)
\end{aligned}$$

where we have introduced $r^2 = -M^2/Q^2$.

VFV Configuration. For the second type of diagram we have

$$F_{_{V,Wn}} = \kappa f_{_V} F_{_{Wn}}, \qquad F_{_{A,Wn}} = \kappa g_{_A} F_{_{Wn}}, \qquad (336)$$

with the following assignment of variables:

$$\begin{aligned}
p_1^2 &= -m_f^2, & p_2^2 &= -m_f^2, & (p_1+p_2)^2 = Q^2,\\
m_1 &= m_3 = M, & m_2 &= m_{f'}.
\end{aligned}$$

Table 2: Constants of FVF abelian-vertices

	λ_3	λ_4	m_2	λ_1	λ_2	f_V	g_A	κ
$\gamma f\overline{f}$	1	0	0	1	0	1	0	$Q_f^3 s_\theta^3$
$\gamma f\overline{f}$	1	0	M_0	v_f	a_f	$\sigma_f^{(2)}$	$2v_f a_f$	$\frac{1}{4}Q_f\frac{s_\theta}{c_\theta^2}$
$\gamma f\overline{f}$	1	0	M	1	1	2	2	$\frac{1}{8}Q_{f'}s_\theta$
$Zf\overline{f}$	v_f	a_f	0	1	0	v_f	a_f	$\frac{1}{2}Q_f^2\frac{s_\theta^2}{c_\theta}$
$Zf\overline{f}$	v_f	a_f	M_0	v_f	a_f	$\sigma_f^{(2)}v_f + 2v_f a_f^2$	$\sigma_f^{(2)}a_f + 2v_f^2 a_f$	$\frac{1}{8}\frac{1}{c_\theta^3}$
$Zf\overline{f}$	$v_{f'}$	$a_{f'}$	M	1	1	$2\sigma_{f'}$	$2\sigma_{f'}$	$\frac{1}{16}\frac{1}{c_\theta}$

In this case, we easily find the important combinations as $f_V = \lambda_1^2 + \lambda_2^2$ and $g_A = 2\lambda_1\lambda_2$, so that we may write

$$\begin{aligned} F_{Wn}\left(Q^2\right) &= -4\left(3+n-4\right)\, C_{24} - 2Q^2\left(C_0 + C_{11} + C_{23}\right) \\ &= -12\left(C_{24} - \frac{1}{6}\right) - 2Q^2\left(C_0 + C_{11} + C_{23}\right). \end{aligned} \tag{337}$$

Within the SM we have $\lambda_{1,2} = 1$ while $\kappa = \dfrac{1}{4}\left(s_\theta, c_\theta\right)\left(-I_f^{(3)}\right)$ for the $(\gamma, Z)\, f\overline{f}$ vertices. For the VFV configuration and in the limit of massless fermions we introduce another subtracted B_0-function:

$$b_f\left(Q^2\right) = B_0\left(Q^2; M, M\right) - B_0\left(0; 0, M\right). \tag{338}$$

In terms of this function and of

$$C_{MM}(Q^2) = C_0\left(0, 0, Q^2; M, 0, M\right), \tag{339}$$

we obtain all $C_{ij}\left(0, 0, Q^2; M, 0, M\right)$-functions

$$C_{11} = \left(r^2 - 1\right) C_{MM}(Q^2) - \frac{1}{Q^2} b_f\left(Q^2\right),$$

$$C_{23} = r^2\left(1-2r^2\right)C_{_{MM}}(Q^2) - \frac{1-2r^2}{Q^2}b_f\left(Q^2\right) - \frac{1}{2Q^2},$$

$$C_{24} = \frac{1}{4}\Delta\left(M\right) + \frac{1}{2} + \frac{1}{2}r^4Q^2C_{_{MM}}(Q^2) + \frac{1}{4}\left(1-2r^2\right)b_f\left(Q^2\right). \quad (340)$$

CC Configuration. Also very important is the charged current (CC) case, a vertex with an external W boson line. We will give the explicit expression for the $W^+d\overline{u}$ couplings, where u and d denote arbitrary partners in the isodoublet. First, we have two abelian diagrams in the massless limit. They can be cast in the

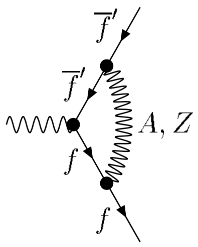


Figure 14: CC $F'VF$ vertices.

following form:

$$F_{_V} = F_{_A} = \frac{1}{\sqrt{2}}W^a_{ud}\left(Q^2\right), \quad (341)$$

where the common function of Q^2 is

$$W^a_{ud}\left(Q^2\right) = \frac{1}{2}Q_u\,Q_d s^2_\theta\,F_{_{Aa}}\left(Q^2\right) + \frac{1}{8\,c^2_\theta}\left(v_u + a_u\right)\,\left(v_d + a_d\right)\,F_{_{Za}}\left(Q^2\right). \quad (342)$$

A reduction of higher-order form factors is again achieved with the help of Eq.(332). The remaining four non-abelian diagrams that survive the massless limit are again written as

$$F_{_V} = F_{_A} = \frac{1}{\sqrt{2}}W^n_{ud}\left(Q^2\right), \quad (343)$$

where the function $W^n_{ud}\left(Q^2\right)$ is

$$W^n_{ud}\left(Q^2\right) = \left(v_u + a_u - v_d - a_d\right)F_{_{Zn}}\left(Q^2\right) + 2\left(Q_u - Q_d\right)s^2_\theta F_{_{An}}\left(Q^2\right). \quad (344)$$

The non-abelian form factors are expressible through the auxiliary function

$$V_{ab}\left(Q^2\right) = 3\,C_{24} - \frac{1}{2} + \frac{1}{2}Q^2(C_0 + C_{11} + C_{23}), \quad (345)$$

where all C_{ij} functions have arguments $\{0, 0, Q^2; M_a, 0, M_b\}$. The result for the form factors is

$$F_{Zn}\left(Q^2\right) = V_{WZ}\left(Q^2\right), \qquad F_{An}\left(Q^2\right) = V_{WA}\left(Q^2\right). \tag{346}$$

It can be easily proven that V_{ab} is symmetric in its two indices, i.e. $V_{WB} = V_{BW}$. In this case, the reduction is given by

$$\begin{aligned}
C_{11} &= \left(r_b^2 - 1\right) C_{ab}\left(Q^2\right) - \frac{1}{Q^2}[B_0\left(Q^2; M_a, M_b\right) - B_0\left(0; 0, M_a\right)], \\
C_{23} &= r_a^2\left(1 - 2\,r_b^2\right) C_{ab}\left(Q^2\right) + \frac{1}{Q^2}\,[\left(r_a^2 + r_b^2 - 1\right) B_0\left(Q^2; M_a, M_b\right) \\
&\quad - r_a^2 B_0\left(0; 0, M_a\right) + \left(1 - r_b^2\right) B_0\left(0; 0, M_b\right) - \frac{1}{2}], \\
C_{24} &= \frac{1}{2} Q^2\, r_a^2 r_b^2 C_{ab}\left(Q^2\right) + \frac{1}{4} + \frac{1}{4}\left(1 - r_a^2 - r_b^2\right) B_0\left(Q^2; M_a, M_b\right) \\
&\quad + \frac{1}{4}\,[r_a^2 B_0\left(0; 0, M_a\right) + r_b^2 B_0\left(0; 0, M_b\right)],
\end{aligned} \tag{347}$$

where we have introduced the universal function,

$$C_{ab}\left(Q^2\right) = C_0\left(0, 0, Q^2; M_a, 0, M_b\right), \qquad r_{a,b}^2 = -\frac{M_{a,b}^2}{Q^2}. \tag{348}$$

The auxiliary function becomes

$$\begin{aligned}
V_{ab}\left(Q^2\right) &= \frac{1}{2} Q^2\left(r_a^2 + r_b^2 + r_a^2 r_b^2\right) C_{ab}\left(Q^2\right) - \frac{1}{4}\left(1 + r_a^2 + r_b^2\right) B_0\left(Q^2; M_a, M_b\right) \\
&\quad + \frac{1}{4}\left(2 + r_a^2\right) B_0\left(0; 0, M_a\right) + \frac{1}{4}\left(2 + r_b^2\right) B_0\left(0; 0, M_b\right).
\end{aligned} \tag{349}$$

So far, we have discussed only the massless approximation. To go further with our derivation we need to consider the m_t-dependence.

10.2 The m_t-dependent part in $\xi = 1$

For massless fermions the $Zf\overline{f}$ couplings receive a correction factor,

$$\begin{aligned}
F_V &= \frac{1}{2} Q_f^2 \frac{s_\theta^2}{c_\theta} v_f F_{Aa}\left(Q^2\right) + \frac{1}{8\,c_\theta^3} v_f\left(v_f^2 + 3a_f^2\right)\, F_{Za}\left(Q^2\right) \\
&\quad + \frac{1}{8\,c_\theta}\left(v_{f'} + a_{f'}\right) F_{Wa}\left(Q^2\right) - \frac{1}{2} c_\theta I_f^{(3)} F_{Wn}\left(Q^2\right),
\end{aligned} \tag{350}$$

where $F_{_{Va}}(Q^2)$ and $F_{_{Wn}}(Q^2)$ are defined by Eq.(332) and Eq.(337), respectively. Similarly, we obtain

$$\begin{aligned}F_{_A} &= \frac{1}{2}Q_f^2\frac{s_\theta^2}{c_\theta}a_f F_{_{An}}(Q^2) + \frac{1}{8\,c_\theta^3}a_f\left(3v_f^2+a_f^2\right)F_{_{Za}}(Q^2)\\ &\quad +\frac{1}{8\,c_\theta}\left(v_{f'}+a_{f'}\right)F_{_{Wa}}(Q^2) - \frac{1}{2}c_\theta I_f^{(3)}F_{_{Wn}}(Q^2)\,. \qquad (351)\end{aligned}$$

When we consider the $Zb\bar{b}$ vertex there will be two additional diagrams involving internal ϕ-lines. Collecting all terms we end up with the following expressions where m_b has been ignored everywhere, but not m_t:

$$\begin{aligned}F_{_V}^b &= V_b(Q^2)+W_b(Q^2)\,, \quad F_{_A}^b = A_b(Q^2)+W_b(Q^2)\,, \qquad (352)\\ V_b(Q^2) &= \frac{1}{2}Q_b^2\frac{s_\theta^2}{c_\theta}v_b F_{_{Aa}}(Q^2) + \frac{1}{8\,c_\theta^3}v_b\left(v_b^2+3a_b^2\right)F_{_{Za}}(Q^2)\,,\\ A_b(Q^2) &= \frac{1}{2}Q_b^2\frac{s_\theta^2}{c_\theta}a_b F_{_{Aa}}(Q^2) + \frac{1}{8\,c_\theta^2}a_b\left(3v_b^2+a_b^2\right)F_{_{Za}}(Q^2)\,,\\ W_b(Q^2) &= \frac{v_t+a_t}{8c_\theta}F_{_{Wa}}^{(1)}(Q^2) + \frac{v_t-a_t}{2c_\theta}F_{_{Wa}}^{(2)}(Q^2) + \frac{c_\theta}{4}F_{_{Wn}}^{(1)}(Q^2) - \frac{s_\theta^2}{c_\theta}F_{_{Wn}}^{(2)}(Q^2)\,.\end{aligned}$$

It can be seen that four different combination of C_{ij}-functions are needed in this case. They are defined by

$$\begin{aligned}F_{_{Wa}}^{(1)}(Q^2) &= \frac{m_t^4}{M_{_W}^2}C_0 + 4C_{24} - 2 + 2Q^2\left(C_{11}+C_{23}\right)\,,\\ F_{_{Wa}}^{(2)}(Q^2) &= \frac{1}{2}m_t^2C_0 + \frac{1}{2}w_t\left[C_{24} - \frac{1}{4} + \frac{Q^2}{2}\left(C_{12}+C_{23}\right)\right]\,, \qquad (353)\end{aligned}$$

with $w_t = m_t^2/M_{_W}^2$ and where all functions have arguments $\{0,0,Q^2;m_t,M_{_W},m_t\}$, and

$$\begin{aligned}F_{_{Wn}}^{(1)}(Q^2) &= -(12+w_t)C_{24} + 2 - 2Q^2(C_0+C_{11}+C_{23}),\\ F_{_{Wn}}^{(2)}(Q^2) &= -\frac{1}{2}\,w_t\left(M_{_W}^2\,C_0 + \frac{1}{2}C_{24}\right)\,, \qquad (354)\end{aligned}$$

where all functions have arguments $\{0,0,Q^2;M_{_W},m_t,M_{_W}\}$.

For the $\gamma b\bar{b}$ vertex we obtain

$$V_b(Q^2) = Q_b^3\,s_\theta^3F_{_{Aa}}(Q^2) + \frac{1}{4}Q_b\frac{s_\theta}{c_\theta^2}\left(v_b^2+a_b^2\right)F_{_{Za}}(Q^2)\,, \qquad (355)$$

$$
\begin{aligned}
A_b\left(Q^2\right) &= \frac{1}{2}Q_b\frac{s_\theta}{c_\theta^2}v_b a_b F_{_{Za}}\left(Q^2\right), \\
W_b\left(Q^2\right) &= s_\theta\left\{Q_u\left[\frac{1}{4}F^{(1)}_{_{Wa}}\left(Q^2\right)+F^{(2)}_{_{Wa}}\left(Q^2\right)\right]+\frac{1}{4}F^{(1)}_{_{Wn}}\left(Q^2\right)+F^{(2)}_{_{Wn}}\left(Q^2\right)\right\}.
\end{aligned}
$$

As a result, the vector and axial–vector current vertices are given and we may turn to the scalar current.

11 One-loop renormalization of the electric charge

To begin the actual discussion on renormalization of the SM we consider the one-loop renormalization of the electric charge. We will use the fine structure constant α, defined through the residue of the pole at zero momentum transfer of charged particles scattering, for example, electron–muon scattering. First, we introduce the problem by discussing some of its aspects in the t'Hooft–Feynman gauge and subsequently we will prove the gauge invariance of the result.

11.1 Electric charge renormalization: the $\xi = 1$ gauge

An unpleasant feature that shows up in the SM is that the Z–γ transition does not vanish at $p^2 = 0$, i.e. $\Sigma_{3Q}(0) \neq 0$, due to bosonic loops. Accordingly, we should add the corresponding contribution to the one-loop renormalization of the electric charge:

$$
4\,\pi\alpha = g^2 s_\theta^2\,\left(1+\delta\alpha\right), \quad \delta\alpha = \delta_{_{\rm WF}} + \delta_{_V} + \delta_\gamma + \delta_{\rm mix}, \tag{356}
$$

where $\delta_{\rm WF}$ is the wave-function factor for the external fermions, δ_V derives from the $\gamma f\overline{f}$ vertices at zero momentum transfer and δ_γ and $\delta_{\rm mix}$ from the photon self-energy and from the γ–Z transitions again evaluated at zero momentum transfer.

There is nothing wrong with the one-loop procedure illustrated in Fig. 15, but we prefer to present an alternative derivation that will become useful whenever we improve upon the pure one-loop

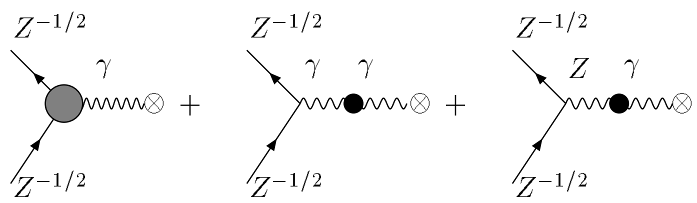


Figure 15: The electron scattering in the Coulomb field.

calculation. Let us consider the Z–γ transition in the R_ξ gauge. For $p^2 = 0$ we obtain

$$\frac{\Sigma_{3Q}(0)}{M^2} = -\frac{1}{2}\left(\xi^2+3\right)\Delta\left(M\right) + \frac{1}{2}\left(1+\frac{3}{\xi^2-1}\right)\xi^2\ln\xi^2 - \frac{1}{4}\xi^2 - \frac{5}{4}. \quad (357)$$

Consider now the Lagrangian written in terms of the non-diagonal fields B^a_μ and B^0_μ. The mass term originates from the interaction with the Higgs field, through the term

$$\mathcal{L}_2 = -\frac{1}{2}g^2\langle v^2\rangle W^+_\mu W^-_\mu - \frac{1}{4}\langle v^2\rangle\left(gB^3_\mu + g'B^0_\mu\right)^2, \quad (358)$$

where $\langle v\rangle$ denotes the Higgs vacuum expectation value. The whole procedure amounts to defining a diagonalization in the neutral sector at $p^2 = 0$ and, after the inclusion of one-loop corrections, respecting gauge parameter independence. This is best accomplished by a re-definition of the $SU(2)$ coupling constant g:

$$g = \overline{g}\,\left(1 - \Gamma\,\overline{g}^2\right), \quad (359)$$

where Γ is a constant yet to be specified. As we did before, the weak mixing angle is introduced in terms of g':

$$g' = -\frac{s_\theta}{c_\theta}\overline{g}, \quad (360)$$

together with the physical fields

$$\begin{pmatrix} Z \\ A \end{pmatrix} = \begin{pmatrix} c_\theta & -s_\theta \\ s_\theta & c_\theta \end{pmatrix}\begin{pmatrix} B_3 \\ B_0 \end{pmatrix}. \quad (361)$$

The bare W mass is now defined by the following equation: $\overline{g}^2\langle v^2\rangle = 2M^2$. Thus, for the Lagrangian we obtain

$$\mathcal{L}_2 = -M^2\,W^+_\mu W^-_\mu - \frac{1}{2}\frac{M^2}{c^2_\theta}\,Z_\mu Z_\mu + \overline{g}^2M^2\Gamma\left(2\,W^+_\mu W^-_\mu + Z_\mu Z_\mu + \frac{s_\theta}{c_\theta}\,Z_\mu A_\mu\right). \quad (362)$$

The new terms of order $\overline{g}^2$ will contribute to the self-energies, resulting in a shift

$$\begin{aligned}\Sigma_{3Q}(p^2) &\rightarrow \overline{\Sigma}_{3Q}(p^2) = \Sigma_{3Q}(p^2) + 16\,\pi^2 M^2 \Gamma, \\ \Sigma_{33;WW}(p^2) &\rightarrow \overline{\Sigma}_{33;WW}(p^2) = \Sigma_{33;WW}(p^2) + 32\,\pi^2 M^2 \Gamma. \end{aligned} \tag{363}$$

The choice of Γ is almost immediate: we observe that

$$\Sigma_{3Q}(0)|_{\xi=1} = -2\,M^2\Delta\,(M) \tag{364}$$

and choose

$$\Gamma = \frac{1}{8\,\pi^2}\Delta\,(M)\,, \tag{365}$$

which is ξ-independent by construction. For arbitrary ξ we still do not have $\overline{\Sigma}_{3Q}(0) = 0$:

$$\frac{1}{M^2}\overline{\Sigma}_{3Q}(0) = -\frac{1}{2}\left(\xi^2-1\right)\Delta\,(M)+\frac{1}{2}\left(1+\frac{3}{\xi^2-1}\right)\xi^2\ln\xi^2-\frac{1}{4}\xi^2-\frac{5}{4}. \tag{366}$$

However, if we choose the $\xi = 1$ gauge, then—here and only here—we have $\overline{\Sigma}_{3Q}(0) = 0$. The shift $g \rightarrow \overline{g}$ will introduce new terms also in other sectors of the Lagrangian. For our present purposes the relevant ones come from the fermionic part which, written for an arbitrary isodoublet, becomes

$$\mathcal{L}_f = \frac{i}{4}\overline{\psi}\gamma_\mu\left(g_2 B^0_\mu + g B^a_\mu \tau_a\right)\,\gamma_+\psi + \frac{i}{4}g_3 B^0_\mu\,\overline{d}\gamma_\mu\gamma_- d + \frac{i}{4}g_4 B^0_\mu\,\overline{u}\gamma_\mu\gamma_- u. \tag{367}$$

We put

$$g_i = -\alpha_i\frac{s_\theta}{c_\theta}\overline{g}, \quad i = 2, 3, 4, \tag{368}$$

and obtain the solution

$$\alpha_2 = 1 - 2\,Q_u = -1 - 2\,Q_d\,, \quad \alpha_3 = -2\,Q_d\,, \quad \alpha_3 = -2\,Q_u\,. \tag{369}$$

From $\mathcal{L}_f$ we derive new vertices which will be called *special vertices*:

$$\begin{aligned}\{A;Z\}f\overline{f} &= -\,(2\pi)^4\,i\,\frac{i}{2}\overline{g}^3\{s_\theta;c_\theta\}I^{(3)}_f\Gamma\gamma_\mu\gamma_+\,, \\ Wu\overline{d} &= -\,(2\pi)^4\,i\,\frac{i}{2\sqrt{2}}\overline{g}^3\Gamma\,\gamma_\mu\gamma_+\,. \end{aligned} \tag{370}$$

In our convention a *special* vertex will always be denoted by an open box. (Fig. 16) The introduction of special vertices will prove to be crucial in showing the ultraviolet finiteness of many results. Later on we will introduce also special trilinear vertices, for example, $\gamma(Z)W^+W^-$.

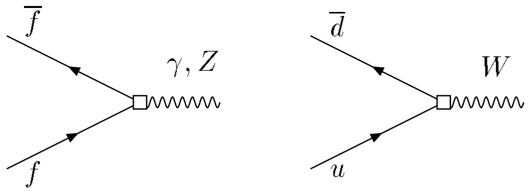


Figure 16: Special vertices.

Once all relevant one-loop terms are computed then the renormalization condition at zero momentum transfer gives

$$4\,\pi\alpha = g^2 s_\theta^2\,(1+\delta\alpha)\,, \qquad \delta\alpha = \delta_{\rm WF} + \delta_{V} + \delta_\gamma\,. \tag{371}$$

Each correction will be split into an e.m. part and a genuinely weak part, $\delta_i = \delta_i^{\rm e.m.} + \delta_i^{\rm w}$. There is a QED Ward identity (Fig. 17) which can be written as follows:

$$\delta_{\rm WF}^{\rm e.m.} + \delta_{V}^{\rm e.m.} = 0\,, \tag{372}$$

and we are left with the purely weak sector. For definiteness we study the scattering of two charged particles, $ff' \to ff'$. The corresponding wave function factor for the whole process is

$$\delta_{\rm WF}^{\rm w} = 2\left(W_{V}^{{\rm w},f} + W_{V}^{{\rm w},f'}\right). \tag{373}$$

In the limit where we ignore all fermion masses but m_t the result is rather simple. The wave-function renormalization factors to be associated with external fermion lines read as follows:

$$\{W_{V}^{{\rm w},f}\,;\,W_{A}^{f}\} = \frac{g^2}{128\pi^2}\Bigl[\frac{1}{c_\theta^2}\,\{\sigma_f^{(2)}\,;\,2v_f a_f\}\left(-\Delta\,(M_{0}) + \frac{1}{2}\right) + w_{W}^{(1)}\Bigr], \tag{374}$$

with $w_W^{(1)}$ given by Eq.(**??**). The non-e.m. vertex corrections, evaluated at $p^2 = 0$, can be cast in the form

$$F_{V}^{\rm w}(0) = \frac{ig^3}{16\pi^2}s_\theta\Bigl[\frac{Q_f}{4c_\theta^2}\sigma_f^{(2)}F_{Za}^{(1)}\,(0) + \frac{Q_{f'}}{4}F_{Wa}^{(1)}\,(0) - \frac{I_f^{(3)}}{2}F_{Wn}^{(1)}\,(0) - 8\,\pi^2 I_f^{(3)}\Gamma\Bigr]. \tag{375}$$

We can write

$$\delta_{V}^{\mathrm{w}} = \frac{g^2}{16\pi^2}\Big[\frac{1}{4c_\theta^2}\sigma_f^{(2)} F_{Za}^{(1)}(0) + \frac{Q_{f'}}{4Q_f} F_{Wa}^{(1)}(0) - \frac{I_f^{(3)}}{2Q_f} F_{Wn}^{(1)}(0) - 16\pi^2 \frac{I_f^{(3)}}{2Q_f}\Gamma\Big], \quad (376)$$

where the $F_{\mathrm{cl}}^{(1)}(0)$ functions are defined by Eqs.(**??**), (**??**) and (**??**) to be the abelian W and Z clusters and the non-abelian W cluster.

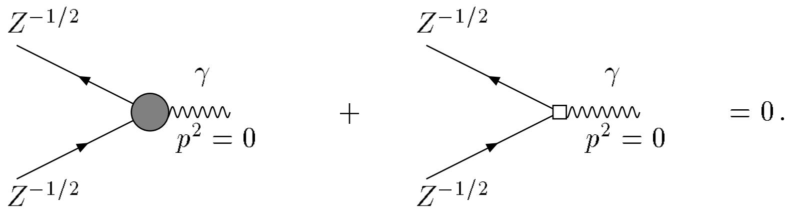


Figure 17: $U(1)$ Ward identity.

We repeat here, for completeness, the definition of cluster of vertices, although already given elsewhere in the book. A neutral current vertex is termed abelian if it contains one vector boson internal line with no trilinear vector couplings, and non-abelian otherwise. A cluster is obtained by replacing vector boson internal lines with ϕ-lines in all possible ways that are allowed by Feynman rules. Collecting Eqs.(373), (374) and (376) we can prove *The validity of the $U(1)$ Ward identity contained in Eq.(372) (Fig. 17) is naturally extended in the full SM to include the non e.m. parts, i.e.*

$$\delta_{\mathrm{WF}}^{\mathrm{w}} + \delta_{V}^{\mathrm{w}} = 0\,. \quad (377)$$

This theorem allows us to write the complete $\gamma f\overline{f}$ interaction as

$$\begin{aligned} V_\mu^{\gamma f\overline{f}}(p^2) &= (2\pi)^4\, i\, \frac{ig^3 s_\theta}{16\,\pi^2}\, \bar{v}\gamma_\mu[F_V(p^2) + G_A(p^2)\gamma_5]u, \\ F_V(p^2) &= F_V^{\mathrm{vert}}(p^2) - F_V^{\mathrm{vert}}(0)\,, \quad G_V(p^2) = G_A^{\mathrm{vert}}(p^2) - G_A^{\mathrm{vert}}(0)\,, \quad (378) \end{aligned}$$

where the superscript 'vert' indicates that wave-function renormalization factors are excluded.

To summarize, we find that in the $\xi = 1$ gauge the non-self-energy corrections are ultraviolet finite, there is no parity violation in the e.m. current at low energy and the renormalized neutrino charge is zero.

Having performed the electric charge renormalization in the renormalizable $\xi = 1$ gauge we must enquire about the gauge independence of the procedure. Let us consider the $\gamma b\overline{b}$ interaction in the R_ξ gauge and let us fix $p_\gamma = 0$, in order to see how gauge parameter independence is achieved, even in the presence of a heavy fermion. The first ingredient that we need is given by

$$\begin{aligned}\Lambda_\mu &= (2\pi)^4\, i\Big\{\frac{1}{16\pi^2}[V^{Za}_\mu(0) + V^{Wa}_\mu(0) + V^{Wn}_\mu(0)] + 2igs_\theta Q_b\gamma_\mu\left(W^{\mathrm{w},b}_{V} + W^{b}_{A}\gamma_5\right)\Big\}\\ &= (2\pi)^4\, i\gamma_\mu\Big\{\frac{i}{16\pi^2}[g^3Q_f\frac{s_\theta}{4c^2_\theta}\left(\sigma^{(2)}_b + 2v_ba_b\gamma_5\right)F_{Za}(0)\\ &\quad +g^3Q_t\frac{s_\theta}{4}\gamma_+F_{Wa}(0) + g^3\frac{s_\theta}{4}\gamma_+F_{Wn}(0)] + 2igs_\theta Q_b\gamma_\mu\left(W^{\mathrm{w},b}_{V} + W^{b}_{A}\gamma_5\right)\Big\},\end{aligned} \tag{379}$$

which represents the sum in the R_ξ gauge of vertex diagrams (abelian and non-abelian, W and Z clusters) and of wave-function factors (Fig. 18).

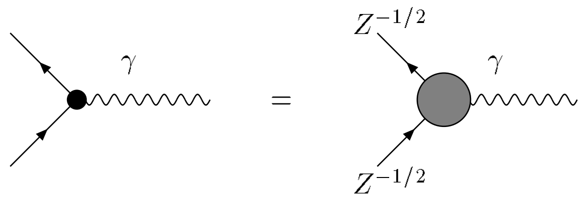


Figure 18: Λ_μ correction, including vertex diagrams and wave-function factors.

Using the low-energy limits of the scalar functions $F_{\mathrm{cl}}(0)$, and wave-function renormalization factors for b-quarks in the R_ξ gauge, we find $\gamma_\mu(\mathcal{V} + \mathcal{A}\gamma_5)$ and, moreover, $\mathcal{V} = \mathcal{A} = \Lambda$, with

$$\Lambda = (2\pi)^4\, i\frac{ig^3s_\theta Q_f}{32\pi^2}\Big[\left(\frac{1-\xi^2}{4} - 1\right)\Delta(M) + \frac{1}{4}\xi^2\ln\xi^2 - \frac{1}{8}\left(\xi^2 + 5 - \frac{6\xi^2}{\xi^2-1}\ln\xi^2\right)\Big]. \tag{380}$$

If we decompose Λ_μ into Q and L components, according to the equation

$$\Lambda_\mu = \Lambda_Q\,\gamma_\mu + \Lambda_L\gamma_\mu\gamma_+\,, \tag{381}$$

then only Λ_L is non-zero and gauge parameter dependent, as can most easily be seen by using B-fields instead of the physical ones.

This particular combination of one-loop corrections is zero in the pure QED sector, while here it differs from zero even in the $\xi = 1$ gauge when we compute it prior to the field re-diagonalization. In general we find a ξ-dependent expression which, however, is quite simple if compared with $F_k(0)$ or $W^{\mathrm{w},b}_{V,A}$ separately.

In the vertex corrections that we have shown the fermion wave-function factors are not yet included. We observe that the vector and the axial–vector parts are different and, moreover, they also depend on ξ_Z, M_0 and m_t. All these dependences cancel when we include the appropriate corrections for the external fermions, and in turn this is related to the fact that the only gauge parameter dependence of vector boson transitions in the neutral sector—the missing ingredients—comes from W and charged ϕ loops.

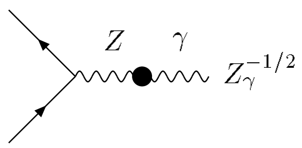


Figure 19: The transition T_μ that accounts for the $Z-\gamma$ mixing and for the photon wave-function factor.

In order to complete the one-loop $\gamma f\overline{f}$ interaction we also need the Z–γ transition. The reason for this is simple: in the R_ξ gauge the transition $\overline{\Sigma}_{3Q}(0)$ is not zero. Therefore, we must consider

$$T_\mu = (2\pi)^4\, i\, \frac{ig^3}{32\,\pi^2}\,\gamma_\mu \left[s^3_\theta Q_f \Pi_{\gamma\gamma}(0) + s_\theta \frac{\overline{\Sigma}_{3Q}(0)}{M^2}\left(-\frac{1}{2} - 2s^2_\theta Q_f - \frac{1}{2}\gamma_5\right)\right]. \tag{382}$$

The transition T_μ, which accounts for the Z–γ mixing and for the photon wave-function factor, splits naturally into two parts (Fig. 19),

$$T_\mu = (2\pi)^4\, i\, \frac{ig^3}{16\,\pi^2}\, s_\theta\, \left(s^2_\theta T_Q\,\gamma_\mu + T_L\,\gamma_\mu\gamma_+\right), \tag{383}$$

where the $T_{Q,L}$ form factors are

$$T_Q = \frac{3}{2}\Delta(M) + \frac{1}{3}, \tag{384}$$

$$T_L = \frac{\xi^2-1}{8}\Delta\,(M) + \frac{1}{3} - \frac{1}{8}\xi^2 \ln\xi^2 + \frac{1}{16}\left(\xi^2 + 5 - 6\frac{\xi^2}{\xi^2-1}\ln\xi^2\right)\Bigg].$$

As before, only T_L is gauge parameter dependent. From the special vertex we also obtain

$$S_\mu = (2\pi)^4\, i\, \frac{ig^3}{32\,\pi^2}\, s_\theta\, \Delta\,(M)\ \gamma_\mu\, \gamma_+\,. \tag{385}$$

It can now be proved that the complete—one-loop—interaction $\gamma f \overline{f}$ is given in terms of transitions by

$$\Lambda_\mu + T_\mu + S_\mu = (2\pi)^4\, i\, \frac{ig^3}{16\pi^2}\, s_\theta^3 Q_f \left[\frac{1}{2}\,\Pi_{\gamma\gamma}(0) - \frac{1}{M^2}\,\overline{\Sigma}_{3Q}(0)\right]\gamma_\mu\,. \tag{386}$$

With this result we have been able to prove the following theorem:

The low-energy axial–vector coupling of the photon is zero in the arbitrary R_ξ gauge. This is evident from Eq.(386). The renormalization of the electric charge, $e = g s_\theta$, in the SM is gauge parameter independent. Indeed, using $\Pi_{\gamma\gamma}(0)$ and $\overline{\Sigma}_{3Q}(0)$, we obtain that in the sum the ξ-dependent terms cancel and the final result is

$$-\frac{1}{2}\,\Pi_{\gamma\gamma}(0) + \frac{1}{M^2}\,\overline{\Sigma}_{3Q}(0) = -\frac{3}{2}\,\Delta\,(M) - \frac{1}{3}. \tag{387}$$