A Frontier in MultiScale - MultiLoop Integrals: the Algebraic - Numerical Method

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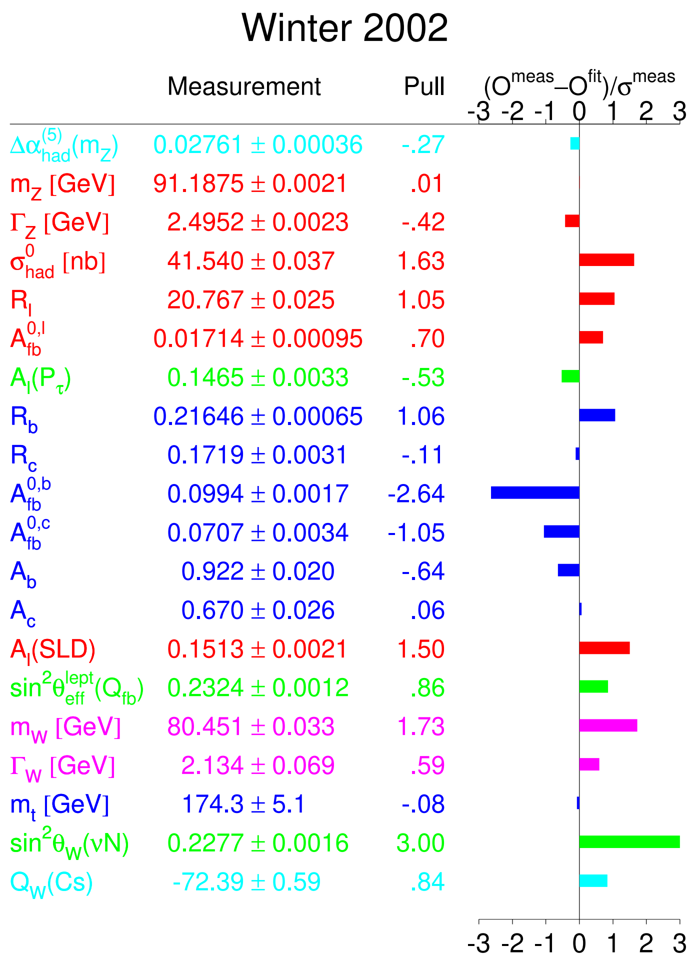
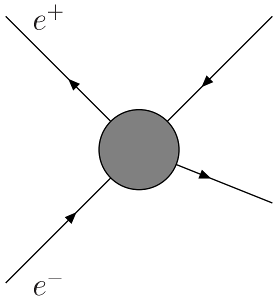


Figure 1: Milestones of EXP precision

This is why high Precision in High Energy Physics

PRESENT STATUS of **RC**



one-loop $\Longrightarrow$ **ALL**

two-loop $\Longrightarrow$ **LO + NLO** + re-summations

IS QED, **SF-method** (or **equivalent**)
FS QED, **available**
IS $\otimes$ FS QED

Fine Points in QED for $2 \to 2$

- **s-channel** $\mathcal{O}\left(\alpha^2 L^n\right)$, $n = 0, 1, 2$ (Calculations), $\mathcal{O}\left(\alpha^3 L^3\right)$ important for Z **lineshape**
- Differences and uncertainties amount to at most ± 0.1 MeV on M_Z and Γ_Z and $\pm 0.01\%$ on $\sigma^0_{\rm h}$ (`MIZA`, `TOPAZ0` and `ZFITTER`)
- **non-annihilation** (Bhabha) both **SF** and **PS** have been analyzed, uncertainty estimated to be **0.061%** (`BHLUMI`)

FULL two-loop EW needed for **GigaZ** ($10^9 Z$ events), **Numerical** evaluation of **FD**?

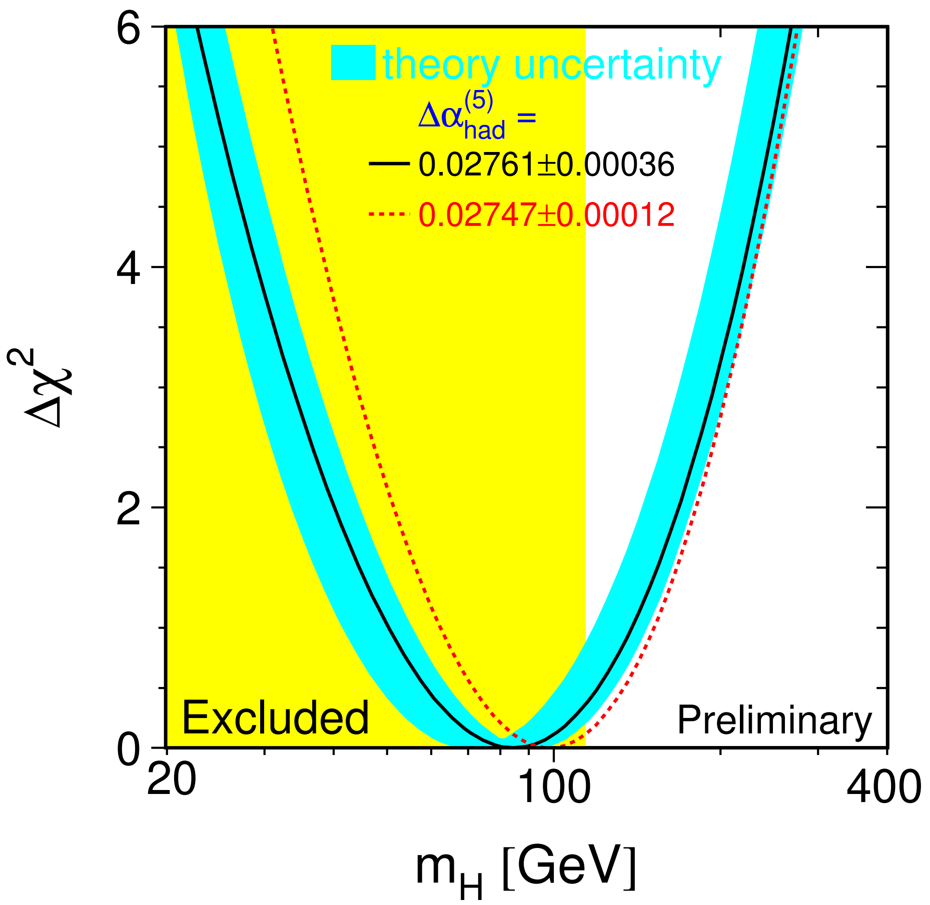
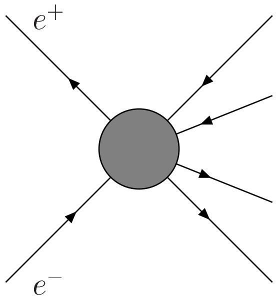


Figure 2: BlueBand

Theoretical Uncertainty not **yet** satisfactory

requires a **full two-loop** analysis of the **SM**
a **real** $\rightarrow$ MultiScale $\leftarrow$ problem



$e^+e^- \to 4\,$**f Born** $\Longrightarrow$ **ALL** processes
$\mathcal{O}(\alpha)$ EW $\Longrightarrow$ **only in DPA for** WW **signal**
$e^+e^- \to 4\mathrm{f} + \gamma \Longrightarrow$ **ALL**
$e^+e^- \to 6\mathrm{f} \Longrightarrow$ **starting**

$\boxed{\textbf{Fine Points in QED for } 2 \to 4}$

for $e^+e^- \to WW \to 4\,f \;\to$ **see DPA**
for generic $e^+e^- \to 4\,f \;\to$ **s – channel SF**
$\to$ *i.e.* **LL approximation**

♢ the latter strictly applicable only if **ISR** can be separated (may lead to **excess** of **ISR**)

♢ **preliminar** investigations towards **non–s SF** by `GRACE` and `SWAP`, **analytical** approach in `hep-ph/0108255`.

$\boxed{\textbf{Perspectives}}$

- Precision Tests:

$$\frac{\Delta\sigma_{WW}}{\sigma_{WW}}(\text{LEP2}) = 1\%, \quad \frac{\Delta\sigma_{WW}}{\sigma_{WW}}(\text{TESLA}) < 0.5\%,$$
$$\Delta M_W(\text{LEP2}) = 30\,\text{MeV}, \quad \Delta M_W(\text{TESLA}) = 15\,\text{MeV},$$

- High precision in M_W, m_{top} highly desirable to

  a) further constrain M_{higgs} in the **SM**;
  b) (dis-)favour **SUSY**

Progress in $e^+e^- \to 4\,\text{f}$ in recent years

- $\mathcal{O}(\alpha)$ corrections (`YFSWW, RacoonWW`);
- Full **ME** calculations (`GRACE, PHEGAS, WRAP, RacoonWW`

For future Colliders

- ☐ Radiative Corrections in the **Threshold Region** and in the **High Energy Range**

$\Updownarrow$

- ☐ **Full one-loop** Calculations are needed, including **Single**W

$\Updownarrow$

- ☐ **many** technical difficulties and

  a) (practical) solution of **gauge - invariance** problem;

  b) better control of **QCD** corrections;

  c) (leading) **two-loop** effects (mostly Sudakov log's)

Therefore there is **NO** additional need to explain

the need for **Fast** & **Efficient** ways

to compute **Multi l&L** diagrams

$\cdots\cdots$ *and*

for the **general** $\searrow$ case the **analytical way**[1] not feasible

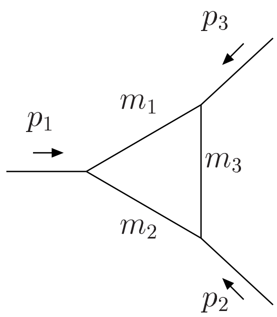
arbitray int/ext masses

although a lot can be done

$\cdots\cdots$ *moreover*

what is so special about the **AW**???

[1] K. G. Chetyrkin, Z. Bern, L. Dixon, D. A. Kosower, J. Fleischer, F. Jegerlehner, O. V. Tarasov, S. G. Gorishnii, S. A. Larin, L. R. Surguladze, F. V. Tkachov, D. J. Broadhurst, A. G. Grozin, V. V. Belokurov, N. I. Usyukina, D. I. Kazakov, A. V. Kotikov, A. I. Davydychev, V. A. Smirnov, M. Caffo, H. Czyz, S. Laporta,d E. Remiddi, T. Gehrmann J. Fleischer, V. A. Smirnov, A. Frink, J. G. Korner, D. Kreimer, K. Schilcher, J. B. Tausk, O. L. Veretin, A. L. Kataev, G. Degrassi, P. Gambino , A. Freitas, S. Heinemeyer, W. Hollik, W. Walter, G. Weiglein, S. Heinemeyer, T. Binoth, G. Heinrich, F. A. Berends, S. Groote, J. G. Korner A. Ghinculov, Y. Yao, J. A. Vermaseren, S. Bauberger, M. Bohm, M. Buza, J. van der Bij, M. Veltman, $\cdots\cdots$



▷ It is the sum of 12 **di-logarithms**;

▷ (Few) More complex diagrams $\Longrightarrow$ **Nielsen - Polilog-arithms**;

▷ (Few) (More complex)^2 diagrams $\Longrightarrow$ **Higher Tran-scendental Functions**;

what is so special about $S_{n,p}(z)$?

▷ Analytical properties explicit,

▷ An **EXPANSION** available, like in

$$\mathrm{Li}_2\,(z) \;=\; \sum_{n=0}^{\infty} B_n\,\frac{\left[-\,\ln(1-z)\right]^{n+1}}{(n+1)\,!},$$

expansion parameter with the same **cut** of $\mathrm{Li}_2\,(z)$
$B_n \equiv$ Bernoulli numbers
$n = 14$ gives 13 decimals accuracy

The same is **NOT** obviously true for

Appell & **Kampé de Fériet** functions

or **nested sums**

although there are interesting novelties, e.g.

multi-dimensional **harmonic polylogarithms**[2]

A **generic** (scalar) N_L - Legs, n_l - Loops diagram

$$G = \left(i\,\pi^{n/2}\right)^{n_l}\,\Gamma\left(N_L - \frac{n}{2}\,n_l\right)\int\frac{dx_G\,\delta\,(1-x_G)}{U^{n/2}\,(V - i\,\delta)^{N_L - nn_l/2}},$$

$$\equiv \int_S \prod_{i=1}^{N_l - 1} dx_i\,I_G(\{x_i\})$$

$$dx_G = \prod_{i=1}^{N_l} dx_i, \qquad x_G = \sum_{i=1}^{N_l} x_i.$$

- □ Beyond **one-loop** and for the **general** mass case **polylogarithms** do not suffice! $\Longrightarrow$

  AW is in general possible only for **small #** of **Scales** (masses and momenta or invariants) like in

[2] E. Remiddi and J. A. Vermaseren, Int. J. Mod. Phys. A **15** (2000) 725 [arXiv:hep-ph/9905237]

a) **QCD**, all masses $\equiv 0$;

b) **QED**, only $m_e \neq 0$ and 0 or **mass shell** for external scales;

c) **SM** $\oplus$ simplifying conditions, e.g. $m/m_{\rm top} \ll 1$.

- In the general case (e.g. **SM**) **AW** not enough already for the **sunset** two-loop topology, i.e.

$$\text{Sunset} \;\equiv\; \sum \text{generalized hypergeometric series}$$

i.e. accessible for **small or large** p^2 but not in the **intermediate** region (**Thresholds!!**)

- Furthermore, **tensor integrals** cannot be fully *scalarized* but a larger basis of **pseudo-scalar** master integrals is needed.

Alternative strategy: Numerical Way (I)

- **Berends, Böhm** *et al*

  Dispersion Relations $\rightarrow$ one-dimensional **integral representation** for two-loop **self-energies**.

- **Caffo, Czyz** and **Remiddi**

  All diagrams $\rightarrow$ (limited) number of **MI** $\equiv$ **master integrals MI through** differential equations in p_i^2

  Results available for **sunset**, **difficulty** in detrmining the **initial conditions** to be provided by other methods.

- **Shimizu** *et al*

  Fully **Numerical** integration below thresholds

  Above thresholds, numerical integration is done after

  a) **Symmetric Method**, the integrand is replaced by a symmetric sum so as to eliminate the singularity;

  b) **Hybrid Method** Integrals for a few variables analytically until an integrable function is obtained,

  Applied to **planar** and **non-planar** two-loop vertices.

□ **Soper**

Calculations (in **QCD**) are performed using **Numerical** integration over FS particles and

Analytical integration over virtual particles,

it requires to deform some of the integration contours into the complex plane.

□ **Ghinculov and Yao**

All diagrams $\rightarrow$ generalized sunset

$$\text{Any diagram} = \int_{\text{ext}} \text{generalized sunset}$$

One-dimensional integral representation for the 10 **generalized sunset MI** needed in a renormalizable theory $\oplus$ numerical integration over the **ext** Feynman parameters.

Alternative strategy: Numerical Way (II)

TOPSIDE Collaboration[3]:

A. Ferroglia (NYU)
G. Passarino (Torino)
M. Passera (Padova)
S. Uccirati (Torino)

The method $\Longrightarrow$ seek algorithm(s) for smoothness that avoid spurious singularites

$$G = \frac{1}{B_G} \int_S dx\, \mathcal{F}[I_G]$$

- ▷ $\mathcal{F}$ is a functional of I_G that makes the whole integrand smooth in **parametric space**,
- ▷ B_G is a function of masses and momenta that contains the **TRUE** singularities of G.

Per lenitatem ad astra

[3]TOrino Project Supporting Intensive Diagrams Evaluation

Paradigms

- A **coherent** approach to a multi scale $\oplus$ loop $\oplus$ leg calculation requires a pragmatic mind often to the exclusion of wonderful generalizations;
- A succesful framework requires a working algorithm and one has to think in terms of his algorithm - a sort of primordial language - without having to translate. No algorithm has more complexity than English language, assuming that one thinks in English;
- Formulae & algorithms must coexist in the same project and one should not forget that alternatives of today may become the orthodoxy of tomorrow;
- Although many auxiliary features are far from satisfactory, especially the interface between symbolic algebra and numerical integration, one should not just seat & wait, many times handmade products, crafts such as `TOPAZ0` & `ZFITTER`, have been simple yet powerful tools. Clearly, one should not underestimate the present calculational crisis or even the lack of professional approach in the theoretical community (from software engineering point of view) and one should consider his present work as located in a transitional phase, as long as it is adaptable to an almost immediate resuming in the future.

Prolegomena: Bernstein-Tkachov theorem:

▷ for any finite set of polynomials

$$V_i(x) \quad \text{where} \quad x = (x_1, \dots, x_N)$$

is a vector of **Feynman parameters**, there exists an identity of the following form:

$$\mathcal{P}\,(x, \partial) \prod_i V_i^{\mu_i+1}(x) \;=\; B \prod_i V_i^{\mu_i}(x).$$

▷ where $\mathcal{P}$ is a polynomial of

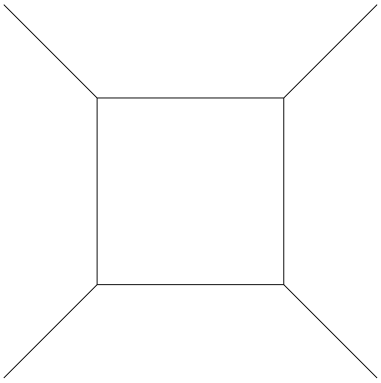
$$x \quad \text{and} \quad \partial_i = \partial/\partial x_i$$

▷ B and all coefficients of $\mathcal{P}$ are polynomials of μ_i and of the coefficients of $V_i(x)$.

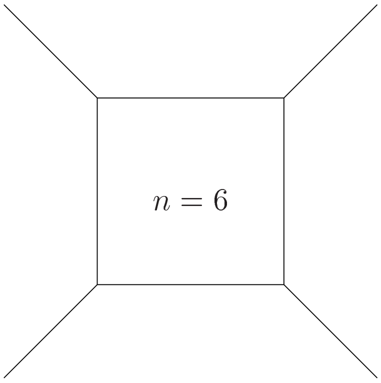
▷ **Iterative applications** of the functional relations, followed by **integration by parts**, allow us to express the integrand in **parametric space** as a combination of

$$(\text{polynomials})^N \;\times\; \text{logarithms of the same polynomials}$$

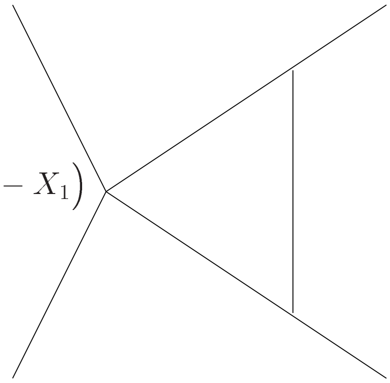
Diagrammatica of **BT** functional relations:



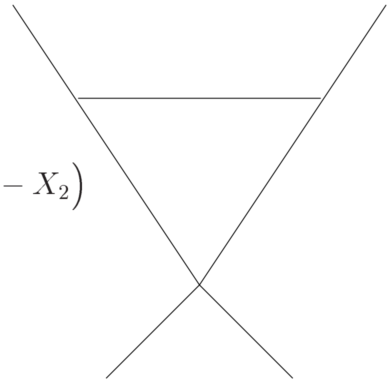
$= \frac{1}{2\,B_4}\Big\{ \; -$



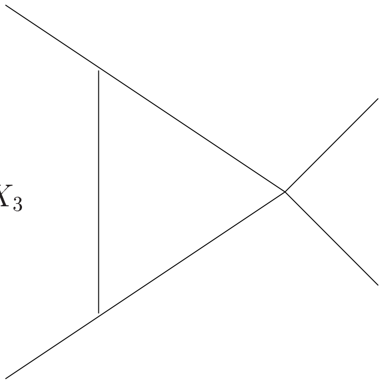
$+ \; \Big(X_2 - X_1\Big)$



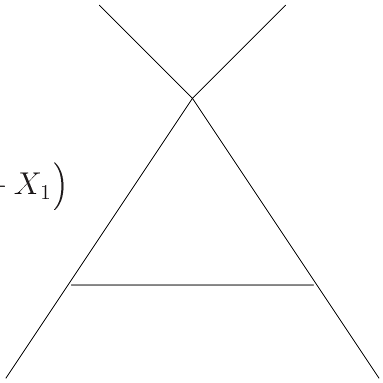
$+ \; \Big(X_3 - X_2\Big)$



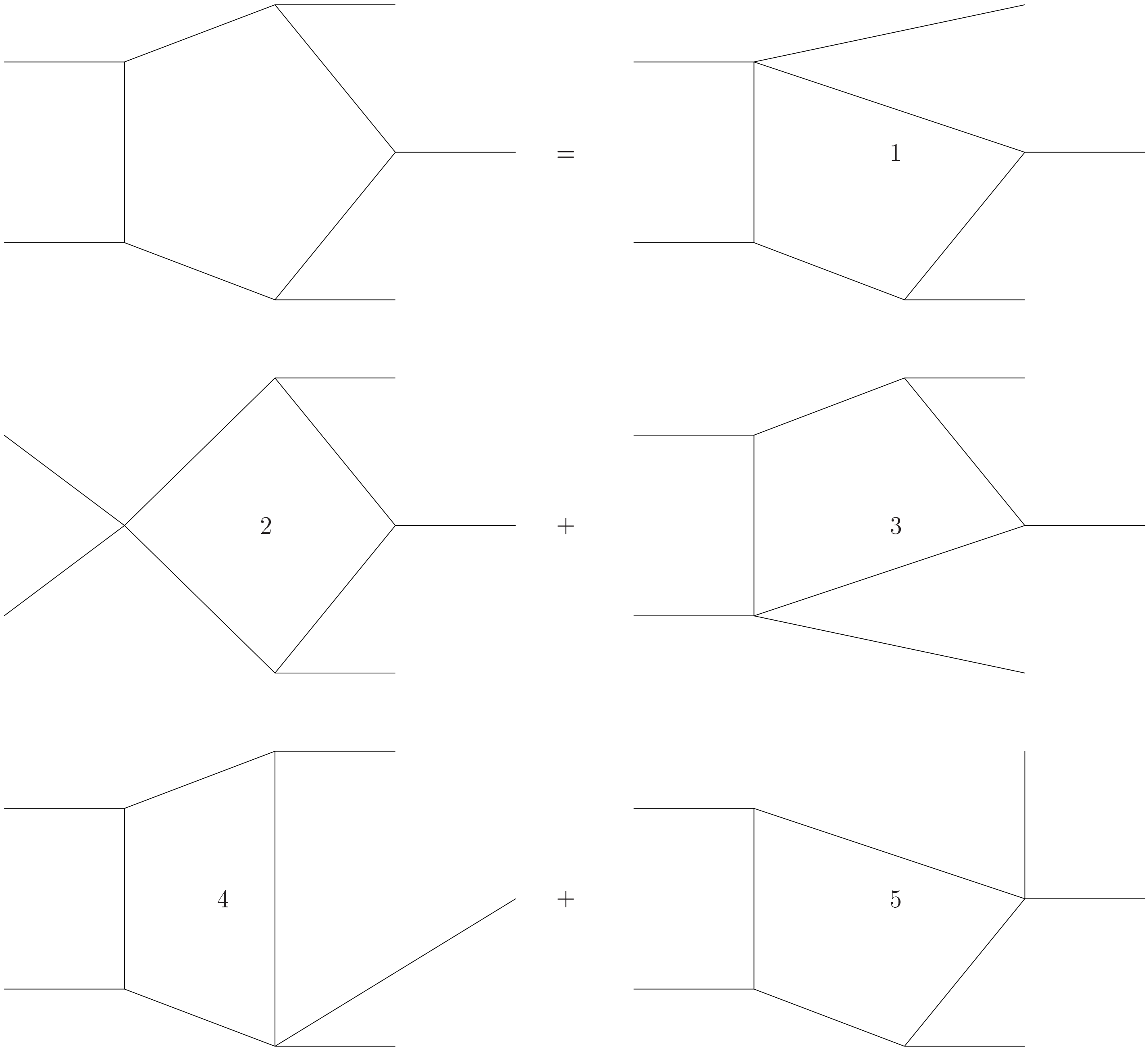
$- \; X_3$



$+ \; \Big(1 + X_1\Big)$



$\Big\}$


=
1
2
+
3
4
+
5

The **Pentagon** is a sum of 4 **Boxes**

with weights:

$$
\begin{aligned}
w_1 &= \frac{1}{2}\frac{1+X_1}{B_5}, \quad w_2 = \frac{1}{2}\frac{X_2-X_1}{B_5}, \quad w_2 = \frac{1}{2}\frac{X_3-X_2}{B_5}, \\
w_4 &= \frac{1}{2}\frac{X_4-X_3}{B_5}, \quad w_5 = -\frac{1}{2}\frac{X_4}{B_5},
\end{aligned}
$$

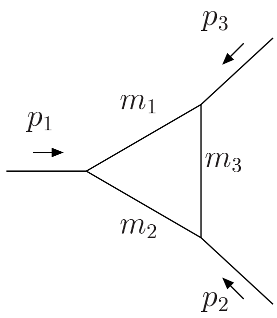
- ☐ Proving it is a **one-line** exercise with **one BT** iteration (cfr. with **several** pages in the original proof).
- ☐ The meaning of **weights** is transparent: $B_5 = 0$ gives the leading **Landau singularity** of the pentagon (more later).
- ☐ Generalization to **more** legs is doable with some extra nicety due to the vanishing of **Gram's determinants** in $n = 4$.

The fully **automatized** interface

Diagrammatica $\Leftrightarrow$ **Numerica**

is **needed** and still **unsatisfactory**

Examples

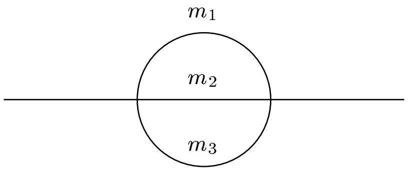


$$C_0 \;=\; \int_0^1 dx_1 \int_0^{x_1} dx_2\, V^{-1-\epsilon/2}(x_1,x_2),$$

$$V(x_1,x_2) \;=\; x^t\, H\, x + 2\, K^t\, x + L - i\,\delta,$$

$$B_3 \;=\; L - K^t\, H^{-1}\, K, \qquad X = H^{-1}\, K.$$

$$\begin{aligned} C_0 \;=\; & \frac{1}{B_3}\,\{\frac{1}{2} + \int_0^1 dx_1\,[\int_0^{x_1} dx_2\, \ln\, V(x_1,x_2) \\ +\; & \frac{1}{2}\,(X_1 - X_2)\, \ln\, V(x_1,x_1) + \frac{1}{2}\, X_2\, \ln\, V(x_1,0) \\ -\; & \frac{1}{2}\,(1 + X_1)\, \ln\, V(1,x_1)]\}. \end{aligned}$$

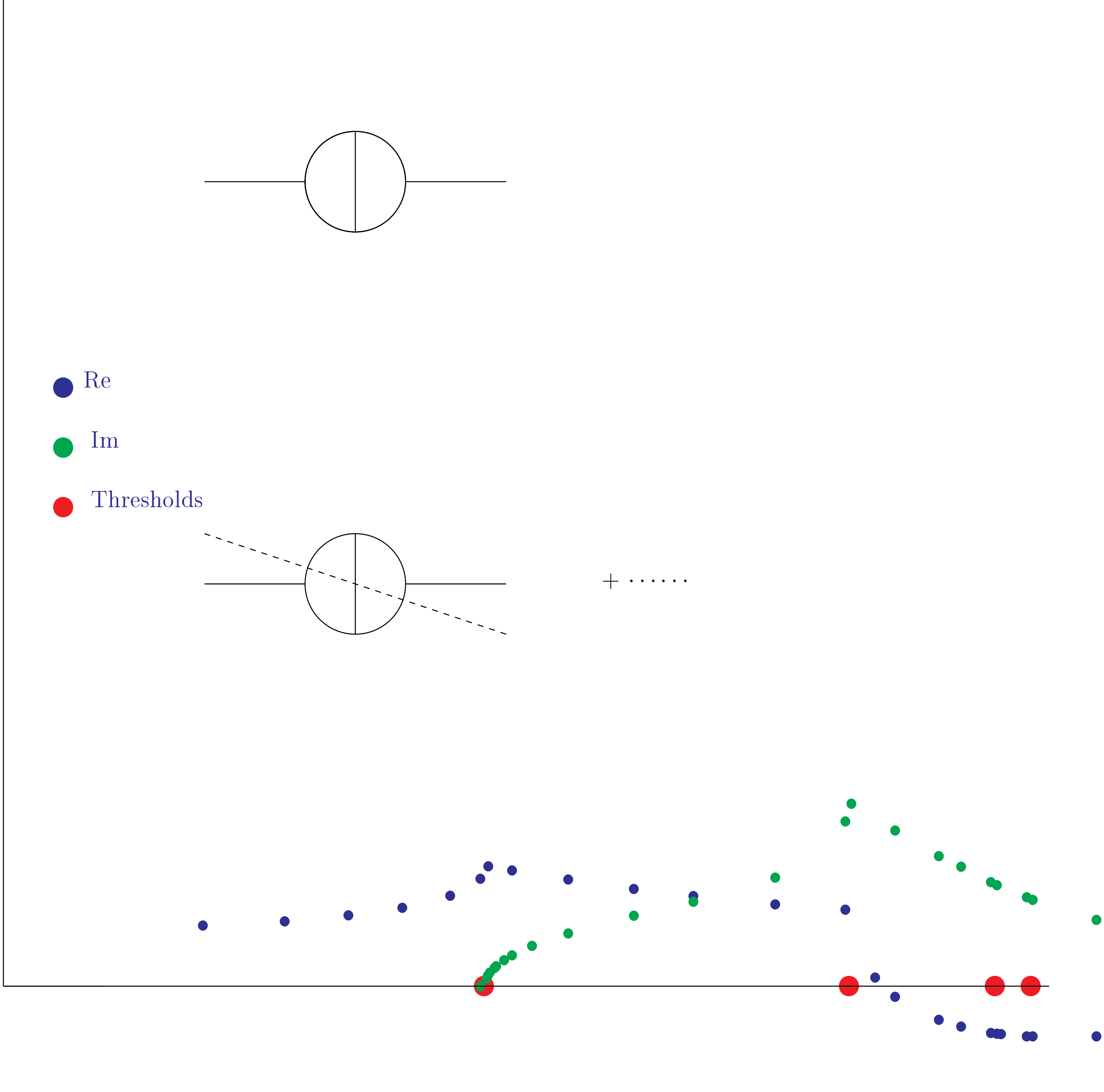


$$S_{111} \; = \; \sum_{i=1}^{3} \, m_i^2 \, S_{111}^i + S_{111}^p,$$

$$\begin{aligned} S_{111}^{3\text{fin}} \; &= \; \int_0^1 \, dx \int_0^1 \, dy \, [\ln \xi(x,y) + \left(\frac{\ln \xi(x,y)}{y-1} \right)_+] \\ &+ \; \ln \mu_3^2 \, \left(\ln \mu_3^2 - 4 \right) + \frac{13}{2} + \frac{1}{2} \zeta(2), \\ S_{111}^{p\text{fin}} \; &= \; \int_0^1 \, dx \int_0^1 \, dy \, y \, \ln \xi(x,y) + \frac{3}{8}, \end{aligned}$$

$$\begin{aligned} \xi(x,y) \; &= \; -y \, \left(y - 1 + \mu_3^2 \right) \, x^2 + [y \, \left(y - 1 + \mu_3^2 \right) \\ &+ \; (1-y) \, \left(\mu_2^2 - \mu_1^2 \right)] \, x + (1-y) \, \mu_1^2. \end{aligned}$$

Why a difficult problem?



Why a difficult problem (II) ?

▷ Try to compute C_0 around the **anomalous threshold** in $W^* \to \bar{u}d$, (**abelian** W exchange)

$$s_{\rm anom\,thr} = \frac{(m_u^2 - m_d^2)^2}{M_W^2}$$

▷ **AT** $\to$ $A_{\rm abs}$ is not directly related through **unitarity** to physical processes. In principle we may have a **logarithmic singularity** in the **real part** and a **jump** in the **imaginary part**. The **LS** is **maximally equal** to $\ln\delta$, the **jump** arises because some argument in the integrand moves rapidly around the cut. **However** this is **Not** necessarily the case; we write:

$$C_0 = \int_0^1 dx_1 \int_0^{x_1} dx_2 \left[(x-X)^t H (x-X) + B_3\right]^{-1},$$

▷ and **unless** $0 < X_2 < X_1 < 1$ there is **no** divergence for $B_3 = 0$. Actually, this is what happens for $W^* \to \bar{u}d$ and **realistic** masses.

Around anomalous threshold

$\sqrt{s} - \sqrt{s}_{\rm anom\, thr} = -5\,\mathrm{GeV}$		
Source	Re	Im
ANALYTICAL ('t Hooft - Veltman)	2.49196916	-6.10507293
NUMERICAL I (one BT)	2.49201730	-6.10506938
NUMERICAL II (two BT)	2.49487588	-6.10507303
$\sqrt{s} - \sqrt{s}_{\rm anom\, thr} = -200\,\mathrm{MeV}$		
Source	Re	Im
ANALYTICAL	2.55665099	-6.16483833
NUMERICAL I	2.55667167	-6.16483680
NUMERICAL II	4.52680950	-6.16483834
$\sqrt{s} - \sqrt{s}_{\rm anom\, thr} = 0\,\mathrm{MeV}$		
Source	Re	Im
ANALYTICAL	unstable†	-6.16619605
NUMERICAL I	divergent	divergent
NUMERICAL II	divergent	divergent
NUMERICAL EXP	2.55774480	-6.16742386
$\sqrt{s} - \sqrt{s}_{\rm anom\, thr} = +200\,\mathrm{MeV}$		
Source	Re	Im
ANALYTICAL	2.56203382	-6.16979661
NUMERICAL I	2.56196036	-6.16980269
NUMERICAL II	4.54553765	-6.16979660

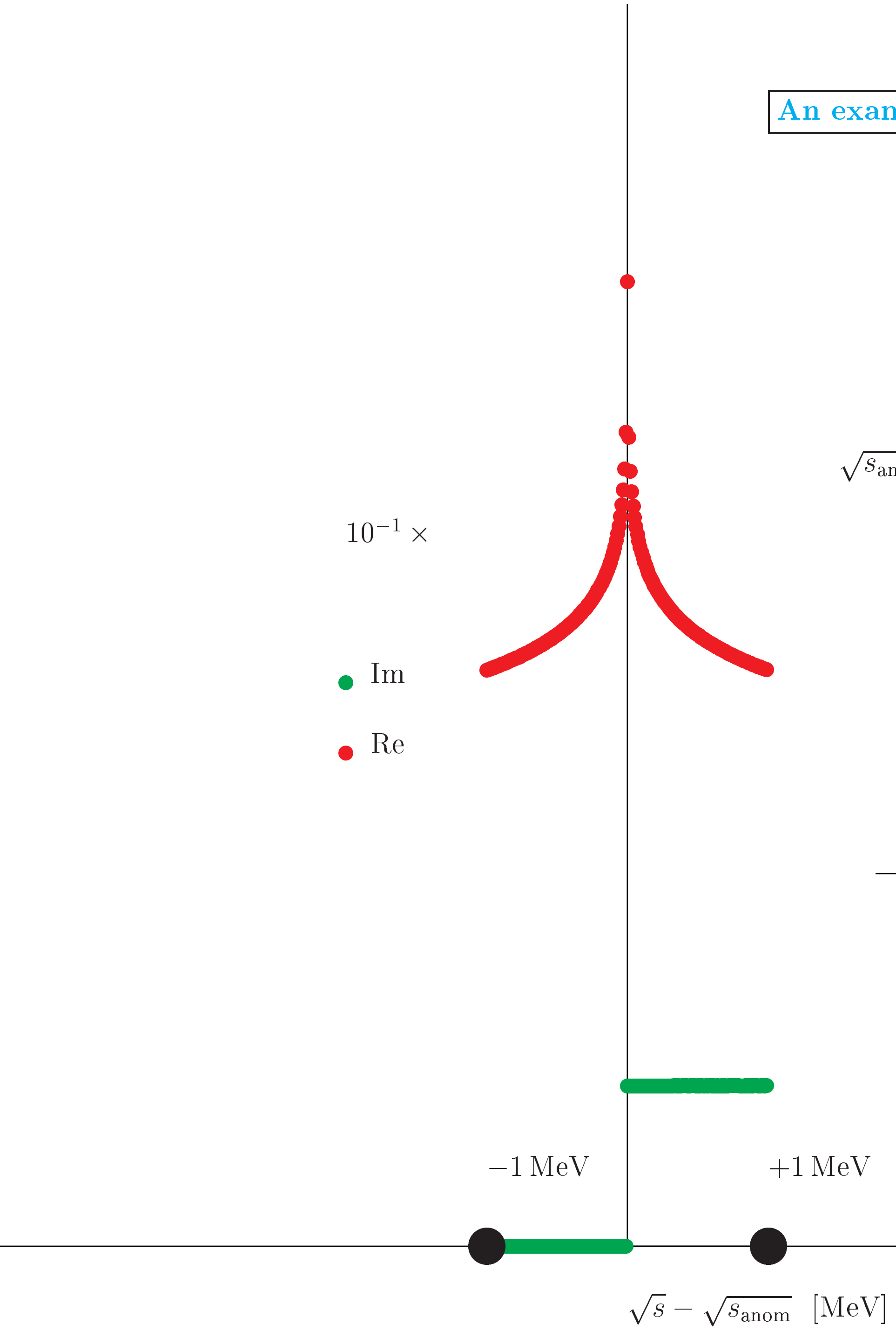
† **in my coding!**

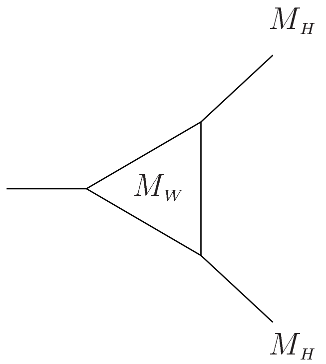
An example where a singularity occurs

$$M_H = \sqrt{3}\,M_W$$

required $\sqrt{2}\,M_W < M_H < 2\,M_W$

$$\sqrt{s_{\rm anom}} = \frac{M_H}{M_W}\,\sqrt{4\,M_W^2 - M_H^2} < 2\,M_H \;!!!$$





Numerical Evaluation requires

Precise knowledge of analytical structure
(si vis pacem para bellum)

$$\Updownarrow\ \Updownarrow\ \Updownarrow$$

Solution of **Landau - Bjorken** equations[4].

$$C_0 \searrow$$

▷ **leading singularity** $\equiv B_3 = 0 \equiv$ anomalous threshold

$$\begin{aligned} s \;&=\; \frac{1}{2\,m_2^2}\,\{2\,m_2^2\,(m_1^2+m_3^2) \\ &-\; (m_1^2+m_2^2-M_1^2)\,(m_3^2+m_2^2-M_2^2) \\ &\pm\; [\lambda(M_1^2,m_1^2,m_2^2)\lambda(M_2^2,m_3^2,m_2^2)]^{1/2}\}, \end{aligned}$$

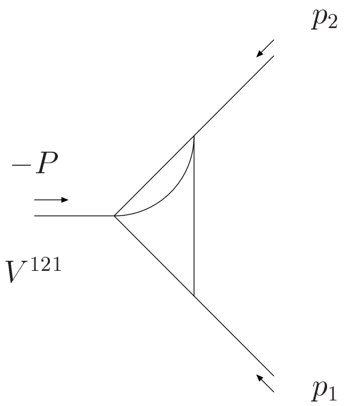
$$(p_1+p_2)^2 \;=\; -s, \qquad p_i^2 = -M_i^2.$$

▷ **sub-leading singularites** $\equiv$ zeros of

$$\lambda(s,m_1^2,m_3^2), \quad \lambda(M_1^2,m_1^2,m_2^2), \quad \lambda(M_2^2,m_2^2,m_3^2).$$

$\equiv$ normal & pseudo thresholds.

[4] In principle also non-Landau or second-type singularities should be considered



$$
\alpha_1\,(q_1^2+m_1^2)=0, \qquad \alpha_2\,((q_1-q_2)^2+m_2^2)=0,
$$
$$
\alpha_3\,((q_2-p_2)^2+m_3^2)=0, \qquad \alpha_4\,((q_2-P)^2+m_4^2)=0,
$$

$$
\begin{aligned}
\alpha_1 q_{1\mu}+\alpha_2(q_1-q_2)_\mu &= 0,\\
-\;\alpha_2(q_1-q_2)_\mu+\alpha_3\,(q_2-p_2)_\mu+\alpha_4\,(q_2-P)_\mu &= 0.
\end{aligned}
$$

Leading $\searrow$

$$
\begin{aligned}
P^2 &= \frac{1}{2\,m_3^2}\,[(-p_2^2-(m_1+m_2)^2+m_3^2)\,(-p_1^2+m_3^2-m_4^2)\\
&+\;2\,m_3^2\,(p_1^2+p_2^2)\pm(\lambda_1\,\lambda_2)^{1/2}]\\
\lambda_1 &= \lambda(-p_1^2,m_3^2,m_4^2) \qquad \lambda_2=\lambda(-p_2^2,(m_1+m_2)^2,m_3^2)
\end{aligned}
$$

Solution for the BT - polynomial $\mathcal{P}$ known

$\forall$ **All Quadrics** $\equiv$ **all** $1l\&nL$ diagrams

Extension to **Cubics** available, example:

$$C(x_1, x_2) = \sum_{i=0}^{3} \sum_{j=0}^{3-i} h_{ij}\, x_1^i\, x_2^j,$$

1. $x_i = x_i' - X_i,$
2. $x_i' = A_{ij}\, x_j'', \quad A_{ij} = A_{ji},$

$$C(x, y) \;\rightarrow\; C_{\rm can} + \lambda = a\, x^3 + b\, y^3 + c\, x^2\, y + d\, x\, y^2 + \lambda.$$

Therefore

$$\begin{aligned}
\int_S dx\, dy C^{-\alpha}(x, y) &= \int_{S'} dx\, dy\, [C_{\rm can} + \lambda]^{-\alpha} \\
&= \frac{1}{3} \int_{S''} dx\, dy\, [x\, C_y + \lambda]^{-\alpha} \\
&= \frac{1}{3\,\lambda} \int_{S''} dx\, dy\, (1 - \frac{x\partial_x}{1-\alpha})\, [x\, C_y + \lambda]^{1-\alpha} \\
C_y &= b\, y^3 + d\, y^2 + c\, y + a.
\end{aligned}$$

but $\cdots\cdots$ this is purely **academical**

Sunset topology (**the easiest**) requires

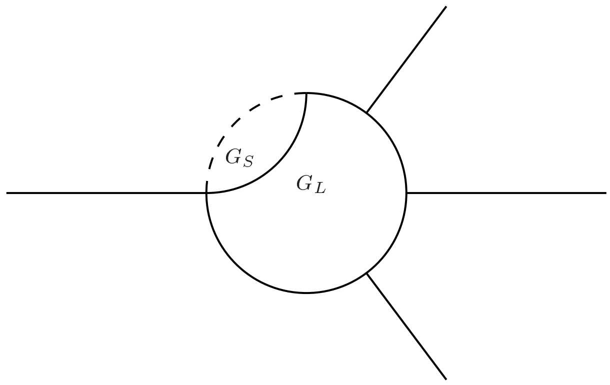
degree $= 4$

in van der Bij - Veltman parametrization

Furthermore, **around - threshold** behavior remains unsolved in **BT** approach,

therefore

Beyond **quadrics** $\Longrightarrow$ *Minimal* BT - algorithm



$$G = \int dx_S \frac{1}{B(x_S)} \int dx_L \mathcal{F}[I_G(x_S, x_L)].$$

- ▷ Analytical structure of G_S simpler,
- ▷ x_S integration with **contour - distortion**, always possible **unless** a **pinch** (or an **end-point**) singularity occur $\cdots\cdots$ but this is a real (Landau) singularity.

However

$\forall$ two-loop two-point functions **2l&2L**

$\forall$ two-loop three-point functions **2l&3L** (but one) **contour - distortion** is not needed $\Longrightarrow$

$\forall$ of them a **parametrization** can be found that allows for a **BT** procedure with $B \equiv B(\mathbf{parameters})$, e.g. the integrand can be transformed, $\{x\} \to \{z\}$, y, into a special form where the **BT**denominator B is parameter - independent !

- ▷ These $2l$ diagrams are no more involved that a $1l$ diagram,
- ▷ Normal BT $\oplus$ integration by parts. $B = 0 \to$ **non-leading** LS.

How to proceed around $B_G \approx 0$?

▷ *$2l\&2L$* and some of the *$2l\&3L$* integrands are *simple* enough so that we can

1. split the integrand in various pieces,
2. apply to some of them *simpler* BT - iterations, e.g.

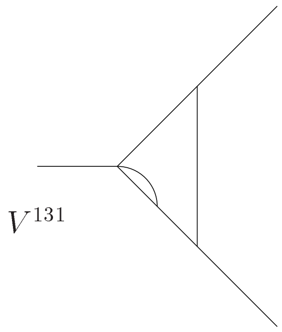
$$-\frac{1}{\epsilon}\,\partial_z\,(z^2+cy)^{-\epsilon} \;=\; 2\,z\,(z^2+cy)^{-1-\epsilon},$$

producing simple results,

▷ express the rest in terms of **hypergeometric** functions which are then expanded around $\epsilon = 0$ and

▷ allow to isolate the **eventual** singularity from a **remainder** that can be safely computed by numerical integration.

▷ Another example:



$$V^{131} = -\left(\frac{\mu^2}{\pi}\right)^{\epsilon} \Gamma\,(1+\epsilon) \int_0^1 dx \int_0^1 dy \int_0^y dz_1 \int_0^{z_1} dz_2$$
$$\times\ [x(1-x)]^{-\epsilon/2}\,(1-y)^{\epsilon/2-1}\,\chi_{131}^{-1-\epsilon},$$

where we rewrite χ_{131} as

$$\chi_{131} = A_{x;131}\,(1-y) + B_{131}(z_1, z_2),$$
$$B_{131}(z_1, z_2) = z^t H z + 2K^t z + m_3^2, \quad A_{x;131} = m_x^2 - m_3^2.$$

The innermost integral becomes now

$$J_{131} = \int_0^{1-z_1} dy\, y^{\epsilon/2-1}\,[A_{x;131}\, y + B_{131}(z_1, z_2)]^{-1-\epsilon}.$$

$$\Rightarrow \quad \text{hypergeometric} \quad \Rightarrow$$

$$I = \int_0^Y dy\, y^{\epsilon/2-1}\,(a\, y + b - i\,\delta)^{-1-\epsilon}$$

$$I \;=\; 2\,\frac{Y^{\epsilon/2}}{\epsilon}\,b^{-1-\epsilon}\,{}_2F_1(1+\epsilon\,,\,\frac{\epsilon}{2}\,;\,1+\frac{\epsilon}{2}\,;\,-\frac{a}{b}\,Y).$$

$$\Rightarrow$$

$$I \;=\; \frac{2}{\epsilon}\,a^{-\epsilon/2}\,b^{-1-\epsilon/2} - \frac{1}{2\,b}\,\ln(1+\frac{b}{a\,Y}) + \mathcal{O}\,(\epsilon)\,.$$

▷ Giving an expression to be integrated **numerically**

$$J_{131} \;=\; \frac{2}{\epsilon}\,(1-\frac{\epsilon}{2}\,\ln A_{x;131})\,C_0 \\ \quad -\; \frac{1}{2\,B_{131}(z_1,z_2)}\,\ln(1+Q_{x;131}).$$

$$Q_{131} \;=\; \frac{B_{131}(z_1,z_2)}{A_{x;131}\,(1-z_1)}.$$

In general other techniques are used, an example:

$$
I(\alpha) \;=\; \int_0^1 dx\,[\chi_B(x) - i\,\delta]^{-\alpha}.
$$

$$
\chi_B(x) \;=\; -P^2\,(x-a)^2 + \frac{\lambda}{4\,P^2} - i\,\delta, \quad a = \frac{1}{2}\,\frac{P^2 + m_2^2 - m_1^2}{P^2}.
$$

where is the singularity ?

▷ Let $x_{\pm}$ be the roots of $\chi_B - i\,\delta = 0$;

▷ whenever they are real and $\in [0,1]$ the integration contour **can be distorted** and the **integral computed**

▷ **unless** a **pinch** will occur, which happens for

$$
\lambda(-P^2, m_1^2, m_2^2) \;=\; 0.
$$

What to do around $\lambda = 0$?

▷ Assume $P^2 = -s$, $s \geq 0$ and $0 \leq a \leq 1$ and obtain

$$
\begin{aligned}
I(\alpha) \;&=\; \Big(\frac{\rho}{s}\Big)^{\alpha} \sum_{X=a}^{1-a} \int_0^X dx\,(1 + \rho\,x^2)^{-\alpha} \\
&=\; \Big(\frac{\rho}{s}\Big)^{\alpha} \sum_{X=a}^{1-a} J_X(\alpha, \rho),
\end{aligned}
$$

▷ where $\rho = -4\, s^2/\lambda$.

▷ We are now interested in the behavior for $|\rho| \to \infty$. Consider the case $\lambda < 0$:

$$
\begin{aligned}
j_X(\alpha, s) &= \int_0^\infty d\rho\, \rho^{s-1} \\
J_X(\alpha, \rho) &= \frac{B(s, \alpha - s)}{1 - 2\, s}\, X^{1-2\, s},
\end{aligned}
$$

which is defined for

$$
0 < \mathrm{Re}\, s < \min\{\frac{1}{2}, \mathrm{Re}\, \alpha\}, \qquad |\arg x^2| < \pi
$$

▷ The inverse Mellin transform will be

$$
J_X(\alpha, \rho) = \frac{1}{2\, \pi\, i} \int_{-i\,\infty}^{+i\,\infty} ds\, \rho^{-s}\, j_X(\alpha, s).
$$

The integration contour will be closed in the right s half-plane and **leading behavior** comes from the pole at $s = 1/2$,

$$
\begin{aligned}
J_X(\alpha, \rho) &= -\frac{1}{2\, \pi\, i}\, \frac{B(1/2, \alpha - 1/2)}{2} \int_{-i\,\infty}^{+i\,\infty} ds\, [\frac{\rho^{-s}}{s - \frac{1}{2}} + \ldots] \\
&= \frac{1}{2}\, B(\frac{1}{2}, \alpha - \frac{1}{2})\, \rho^{-1/2} + \ldots, \quad 1 < \rho < \infty.
\end{aligned}
$$

▷ Therefore we obtain the **leading term** in an expansion which is valid in the strip

$$0 \; < \; -\lambda < 4\,s^2,$$

▷ The **analytical continuation** for $\lambda > 0$ follows from the $i\,\delta$ prescription,

$$I(\alpha) \;\sim\; 2^{2\,\alpha-1}\,B(\frac{1}{2},\alpha-\frac{1}{2})\,(-\frac{\lambda+i\,\delta}{s})^{1/2-\alpha}, \quad \text{for} \quad \lambda \to 0.$$

▷ Note that for the **pseudo-threshold**, $s = (m_1 - m_2)^2$, we have that $a \notin [0,1]$. The complete expansion reads as follows:

$$J_X(\alpha,\rho) \;=\; \frac{1}{2}\,B(\frac{1}{2},\alpha-\frac{1}{2})\,\rho^{-1/2} + J_X^{\rm reg}(\alpha,\rho),$$

$$\begin{aligned} J_X^{\rm reg}(\alpha,\rho) \;&=\; \sum_{k=0}^{\infty}(-1)^k\,\frac{\Gamma\,(\alpha+k)}{\Gamma\,(k+1)}\,\frac{X^{1-2\,(\alpha+k)}}{1-2\,(\alpha+k)}\,\rho^{-\alpha-k} \\ &=\; -\frac{1}{2}\,X\,(\rho\,X^2)^{-\alpha}\,\frac{\Gamma\,(\alpha-1/2)}{\Gamma\,(\alpha+1/2)} \\ &\times\; {}_2F_1(\alpha\,,\,\alpha-\frac{1}{2}\,;\,\alpha+\frac{1}{2}\,;\,-\rho^{-1}). \end{aligned}$$

Any **realistic** numerical strategy must be able to deal with

- ▷ **IR** configurations
- ▷ **Collinear** configurations
- ▷ **Sudakov** configurations

The **numerical** way to **IR** configurations:

A necessary condition for the presence of infrared divergences is that

the Landau equations are fulfilled.

▷ One - loop is easy: $C_0 \Rightarrow B_3 = 0$ when

$$a \;=\; b = f = m^2, \qquad c = -\,e = s - 2\,m^2, \quad d = -2\,m^2,$$

$$\Updownarrow$$

$$[1 + P_x \frac{\partial}{\partial_x} + P_y \frac{\partial}{\partial_y}]\, V^{\mu+1}(x, y) \;=\; 0,$$

solution $\searrow$

$$P_{\{x;y\}} \;=\; \frac{\{1-x;-y\}}{2(\mu+1)}.$$

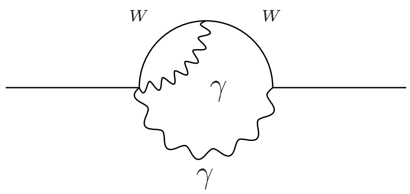
$$\int_0^1 dx \int_0^x dy\, V^{-1-\varepsilon/e}(x,y) \;=\; -\frac{1}{\varepsilon}\int_0^1 dx\, V^{-1-\varepsilon/2}(x,x).$$

In the final answer we must keep trace of all n-dependent factors,

$$\begin{aligned} C_0 \;&=\; -\,\mu^{\varepsilon}\pi^{-\varepsilon/2}\,\Gamma\left(1+\frac{\varepsilon}{2}\right)\frac{1}{\varepsilon}\int_0^1 dx\, V^{-1-\varepsilon/2}(x,x) \\ &=\; -\,\frac{1}{2}\,\mu^{\varepsilon}\,\frac{1}{\bar{\varepsilon}}\int_0^1 dx\, V^{-1-\varepsilon/2}(x,x), \end{aligned}$$

$$\begin{aligned} C_0^{\rm IR} \;&=\; \frac{1}{s-4\,m^2}\{-\frac{1}{2}\frac{1}{\hat{\varepsilon}}[1+\frac{1}{2}\int_0^1 dx\,\ln\frac{V(x,x)}{m^2}]-\frac{1}{2} \\ &+\;\frac{1}{4}\ln\frac{m^2}{\mu^2}\left(1+\frac{1}{2}\ln\frac{m^2}{\mu^2}\right)-\frac{1}{4}\int_0^1 dx\,\ln\frac{V(x,x)}{\mu^2} \\ &\times\;[3+\frac{1}{2}\ln\frac{V(x,x)}{\mu^2}]\} \end{aligned}$$

▷ **(Almost)** all *on-shell* derivatives of $2l\,\&\,2L$ functions are easy enough to allow for BT treatment with a consequent and straightforward analytical evaluation:



$$S_p^{121} = -\frac{2}{m^2}\left(\frac{\mu^2}{\pi\, m^2}\right)^{\epsilon}\Gamma(\epsilon)\int_0^1 dx\int_0^1 dy\int_0^y dz\,[x(1-x)]^{-\epsilon/2}$$
$$\times\;(1-y)^{\epsilon/2}\{z(1-z)\,[(1+\epsilon)\,(z^2-\frac{1}{x})\,\chi_{121}^{-2-\epsilon}$$
$$+\;\frac{\epsilon}{2}\,\chi_{121}^{-1-\epsilon}] + z\,(3\,z-2)\,\chi_{121}^{-1-\epsilon}\}|_{\rm os},$$

$$\chi_{{\rm os};121} = (z-1)^2 + \frac{1}{X}(1-y), \qquad X = \frac{x}{1-x}.$$

First step: **BT** algorithm

$$[1+\frac{1}{1+\epsilon}(y\partial_y + z\partial_z)]\,\chi_{{\rm os};121}^{-1-\epsilon} = -(z^2-\frac{1}{x})\,\chi_{{\rm os};121}^{-2-\epsilon},$$
$$z(1-z)\,\chi_{{\rm os};121}^{-1-\epsilon} = \frac{1}{2\epsilon}\,z\partial_z\,\chi_{{\rm os};121}^{-\epsilon}.$$

Second step: integration by parts

$$S_p^{121} = \frac{1}{2m^2}\left(\frac{\mu^2}{\pi\, m^2}\right)^{\epsilon}\Gamma(\epsilon)\int_0^1 dx\,[x(1-x)]^{-\epsilon/2}(\mathcal{S}_1-\mathcal{S}_2),$$

$$\mathcal{S}_1 = \int_0^1 dy\,(1-y)\,y^{-\epsilon/2-1}\,(y+X')^{-\epsilon},$$

$$\mathcal{S}_2 = \int_0^1 dy\int_y^1 dz\,y^{\epsilon/2-1}\,(z^2+X'\,y)^{-\epsilon},$$

Third step: another **BT** algorithm

$$\begin{aligned} I(\mu,\nu) &= \int_0^1 dy\int_0^1 dz\,y^{\mu}\,z^{\nu}\,(z+\frac{y}{X})^{-\epsilon} \\ &= \frac{1}{2+\mu+\nu-\epsilon}\int_0^1 dy\,[y^{\mu}\,(1+\frac{y}{X})^{-\epsilon}+y^{\nu}\,(y+\frac{1}{X})^{-\epsilon}]. \end{aligned}$$

Fourth step: **hypergeometric** functions, x-integration and Laurant expansion

$$\begin{aligned} S_p^{121} &= \frac{1}{m^2}(\frac{\mu^2}{\pi\, m^2})^{\epsilon}[-\frac{2}{\epsilon^2}-2\,(2-\gamma)\,\frac{1}{\epsilon}-7+\gamma\,(4-\gamma) \\ &- \frac{1}{2}\zeta(2)+\mathcal{O}(\epsilon)]. \end{aligned}$$

More complex diagrams require a new algorithm

- ▷ We **regularize** in n-dimensions and isolate the **infrared poles** by using the technique of

iterated sector decomposition (or **defoliation**). [5]

- ▷ A k/n defoliation is defined by the following relation

$$\int_0^1 \prod_{i=1}^{k} dx_i \prod_{j=k+1}^{n} dy_j \, F(\{x\}\,;\,\{y\}) = \sum_{l=1}^{k} \int_0^1 \prod_{i=1}^{k} dx_i \, x_l^{k-1} \times \prod_{j=k+1}^{n} dy_j \, F(x_l\, x_1, \dots, x_l, \dots x_l\, x_k\,;\,\{y\}).$$

- ▷ A chain of defoliations will stop when all the singularites of F which are located at the vertices of the **simplex** $[0,1]^k$ have been factorized, $F = x_l^{\mu_l}\, F$.
- ▷ To find the **sectors**, although far from unique, is **not** the difficult part of the evaluation. The resulting expression will contain **infrared poles** as well as an **infrared finite** part.
- ▷ The latter and the residues are given by integrals of **inverse powers** of polynomials that are not (usually) **positive defined**. BT - iterations or Mellin - Barnes **splittings** are still required.

[5] T. Binoth and G. Heinrich, Nucl. Phys. B **585** (2000) 741 [arXiv:hep-ph/0004013]

Again understanding from **simple examples**:

$$C_0\left(p_1^2, p_2^2, P^2; m, 0, m\right) \;=\; \mu^{\epsilon}\pi^{-\epsilon/2} \int_0^1 dx \int_0^x dy\, V^{-1-\epsilon/2}(x,y),$$

$$\begin{aligned} V(x,y) \;&=\; -p2^2\, x^2 - p_1^2\, y^2 + (p_1^2 + p_2^2 - P^2)\, x\, y \\ &+\; (p_2^2 - m^2)\, x - (p_2^2 - P^2 - m^2)\, y + m^2. \end{aligned}$$

▷ For $m_2 = 0$ and $m_1 = m_3 = m$ the **anomalous threshold** condition is satisfied if

$$p_1^2 \;=\; p_2^2 = -m^2 \quad \text{and} \quad P^2 \quad \text{is uncontrained}$$

$$\Updownarrow$$

$$V(x,y) \;=\; m^2\,(1-x)^2 + m^2\, y^2 + (P^2 + 2\, m^2)\,(1-x)\, y,$$

▷ showing that $V = 0$ for $x = 1, y = 0$. With $x = 1 - x', y = (1 - x')\, y'$ we

1. split the integration by using

$$1 \;=\; \theta(x-y) + \theta(y-x)$$

2. re-map the two integrals in the interval $[0,1]^2$ and obtain

$$
\begin{aligned}
C_0^{IR} &= \mu^{\epsilon}\,\pi^{-\epsilon/2} \int_0^1 dx \int_0^1 dy \sum_{z=x,y} (z^{-1-\epsilon} - z^{-\epsilon})\,\chi_z^{-1-\epsilon/2}, \\
\chi_x &= m^2\,(1 + x^2\,y^2) + y\,(1-x)\,(2\,m^2 + P^2 + m^2\,y) \\
\chi_y &= m^2\,(1 + x^2\,y^2) + x\,(1-x\,y)\,(P^2 - 2\,m^2) \\
&+ m^2\,x\,(x - 2\,y).
\end{aligned}
$$

▷ The infrared pole can now be extracted as

$$
\begin{aligned}
P_{IR} &= 2\,\mu^{\epsilon}\,\pi^{-\epsilon/2} \int_0^1 dx \int_0^1 dy\, x^{-1-\epsilon}\,\chi_{IR}^{-1-\epsilon/2} \\
&= -\frac{2}{\epsilon}\,\mu^{\epsilon}\,\pi^{-\epsilon/2} \int_0^1 dy\,\chi_{IR}^{-1-\epsilon/2}, \\
\chi_{IR}(y) &= m^2\,y^2 + (P^2 + 2\,m^2)\,y + m^2.
\end{aligned}
$$

Strategies for $P^2 = -\,s,\, s > 0:$

▷ Away from $s \approx 4\,m^2$ and $m^2 \ll s$: one **BT** iteration is enough and **numerical integration** follows;

▷ Around threshold: with $\beta^2 = 1 - 4\,m^2/s$ rewrite

$$\chi_{IR}(y) \;=\; m^2\,(y - \frac{1+\beta^2}{1-\beta^2})^2 + \frac{\beta^2}{\beta^2-1}\,s$$

and apply Mellin - Barnes techniques to obtain the $\beta \to 0$ behavior.

▷ For $s \gg m^2$ it is better to isolate the leading behavior:

$$\begin{aligned}
\int_0^1 dy\,\chi_{IR}^{-1} &= \frac{1}{4\,m^2}\,\frac{\Gamma\,(1/2)}{2\,\pi\,i}\,\int_{-i\,\infty}^{+i\,\infty} dt \\
&\times \frac{\Gamma\,(-t)\;\Gamma^2\,(t+1)}{\Gamma\,(t+3/2)}\,(-\frac{s}{4\,m^2})^t \\
&= -\frac{1}{s}\sum_{n=0}^{\infty}\frac{\Gamma\,(1/2)}{\Gamma\,(1/2-n)\;n!}\,(-\frac{4\,m^2}{s})^n \\
&\times [\ln\frac{-\,s}{4\,m^2} + 3\,\psi(n+1) - \psi(\frac{1}{2}-n)].
\end{aligned}$$

About **Reduction**

- ▷ The **standard** procedure is based on the introduction of **scalar functions** and of **tensor integrals** that are reduced to linear combinations of scalar ones.
- ▷ The **new** procedure is different since the **functional relation** can be written in a more general form starting from

$$
\begin{aligned}
&\int_S dx\, Q(x) \prod_i V_i^{\mu_i}(x), \\
Q(x) \;&=\; \text{an arbitrary polynomial of } x.
\end{aligned}
$$

- ▷ Hence we are led to consider **all integrals** in **parametric space** with no **pre-reduction** in **momentum space**.
- ▷ Furthermore, one can get rid of $Q(x)$ by using

$$
\int_S dx\,(x_i + X_i)\, V^\mu \;=\; \frac{1}{2\,(\mu+1)} \int_S dx\, H^{-1}_{ij}\, \partial_j\, V^{\mu+1},
$$

where X_i is the x_i **co-factor** in **BT** theorem.

There is no **real** Multiloop calculation without the γ^5 problem

Highlights in a systematic approach:[6]

▷ given the well-known *non-regularization* of fermion loops,

▷ if we insist (and we do) on the **trace condition** $\Rightarrow$ **gauge invariance** must be broken in order to obtain a **pseudo-regularization**.

▷ A systematic treatment is needed for the so-called **evanescent operators**; given

$$\{\gamma^\mu, \gamma^\nu\} \;=\; 2\,\delta^{\mu\nu}\,I, \qquad \mathrm{Tr}\,I = 4,$$

$$\delta_{\mu\nu} \;=\; \hat{\delta}_{\mu\nu} + \bar{\delta}_{\mu\nu}, \quad \bar{\delta}_{\mu\alpha}\,\hat{\delta}_{\alpha\nu} = 0, \quad \bar{\delta}_{\mu\mu} = n-4, \quad \hat{\delta}_{\mu\mu} = 4,$$

[6] G. Bonneau, Nucl. Phys. B **177** (1981) 523.
P. Breitenlohner and D. Maison, Commun. Math. Phys. **52** (1977) 11.

▷ Evanescent operators $\equiv \mathcal{E}$ where

$$\bar{p}\cdot\bar{p} = \epsilon\,\hat{p}\cdot\hat{p}\,\mathcal{E}, \qquad \mathcal{E} = \frac{\bar{p}\cdot\bar{p}}{\hat{p}\cdot\hat{p}\,\epsilon},$$

▷ **Evanescent terms are** formally zero in the limit $\epsilon \to 0$ but their effect must be **analyzed carefully** due to the presence of **poles** at $\epsilon = 0$.

▷ Due to **pseudo-regularization** Green's functions do **not** fulfill the corresponding **Ward identities** which must be restored by hand. The first step in the correct treatment of the theory consists in **renormalizing away the poles** at $\epsilon = 0$. This we do in the MS-scheme since **finite-renormalization** can always be done later on.

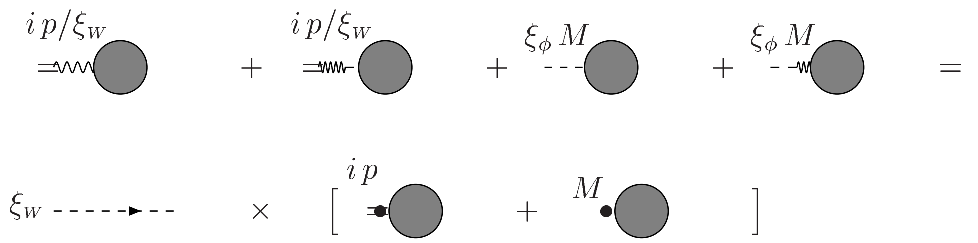
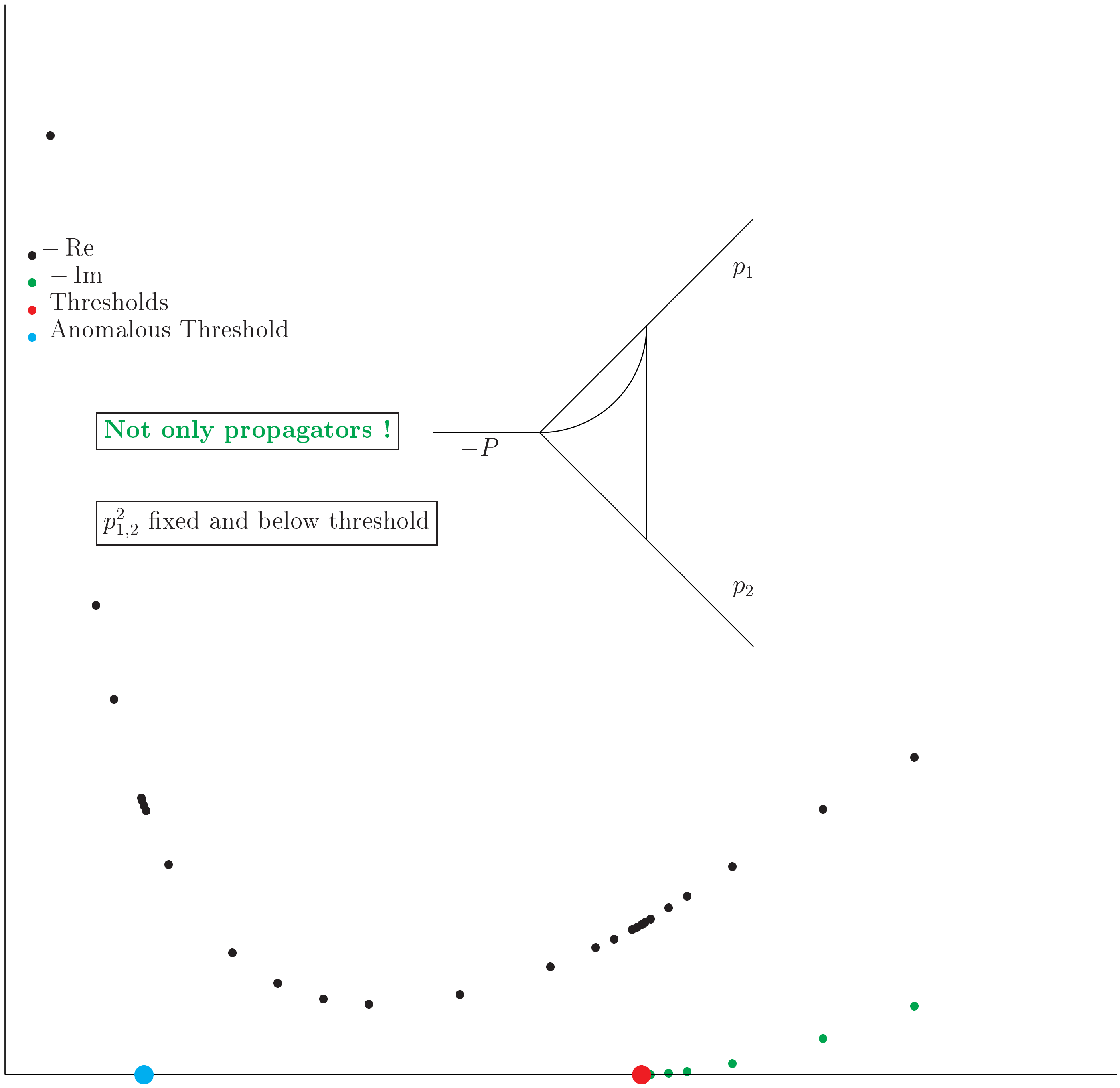
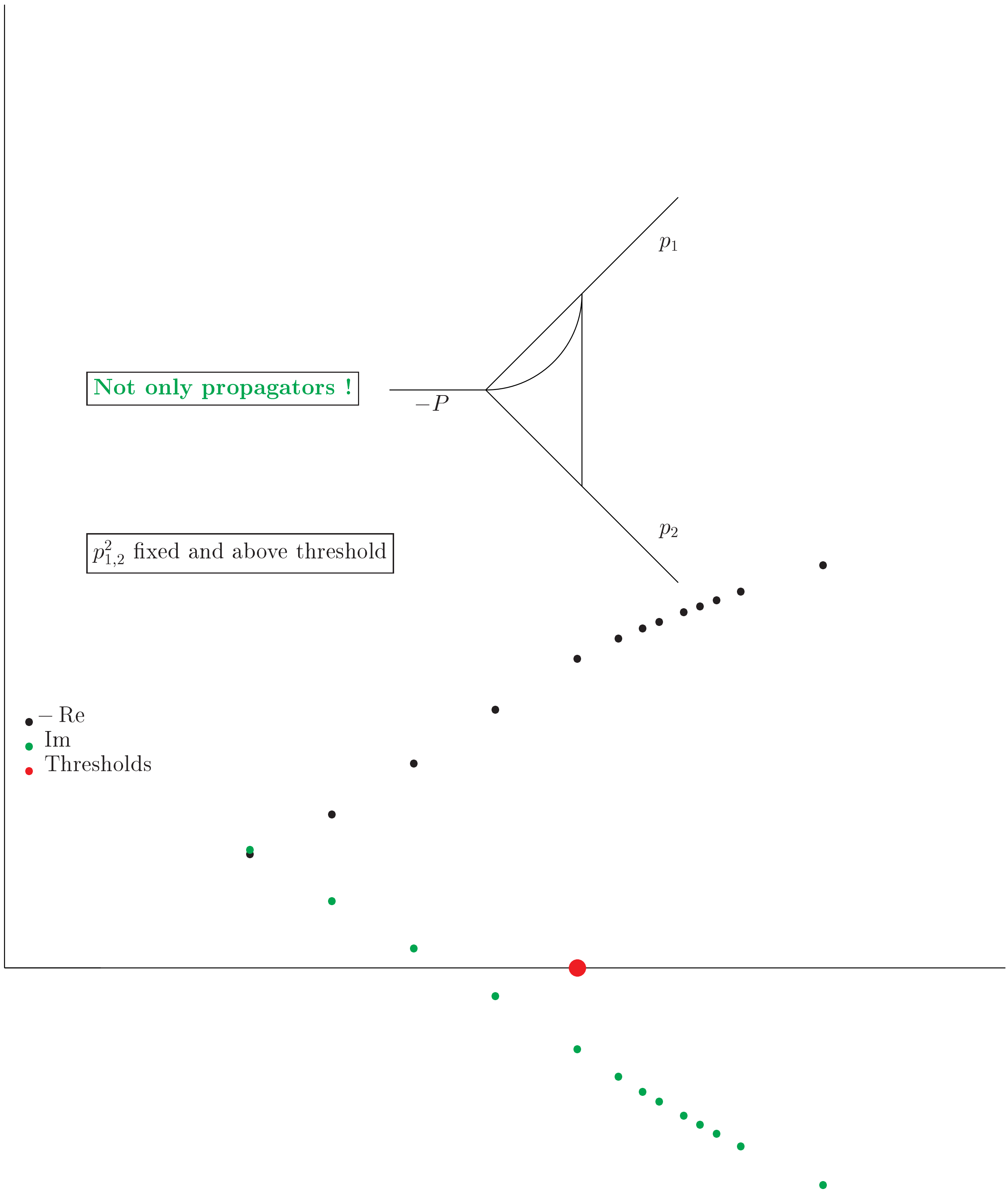
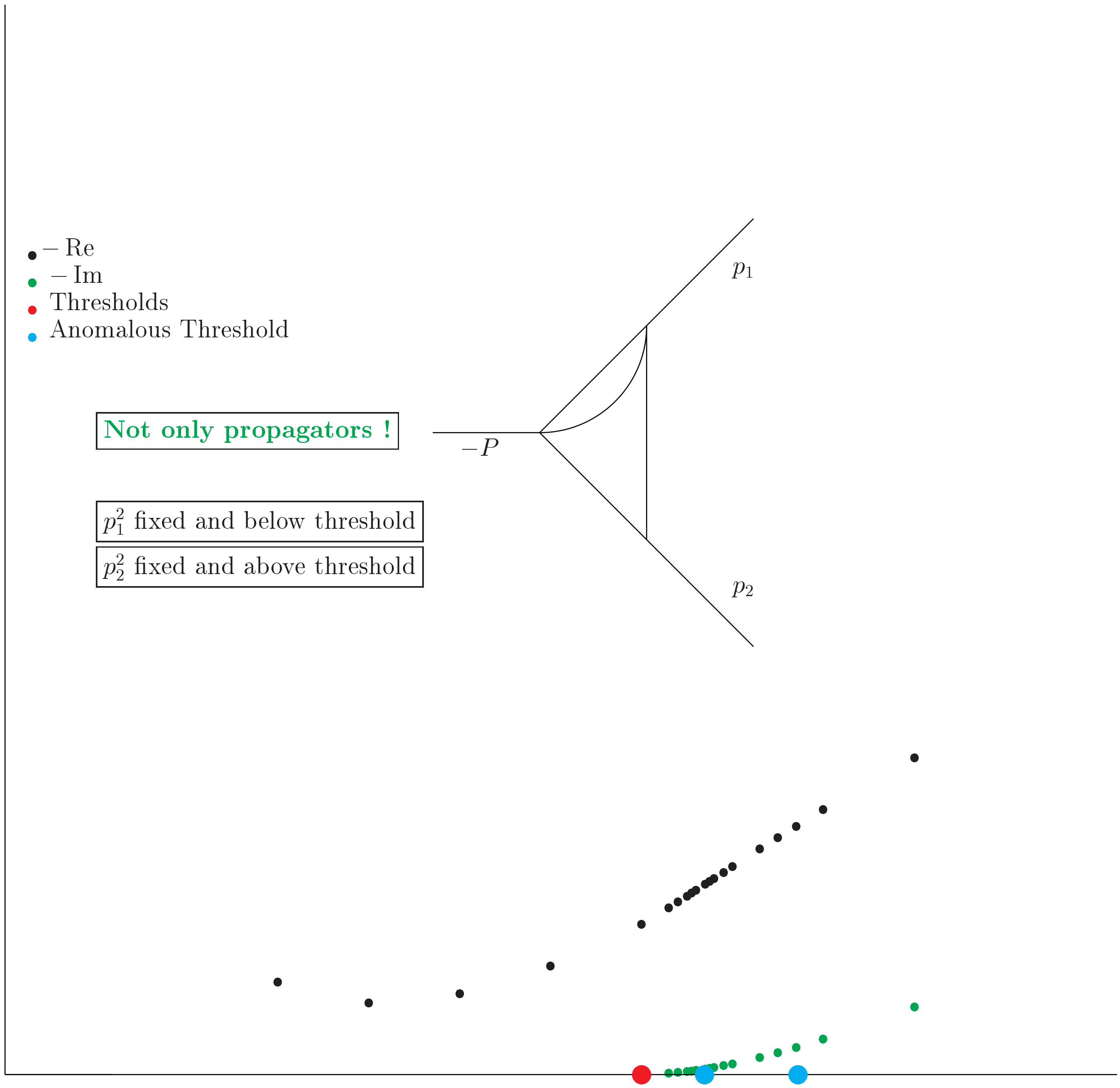


Figure 3: Ward identities for self-energy diagrams. The symbol = indicates multiplication by $i\,p_\mu$ and a dot indicates a truncated Green function.



Anomalous Threshold may occur below normal ones!!!


Not only propagators !
p_1
$-P$
p_2
$p_{1,2}^2$ fixed and above threshold
– Re
Im
Thresholds



– Re
– Im
Thresholds
Anomalous Threshold
Not only propagators !
p_1^2 fixed and below threshold
p_2^2 fixed and above threshold
$-P$
p_1
p_2

So far ......

- ▷ **2l&2/3L** functions are evaluated using the **Korobov-Conroy** number theoretic method with a **Monte-Carlo** error estimate,
- ▷ while **1l&nL** are evaluated with a globally adaptive algorithm (**van Doreen** & **de Ridder, Genz** & **Malik**) for integrals of a vector of similar functions, to assist calculations based on **Form Factor** reduction.
- ▷ A `Form` (**Vermaseren**) code generates all **Standard Model 2l&2L** functions available, i.e. **SM** rules are built-in. Interface is still handmade.
- ▷ At this moment `TOPSIDE` project is mostly in the phase of getting the (right) *numbers* from all two-loop diagrams with **Keyword** $\equiv$ **Everywhere**.
- ▷ The final strategy will be to **map** as many integrals as possible in one integral,

$$\text{Process} \equiv \int_{\text{momenta}} \sum \text{diagrams}$$

$$\equiv \int_{S_{\max}} \sum_{ij} Q_{ij}(x_{\text{sub}_i} \in \{x_{\max}\})\, V_i^{\mu_i}(x_{\text{sub}_i} \in \{x_{\max}\}).$$

Example: $Z \to \overline{f} f$, one-loop

$$\{x_{\max}\} = \{x_1, x_2\},$$

$$Diagrams = \begin{cases} \textbf{WFnormalization} & \equiv \textbf{two} - \textbf{point} & x_{\mathrm{sub}_i} = x_1 \\ \textbf{Vertices} & \equiv \textbf{three} - \textbf{point} & x_{\mathrm{sub}_i} = x_{\max} \end{cases}$$

$$S_{\max} = \{0 \le x_{1,2} \le 1\}$$

$$\begin{aligned} Q_{ij} &= j\textbf{th component of} \\ & \quad i - \textbf{diagram, without mom-reduction, e.g.} \end{aligned}$$

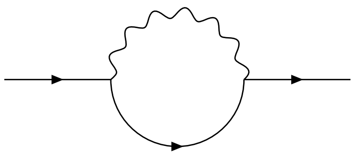


Figure 4: Fermionic self-energy in QED.

$$\begin{aligned} \Sigma &= -\, i e^2\, \pi^{2-\epsilon/2}\, \mu^{\epsilon}\, \Gamma\left(\frac{\epsilon}{2}\right) \int_0^1 dx\, \left[i\, (2-\epsilon)\, x\not{p} + (4-\epsilon)\, m_e\right] V^{-\epsilon/2}, \\ V(x) &= (1-x)\, \left(x\, p^2 + m_e^2\right). \end{aligned}$$

Preliminar considerations about **CPU - time**

▷ **Keyword** $\equiv$ **Multipurpose**: the aim is to get results **everywhere**;

▷ During the last 4 **years** I gained a factor 80 in **CPU - time** by changing 3 Workstations;

▷ **Warning**: $t_{N\,\text{diagrams}} \ll N \times t_{1\,\text{diagram}}$

$\mathrm{Re}\, S^{221}$	0.480(2)	0.4786(9)	0.4788(2)	0.4790(1)
$\mathrm{Im}\, S^{221}$	0.372(1)	0.3695(4)	0.3703(2)	0.3704(1)
CPU time [sec]	0.81	12.10	46.20	89.60

Table 1: S^{221} at $-1\,\mathrm{GeV}$ from the second two-particle normal threshold, on a alpha EV6 (21264), 500 MHz