Precision Physics Near LEP Shutdown and Evolutionary Developments

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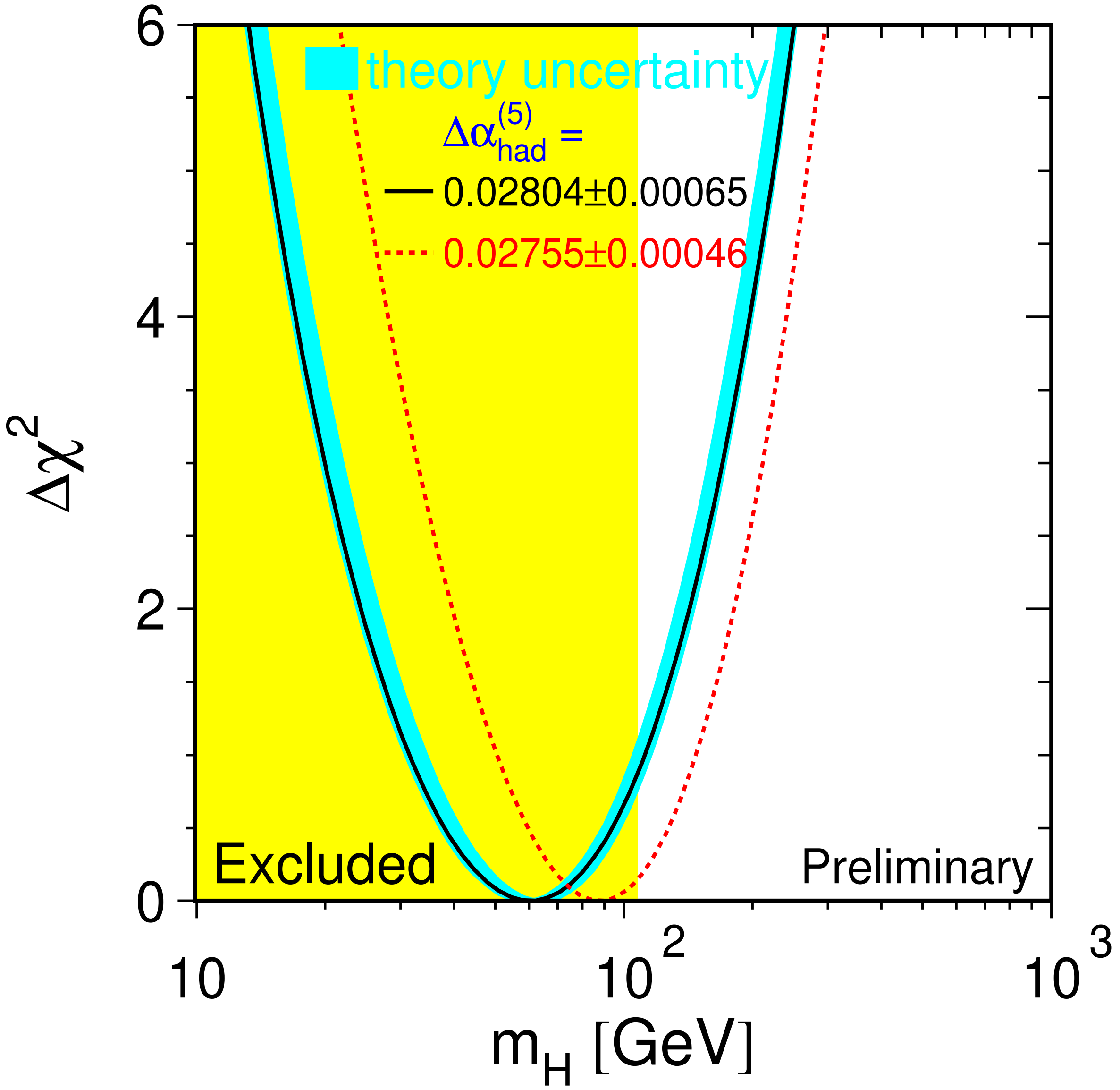
Abstract

Something from the past, a lot from the present,
a hint at the future.

Experimental Progress

- The Exp. progress made in the **1990's** is remarkable in many ways. With the **right experiments** we can reveal the fine details of high energy phenomena.
- Near **LEP** Shutdown we known that physics at the **microscopic scale** has a basis as **rationale** as chemistry.
- Everything moves, **not by magic**, but because there is a **mechanism at work**.
- Everything is encouraging for the major **unsolved** aspects of the breaking of electroweak gauge symmetry.



The blue band **a.d. 2000**

The most celebrated figure of LEP era

It is interesting to examine the **environment** in which the **technology** for producing this figure has been developed. One way is through the **Kühn tower**.

the Pushers: **higly-specialized experts, QED, QCD etc**

△ △ △

the Smugglers: **those who assemble a code mooncursing it**

△ △ △

the Users: **the experimental community at large**

∇ Some sort of **universal language** has been created to allow a **meaningful** communication

The Pseudo-Observables

- The **PO** are related to measured **cross-sections** and **asymmetries** by some deconvolution or **unfolding** procedure.

- The concept itself of **pseudo-observability** is rather difficult to define.

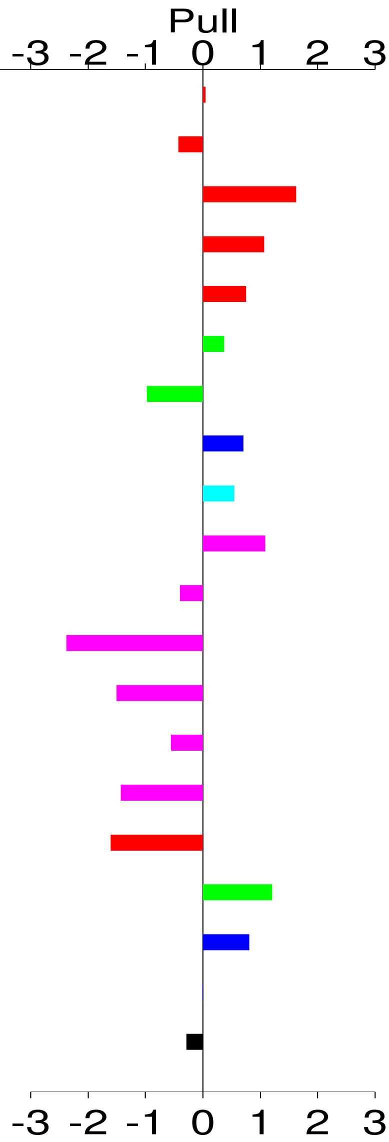
- One way to introduce it is to say that the experiments **measure** some **primordial** (basically **cross-sections** and thereby **asymmetries** also) quantities which are then

- **reduced** to secondary quantities under some set of **specific assumptions**.

- Within these assumptions the secondary quantities, the **PO**, also deserve the label of **observability**.

Osaka 2000

	Measurement	Pull
m_Z [GeV]	91.1875 ± 0.0021	.05
Γ_Z [GeV]	2.4952 ± 0.0023	-.42
σ^0_{hadr} [nb]	41.540 ± 0.037	1.62
R_l	20.767 ± 0.025	1.07
$A^{0,l}_{fb}$	0.01714 ± 0.00095	.75
A_e	0.1498 ± 0.0048	.38
A_τ	0.1439 ± 0.0042	-.97
$\sin^2\theta^{lept}_{eff}$	0.2321 ± 0.0010	.70
m_W [GeV]	80.427 ± 0.046	.55
R_b	0.21653 ± 0.00069	1.09
R_c	0.1709 ± 0.0034	-.40
$A^{0,b}_{fb}$	0.0990 ± 0.0020	-2.38
$A^{0,c}_{fb}$	0.0689 ± 0.0035	-1.51
A_b	0.922 ± 0.023	-.55
A_c	0.631 ± 0.026	-1.43
$\sin^2\theta^{lept}_{eff}$	0.23098 ± 0.00026	-1.61
$\sin^2\theta_W$	0.2255 ± 0.0021	1.20
m_W [GeV]	80.452 ± 0.062	.81
m_t [GeV]	174.3 ± 5.1	-.01
$\Delta\alpha^{(5)}_{had}(m_Z)$	0.02804 ± 0.00065	-.29



Those who paved the road

and, **here**, I don't mean all those who gave **seminal contributions**!

- **Sirlin and collaborators**,
- **Veltman and collaborators**,
- the german connection: **Hollik, Kühn, Jegerlehner and collaborators,**
- the dutch connection: **Berends, Kleiss, Beenakker and collaborators,**
- the former east block: **Bardin, Riemann and collaborators,**
- the Krakow - Knoxville collaboration **Jadach, Ward and collaborators**
- the italian gang

a quick example ......

- The explicit formulae for the $Zf\overline{f}$ vertex are always written starting from a **Born-like** form of
  - ▷ a pre-factor × fermionic current,

- where the Born parameters are promoted to effective, scale-dependent parameters,

$$\rho_Z^f \gamma_\mu \left[\left(I_f^{(3)} + i\, a_L \right) \gamma_+ - 2\, Q_f \kappa_Z^f s^2 + i\, a_Q \right] = \gamma_\mu \left(\mathcal{G}_V^f + \mathcal{G}_A^f \gamma_5 \right), \tag{1}$$

  where $\gamma_+ = 1 + \gamma_5$ and $a_{Q,L}$ are the SM imaginary parts.

- By definition, the total and partial widths of the Z boson include also **QED** and **QCD** corrections. The partial decay width is therefore described by the following expression:

$$\Gamma_f \equiv \Gamma\left(Z \to f\overline{f}\right) = 4\, cf.\ \Gamma_0 \left(|\mathcal{G}_V^f|^2\, R_V^f + |\mathcal{G}_A^f|^2\, R_A^f \right) + \Delta_{\rm EW/QCD}\,, \tag{2}$$

  where $cf. \; = 1$ or 3 for leptons or quarks ($f = l, q$), and

  - ▷ R_V^f and R_A^f describe the final state QED and QCD corrections and take into account the fermion mass m_f.

- The last term,

$$\Delta_{\rm EW/QCD} = \Gamma^{(2)}_{\rm EW/QCD} - \frac{\alpha_S}{\pi}\, \Gamma^{(1)}_{\rm EW}\,, \tag{3}$$

  accounts for the **non-factorizable** corrections.

- The standard partial width, Γ_0, is

$$\Gamma_0 = \frac{G_F M_Z^3}{24\sqrt{2}\,\pi} = 82.945(7)\,\text{MeV}. \tag{4}$$

- The peak hadronic and leptonic cross-sections are defined by

$$\sigma_h^0 = 12\pi\,\frac{\Gamma_e \Gamma_h}{M_Z^2 \Gamma_Z^2} \qquad \sigma_\ell^0 = 12\pi\,\frac{\Gamma_e \Gamma_l}{M_Z^2 \Gamma_Z^2}\,, \tag{5}$$

where Γ_Z is the total decay width of the Z boson, i.e, the sum of all partial decay widths.

- The effective electroweak **mixing** angles (*effective sinuses*) are always defined by

$$4\,|Q_f|\,\sin^2\theta^f_{\text{eff}} = 1 - \frac{\text{Re}\,\mathcal{G}^f_V}{\text{Re}\,\mathcal{G}^f_A} = 1 - \frac{g^f_V}{g^f_A}\,, \tag{6}$$

where we define

$$g^f_V = \text{Re}\,\mathcal{G}^f_V, \qquad g^f_A = \text{Re}\,\mathcal{G}^f_A. \tag{7}$$

- The forward-backward asymmetry A_{FB} is defined via

$$A_{\text{FB}} = \frac{\sigma_{\text{F}} - \sigma_{\text{B}}}{\sigma_{\text{F}} + \sigma_{\text{B}}}\,, \qquad \sigma_{\text{T}} = \sigma_{\text{F}} + \sigma_{\text{B}}\,, \tag{8}$$

where σ_{F} and σ_{B} are the cross-sections for forward and backward scattering, respectively.

Admittedly, we are entering a **new era**:

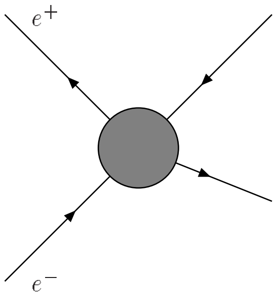
we have been moving from a **mediaeval architecture** of radiative corrections

to the **renaissance** of **LEP period** and now one should not indulge in deciphering the intimate meaning of the **last digit** in the **last fit** of the **LEPEWWG**.

A new **frontier** is at the horizon: most likely it is goodbye to the one man show. Running a new **RC** project will be a little like running an experiment.

PRESENT STATUS of RC

PCP Report hep-ph/9902452



one-loop $\Longrightarrow$ **ALL**

two-loop $\Longrightarrow$ **LO + NLO**

+ re-summations

IS QED, **SF-method** (or **equivalent**)
FS QED, **available**
IS $\otimes$ **FS QED**

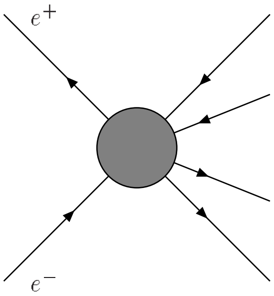
Fine Points in QED for $2 \to 2$

s-channel $\mathcal{O}\left(\alpha^2 L^n\right)$, $n = 0, 1, 2$ (Calculations), $\mathcal{O}\left(\alpha^3 L^3\right)$
important for Z **lineshape**

Differences and uncertainties amount to at most $\pm 0.1\,\mathrm{MeV}$ on M_Z and Γ_Z and $\pm 0.01\%$ on σ^0_{h} (**MIZA, TOPAZ0** and **ZFITTER**) LEP EWWG/LS 2000-01

non-annihilation (Bhabha) both **SF** and **PS** have been analyzed, uncertainty estimated to be **0.061%** (**BHLUMI**)

FULL two-loop EW needed for **GigaZ** ($10^9 Z$ events),
Numerical evaluation of **FD**?



$e^+e^- \to 4\,\mathbf{f}$ **Born** $\Longrightarrow$ **ALL** processes

$\mathcal{O}(\alpha)$ EW $\Longrightarrow$ **only in DPA for** WW **signal**

$e^+e^- \to 4\mathrm{f} + \gamma \Longrightarrow$ **ALL**

hep-ph/0005309

Fine Points in QED for $2 \to 4$

for $e^+e^- \to WW \to 4\,f \;\to$ **see DPA**

for generic $e^+e^- \to 4\,f \;\to$ **s − channel SF**

$\to$ *i.e.* **LL approximation**

◇ the latter strictly applicable only if **ISR** can be separated (may lead to **excess** of **ISR**)

◇ **preliminar** investigations towards **non–s SF** by **GRACE** and **SWAP** **hep-ph/0005309**

Bulk of the effect but *ad hoc*

How well do we know realistic observables in $e^+e^- \to 2\mathrm{f}$?

- Pretty **well** around the Z **peak**
- Reasonably well **up to LEP 2 energies**, but **fermion-pair** corrections are not really under control.

Let's see the general state of PP

- Let's start from a simple case, e^+e^- **PP**-corrections to $e^+e^- \to \overline{b}b$.
  - ▷ diagrams of the first row are termed **Multi-Peripheral**, or **MP**;
  - ▷ diagrams of the second row are termed **Initial State Singlet**, or **ISS**;
  - ▷ diagrams of the third row are termed **Initial State Non-Singlet**, or **ISNS**;
  - ▷ diagrams of the fourth row are termed **Final State**, or **FS**.

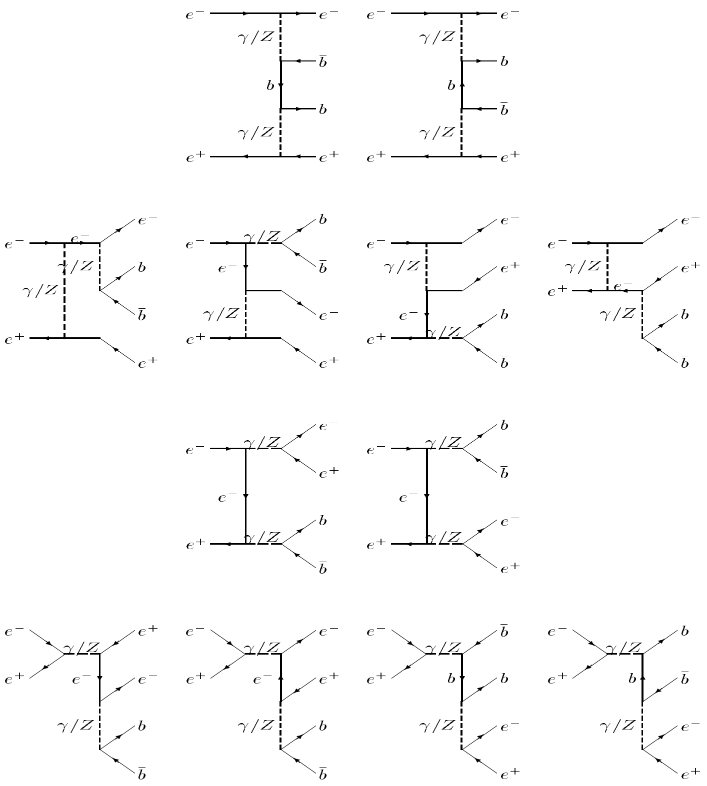


Figure 12: *Complete set of diagrams for the process* $e^+e^- \to e^+e^-b\bar{b}$.

26

- Note that we include both γ and Z exchange.
- On top of **real pair production** one has to include **virtual** e^+e^- pairs.

Let's compare as a typical example the sum of **virtual** and **real** pair corrections for **primary** hadrons and **primary** muons at $\sqrt{(s)} = 189$ **GeV**.

Table 1: Relative virtual + real pair corrections in per mill of the total cross-section for the process $e^+e^- \to$hadrons at $\sqrt{s} = 189$ GeV for different signal definitions. A – means that this definition is not accessible in the respective program.

definition	GENTLE	ZFITTER	TOPAZ0	GENTLE	ZFITTER	TOPAZ0
R_{cut}	0.01			0.7225		
A) diagram, $s' = M^2_{\mathrm{prop}}$	14.72	15.76	16.42	-1.13	-0.92	-0.98
B) diagram, $s' = M^2_{\mathrm{inv}}$	14.74	15.82	–	-1.45	-1.20	–
C) cuts, $P_{\mathrm{cut}} = 0.15$	14.64	–	–	-1.45	–	–
D) cuts, $P_{\mathrm{cut}} = 0.10$	14.45	–	–	-1.45	–	–
$\delta_{\mathrm{B}} - \delta_{\mathrm{A}}$	0.02	0.06	–	-0.32	-0.29	–
$\delta_{\mathrm{B}} - \delta_{\mathrm{C}}$	0.10	–	–	< 0.0001	–	–
$\delta_{\mathrm{B}} - \delta_{\mathrm{D}}$	0.29	–	–	< 0.0001	–	–

Table 2: Relative virtual + real pair corrections in per mill of the total cross-section for the process $e^+e^- \to \mu^+\mu^-$ at $\sqrt{s} = 189$ GeV for different signal definitions. A – means that this definition is not accessible in the respective program.

definition	GENTLE	ZFITTER	TOPAZ0	GENTLE	ZFITTER	TOPAZ0
R_{cut}	0.01			0.7225		
A) diagram, $s' = M^2_{\mathrm{prop}}$	10.73	11.73	12.26	-1.37	-1.00	-0.98
B) diagram, $s' = M^2_{\mathrm{inv}}$	10.84	11.96	–	-2.58	-2.05	–
C) cuts, $P_{\mathrm{cut}} = 0.15$	10.72	–	–	-2.58	–	–
D) cuts, $P_{\mathrm{cut}} = 0.10$	10.57	–	–	-2.58	–	–
$\delta_{\mathrm{B}} - \delta_{\mathrm{A}}$	0.11	0.23	–	-1.21	-1.05	–
$\delta_{\mathrm{B}} - \delta_{\mathrm{C}}$	0.12	–	–	< 0.0001	–	–
$\delta_{\mathrm{B}} - \delta_{\mathrm{D}}$	0.27	–	–	< 0.0001	–	–

The comparison between **GENTLE, ZFITTER**, and **TOPAZ0** for the **diagram-based** definition reveals **maximum differences** of **1.7 (1.5) per mill** for **inclusive hadrons (muons)** and **0.2 (0.4) per mill** for **high** s' **hadrons (muons)**.

$\boxed{\textbf{caveat: PP to be completed}}$

expressly at **higher energies** and above **vector boson** thresholds.

$\boxed{\textbf{First evolutionary development:}}$

- **4f** **MC** to be **fully** interfaced with **2f** **MC**

Imagine: **LEP** will be running another year ......

NEXT: There are several 4fprocesses (those with t-channel γs) that are not dominated by annihilation

example: single-W production, **two-photon** processes, etc

$\frown$ $\boxed{\text{How to \textbf{include} the \textbf{bulk} of QED \textbf{RC}?}}$

$\boxed{\textbf{Second evolutionary development:} \Longrightarrow}$

Is multi-γ **radiation** a **one**-scale or
a **multi**-scale **convolution** phenomenon?

$$\sigma\left(p_+p_- \to q_1\dots q_n + \mathbf{QED}\right) \stackrel{?}{=} \int dx_+\,dx_-\,D(x_+,?)D(x_-,?) \times \sigma\left(x_+p_+x_-p_- \to q_1\dots q_n\right)$$

? to be guessed

How are **standard SFs** related to **exact YFS exponentiation**?

▷ In **standard YFS** treatment,

$$\sigma\left(p_+ + p_- \to \sum_{i=1,2l} q_i + \sum_{j=1,n} k_j\right) \sim \int dPS_q \mid M_0\mid^2 \times E\left(p_+ + p_- - \sum_i q_i\right),$$

$$E(K) = \frac{1}{(2\,\pi)^4}\int d^4x\,\exp(iK\cdot x)\,E(x),$$
$$E(x) = \exp\left\{\frac{\alpha}{2\,\pi^2}\int d^4k e^{ik\cdot x}\delta^+(k^2)\mid j^\mu(k)\mid^2\right\}$$

▷ **do not** separate **soft** from **hard** and compute $(n \neq 4, k^2 = 0)$

$$I = \int d^n k\,e^{ik\cdot x}\frac{\delta^+(k^2)}{p_i\cdot k\,p_j\cdot k}$$

▷ In **Dimensional Regularization** one has $\forall x^2$

$$
\begin{aligned}
I(x) &= -\pi\,\rho \int_0^1 \frac{du}{P^2}\left(\frac{1}{\hat{\varepsilon}} + 2\,\ln 2 - \ln x^2 - \xi\,\ln\frac{\xi+1}{\xi-1}\right), \\
\xi &= \frac{|x_0|}{r}, \quad \text{with} \quad x_0 \to x_0 + i\delta(\delta \to 0_+), \\
P &= p_j + (\rho p_i - p_j)\,u,
\end{aligned}
$$

$$
(\rho p_i - p_j)^2 \;=\; 0, \quad x_0 = -\frac{P\cdot x}{\sqrt{-P^2}}, \quad r^2 = x_0^2 + x^2.
$$

▷ **Last** integral $\to$ **IR-pole** + collection of Li_2's

▷ $E(K)$ **not** available in close form. **Proposed scheme**

$$
\boxed{\longmapsto \textbf{define} \text{ coplanar approximation}}
$$

$$
\begin{aligned}
I^c_{ij} &\stackrel{\text{def}}{=} -\frac{2}{3}\pi\,\rho_{ij}\,\mathcal{F}_{\text{cp}}\,\frac{1}{p_j^2 - \rho_{ij}^2\,p_i^2}\,\ln\frac{\rho_{ij}^2 p_i^2}{p_j^2}, \\
I^c_{ii} &\stackrel{\text{def}}{=} -\frac{2}{3}\pi\,\rho_{ij}\,\mathcal{F}_{\text{cp}}\,\frac{1}{m_i^2}, \\
\mathcal{F}_{\text{cp}} &= \ln\left\{e^{-\Delta_{\text{IR}}}\,\frac{p_i\cdot x\,p_j\cdot x}{m_i m_j}\right\}, \\
\Delta_{\text{IR}} &= \frac{1}{\hat{\varepsilon}} + \text{constants.}
\end{aligned}
$$

C. Chahine '78

▷ within **coplanar** approximation $\rightsquigarrow$

$$E^{\text{pair}\,<ij>}(K) \stackrel{\text{cp}}{\to} \frac{1}{(2\,\pi)^2}\left\{\frac{e^{-\Delta_{\text{IR}}}}{m_i m_j}\right\}^{-\alpha A_{ij}} \frac{1}{\Gamma^2(\alpha A_{ij})}$$

$$\times \int_0^\infty d\sigma d\sigma' \,(\sigma\sigma')^{\alpha A_{ij}-1}\,\delta^4\left(\sigma p_i + \sigma' p_j - K\right).$$

N.B. that's why *coplanar* $\nearrow$

▷ **Note** that $\alpha A \sim \beta$ for **invariant** $\gg$ **mass**^{2} but

▷ the above **expression** is valid for **all regimes** and $\xrightarrow{\text{see next figure}}$

▷ is **easily** generalized to **n emitters** with the **result** that

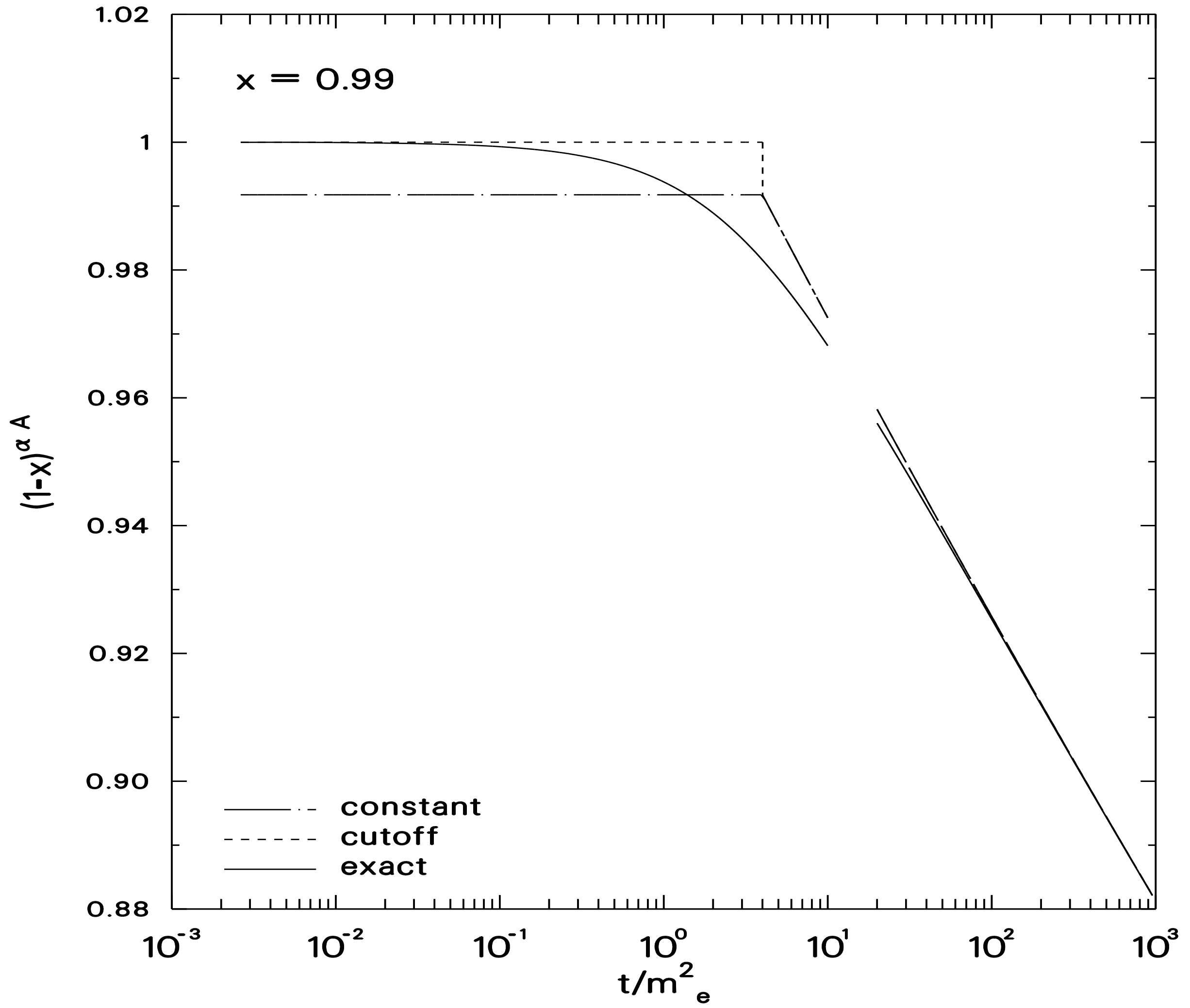
A.Ballestrero,G.P. work in progress

1. In $2 \to n$ **ANY** external charged leg i talks to **ALL** other charged legs,
2. each time with a **known scale** s_{ij},
3. and with a **known TOTAL** weight proportional to

$$x_i^{\alpha\left(A_1^i+\ldots+A_I^i\right)-1} / \,\Gamma\left(\alpha\,\left(A_1^i+\ldots+A_I^i\right)\right), \qquad 0 \le x_i \le 1$$

$\nearrow$ **N.B.** each A has the appropriate sign **in/out, part/antp**

4. where $I(i)$ is the number of **pairs** $< ij >$ with i fixed.



The IR exponent

$$\alpha A = \frac{2\,\alpha}{\pi}\left\{\frac{1+r^2}{1-r^2}\,\ln\frac{1}{r} - 1\right\}, \qquad \frac{m_e^2}{|t|} = \frac{r}{(1-r)^2}$$

$$-A(s,m_e) - A(t,m_e) + A(u,m_e) \overset{\text{not a guess}}{\sim} \frac{2}{\pi}\left[\ln\frac{st}{m_e^2 u} - 1\right]$$

CONCLUSION for QED

- ▷ **SF**-language still **applicable** but
- ▷ **ONE** scale is not **enough** (quite obviously),
- ▷ each external **LEG** brings a **Structure Function**,
- ▷ since **ALL** charged legs talk to each other, **each SF** is **NOT** function of **ONE ad hoc** scale but
- ▷ **ALL** $< ij >$ scales enter into $\mathbf{SF}_i$, **exact** spectral-function is a convolution of **SFs**

$$
\begin{aligned}
E^{\mathrm{pair}\,<ij>}(K) &= \int d^4K'\,\Phi(K')E^{\mathrm{pair}\,<ij>}_{\mathrm{cp}}(K-K'),\\
\Phi(K) &= \frac{1}{(2\,\pi)^4}\int d^4x\;\exp\{i\,K\cdot x+\alpha\,\left(I-I_{\mathrm{cp}}\right)\}\\
&= \delta(K)+\mathcal{O}\,(\alpha)
\end{aligned}
$$

- ▷ **IR-finite** reminders and **virtual** parts can be added according to standard approach, etc etc

$\leadsto$

$$(\mathbf{Multi-Scale\ \ SF})\ \times\ (\mathbf{1}+\mathcal{O}\,(\alpha))$$

Third evolutionary development:

A **systematic** approach to **multi-loop calculation**

- Although a lot can be done by **approximations**:
  - **massless** internal lines,
  - small momenta **expansion**,
  - large momenta **expansion**
  - etc

- do we have a **new, implementable** algorithm for a full scale attack to the problem?

There is a **new** idea around

$\boxed{\textbf{Tkachov Theorem}}$ or

$\boxed{\textbf{generalized Bernstein theorem}}$

$$\mathcal{P}\,(x,\partial)\prod_i V_i^{\mu_i+1}(x) \;=\; B\,V_i^{\mu_i}(x).$$

where $\mathcal{P}$ is a polynomial of $x = (x_1 \cdots x_n)$ (a vector of Feynman parameters) and $\partial_i = \partial/\partial_i$; B and all coefficients of $\mathcal{P}$ are polynomials of μ_i and of the coefficients of $V_i(x)$.

We can do numerical calculations!

However

One-loop versus **multi-loop** is exactly as 2d Ising model versus 3d Ising model

- For one-loop we have a universal master formula (F. V. Tkachov):

  $$V(x) = x^t\, H\, x + 2\, K^t\, x + L,$$

$$\mathcal{P} = 1 - \frac{(x + X)\,\partial_x}{2\,(\mu + 1)},$$

$$X = K^t\, H^{-1}, \qquad B = L - K^t\, H^{-1}\, K.$$

this everybody can do ......

$$
\begin{aligned}
C_0 &= \int_0^1 dx \int_0^x dy V^{-1-\varepsilon/2}(x,y), \\
V(x,y) &= a\,x^2 + b\,y^2 + c\,xy + d\,x + e\,y + f - i\,\varepsilon
\end{aligned}
$$

$$
\begin{aligned}
C_0^d &= 2\int_0^1 dx \int_0^x dy V(x,y) \ln V(x,y) G_2^2 \Delta^{-2}, \\
C_0^s &= \frac{1}{2}\int_0^1 dx V(x,x) \ln V(x,x) G_2 \Delta^{-2}(a_x - a_y) + \frac{3}{4}\int_0^1 dx V(x,x) \ln V(x,x) \Delta^{-1}(a_x - a_y) G_{11}\Delta_1^{-1} \\
&+ \frac{1}{2}\int_0^1 dx V(x,0) \ln V(x,0) G_2 \Delta^{-2} a_y + \frac{3}{4}\int_0^1 dx V(x,0) \ln V(x,0) \Delta^{-1} a_y G_{12}\Delta_2^{-1} \\
&- \frac{1}{2}\int_0^1 dx V(1,x) \ln V(1,x) G_2 \Delta^{-2} a_x - \frac{3}{4}\int_0^1 dx V(1,x) \ln V(1,x) G_2 \Delta^{-1} G_{13}\Delta_3^{-1} \\
&- \frac{1}{2}\int_0^1 dx V(1,x) \ln V(1,x) G_2^2 \Delta^{-2} - \frac{3}{4}\int_0^1 dx V(1,x) \ln V(1,x) \Delta^{-1} a_x G_{13}\Delta_3^{-1}, \\
C_0^0 &= -\frac{1}{4} G_2 \Delta^{-1} a_{3x} \Delta_3^{-1}(b + c + e) - \frac{1}{8} G_2 \Delta^{-1} G_{13}\Delta_3^{-1}(\frac{4}{3}b + c + e) + \frac{1}{12} G_2^2 \Delta^{-2}(3a + b + \frac{3}{2}c + 4d + 2e + 6f) \\
&+ \frac{1}{4}\Delta^{-1} a_x a_{1x}\Delta_1^{-1}(a + b + c + d + e) - \frac{1}{4}\Delta^{-1} a_x a_{3x}\Delta_3^{-1}(b + c + e) + \frac{1}{6}\Delta^{-1} a_x G_{11}\Delta_1^{-1}(a + b + c + \frac{3}{4}d + \frac{3}{4}e) \\
&- \frac{1}{8}\Delta^{-1} a_x G_{13}\Delta_3^{-1}(\frac{4}{3}b + c + e) - \frac{1}{4}\Delta^{-1} a_y a_{1x}\Delta_1^{-1}(a + b + c + de) + \frac{1}{4}\Delta^{-1} a_y a_{2x}\Delta_2^{-1}(a + d) \\
&- \frac{1}{6}\Delta^{-1} a_y G_{11}\Delta_1^{-1}(a + b + c + \frac{3}{4}d + \frac{3}{4}e) + \frac{1}{6}\Delta^{-1} a_y G_{12}\Delta_2^{-1}(a + \frac{3}{4}d) + \frac{1}{4} V(0,0) \ln V(0,0) \Delta^{-1} a_x a_{1x}\Delta_1^{-1} \\
&- \frac{1}{4} V(0,0) \ln V(0,0) \Delta^{-1} a_y a_{1x}\Delta_1^{-1} + \frac{1}{4} V(0,0) \ln V(0,0) \Delta^{-1} a_y a_{2x}\Delta_2^{-1} - \frac{1}{4} V(1,0) \ln V(1,0) G_2 \Delta^{-1} a_{3x}\Delta_3^{-1} \\
&- \frac{1}{4} V(1,0) \ln V(1,0) \Delta^{-1} a_x a_{3x}\Delta_3^{-1} - V(1,0) \ln V(1,0) \Delta^{-1} a_y a_{2x}\Delta_2^{-1} - \frac{1}{4} V(1,0) \ln V(1,0) \Delta^{-1} a_y G_{12}\Delta_2^{-1} \\
&+ \frac{1}{4} V(1,1) \ln V(1,1) G_2 \Delta^{-1} a_{3x}\Delta_3^{-1} + \frac{1}{4} V(1,1) \ln V(1,1) G_2 \Delta^{-1} G_{13}\Delta_3^{-1} - \frac{1}{4} V(1,1) \ln V(1,1) \Delta^{-1} a_x a_{1x}\Delta_1^{-1} \\
&+ \frac{1}{4} V(1,1) \ln V(1,1) \Delta^{-1} a_x a_{3x}\Delta_3^{-1} - \frac{1}{4} V(1,1) \ln V(1,1) \Delta^{-1} a_x G_{11}\Delta_1^{-1} + \frac{1}{4} V(1,1) \ln V(1,1) \Delta^{-1} a_x G_{13}\Delta_3^{-1} \\
&+ \frac{1}{4} V(1,1) \ln V(1,1) \Delta^{-1} a_y a_{1x}\Delta_1^{-1} + \frac{1}{4} V(1,1) \ln V(1,1) \Delta^{-1} a_y G_{11}\Delta_1^{-1}.
\end{aligned}
$$

few **Gram** determinants etc

$$
\begin{aligned}
a_x &= 2db - ce, \\
a_y &= 2ea - dc, \\
a_{1x} &= (d+e)/2, \\
a_{2x} &= d/2, \\
a_{3x} &= (c+e)/2, \\
\Delta &= f G_2 - (bd^2 - cde + ae^2), \\
\Delta_1 &= f G_{11} - \frac{1}{4}(d+e)^2, \\
\Delta_2 &= f G_{12} - \frac{1}{4}d^2, \\
\Delta_3 &= (a+d+f) G_{13} - \frac{1}{4}(c+e)^2, \\
G_2 &= 4ab - c^2, \\
G_{11} &= a + b + c, \\
G_{12} &= a, \\
G_{13} &= b,
\end{aligned}
$$

IR-divergent C_0 - functions?

easy, you can prove that for

$$
\begin{aligned}
a &= m^2, \quad b = m^2, \quad c = s - 2\,m^2, \\
d &= -2\,m^2, \quad e = s - 2\,m^2, \quad f = m^2,
\end{aligned}
$$

$$
\Longrightarrow B = 0, \quad \text{or}
$$

$$
\Big[1 + P_x \frac{\partial}{\partial_x} + P_y \frac{\partial}{\partial_y}\Big] V^{\mu+1}(x,y) = 0,
$$

$$
P_x = \frac{1}{2(\mu+1)}\,(1-x)\,, \qquad P_y = -\frac{1}{2(\mu+1)}\,y
$$

This we use to write

$$
\begin{aligned}
\int_0^1 dx \int_0^x dy\, V^{-1-\varepsilon/e}(x,y) &= \frac{1}{\varepsilon}\int_0^1 dx\, V^{-1-\varepsilon/e}(x,x), \\
\Big[1 + \frac{1}{2\,(\mu+1)}\Big(\frac{1}{2} - x\Big)\Big] V^{1+\mu}(x,x) &= -\frac{1}{4}\,\big(s - 4\,m^2\big)\,V^{\mu}(x,x) \\
C_0^{\rm IR} &= \frac{1}{s-4m^2}\Big\{\big(-\frac{1}{2}+\frac{1}{\varepsilon}\big) + \int_0^1 dx\,\ln V(x,x)\big)\big(-\frac{3}{4}+\frac{1}{2\,\varepsilon}\big) \\
&+ \frac{1}{8}\,\ln V(0,0)\big(1-\frac{2}{\varepsilon}\big) + \frac{1}{8}\,\ln V(1,1)\big)\big(1-\frac{2}{\varepsilon}\big)\Big\}.
\end{aligned}
$$

$\boxed{\text{massless pentagon?}}$

easy, (also for the **massive** one) you can prove that

$$
\begin{aligned}
E_0 &= \int_0^1 dx_1 \int_0^{1-x_1} dx_2 \int_0^{1-x_1-x_2} dx_3 \int_0^{1-x_1-x_2-x_3} dx_4 \\
&\times \left[x^t\, H^{-1}\, x + 2\, K^t\, x\right]^{-3-\varepsilon/2},
\end{aligned}
$$

$$
H = \frac{1}{2}\begin{pmatrix} 0 & -s_{51} & s_{12} - s_{34} & s_{45} \\ - & -2\, s_{51} & -s_{34} - s_{51} & s_{23} - s_{51} \\ - & - & -2\, s_{34} & -s_{34} \\ - & - & - & 0 \end{pmatrix}
$$

$$
\begin{aligned}
K_1 &= 0, \quad K_2 = \frac{1}{2}\, s_{51}, \quad K_3 = \frac{1}{2}\, s_{34}, \quad K_4 = 0, \\
S_{ij} &= -\,(p_i + p_j)^2\,.
\end{aligned}
$$

Now we raise the exponent,
integrate by parts, etc

Here $B = \frac{1}{16} s_{12} s_{23} s_{34} s_{45} s_{51}$

Moving to multi-loop

- Any diagram with N_L **legs** and n_l **loops** is

$$
\begin{aligned}
G &= (-1)^N_L \left(\frac{i}{\pi^{n/2}}\right)^{n_l} \Gamma\left(N_L - \frac{n}{2}\,n_l\right) \int \frac{dz_G\,\delta\left(1-z_G\right)}{U^{n/2}\,\left(V - i\,\varepsilon\right)^{N_L - n n_l/2}}, \\
V &= \sum_i m_i^2\, z_i - \sum_i q_i^2\, z_i - \frac{1}{U} \sum_{ij} B_{ij}\, q_i \cdot q_j\, z_i z_j, \\
U &= \sum_T \prod_{z_i \in T} z_i = \det\left(U_{rs}\right), \quad U_{rs} = \sum_i z_i \eta_{ir} \eta_{is}.
\end{aligned}
$$

- η_{is} is the projection of **line** i along the **loop** s. T is a co-tree.
- **UV**-singularites come from U so that, for finite diagrams, one should raise only the factor

$$
\tilde{V} = U\left[\sum_i m_i^2\, z_i - \sum_i q_i^2\, z_i\right] - \sum_{ij} B_{ij}\, q_i \cdot q_j\, z_i z_j.
$$

a two-loop primer?

$$
\begin{aligned}
N_L &= 3, \qquad U^{-3+3/2\varepsilon}\, \tilde{V}^{1-\varepsilon}, \\
N_L &= 4, \qquad U^{-2+3/2\varepsilon}\, \tilde{V}^{-\varepsilon}, \\
N_L &= 5, \qquad U^{-1+3/2\varepsilon}\, \tilde{V}^{-1-\varepsilon}.
\end{aligned}
$$

Comments:

- For **two-loops** U is **quadratic** and $\tilde{V}$ is **cubic**,
- so, we have to construct a Bernstein functional relation for a **quintic !** (or higher) polynomial.
- Try to do it in **FORM** and you will be told **Input expansion buffer overflow. Try something else.**
- For $N_L = 3, 4$, we have $\tilde{V}$ to a positive power and it is enough to **regularize** G with some K_S operation ($S \in G$).
- For $N_L > 5$, we have $\tilde{V}$ to a negative power, U to a positive power (no **UV** divergency), so we could raise a **cubic**.
- For **self-energies** at $q^2 = 0$ it is easy
  - Consider, for example, $n_l = 2, N_L = 3$, perform a **projective** transformation

$$z_i = A_i u_i / \sum_j A_j u_j, \qquad A_i = \frac{1}{m_i^2},$$

  - to obtain $\tilde{V} = -U$ **quadratic** and,

$$\begin{aligned}\tilde{V} &= -xy\Big(\frac{1}{m_1^2 m_2^2} - \frac{1}{m_1^2 m_3^2} + \frac{1}{m_2^2 m_3^2}\Big) - x\frac{1}{m_1^2 m_2^2} \\ &+ x^2 \frac{1}{m_1^2 m_2^2} + y\Big(\frac{1}{m_1^2 m_2^2} - \frac{1}{m_1^2 m_3^2}\Big) + y^2 \frac{1}{m_2^2 m_3^2}\end{aligned}$$

  - We know how to raise a **quadratic**
- $N_L = 2\,(n_l + 1)$ is nice, a single polynomial of degree $n_l + 1$

Of course, **vacuum diagrams** are also easy. At **two-loop** level we have

$$G_2^{\rm vac} = (4\,\pi)^{\varepsilon-4}\,\Gamma\left(\varepsilon-1\right)\,\left(K^t H^{-1} K\right)^{-2}\,\int_0^1 dx\,\int_0^x dy$$

$$\times\;\left[1-\frac{\left(x+K^t H^{-1}\right)\partial}{\varepsilon-2}\right]\left[1-\frac{\left(x+K^t H^{-1}\right)\partial}{\varepsilon}\right]U^{\varepsilon/2}(x,y),$$

where U has been presented above.

- □ The same is, obviously true, for **tadpoles**.

- □ We are still missing a **real** multi-loop example, **see below**.

Additional problems

- $\tilde{V}$ is an highly incomplete polynomial, few **non-zero** coefficients,

- $\mathcal{P}$ can be found via a direct study of linear systems of n equations ($n \gg 100$) in m variables and very often the matrix of coefficients has **rank** $< \mathbf{n}$, so the system is **generally** impossible and,

- you need to go to higher n, m, i.e. beyond $n_{\rm min} = \min\{n < m\}$ (try one-loop master formulae with $K = L = 0$!) and your **algebraic manipulation** will exhaust your computer before you find a solution.

  ▷ A simple **counting**

$$\mathcal{P} \;=\; P_n + P^i_{n+1}\partial_i + P^{ij}_{n+2}\partial_i\partial_j + \ldots$$

  ▷ **Four variables** and a **cubic** V require

  1. $n = 2, n_{\rm eq} = 126, n_{\rm var} = 155$ for first derivatives
  2. $n = 1, n_{\rm eq} = 330, n_{\rm var} = 415$ for second derivatives
  3. $n = 0, n_{\rm eq} = 715, n_{\rm var} = 871$ for third derivatives

  ▷ Simple examples already show that, for realistic polynomials V, one has to go **beyond second order in derivatives**.

- relation between $B = 0$ and **Landau singularities** still to be examined, although $B = 0$ contains **all** singularities of G, **physical and spurious** ones.

- $L - K^t H^{-1} K$ (or B in general) is not a **Gram** determinant but its spurious zeros (**if any**) are a vexation, as much as those of **Gram** determinants in **PV** reduction.

- For $C_0\left(-m_a^2, -m_b^2, -s; m_b, M, m_a\right)$ we obtain

$$
\begin{aligned}
G_{11} &= -\frac{1}{4}\,\lambda\left(s, m_a^2, m_b^2\right), \\
G_{12} &= -\frac{1}{4}\,\lambda\left(M^2, m_a^2, m_b^2\right), \\
G_{13} &= -\frac{1}{4}\,\lambda\left(M^2, m_a^2, m_b^2\right), \\
G_2 &= \left[\left(m_a^2 - m_b^2\right)^2 - M^2 s\right] \\
&\times \left[s + M^2 - 2\left(m_a^2 + m_b^2\right)\right].
\end{aligned}
$$

spurious singularities?