MultiLoops & MultiLegs
Algebraic - Numerical Approach

Giampiero Passarino

Dipartimento di Fisica Teorica, Università di Torino, Italy

INFN, Sezione di Torino, Italy

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TOPSIDE Collaboration*

A. Ferroglia (NYU)
G. Passarino (Torino)
M. Passera (Padova)
S. Uccirati (Torino)

*TOrino Project Supporting Intensive Diagrams Evaluation

There is **NO** need to explain the need for

Fast & **Efficient** ways to compute **ML&L** diagrams

$\cdots\cdots$ *and*

for the **general** $\searrow$ case the **analytical way**[1] not feasible

arbitray int/ext masses

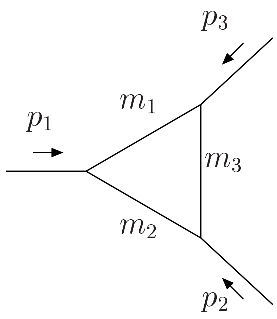
although a lot can be done

$\cdots\cdots$ *moreover*

what is so special about the

AW???

[1] K. G. Chetyrkin, Z. Bern, L. Dixon, D. A. Kosower, J. Fleischer, F. Jegerlehner, O. V. Tarasov, S. G. Gorishnii, S. A. Larin, L. R. Surguladze, F. V. Tkachov, D. J. Broadhurst, A. G. Grozin, V. V. Belokurov, N. I. Usyukina, D. I. Kazakov, A. V. Kotikov, A. I. Davydychev, V. A. Smirnov, M. Caffo, H. Czyz, S. Laporta,d E. Remiddi, T. Gehrmann J. Fleischer, V. A. Smirnov, A. Frink, J. G. Korner, D. Kreimer, K. Schilcher, J. B. Tausk, O. L. Veretin, A. L. Kataev, G. Degrassi, P. Gambino , A. Freitas, S. Heinemeyer, W. Hollik, W. Walter, G. Weiglein, S. Heinemeyer, T. Binoth, G. Heinrich, F. A. Berends, S. Groote, J. G. Korner A. Ghinculov, Y. Yao, J. A. Vermaseren, S. Bauberger, M. Bohm, M. Buza, J. van der Bij, M. Veltman, $\cdots\cdots$



- ▷ It is the sum of 12 **di-logarithms**;
- ▷ (Few) More complex diagrams $\Longrightarrow$ **Nielsen - Polilogarithms**;
- ▷ (Few) (More complex)^2 diagrams $\Longrightarrow$ **Higher Transcendental Functions**;

what is so special about $S_{n,p}(z)$?

- ▷ Analytical properties explicit,
- ▷ An **EXPANSION** available, like in

$$\mathrm{Li}_2(z) = \sum_{n=0}^{\infty} B_n \frac{\left[-\ln(1-z)\right]^{n+1}}{(n+1)!},$$

expansion parameter with the same **cut** of $\mathrm{Li}_2(z)$

$B_n \equiv$ Bernoulli numbers

$n = 14$ gives 13 decimals accuracy

The same is **NOT** obviously true for

Appell & **Kampé de Fériet** functions

or **nested sums**

although there are interesting novelties, e.g.

multi-dimensional **harmonic polylogarithms**[2]

Alternative strategy: Numerical Way

$$G = \left(i\,\pi^{n/2}\right)^{n_l}\,\Gamma\left(N_L - \frac{n}{2}\,n_l\right)\int \frac{dx_G\,\delta\,(1-x_G)}{U^{n/2}\,(V-i\,\delta)^{N_L-n n_l/2}},$$

$$\equiv \int_S \prod_{i=1}^{N_l-1} dx_i\, I_G(\{x_i\})$$

$$dx_G = \prod_{i=1}^{N_l} dx_i, \qquad x_G = \sum_{i=1}^{N_l} x_i.$$

The method $\Longrightarrow$ seek algorithm(s) for smoothness

$$G = \frac{1}{B_G}\int_S dx\,\mathcal{F}[I_G]$$

[2] E. Remiddi and J. A. Vermaseren, Int. J. Mod. Phys. A **15** (2000) 725 [arXiv:hep-ph/9905237]

- ▷ $\mathcal{F}$ is a functional of I_G that makes the whole integrand smooth in **parametric space**,
- ▷ B_G is a function of masses and momenta that contains the **TRUE** singularities of G.

Per lenitatem ad astra

Bernstein-Tkachov theorem:

- ▷ for any finite set of polynomials

$$V_i(x) \qquad \text{where} \qquad x = (x_1, \ldots, x_N)$$

is a vector of **Feynman parameters**, there exists an identity of the following form:

$$\mathcal{P}\ (x, \partial) \prod_i V_i^{\mu_i+1}(x) \ = \ B \prod_i V_i^{\mu_i}(x).$$

- ▷ where $\mathcal{P}$ is a polynomial of

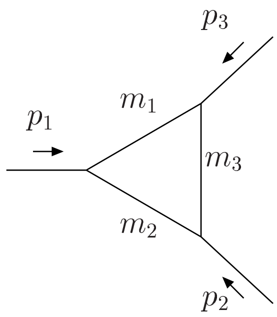
$$x \qquad \text{and} \qquad \partial_i = \partial / \partial x_i$$

- ▷ B and all coefficients of $\mathcal{P}$ are polynomials of μ_i and of the coefficients of $V_i(x)$.

▷ **Iterative applications** of the functional relations, followed by **integration by parts**, allow us to express the integrand in **parametric space** as a combination of

$$(\text{polynomials})^N \quad \times \quad \text{logarithms of the same polynomials}$$

Examples

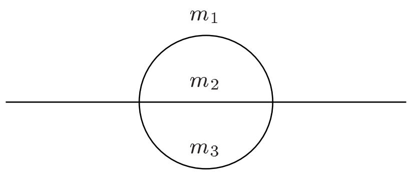


$$C_0 \;=\; \int_0^1 dx_1 \int_0^{x_1} dx_2\, V^{-1-\epsilon/2}(x_1, x_2),$$

$$V(x_1, x_2) \;=\; x^t\, H\, x + 2\, K^t\, x + L - i\,\delta,$$

$$B_3 \;=\; L - K^t\, H^{-1}\, K, \qquad X = H^{-1}\, K.$$

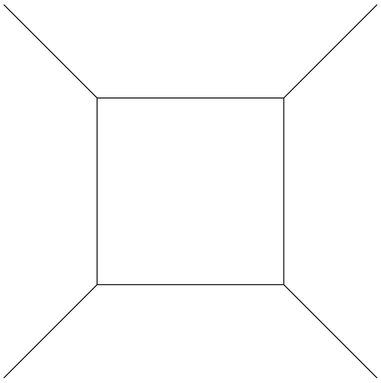
$$
\begin{aligned}
C_0 &= \frac{1}{B_3}\{\frac{1}{2} + \int_0^1 \, dx_1 \, [\int_0^{x_1} \, dx_2 \, \ln \, V(x_1, x_2) \\
&+ \frac{1}{2}(X_1 - X_2) \, \ln \, V(x_1, x_1) + \frac{1}{2} X_2 \, \ln \, V(x_1, 0) \\
&- \frac{1}{2}(1 + X_1) \, \ln \, V(1, x_1)]\}.
\end{aligned}
$$



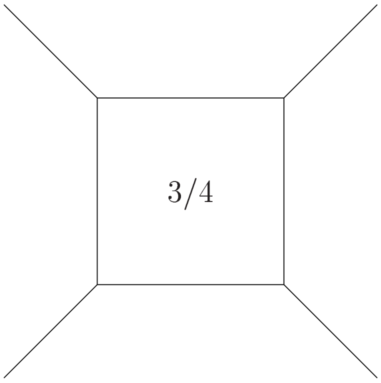
$$
S_3 = \sum_{i=1}^{3} m_i^2 \, S_{3i} + S_{3p},
$$

$$
\begin{aligned}
S_{33}^{\rm fin} &= \int_0^1 \, dx \, \int_0^1 \, dy \, [\ln \xi(x,y) + \left(\frac{\ln \xi(x,y)}{y-1}\right)_+] \\
&+ \ln \mu_3^2 \, \left(\ln \mu_3^2 - 4\right) + \frac{13}{2} + \frac{1}{2}\zeta(2), \\
S_{3p}^{\rm fin} &= \int_0^1 \, dx \, \int_0^1 \, dy \, y \, \ln \xi(x,y) + \frac{3}{8}, \\
\xi(x,y) &= -y \, \left(y - 1 + \mu_3^2\right) \, x^2 + [y \, \left(y - 1 + \mu_3^2\right) \\
&+ (1-y) \, \left(\mu_2^2 - \mu_1^2\right)] \, x + (1-y) \, \mu_1^2.
\end{aligned}
$$

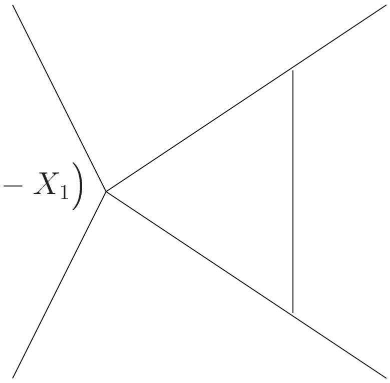
Diagrammatica of **BT** functional relations:



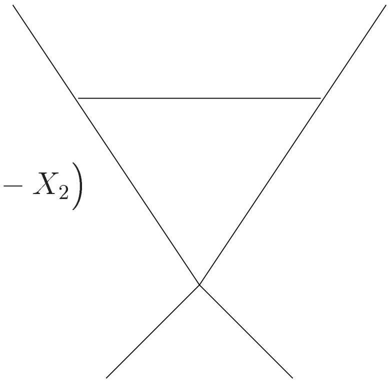
$$= \frac{1}{2\,B_4}\Big\{ \quad -$$



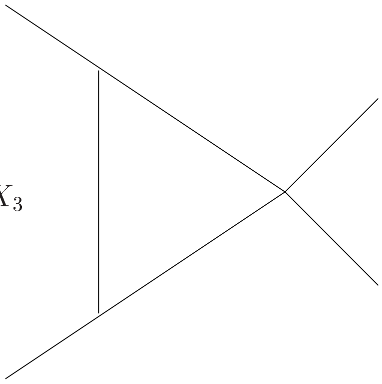
$$+ \quad \Big(X_2 - X_1\Big)$$



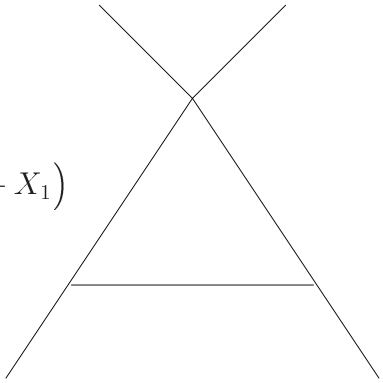
$$+ \quad \Big(X_3 - X_2\Big)$$



$$- \quad X_3$$

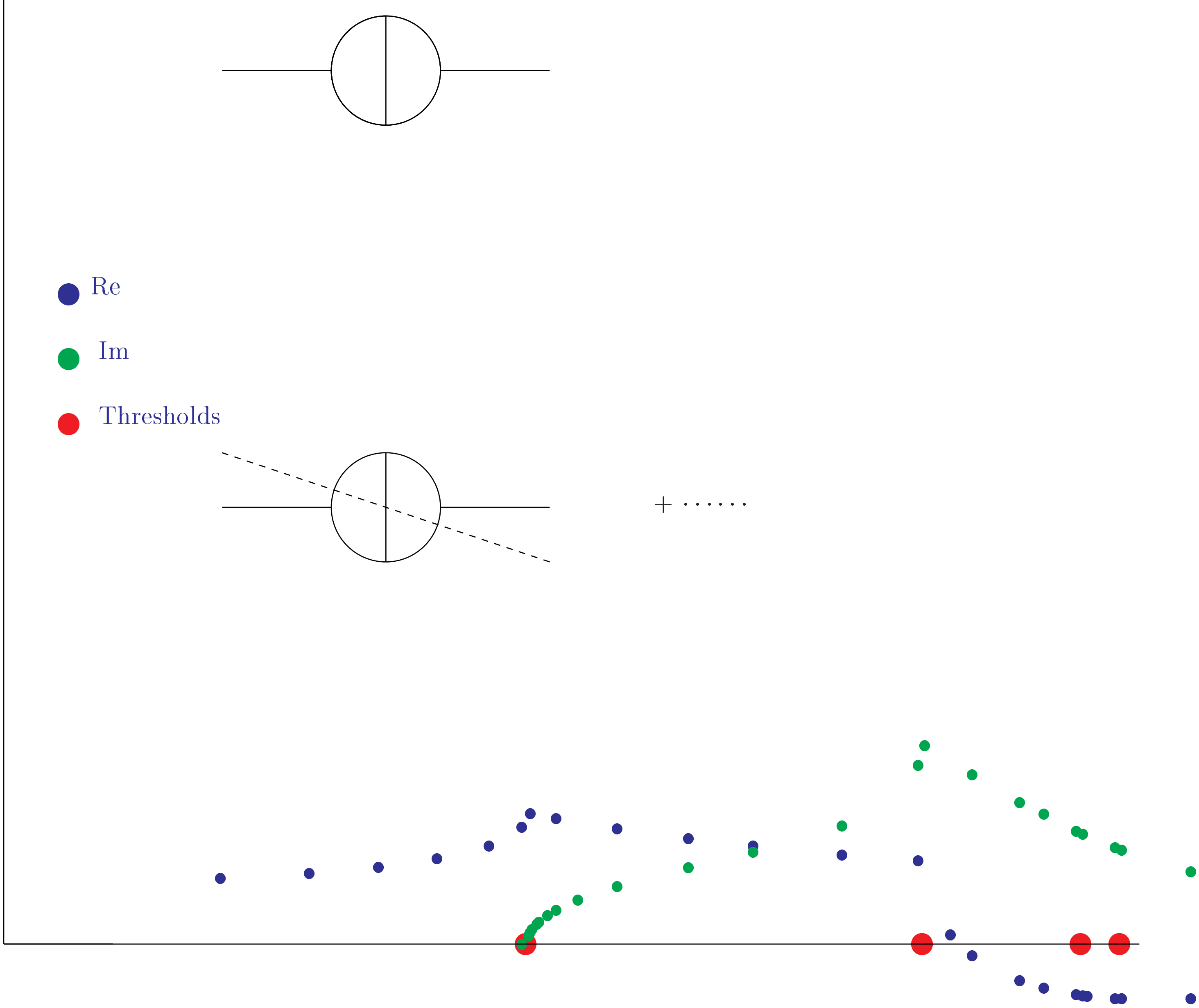


$$+ \quad \Big(1 + X_1\Big)$$



$$\Big\}$$

Why a difficult problem?



Why a difficult problem (II) ?

▷ Try to compute C_0 around the **anomalous threshold** in $W^* \to \bar{u}d$, (**abelian** W exchange)

$$s_{\text{anom thr}} = \frac{(m_u^2 - m_d^2)^2}{M_W^2}$$

AT $\to$ A_{abs} is not directly related through **unitarity** to physical processes

In principle we may have a **logarithmic singularity** in the **real part**

and a **jump** in the **imaginary part**

The **LS** is **maximally equal** to $\ln\,\delta$

the **jump** arises because some argument in the integrand moves rapidly around the cut.

However this is **Not** necessarily the case

$$\Longrightarrow$$

We write

$$C_0 = \int_0^1 dx_1 \int_0^{x_1} dx_2 \left(x^t\,H\,x + 2\,K^t\,x + L\right)^{-1}$$

$$= \int_0^1 dx_1 \int_0^{x_1} dx_2 \left[(x-X)^t\,H\,(x-X) + B_3\right]^{-1},$$

and **unless** $0 < X_2 < X_1 < 1$

there is **no** divergence for $B_3 = 0$

Actually, this is what happens for $W^* \to \bar{u}d$

and **realistic** masses.

$\triangleright$ Here is how it goes (**solution** $\equiv$ **expansion**, e.g. through Mellin-Barnes **splitting**):

Around anomalous threshold

$\sqrt{s}-\sqrt{s}_{\mathrm{anom\,thr}} = -5\,\mathrm{GeV}$		
Source	Re	Im
ANALYTICAL ('t Hooft - Veltman)	2.49196916	-6.10507293
NUMERICAL I (one BT)	2.49201730	-6.10506938
NUMERICAL II (two BT)	2.49487588	-6.10507303
$\sqrt{s}-\sqrt{s}_{\mathrm{anom\,thr}} = -200\,\mathrm{MeV}$		
Source	Re	Im
ANALYTICAL	2.55665099	-6.16483833
NUMERICAL I	2.55667167	-6.16483680
NUMERICAL II	4.52680950	-6.16483834
$\sqrt{s}-\sqrt{s}_{\mathrm{anom\,thr}} = 0\,\mathrm{MeV}$		
Source	Re	Im
ANALYTICAL	unstable†	-6.16619605
NUMERICAL I	divergent	divergent
NUMERICAL II	divergent	divergent
$\sqrt{s}-\sqrt{s}_{\mathrm{anom\,thr}} = +200\,\mathrm{MeV}$		
Source	Re	Im
ANALYTICAL	2.56203382	-6.16979661
NUMERICAL I	2.56196036	-6.16980269
NUMERICAL II	4.54553765	-6.16979660

† **in my coding!**

Around normal threshold

$\sqrt{s}-\sqrt{s}_{\mathrm{norm\,thr}} = -1\,\mathrm{GeV}$		
Source	Re	Im
NUMERICAL I (one BT)	-0.597864577	-3.33890433
NUMERICAL II (two BT)	-0.597883176	-3.33883950
$\sqrt{s}-\sqrt{s}_{\mathrm{norm\,thr}} = -200\,\mathrm{MeV}$		
Source	Re	Im
NUMERICAL I	-0.506700500	-3.39489874
NUMERICAL II	-0.506973251	-3.39487640
$\sqrt{s}-\sqrt{s}_{\mathrm{norm\,thr}} = +1\,\mathrm{MeV}$		
Source	Re	Im
NUMERICAL I	-0.439171689	-3.40769265
NUMERICAL II	0.123549052	-3.40766495
$\sqrt{s}-\sqrt{s}_{\mathrm{norm\,thr}} = +200\,\mathrm{MeV}$		
Source	Re	Im
NUMERICAL I	-0.433406617	-3.35962164
NUMERICAL II	-0.433100074	-3.35955053

REAL*8 Analytical shows problems !!!

Numerical Evaluation requires

Precise knowledge of analytical structure
(si vis pacem para bellum)

$$\Updownarrow \Updownarrow \Updownarrow$$

Solution of **Landau - Bjorken** equations[3].

$$C_0 \searrow$$

▷ **leading singularity** $\equiv B_3 = 0 \equiv$ anomalous threshold

$$\begin{aligned} s \;=\; & \frac{1}{2\,m_2^2}\{2\,m_2^2\,(m_1^2+m_3^2) \\ - \; & (m_1^2+m_2^2-M_1^2)\,(m_3^2+m_2^2-M_2^2) \\ \pm \; & [\lambda(M_1^2,m_1^2,m_2^2)\lambda(M_2^2,m_3^2,m_2^2)]^{1/2}\}, \end{aligned}$$

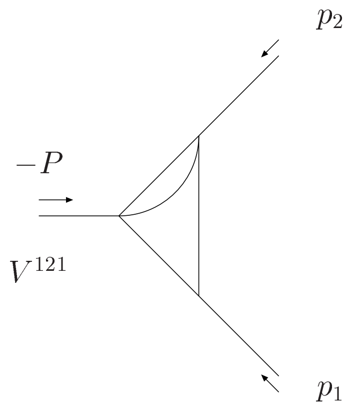
$$(p_1+p_2)^2 \;=\; -s, \qquad p_i^2 = -M_i^2.$$

▷ **sub-leading singularites** $\equiv$ zeros of

$$\lambda(s,m_1^2,m_3^2), \quad \lambda(M_1^2,m_1^2,m_2^2), \quad \lambda(M_2^2,m_2^2,m_3^2).$$

$\equiv$ normal & pseudo thresholds.

[3] In principle also non-Landau or second-type singularities should be considered



$$\alpha_1\,(q_1^2+m_1^2)=0, \qquad \alpha_2\,((q_1-q_2)^2+m_2^2)=0,$$
$$\alpha_3\,((q_2-p_2)^2+m_3^2)=0, \qquad \alpha_4\,((q_2-P)^2+m_4^2)=0,$$

$$\alpha_1 q_{1\mu}+\alpha_2(q_1-q_2)_\mu \;=\; 0,$$
$$-\;\alpha_2(q_1-q_2)_\mu+\alpha_3\,(q_2-p_2)_\mu+\alpha_4\,(q_2-P)_\mu=0.$$

Leading $\searrow$

$$\begin{aligned} P^2 \;&=\; \frac{1}{2\,m_3^2}\,[(-p_2^2-(m_1+m_2)^2+m_3^2)\,(-p_1^2+m_3^2-m_4^2) \\ &+\; 2\,m_3^2\,(p_1^2+p_2^2)\pm(\lambda_1\,\lambda_2)^{1/2}] \end{aligned}$$

$$\lambda_1 \;=\; \lambda(-p_1^2,m_3^2,m_4^2) \qquad \lambda_2=\lambda(-p_2^2,(m_1+m_2)^2,m_3^2)$$

Solution for the BT - polynomial $\mathcal{P}$ known

$\forall$ **All Quadrics** $\equiv$ **all** $2L\&nL$ diagrams

Extension to **Cubics** available

but $\cdots\cdots$

this is purely **academical**

Sunset topology (**the easiest**) requires

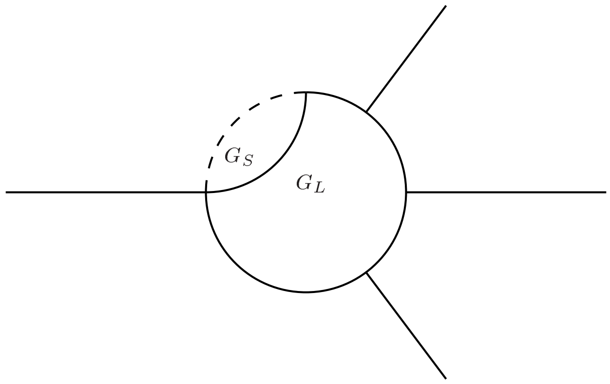
degree $= 4$

in van der Bij - Veltman parametrization

Furthermore, **around - threshold** behavior remains unsolved in **BT** approach,

therefore

Beyond **quadrics** $\Longrightarrow$ *Minimal* BT - algorithm



$$G \;=\; \int dx_S\,\frac{1}{B(x_S)}\,\int dx_L\,\mathcal{F}[I_G(x_S,x_L)].$$

- ▷ Analytical structure of G_S simpler,
- ▷ x_S integration with **contour - distortion**, always possible **unless** a **pinch** (or an **end-point**) singularity occur
  $\cdots\cdots$ but this is a real (Landau) singularity.

However

$\forall$ two-loop two-point functions **2L&2L**

$\forall$ two-loop three-point functions **2L&3L** (but one)

contour - distortion is not needed

$\Longrightarrow$

$\forall$ of them a **parametrization** can be found that allows for a **BT** procedure with $B \equiv B$**(parameters)**, e.g. the integrand can be transformed

$$\{x\} \to \{z\}\,,\, y$$

into a special form:

$$\chi \;=\; z^t\mathcal{H}z + 2\mathcal{K}^t z + k\,(1-y) + L,$$

$$\chi^\mu \;=\; -\frac{1}{b}\left\{1 - \frac{1}{\mu+1}\left[(y-1)\partial_y + \frac{1}{2}(z^t + Z^t)\partial_z\right]\right\}\chi^{\mu+1}$$

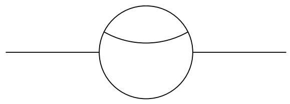
$$Z^t \;=\; \mathcal{K}^t\mathcal{H}^{-1}, \qquad b = \mathcal{K}^t\mathcal{H}^{-1}\mathcal{K} - L.$$

b is y - independent !

- ▷ These $2L$ diagrams are no more involved that a $1L$ diagram,
- ▷ Normal BT $\oplus$ integration by parts. $B = 0 \to$ **non-leading** LS.

How to proceed around $B \approx 0$?

▷ $2L\&2L$ and some of the $2L\&3L$ integrands are *simple* enough so that we can[4]



1. split the integrand in various pieces,

$$S^{131} = -\frac{1}{s}\left(\frac{\mu^2}{\pi\, s}\right)^{\epsilon} \Gamma\left(1+\epsilon\right) \int_0^1 dx \left[x(1-x)\right]^{1+\epsilon/2} J,$$

$$J = \int_0^1 dy \int_{-z_m}^{1-z_m-y} dz\, y^{\epsilon/2-1} \left(1 - z_m - y - z\right) \times \left(z^2 + cy - \frac{1}{4}\lambda\right)^{-1-\epsilon},$$

$$\lambda \equiv \lambda(1, \mu_3^2, \mu_4^2),$$

where $z_m = \mu_{34}^2/2$, $c = \mu_x^2 - \mu_3^2$ and $\mu_x^2 \equiv$ quadratic in x. We will set $\lambda = 0$ whenever possible; J is the sum of three contributions;

[4] example taken from S^{131} around the two-particle cut

2. apply to some of them, e.g.

$$J_1 \;=\; -\int_0^1 dy \int_{-z_m}^{1-z_m-y} dz\, y^{\epsilon/2-1}\, z\, (z^2+cy)^{-1-\epsilon}.$$

▷ *simpler* BT - iterations, e.g.

$$-\frac{1}{\epsilon}\,\partial_z\,(z^2+cy)^{-\epsilon} \;=\; 2\,z\,(z^2+cy)^{-1-\epsilon},$$

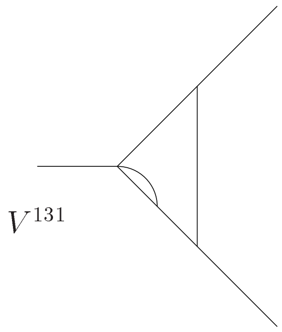
▷ producing simple results

$$\begin{aligned} J_1 \;=\; & \frac{2}{\epsilon}\,\ln\frac{z_m}{1-z_m} + 2\,[\ln^2 z_m - \ln^2(1-z_m)] \\ + \; & \frac{1}{2}\int_0^1 dy\,[\frac{\ln\eta(y,-z_m)}{y} - \frac{\ln\eta(y,1-z_m-y)}{y}]_+, \end{aligned}$$

where $\eta(y,z) = z^2 + cy$.

▷ express the rest in terms of **hypergeometric** functions which are then expanded around $\epsilon = 0$ and allow to isolate the **square root** singularity from a **remainder** that can be safely computed by numerical integration.

▷ Another example:



$$V^{131} = -\left(\frac{\mu^2}{\pi}\right)^{\epsilon}\Gamma\left(1+\epsilon\right)\int_0^1 dx \int_0^1 dy \int_0^y dz_1 \int_0^{z_1} dz_2$$
$$\times\ \left[x(1-x)\right]^{-\epsilon/2}\left(1-y\right)^{\epsilon/2-1}\chi_{131}^{-1-\epsilon},$$

where we rewrite χ_{131} as

$$\chi_{131} = A_{x;131}\left(1-y\right)+B_{131}(z_1,z_2),$$
$$B_{131}(z_1,z_2) = z^t H z + 2K^t z + m_3^2, \quad A_{x;131} = m_x^2 - m_3^2.$$

The innermost integral becomes now

$$J_{131} = \int_0^{1-z_1} dy\, y^{\epsilon/2-1}\left[A_{x;131}\, y + B_{131}(z_1,z_2)\right]^{-1-\epsilon}.$$

$$\Rightarrow \quad \text{hypergeometric} \quad \Rightarrow$$

$$I = \int_0^Y dy\, y^{\epsilon/2-1}\left(a\, y + b - i\,\delta\right)^{-1-\epsilon}$$

$$I \;=\; 2\,\frac{Y^{\epsilon/2}}{\epsilon}\,b^{-1-\epsilon}\;{}_2F_1(1+\epsilon\,,\,\frac{\epsilon}{2}\,;\,1+\frac{\epsilon}{2}\,;\,-\frac{a}{b}\,Y).$$

$$\Rightarrow$$

$$I \;=\; \frac{2}{\epsilon}\,a^{-\epsilon/2}\,b^{-1-\epsilon/2} - \frac{1}{2\,b}\,\ln(1+\frac{b}{a\,Y}) + \mathcal{O}\left(\epsilon\right).$$

$\triangleright$ Giving an expression to be integrated **numerically**

$$\begin{aligned} J_{131} \;&=\; \frac{2}{\epsilon}\,(1-\frac{\epsilon}{2}\,\ln A_{x;131})\,C_0 \\ &-\;\frac{1}{2\,B_{131}(z_1,z_2)}\,\ln(1+Q_{x;131}). \end{aligned}$$

$$Q_{131} \;=\; \frac{B_{131}(z_1,z_2)}{A_{x;131}\,(1-z_1)}.$$

In general other techniques are used, an example:

$$I(\alpha) \;=\; \int_0^1 dx\, [\chi_B(x) - i\,\delta]^{-\alpha}.$$

$$\chi_B(x) \;=\; -P^2\,(x-a)^2 + \frac{\lambda}{4\,P^2} - i\,\delta, \quad a = \frac{1}{2}\,\frac{P^2 + m_2^2 - m_1^2}{P^2}.$$

where is the singularity ?

- $\triangleright$ Let $x_{\pm}$ be the roots of $\chi_B - i\,\delta = 0$;
- $\triangleright$ whenever they are real and $\in [0,1]$ the integration contour **can be distorted** and the **integral computed**
- $\triangleright$ **unless** a **pinch** will occur, which happens for

$$\lambda(-P^2, m_1^2, m_2^2) \;=\; 0.$$

What to do around $\lambda = 0$?

- $\triangleright$ Assume $P^2 = -s$, $s \geq 0$ and $0 \leq a \leq 1$ and obtain

$$I(\alpha) \;=\; (\frac{\rho}{s})^\alpha \sum_{X=a}^{1-a} \int_0^X dx\, (1 + \rho\, x^2)^{-\alpha}$$
$$\;=\; (\frac{\rho}{s})^\alpha \sum_{X=a}^{1-a} J_X(\alpha, \rho),$$

▷ where $\rho = -4\,s^2/\lambda$.

▷ We are now interested in the behavior for $|\rho| \to \infty$. Consider the case $\lambda < 0$:

$$j_X(\alpha, s) \;=\; \int_0^\infty d\rho\,\rho^{s-1}$$
$$J_X(\alpha, \rho) \;=\; \frac{B(s, \alpha - s)}{1 - 2\,s}\,X^{1-2\,s},$$

which is defined for

$$0 < \mathrm{Re}\,s < \min\{\frac{1}{2}, \mathrm{Re}\,\alpha\}, \qquad |\arg x^2| < \pi$$

▷ The inverse Mellin transform will be

$$J_X(\alpha, \rho) \;=\; \frac{1}{2\,\pi\,i}\int_{-i\,\infty}^{+i\,\infty} ds\,\rho^{-s}\,j_X(\alpha, s).$$

The integration contour will be closed in the right s half-plane and **leading behavior** comes from the pole at $s = 1/2$,

$$J_X(\alpha, \rho) \;=\; -\frac{1}{2\,\pi\,i}\,\frac{B(1/2, \alpha - 1/2)}{2}\int_{-i\,\infty}^{+i\,\infty} ds\,[\frac{\rho^{-s}}{s - \frac{1}{2}} + \ldots]$$
$$= \frac{1}{2}\,B(\frac{1}{2}, \alpha - \frac{1}{2})\,\rho^{-1/2} + \ldots, \quad 1 < \rho < \infty.$$

▷ Therefore we obtain the **leading term** in an expansion which is valid in the strip

$$0 \;<\; -\lambda < 4\,s^2,$$

▷ The **analytical continuation** for $\lambda > 0$ follows from the $i\,\delta$ prescription,

$$I(\alpha) \;\sim\; 2^{2\,\alpha-1}\,B(\frac{1}{2},\alpha-\frac{1}{2})\,(-\frac{\lambda+i\,\delta}{s})^{1/2-\alpha}, \quad \text{for} \quad \lambda \to 0.$$

▷ Note that for the **pseudo-threshold**, $s = (m_1 - m_2)^2$, we have that $a \notin [0,1]$. The complete expansion reads as follows:

$$J_X(\alpha,\rho) \;=\; \frac{1}{2}\,B(\frac{1}{2},\alpha-\frac{1}{2})\,\rho^{-1/2} + J_X^{\rm reg}(\alpha,\rho),$$

$$\begin{aligned} J_X^{\rm reg}(\alpha,\rho) \;&=\; \sum_{k=0}^{\infty}(-1)^k\,\frac{\Gamma\,(\alpha+k)}{\Gamma\,(k+1)}\,\frac{X^{1-2\,(\alpha+k)}}{1-2\,(\alpha+k)}\,\rho^{-\alpha-k} \\ &=\; -\frac{1}{2}\,X\,(\rho\,X^2)^{-\alpha}\,\frac{\Gamma\,(\alpha-1/2)}{\Gamma\,(\alpha+1/2)} \\ &\times\; {}_2F_1(\alpha\,,\,\alpha-\frac{1}{2}\,;\,\alpha+\frac{1}{2}\,;\,-\rho^{-1}). \end{aligned}$$

Any **realistic** strategy must be able to deal with

- ▷ **IR** configurations
- ▷ **Collinear** configurations
- ▷ **Sudakov** configurations

The **numerical** way to **IR** configurations:

A necessary condition for the presence of infrared divergences is that

the Landau equations are fulfilled.

▷ One - loop is easy: $C_0 \Rightarrow B_3 = 0$ when

$$a = b = f = m^2, \qquad c = -e = s - 2\,m^2, \quad d = -2\,m^2,$$

$$\Updownarrow$$

$$[1 + P_x \frac{\partial}{\partial_x} + P_y \frac{\partial}{\partial_y}]\, V^{\mu+1}(x,y) \;=\; 0,$$

solution $\searrow$

$$P_{\{x;y\}} = \frac{\{1-x;-y\}}{2(\mu+1)}.$$

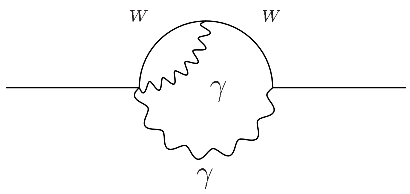
$$\int_0^1 dx \int_0^x dy\, V^{-1-\varepsilon/e}(x,y) = -\frac{1}{\varepsilon}\int_0^1 dx\, V^{-1-\varepsilon/2}(x,x).$$

In the final answer we must keep trace of all n-dependent factors,

$$\begin{aligned} C_0 &= -\mu^{\varepsilon}\pi^{-\varepsilon/2}\,\Gamma\left(1+\frac{\varepsilon}{2}\right)\frac{1}{\varepsilon}\int_0^1 dx\, V^{-1-\varepsilon/2}(x,x) \\ &= -\frac{1}{2}\,\mu^{\varepsilon}\,\frac{1}{\bar{\varepsilon}}\int_0^1 dx\, V^{-1-\varepsilon/2}(x,x), \end{aligned}$$

$$\begin{aligned} C_0^{\mathrm{IR}} = \frac{1}{s-4\,m^2}\{&-\frac{1}{2}\frac{1}{\hat{\varepsilon}}[1+\frac{1}{2}\int_0^1 dx\,\ln\frac{V(x,x)}{m^2}] - \frac{1}{2} \\ &+ \frac{1}{4}\ln\frac{m^2}{\mu^2}\left(1+\frac{1}{2}\ln\frac{m^2}{\mu^2}\right) - \frac{1}{4}\int_0^1 dx\,\ln\frac{V(x,x)}{\mu^2} \\ &\times [3+\frac{1}{2}\ln\frac{V(x,x)}{\mu^2}]\} \end{aligned}$$

▷ **(Almost)** all *on-shell* derivatives of $2L$ & $2L$ functions are easy enough to allow for BT treatment with a consequent and straightforward analytical evaluation:



$$
\begin{aligned}
S_p^{121} &= -\frac{2}{m^2}\left(\frac{\mu^2}{\pi\, m^2}\right)^{\epsilon}\Gamma\left(\epsilon\right)\int_0^1 dx\int_0^1 dy\int_0^y dz\,[x(1-x)]^{-\epsilon/2} \\
&\times\; (1-y)^{\epsilon/2}\{z(1-z)\,[(1+\epsilon)\,(z^2-\frac{1}{x})\,\chi_{121}^{-2-\epsilon} \\
&+\; \frac{\epsilon}{2}\,\chi_{121}^{-1-\epsilon}] + z\,(3\,z-2)\,\chi_{121}^{-1-\epsilon}\}|_{\rm os},
\end{aligned}
$$

$$
\chi_{{\rm os};121} \;=\; (z-1)^2+\frac{1}{X}\,(1-y), \qquad X=\frac{x}{1-x}.
$$

First step: **BT** algorithm

$$
[1+\frac{1}{1+\epsilon}\,(y\partial_y+z\partial_z)]\,\chi_{{\rm os};121}^{-1-\epsilon} \;=\; -(z^2-\frac{1}{x})\,\chi_{{\rm os};121}^{-2-\epsilon},
$$

$$
z(1-z)\,\chi_{{\rm os};121}^{-1-\epsilon} = \frac{1}{2\epsilon}\,z\partial_z\,\chi_{{\rm os};121}^{-\epsilon}.
$$

Second step: integration by parts

$$S_p^{121} = \frac{1}{2m^2}\left(\frac{\mu^2}{\pi\, m^2}\right)^{\epsilon} \Gamma\,(\epsilon)\, \int_0^1 dx\,[x(1-x)]^{-\epsilon/2}\,(\mathcal{S}_1 - \mathcal{S}_2),$$

$$\mathcal{S}_1 = \int_0^1 dy\,(1-y)\,y^{-\epsilon/2-1}\,(y+X')^{-\epsilon},$$

$$\mathcal{S}_2 = \int_0^1 dy \int_y^1 dz\, y^{\epsilon/2-1}\,(z^2+X'\,y)^{-\epsilon},$$

Third step: another **BT** algorithm

$$\begin{aligned} I(\mu,\nu) &= \int_0^1 dy \int_0^1 dz\, y^{\mu}\, z^{\nu}\,(z+\frac{y}{X})^{-\epsilon} \\ &= \frac{1}{2+\mu+\nu-\epsilon}\int_0^1 dy\,[y^{\mu}\,(1+\frac{y}{X})^{-\epsilon} + y^{\nu}\,(y+\frac{1}{X})^{-\epsilon}]. \end{aligned}$$

Fourth step: **hypergeometric** functions, x-integration and Laurant expansion

$$\begin{aligned} S_p^{121} &= \frac{1}{m^2}\,(\frac{\mu^2}{\pi\, m^2})^{\epsilon}\,[-\frac{2}{\epsilon^2} - 2\,(2-\gamma)\,\frac{1}{\epsilon} - 7 + \gamma\,(4-\gamma) \\ &- \frac{1}{2}\zeta(2) + \mathcal{O}\,(\epsilon)]. \end{aligned}$$

More complex diagrams require a new algorithm

▷ We **regularize** in n-dimensions and isolate the **infrared poles** by using the technique of

iterated sector decomposition (or **defoliation**). [5]

▷ A k/n defoliation is defined by the following relation

$$\int_0^1 \prod_{i=1}^{k} dx_i \prod_{j=k+1}^{n} dy_j\, F(\{x\}\,;\,\{y\}) = \sum_{l=1}^{k} \int_0^1 \prod_{i=1}^{k} dx_i\, x_l^{k-1}$$
$$\times \prod_{j=k+1}^{n} dy_j\, F(x_l\, x_1, \dots, x_l, \dots x_l\, x_k\,;\,\{y\}).$$

▷ A chain of defoliations will stop when all the singularites of F which are located at the vertices of the **simplex** $[0,1]^k$ have been factorized, $F = x_l^{\mu_l}\, F$.

▷ To find the **sectors**, although far from unique, is **not** the difficult part of the evaluation. The resulting expression will contain **infrared poles** as well as an **infrared finite** part.

▷ The latter and the residues are given by integrals of **inverse powers** of polynomials that are not (usually) **positive defined**. BT - iterations or Mellin - Barnes **splittings** are still required.

[5] T. Binoth and G. Heinrich, Nucl. Phys. B **585** (2000) 741 [arXiv:hep-ph/0004013]

Again understanding from **simple examples**:

$$
\begin{aligned}
C_0\left(p_1^2,p_2^2,P^2;m,0,m\right) &= \mu^{\epsilon}\pi^{-\epsilon/2}\int_0^1 dx\int_0^x dy\,V^{-1-\epsilon/2}(x,y),\\
V(x,y) &= -p2^2\,x^2-p_1^2\,y^2+(p_1^2+p_2^2-P^2)\,x\,y\\
&+\ (p_2^2-m^2)\,x-(p_2^2-P^2-m^2)\,y+m^2.
\end{aligned}
$$

▷ For $m_2 = 0$ and $m_1 = m_3 = m$ the **anomalous threshold** condition is satisfied if

$$
\begin{aligned}
p_1^2 &= p_2^2 = -m^2 \quad \text{and} \quad P^2 \quad \text{is uncontrained}\\
&\Updownarrow\\
V(x,y) &= m^2\,(1-x)^2+m^2\,y^2+(P^2+2\,m^2)\,(1-x)\,y,
\end{aligned}
$$

▷ showing that $V = 0$ for $x = 1, y = 0$. With $x = 1-x', y = (1-x')\,y'$ we

1. split the integration by using

$$1 = \theta(x-y)+\theta(y-x)$$

2. re-map the two integrals in the interval $[0,1]^2$ and obtain

$$
\begin{aligned}
C_0^{IR} &= \mu^\epsilon\,\pi^{-\epsilon/2}\int_0^1 dx\int_0^1 dy\sum_{z=x,y}(z^{-1-\epsilon}-z^{-\epsilon})\,\chi_z^{-1-\epsilon/2},\\
\chi_x &= m^2\,(1+x^2\,y^2)+y\,(1-x)\,(2\,m^2+P^2+m^2\,y)\\
\chi_y &= m^2\,(1+x^2\,y^2)+x\,(1-x\,y)\,(P^2-2\,m^2)\\
&+\; m^2\,x\,(x-2\,y).
\end{aligned}
$$

▷ The infrared pole can now be extracted as

$$
\begin{aligned}
P_{IR} &= 2\,\mu^\epsilon\,\pi^{-\epsilon/2}\int_0^1 dx\int_0^1 dy\,x^{-1-\epsilon}\,\chi_{IR}^{-1-\epsilon/2}\\
&= -\frac{2}{\epsilon}\,\mu^\epsilon\,\pi^{-\epsilon/2}\int_0^1 dy\,\chi_{IR}^{-1-\epsilon/2},\\
\chi_{IR}(y) &= m^2\,y^2+(P^2+2\,m^2)\,y+m^2.
\end{aligned}
$$

Strategies for $P^2=-\,s,\,s>0$:

▷ Away from $s\approx 4\,m^2$ and $m^2\ll s$: one **BT** iteration is enough and **numerical integration** follows;

▷ Around threshold: with $\beta^2 = 1 - 4\,m^2/s$ rewrite

$$\chi_{IR}(y) \;=\; m^2\,(y - \frac{1+\beta^2}{1-\beta^2})^2 + \frac{\beta^2}{\beta^2-1}\,s$$

and apply Mellin - Barnes techniques to obtain the $\beta \to 0$ behavior.

▷ For $s \gg m^2$ it is better to isolate the leading behavior:

$$\begin{aligned}
\int_0^1 dy\,\chi_{IR}^{-1} \;&=\; \frac{1}{4\,m^2}\,\frac{\Gamma\,(1/2)}{2\,\pi\,i}\,\int_{-i\,\infty}^{+i\,\infty} dt \\
&\times\; \frac{\Gamma\,(-t)\;\Gamma^2\,(t+1)}{\Gamma\,(t+3/2)}\,(-\frac{s}{4\,m^2})^t \\
&=\; -\,\frac{1}{s}\,\sum_{n=0}^{\infty}\frac{\Gamma\,(1/2)}{\Gamma\,(1/2-n)\;n!}\,(-\frac{4\,m^2}{s})^n \\
&\times\; [\ln\frac{-\,s}{4\,m^2} + 3\,\psi(n+1) - \psi(\frac{1}{2}-n)].
\end{aligned}$$

About **Reduction**

- ▷ The **standard** procedure is based on the introduction of **scalar functions** and of **tensor integrals** that are reduced to linear combinations of scalar ones.
- ▷ The **new** procedure is different since the **functional relation** can be written in a more general form starting from

$$
\begin{aligned}
& \int_S dx\, Q(x) \prod_i V_i^{\mu_i}(x), \\
Q(x) \;&=\; \text{an arbitrary polynomial of } x.
\end{aligned}
$$

- ▷ Hence we are led to consider **all integrals** in **parametric space** with no **pre-reduction** in **momentum space**.
- ▷ The **number of terms** in the final expression does not matter and the only relevant requisite is the **smoothness** of the integrand.

A **realistic example** of **Reduction of Obvious**

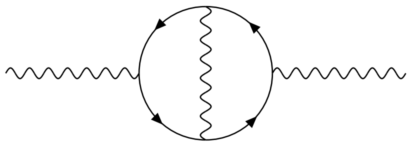


Figure 1: A contribution to the photon self-energy in QED

$$\Pi^{(2)}_{\mu\nu} = \pi^4\, e^4\, \Pi^{(2)}_0\, (\delta_{\mu\nu} - \frac{p_\mu p_\nu}{p^2}), \qquad \Pi^{(2)}_0 = \frac{1}{3-\epsilon}\, S.$$

For the function S we simplify the **obvious** in the numerator and obtain

$$\begin{aligned} S \;=\; & 4\left[(\epsilon-2)\, p^4 + 8\, m^4 + 2\,\epsilon\, p^2 m^2\right] S^{221}(p^2; m, m, 0, m, m) \\ +\; & 2\left[8\, m^2 - (8 - 2\,\epsilon - \epsilon^2)\, p^2\right] B^2_0(p^2; m, m) \\ +\; & 8\,(2-\epsilon)\, p^2\, B^2_1(p^2; m, m) \\ -\; & 16\left[2\, m^2 + (\epsilon-2)\, p^2\right] S^{121}_4(p^2; m, 0, m, m) \\ +\; & 4\,\epsilon\,(2-\epsilon)\, S^{111}(p^2; m, 0, m) \\ +\; & 8\,(\epsilon-2)\, S^{111}(0; m, 0, m) + 8\,(2-\epsilon)\, A_0(m)\, B_0(p^2; m, m). \end{aligned}$$

$G^{\alpha\beta\gamma}$ as in 't Hooft & Veltman

There is no **real** Multiloop calculation without

the γ^5 problem

Highlights in a systematic approach:[6]

▷ given the well-known *non-regularization* of fermion loops,

▷ if we insist (and we do) on the **trace condition** $\Rightarrow$ **gauge invariance** must be broken in order to obtain a **pseudo-regularization**.

▷ A systematic treatment is needed for the so-called **evanescent operators**; given

$$\{\gamma^\mu, \gamma^\nu\} \;=\; 2\,\delta^{\mu\nu}\,I, \qquad \mathrm{Tr}\,I = 4,$$

$$\delta_{\mu\nu} \;=\; \hat{\delta}_{\mu\nu} + \bar{\delta}_{\mu\nu}, \quad \bar{\delta}_{\mu\alpha}\,\hat{\delta}_{\alpha\nu} = 0, \quad \bar{\delta}_{\mu\mu} = n-4, \quad \hat{\delta}_{\mu\mu} = 4,$$

[6] G. Bonneau, Nucl. Phys. B **177** (1981) 523.
P. Breitenlohner and D. Maison, Commun. Math. Phys. **52** (1977) 11.

▷ Evanescent operators $\equiv \mathcal{E}$ where

$$\bar{p}\cdot\bar{p} = \epsilon\,\hat{p}\cdot\hat{p}\,\mathcal{E}, \qquad \mathcal{E} = \frac{\bar{p}\cdot\bar{p}}{\hat{p}\cdot\hat{p}\,\epsilon},$$

▷ **Evanescent terms are** formally zero in the limit $\epsilon \to 0$ but their effect must be **analyzed carefully** due to the presence of **poles** at $\epsilon = 0$.

▷ Due to **pseudo-regularization** Green's functions do **not** fulfill the corresponding **Ward identities** which must be restored by hand. The first step in the correct treatment of the theory consists in **renormalizing away the poles** at $\epsilon = 0$. This we do in the MS-scheme since **finite-renormalization** can always be done later on.

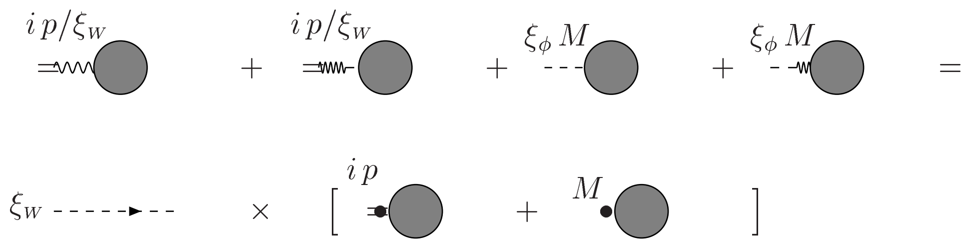


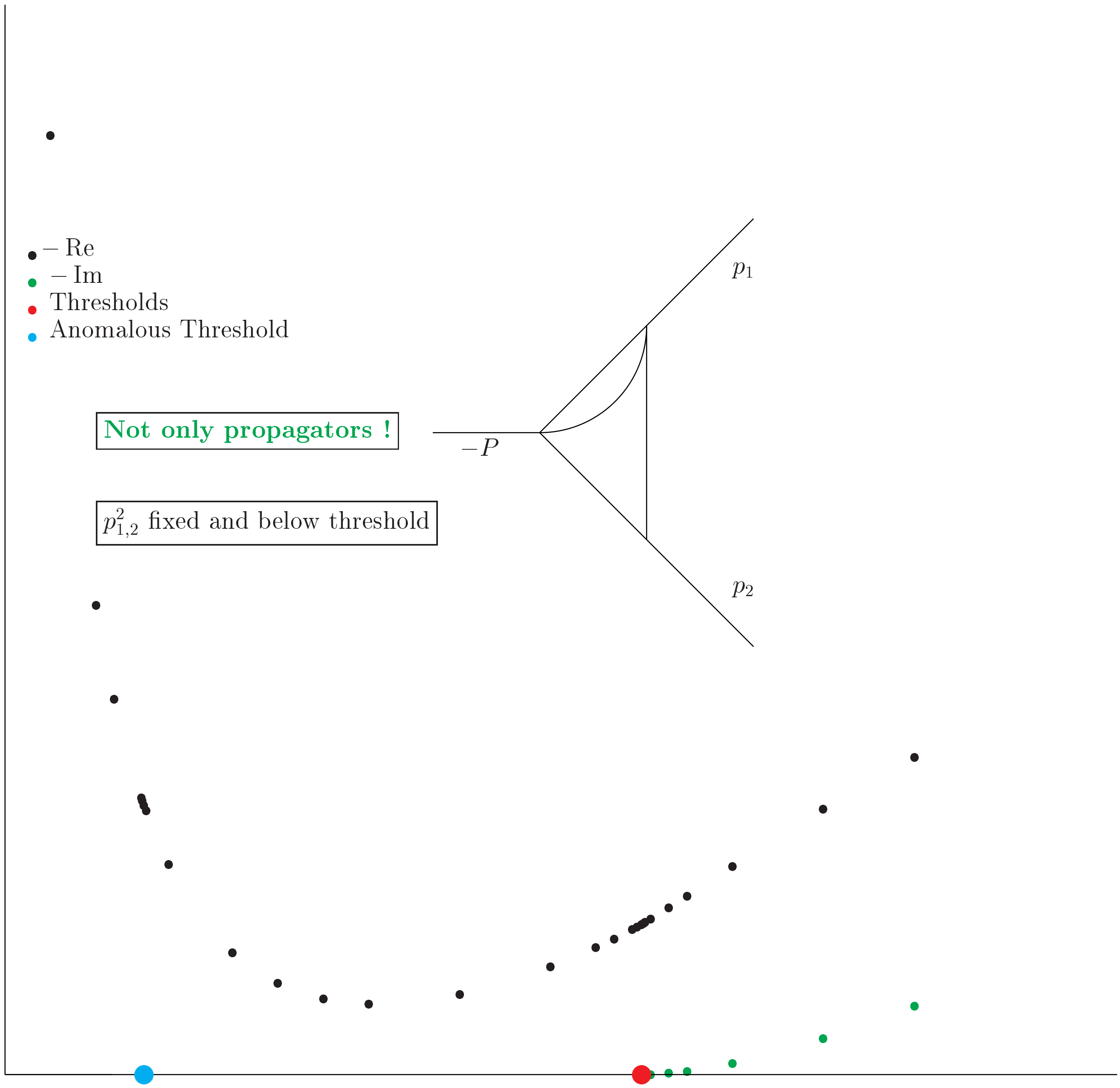
Figure 2: Ward identities for self-energy diagrams. The symbol = indicates multiplication by $i\,p_\mu$ and a dot indicates a truncated Green function.

$s\,[\,\mathrm{GeV}^2]$	$\mathrm{Re}\, S^{221}$	$\mathrm{Re}\, S^{221}$ref. [1]	$\mathrm{Im}\, S^{221}$	$\mathrm{Im}\, S^{221}$ref. [1]
0.1	$-0.287205 \pm 0.20 \times 10^{-4}$ $-0.287237 \pm 0.35 \times 10^{-5}$	$-0.287238 \pm 0.25 \times 10^{-5}$	0.00	0.00
0.5	$-0.294578 \pm 0.15 \times 10^{-4}$ $-0.294590 \pm 0.23 \times 10^{-5}$	$-0.294952 \pm 0.25 \times 10^{-5}$	0.00	0.00
1.0	$-0.304543 \pm 0.21 \times 10^{-4}$ $-0.304522 \pm 0.22 \times 10^{-5}$	$-0.304521 \pm 0.25 \times 10^{-5}$	0.00	0.00
5.0	$-0.452499 \pm 0.17 \times 10^{-4}$ $-0.452518 \pm 0.31 \times 10^{-4}$	$-0.452520 \pm 0.28 \times 10^{-5}$	0.00	0.00
10.0	$-0.488036 \pm 0.59 \times 10^{-3}$ $-0.488224 \pm 0.18 \times 10^{-3}$	$-0.488153 \pm 0.23 \times 10^{-5}$	$-0.353197 \pm 0.69 \times 10^{-3}$ $-0.353082 \pm 0.18 \times 10^{-3}$	$-0.353217 \pm 0.23 \times 10^{-5}$
50.0	$0.173897 \pm 0.49 \times 10^{-4}$ $0.173900 \pm 0.18 \times 10^{-4}$	$0.173901 \pm 0.22 \times 10^{-5}$	$-0.118089 \pm 0.40 \times 10^{-4}$ $-0.118082 \pm 0.20 \times 10^{-4}$	$-0.118080 \pm 0.22 \times 10^{-5}$

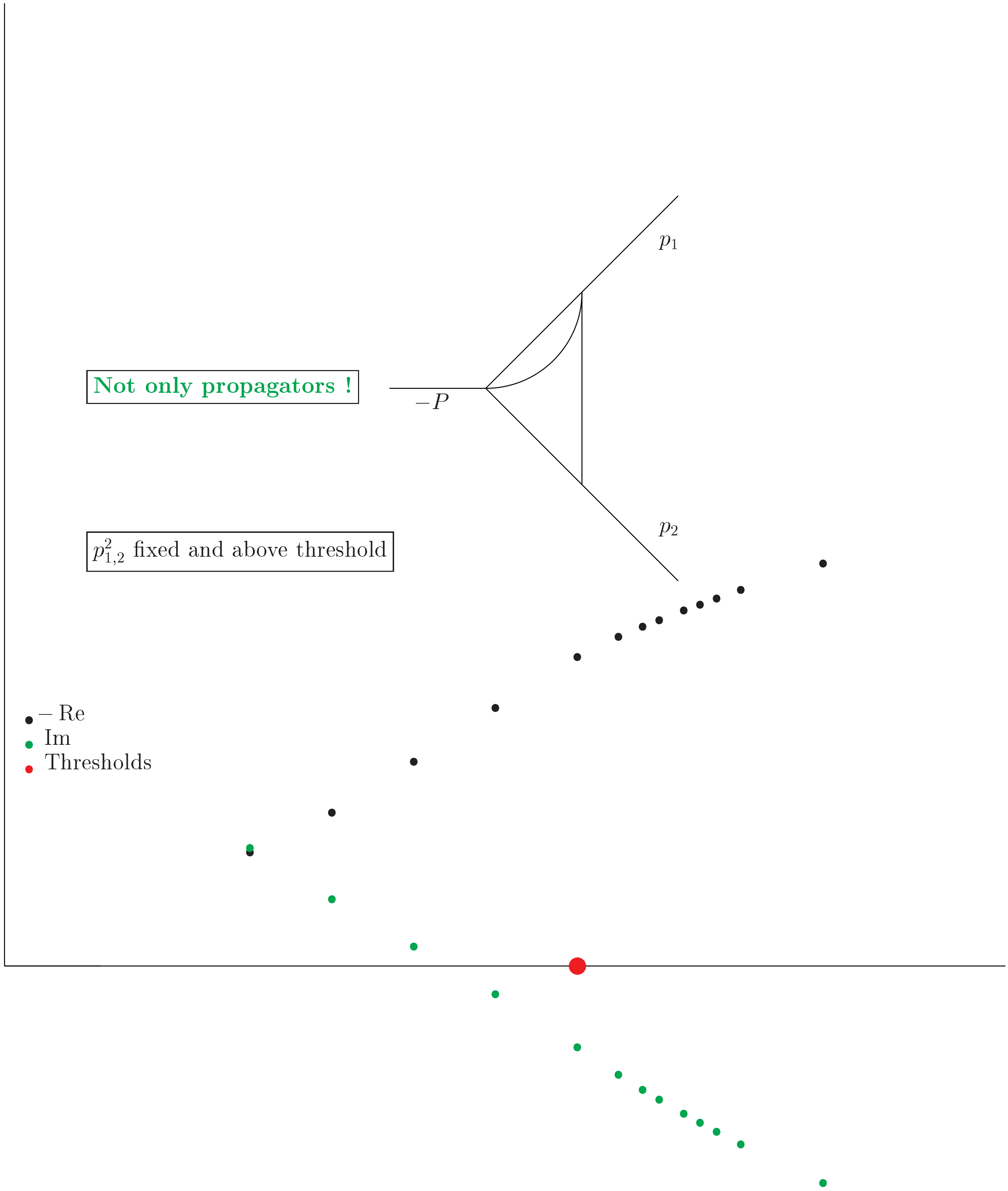
Table 1: S^{221} compared with the numerical result of Tab. 1 in ref. [1]. Here $m_n^2 = n\,\mathrm{GeV}^2$, for $n = 1, \dots, 5$. The second entry refers to a run with a much larger number of points in sub-routine D01GDF.

References

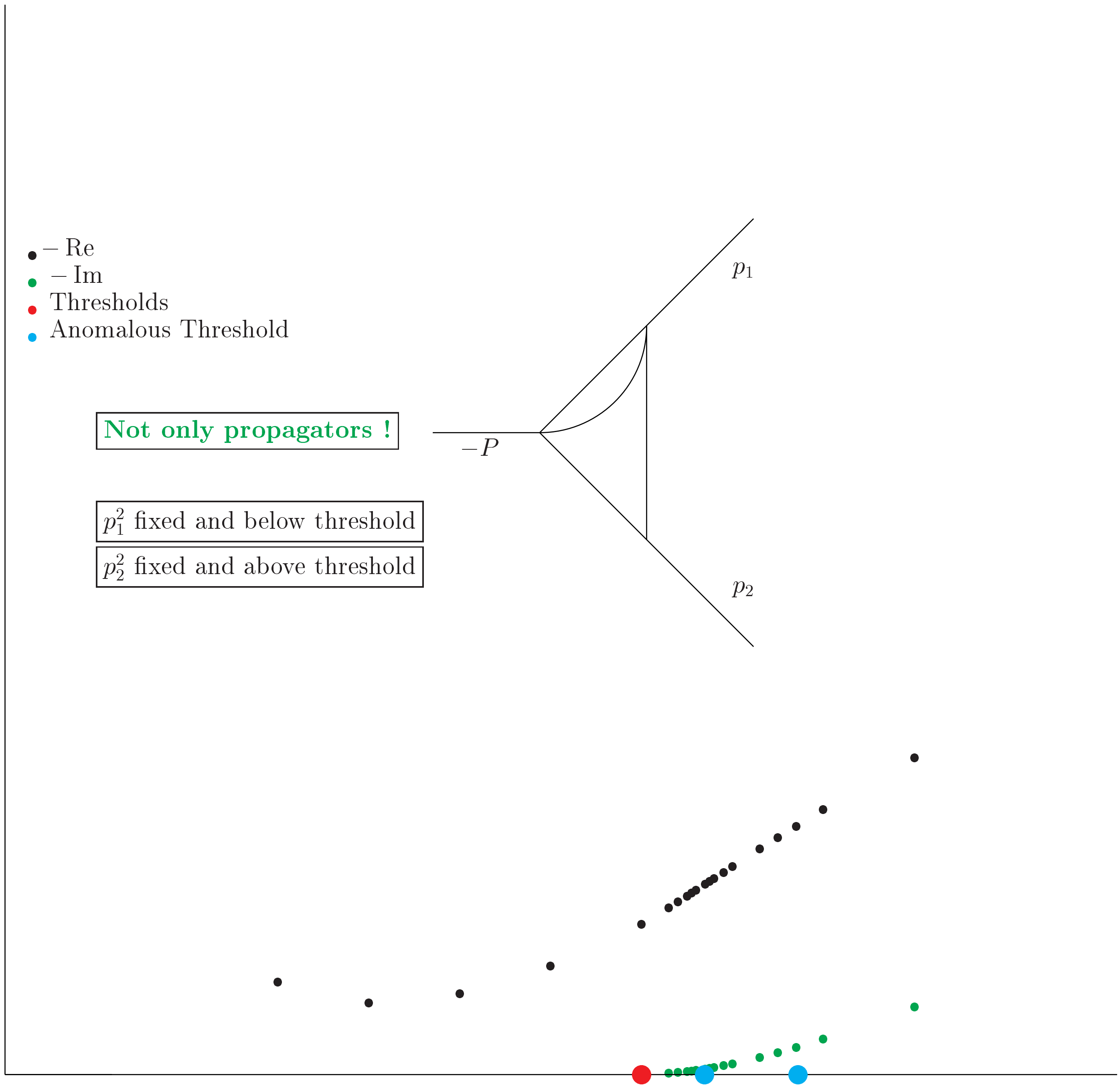
[1] S. Bauberger and M. Bohm, Nucl. Phys. B **445** (1995) 25 [arXiv:hep-ph/9501201].



Anomalous Threshold may occur below normal ones!!!



Not only propagators !
$-P$
p_1
p_2
$p^2_{1,2}$ fixed and above threshold
$-$ Re
Im
Thresholds


– Re
– Im
Thresholds
Anomalous Threshold
Not only propagators !
p_1^2 fixed and below threshold
p_2^2 fixed and above threshold
$-P$
p_1
p_2

Preliminar considerations about **CPU - time**

▷ **Keyword** $\equiv$ **Multipurpose**: the aim is to get results **everywhere**;

▷ During the last 4 **years** I gained a factor 80 in **CPU - time** by changing 3 Workstations;

▷ **Warning**: $t_{N\,\text{diagrams}} \ll N \times t_{1\,\text{diagram}}$

$\mathrm{Re}\, S^{221}$	0.480(2)	0.4786(9)	0.4788(2)	0.4790(1)
$\mathrm{Im}\, S^{221}$	0.372(1)	0.3695(4)	0.3703(2)	0.3704(1)
CPU time [sec]	0.81	12.10	46.20	89.60

Table 2: S^{221} at $-1\,\mathrm{GeV}$ from the second two-particle normal threshold

Conclusions

- ▷ A `Form` code generating all **Standard Model 2L&2L** functions available,

- ▷ All **1L&nL** functions, up to $n = 6$, **available**,

- ▷ All **2L&2L** functions and their *on-shell* derivatives **available**,

- ▷ All **non-IR 2L&3L** functions **available**,

- ▷ All **IR 2L&3L** functions but the *non-planar* vertex (under completion) **available**,

- ▷ **2L&2(3)L** functions are evaluated using the **Korobov-Conroy** number theoretic method with a **Monte-Carlo** error estimate,

- ▷ List of *finite counterterms* in the **TVBM** - scheme under completion