Loops To The Deepest

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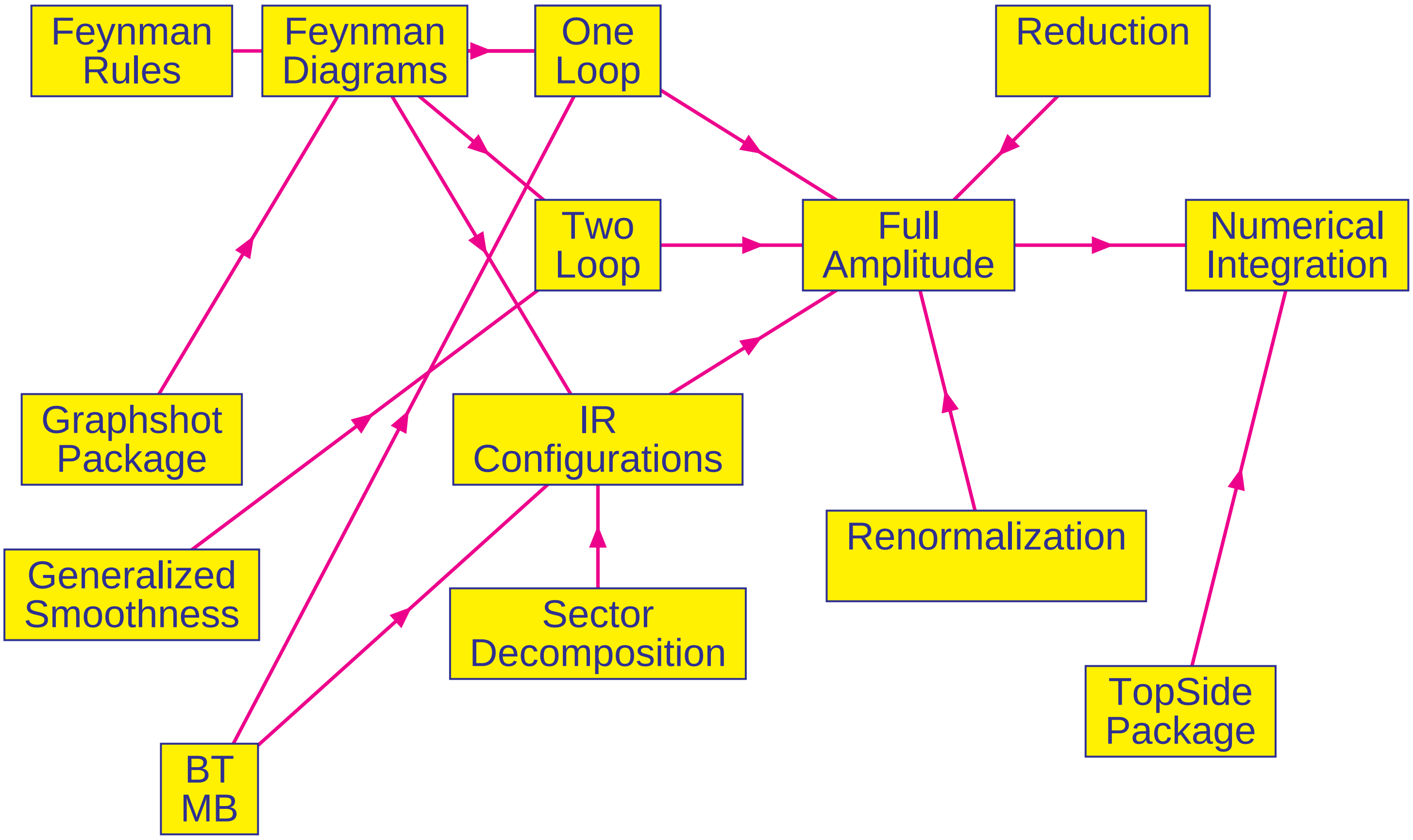
Prolegomena: the overall project



Basics: the problem made explicit

No matter how long you turn it around the problem in any realistic multi-loop, multi-leg calculation is connected with a simple fact:

$$\textbf{(non-)scalar}\ G = \frac{(\textbf{something}\,;\,\int \textbf{something})^{\dagger}}{f(\textbf{parameters})},$$

$$\dagger\textbf{something} \equiv \textbf{HTF or a smooth integrand}.$$

Examples of f are

- Gram determinants in the standard tensor reduction;
- denominators in the IBP - techmique.

Optimal case for $f = 0$:

∇ $\equiv$ solutions of Landay equations

∇ Nature of singularity NOT overestimated

An algorithm for all one-loop diagrams

Any one-loop Feynman diagram G, irrespective of the number N ofvertices, can be expressed as

$$G = \int_S dx\, Q(x)\, V^{\mu}(x),$$

where the integration region is $x_j \geq 0, \Sigma_j\, x_j \leq 1$, with $j = 1, \dots, N-1$, and $V(x)$ is a quadratic polynomial in x,

$$V(x) = x^t\, H\, x + 2\, K^t\, x + L,$$

and $Q(x)$ is also a polynomial that accounts for parametrized tensor integrals. The solution to the problem of determining the polynomial $\mathcal{P}$ is as follows:

$$\mathcal{P} = 1 - \frac{(x - X)^t\, \partial_x}{2\, (\mu + 1)},$$
$$X^t = -\, K^t\, H^{-1}, \qquad B = L - K^t\, H^{-1}\, K,$$

where the matrix H is symmetric.

- C_0

$$\begin{aligned} C_0 &= \frac{1}{B_3}\{\frac{1}{2}+\int_0^1 dx_1\,[\int_0^{x_1} dx_2\,\ln V(x_1,x_2) \\ &- \frac{1}{2}\sum_{i=0}^{2}(X_i - X_{i+1})\,\ln V(i\,\widehat{i+1})]\}, \\ V(\widehat{0\,1}) &= V(1,x_1), \quad V(\widehat{1\,2}) = V(x_1,x_1), \quad V(\widehat{2\,3}) = V(x_1,0), \\ C_0(G_3=0) &= -\frac{1}{2\,b_3}\int_0^1 dx\,\sum_{i=0}^{2}(\mathcal{X}_i - \mathcal{X}_{i+1})\,\ln V(i\,\widehat{i+1}), \end{aligned}$$

- **where** $B_3 = b_3/G_3$ **and** $\mathcal{X} = -\Delta\,K$

- C_0 **at** $\mathcal{O}(\epsilon)$

$$\begin{aligned} C_0(d) &= C_0(4) + \epsilon\,C_0^{(1)} + \mathcal{O}\left(\epsilon^2\right), \\ B_3\,C_0^{(1)} &= -\frac{1}{2}\int dS_2\,[\ln V(x_1,x_2) + \frac{1}{2}\ln^2 V(x_1,x_2)] \\ &+ \frac{1}{8}\int dS_1\,\sum_{i=0}^{2}(X_i - X_{i+1})\,\ln^2 V(i\,\widehat{i+1}), \end{aligned}$$

Evaluation of E_0 when $B_5 \neq 0$

It is a virtue of the BT algorithm that we can show the decomposition of E_0 in five boxes (in $d = 4$) with just one iteration. We obtain

$$E_0 = \frac{1}{4\,B_5} \sum_{i=0}^{4} w_i\,D_0^{(i)},$$

where the weights are

$$w_i = X_i - X_{i+1}, \qquad X_0 = 1, \quad X_5 = 0,$$

and where $B_5 = L - K^t\,H^{-1}\,K$ and $X = -K^t\,H^{-1}$. The further advantage in this derivation is that the nature of the weights is transparent since $B_5 = 0$ corresponds to a Landau singularity of the pentagon. Furthermore the boxes are specified by

$$D_0^{(i)} = \int dS_3\,V^{-2-\epsilon/2}(\widehat{i\;i+1}),$$

where the contractions are

$$\widehat{0\,1} = (1, x_1, x_2, x_3), \quad \widehat{4\,5} = (x_1, x_2, x_3, 0),$$

etc. As long as B_5 is not around zero the derivation for the pentagon is completed since we know how to deal with boxes.

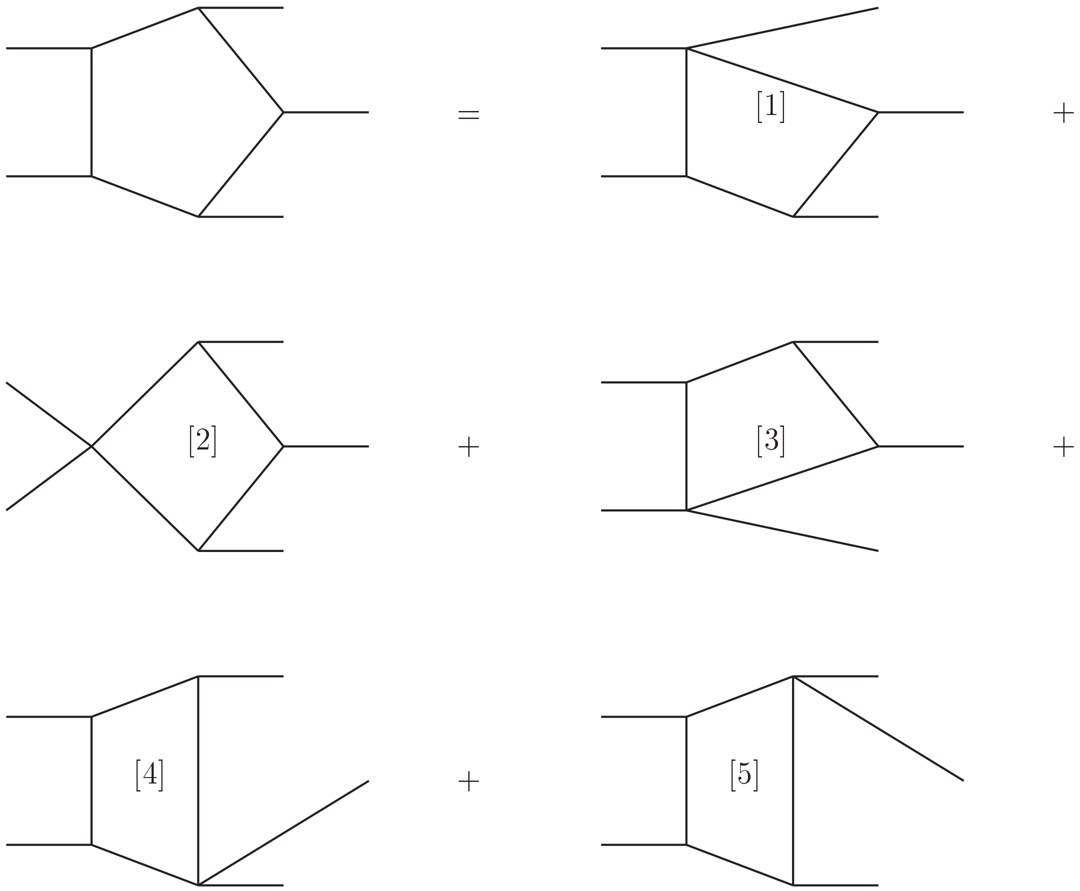
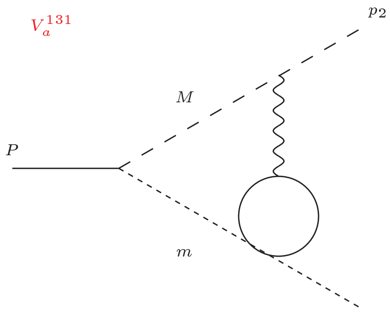


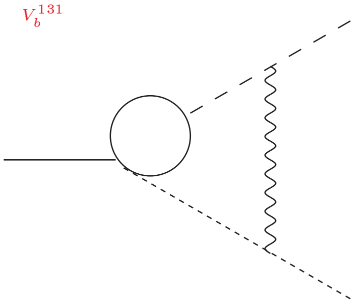
Figure 1: Diagrammatical representation of the BT algorithm for the pentagon. The symbol $[i]$ denotes multiplication of the corresponding box by a factor $w_i/(4\,B_5)$.

Something difficult: IR configurations

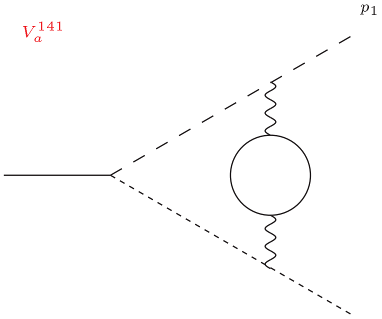
V_a^{131}



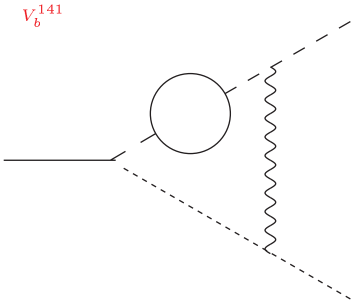
V_b^{131}



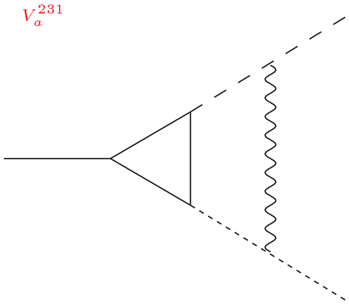
V_a^{141}



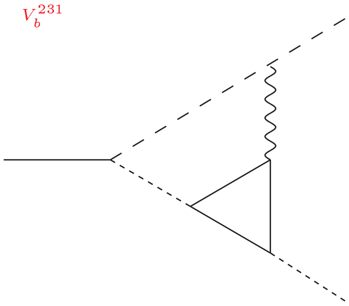
V_b^{141}



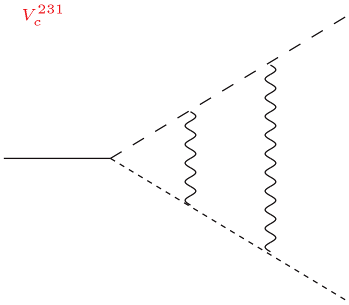
V_a^{231}



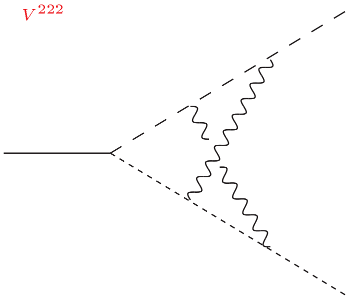
V_b^{231}



V_c^{231}



V^{222}



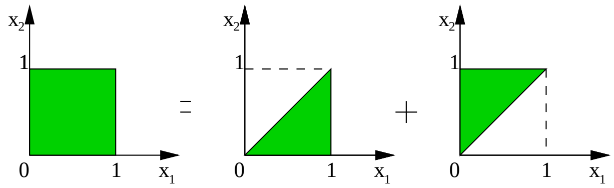
Sector decomposition

In multi-loop diagrams the IR singularities are often overlapping. Procedure:

- **Place the singular points at the hedge of the parameters space and remap variables to the unit cube. Example:**

$$I \;=\; \int_0^1 dx\, dy\, P^{-2-\epsilon}(x,y), \qquad P(x,y) = a\,(1-y)\,x^2 + b\,y$$

- **Decompose the integration domain**



$$\int_0^1 dx \int_0^1 dy \;=\; \int_0^1 dx \int_0^x dy + \int_0^1 dy \int_0^y dx$$

- **Remap the variables to the unit cube**

$$I = \int_0^1 dx\, dy\, x\, P^{-2-\epsilon}(x, xy) + \int_0^1 dx\, dy\, y\, P^{-2-\epsilon}(xy, y),$$

- **Factorize the variables**

$$I = \int_0^1 dx\, dy \left[x^{-1-\epsilon}\, P_1^{-2-\epsilon}(x,y) + y^{-1-\epsilon}\, P_2^{-2-\epsilon}(x,y)\right],$$

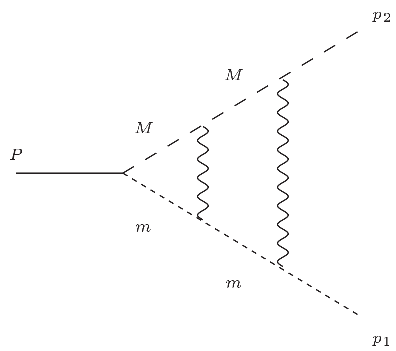
$$P_1 = a\,(1 - x\,y)\,x + b\,y, \qquad P_2 = a\,(1-y)\,x^2 + b$$

- **Iterate the procedure until all polynomials are free from zeros.**
- **Perform a Taylor expansion in the factorized variables and integrate to extract the IR poles:**

$$I_2 = -\frac{\mathbf{1}}{\epsilon} \int_0^1 dx\, P_2^{-2-\epsilon}(x,0) + \int_0^1 dx \int_0^1 dy\, \frac{P_2^{-2}(x,y) - P_2^{-2}(x,0)}{y}.$$

- **If $\mathbf{a}, \mathbf{b} > \mathbf{0}$ we can integrate numerically**

The V_c^{231} diagram (double IR pole)



$$p_1^2 = -m^2 \qquad p_2^2 = -M^2$$

After some work the integrand takes the form:

$$\begin{aligned}\xi^{-2-\epsilon} &= \left[(1-x_2)\,y_1\,\chi(y_2) + x_2\,y_3\,\chi(x_1)\right]^{-2-\epsilon} \\ \chi(x) &= -p^2\,x^2 + (p^2 - m_1^2 + m_2^2)\,x + m_1^2\end{aligned}$$

Points that give rise to IR divergencies

$$y_1 = y_3 = 0, \qquad\qquad y_1 = x_2 = 0$$

Procedure

- **Sector decomposition**
- **BT method to raise power**

Comparison with Bonciani-Mastrolia-Remiddi

$\rightsquigarrow m_1 = m_2 = m = M$ **(=4 GeV)**

$$V_a^{141} \;=\; -\,\pi^{-\epsilon}\,\Gamma^2 \left(1+\frac{\epsilon}{2}\right)\,\left(I_{-1}\,\frac{1}{\epsilon}+I_0\right)$$

Comparison with Davydychev-Kalmykov

$$\begin{cases} m_1 = m_2 = m_3\,(=180\,GeV) \\ \mathbf{m = M = 0} \qquad\qquad \Rightarrow \qquad\qquad \text{Collinear divergencies} \end{cases}$$

$$V_a^{231} \;=\; \pi^{-\epsilon}\,\Gamma^2\,(1+\epsilon/2)\,\left(I_{-2}\,\epsilon^{-2}+I_{-1}\,\epsilon^{-1}+I_0\right)$$

Comparison with Davydychev-Smirnov

$\rightsquigarrow$ **Valid for** $m \ll M, |P|$

$$V_c^{231} \;=\; \pi^{-\epsilon}\,\Gamma\,(2+\epsilon)\,\left(I_{-2}\,\frac{1}{\epsilon^2}+I_{-1}\,\frac{1}{\epsilon}+I_0\right)$$

	$\sqrt{s}\ [GeV]$	$\mathrm{Re}\, I_0 \quad [GeV^{-4}]$	$\mathrm{Im}\, I_0 \quad [GeV^{-4}]$
Our **BMR**	400	$7.6967(5) \times 10^{-6}$ 7.69672×10^{-6}	$-1.6756(4) \times 10^{-6}$ -1.67503×10^{-6}
Our **BMR**	40	$5.09867(7) \times 10^{-4}$ 5.09867×10^{-4}	$-1.37702(6) \times 10^{-4}$ -1.37702×10^{-4}
Our **BMR**	0.4	$-1.64816^{*} \times 10^{-2}$ -1.64816×10^{-2}	0 0
Our **BMR**	0.04	$-1.64337^{*} \times 10^{-2}$ -1.64337×10^{-2}	0 0
	$\sqrt{t}\ [GeV]$	$\mathrm{Re}\, I_0 \quad [GeV^{-4}]$	$\mathrm{Im}\, I_0 \quad [GeV^{-4}]$
Our **BMR**	0.04	$-1.64328^{*} \times 10^{-2}$ -1.64328×10^{-2}	0 0
Our **BMR**	0.4	$-1.63851^{*} \times 10^{-2}$ -1.63851×10^{-2}	0 0
Our **BMR**	40	$-5.42064^{*} \times 10^{-4}$ -5.42064×10^{-4}	0 0
Our **BMR**	400	$-7.71078^{*} \times 10^{-6}$ $-7.71078^{*} \times 10^{-6}$	0 0

	$\sqrt{s}$ $[GeV]$	Re I_0 $[GeV^{-4}]$	Im I_0 $[GeV^{-4}]$
Our **DK**	400	$5.1343(1) \times 10^{-8}$ 5.13445×10^{-8}	$1.94009(8) \times 10^{-8}$ 1.94008×10^{-8}
Our **DK**	300	$5.68801^{*} \times 10^{-8}$ 5.68801×10^{-8}	$-1.61218^{*} \times 10^{-8}$ -1.61218×10^{-8}
Our **DK**	200	$9.36340^{*} \times 10^{-8}$ 9.36340×10^{-8}	$-2.84232^{*} \times 10^{-8}$ -2.84232×10^{-8}
Our **DK**	100	$2.94726^{*} \times 10^{-7}$ 2.94726×10^{-7}	$-9.74218^{*} \times 10^{-8}$ -9.74218×10^{-8}
	$\sqrt{t}$ $[GeV]$	Re I_0 $[GeV^{-4}]$	Im I_0 $[GeV^{-4}]$
Our **DK**	100	$-2.85709^{*} \times 10^{-7}$ -2.85709×10^{-7}	0 0
Our **DK**	200	$-7.61695^{*} \times 10^{-8}$ -7.61695×10^{-8}	0 0
Our **DK**	300	$-3.29938^{*} \times 10^{-8}$ -3.29938×10^{-8}	0 0
Our **DK**	400	$-1.74228^{*} \times 10^{-8}$ -1.74228×10^{-8}	0 0

	$s\ [GeV^2]$	$\mathrm{Re}\, I_0 \quad [GeV^{-4}]$	$\mathrm{Im}\, I_0 \quad [GeV^{-4}]$
Our **DS**	10.2	16.346(5) 16.3459	−18.059(4) −18.0590
Our **DS**	9.2	19.928(6) 19.9189	−22.175(4) −22.1755
Our **DS**	8.2	24.898(8) 24.9015	−27.980(4) −27.9753
Our **DS**	7.2	32.185(8) 32.1805	−36.547(8) −36.5550
Our **DS**	6.2	43.51(1) 43.4927	−50.114(8) −50.1010
Our **DS**	5.2	62.62(2) 62.6575	−73.51(2) −73.5359
Our **DS**	4.2	99.58(3) 99.6039	−120.09(2) −120.086
Our **DS**	3.2	188.04(5) 188.017	−237.07(5) −237.028

$$M = 1\, GeV, \;\; m = 1\, MeV$$

Ranking Two-Loop Diagrams

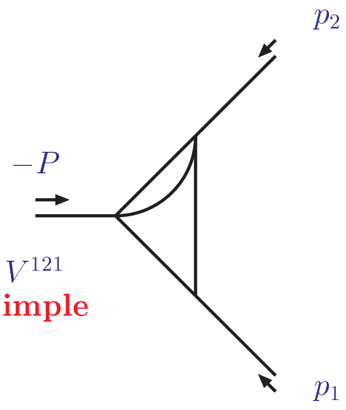
- **Two - point: all Simple**
- **Three - Point: see next figure**

Simple BT suffices

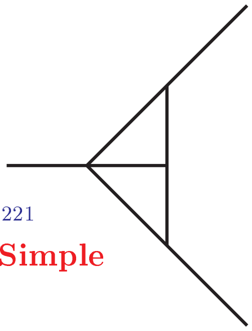
Semi-Simple BT not enough, new kernels required for the smothness algorithm

Difficult Numerical differentiation required

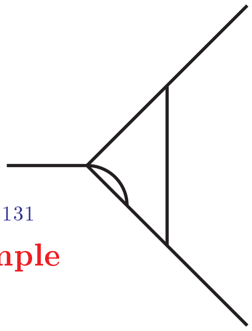
Vertices



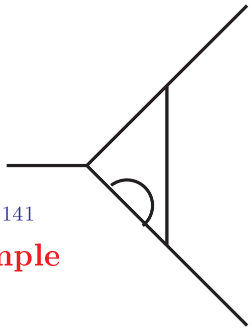
V^{121}
Simple



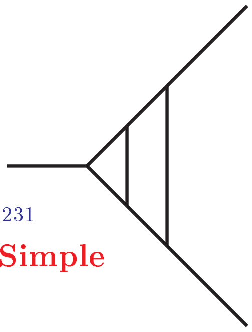
V^{221}
Semi-Simple



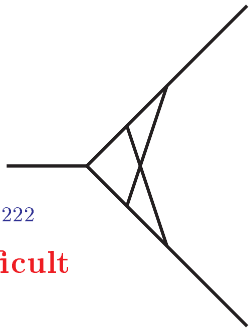
V^{131}
Simple



V^{141}
Simple



V^{231}
Semi-Simple



V^{222}
Difficult

Integrated BT - algorithm

Given a two-loop diagram $G(n_1, \cdots n_I)$ we define the associated δ-diagram

$$G_\delta(0\,;\, n_1, \cdots n_I) = \frac{\mu^{2\epsilon}}{\pi^4} \int d^n q_1\, d^n q_2 \prod_{i=1}^{\alpha} (k_i^2 + m_i^2)^{-n_i-\delta}$$

$$\times \prod_{j=\alpha+1}^{\alpha+\gamma} (k_j^2 + m_j^2)^{-n_j-\delta} \prod_{l=\alpha+\gamma+1}^{\alpha+\gamma+\beta} (k_l^2 + m_l^2)^{-n_k-\delta},$$

and are interested in

$$G(0\,;\, n_1, \cdots n_I) = \lim_{\delta\to 0} G_\delta(0\,;\, n_1, \cdots n_I).$$

Furthermore, a two-loop diagram is carachterized

- **by a degree, $\mathcal{D} = \Sigma\, n_i - 4 + \epsilon$;**
- **by a rank, given by the power of irreducible scalar products present in the numerator.**

Clearly, diagrams with $\mathcal{D} = \epsilon$ are simple to evaluate since, after a Laurent expansion they contain at most integrals of logarithms. The idea is to lower $\mathcal{D}$ by

- **writing IBP (8) and Lorentz identities (1) for**

$$G_\delta(\mathbf{rank}\,;\, 0\,,\, -1\,,\, \cdots\,,\, -1) \quad \textbf{and}$$
$$G_\delta(\mathbf{rank}\,;\, +1\,,\, -1\,,\, \cdots\,,\, -1),$$

with increasing values for the power of irreducible scalar products till we can solve for the δ-scalar diagram of degree $\mathcal{D}$ in favor of (possibly) scalar δ-diagrams of degree $\mathcal{D}-1$ or less. The whole procedure is then iterated till we reach a decomposition in terms of δ-diagrams of degree $\mathcal{D} = \epsilon \,(\mathbf{mod}\, \delta)$. At this point we take the limit $\delta \to 0$ and obtain an expression for the original diagram. The solution will be of the following form:

$$G_\delta(-1\,,\,-1\,,\,\cdots\,,\,-1) \;=\; \delta^{-1} \sum_{\{\mathcal{C}\}} \sum_{\mathbf{rank}} P_{\mathcal{C},\mathbf{r}}\, G_\delta(\mathcal{C}),$$

where $\mathcal{C}$ represents a *contraction*, i.e. at least one power $-1-\delta$ is replaced by $-\delta$ and the P are polynomials in the external parameters. Note that positive powers should be also eliminated since, e.g.

$$G_\delta(0\,;\,1-\delta\,,\,-1-\delta\,,\,-2-\delta)$$

has always to be understood as

$$G_\delta(q_1^2\,;\,-\delta\,,\,-1-\delta\,,\,-2-\delta) + m_1^2\, G_\delta(0\,;\,-\delta\,,\,-1-\delta\,,\,-2-\delta),$$

i.e. with $\mathcal{D} = -1\,(\mathbf{mod}\,\delta)$.

Solution for C_0

– **Equations:**

$$\int d^nq \, \frac{\partial}{\partial q_\mu} \{v_\mu \, [1]^{-\delta} \, [2]^{-1-\delta} \, [3]^{-1-\delta}\} = 0,$$

$$\int d^nq \, \frac{\partial}{\partial q_\mu} \{v_\mu \, [1]^{-1-\delta} \, [2]^{-\delta} \, [3]^{-1-\delta}\} = 0$$

$$\int d^nq \, \frac{\partial}{\partial q_\mu} \{v_\mu \, [1]^{-1-\delta} \, [2]^{-1-\delta} \, [3]^{-\delta}\} = 0$$

$$\int d^nq \, \frac{\partial}{\partial q_\mu} \{v_\mu \, [1]^{1-\delta} \, [2]^{-1-\delta} \, [3]^{-1-\delta}\} = 0$$

$$\int d^nq \, \{p_1 \cdot p_2 \, (p_{1\mu} \, \frac{\partial}{\partial p_{1\mu}} - p_{2\mu} \, \frac{\partial}{\partial p_{2\mu}})$$

$$+ \; p_2^2 \, p_{1\mu} \, \frac{\partial}{\partial p_{2\mu}} - p_1^2 \, p_{2\mu} \, \frac{\partial}{\partial p_{1\mu}}\} \, [1]^{-\delta} \, [2]^{-1-\delta} \, [3]^{-1-\delta} = 0,$$

$$\int d^nq \, \{p_1 \cdot p_2 \, (p_{1\mu} \, \frac{\partial}{\partial p_{1\mu}} - p_{2\mu} \, \frac{\partial}{\partial p_{2\mu}})$$

$$+ \; p_2^2 \, p_{1\mu} \, \frac{\partial}{\partial p_{2\mu}} - p_1^2 \, p_{2\mu} \, \frac{\partial}{\partial p_{1\mu}}\} \, [1]^{-1-\delta} \, [2]^{-\delta} \, [3]^{-1-\delta} = 0,$$

$$\int d^nq \, \{p_1 \cdot p_2 \, (p_{1\mu} \, \frac{\partial}{\partial p_{1\mu}} - p_{2\mu} \, \frac{\partial}{\partial p_{2\mu}})$$

$$+ \; p_2^2 \, p_{1\mu} \, \frac{\partial}{\partial p_{2\mu}} - p_1^2 \, p_{2\mu} \, \frac{\partial}{\partial p_{1\mu}}\} \, [1]^{-\delta} \, [2]^{-1-\delta} \, [3]^{-1-\delta} = 0,$$

$$\int d^nq \, \{p_1 \cdot p_2 \, (p_{1\mu} \, \frac{\partial}{\partial p_{1\mu}} - p_{2\mu} \, \frac{\partial}{\partial p_{2\mu}})$$

$$+ \; p_2^2 \, p_{1\mu} \, \frac{\partial}{\partial p_{2\mu}} - p_1^2 \, p_{2\mu} \, \frac{\partial}{\partial p_{1\mu}}\} \, [1]^{-\delta} \, [2]^{-1-\delta} \, [3]^{-\delta} = 0,$$

$$\int d^nq \, \{p_1 \cdot p_2 \, (p_{1\mu} \, \frac{\partial}{\partial p_{1\mu}} - p_{2\mu} \, \frac{\partial}{\partial p_{2\mu}})$$

$$+ \ p_2^2\, p_{1\mu}\, \frac{\partial}{\partial p_{2\mu}} - p_1^2\, p_{2\mu}\, \frac{\partial}{\partial p_{1\mu}} \} \, [1]^{1-\delta}\, [2]^{-1-\delta}\, [3]^{-1-\delta} = 0,$$

– Solution

$$C_{0\delta}(-1,-1,-1) \ = \ \delta^{-1}\, \frac{m^2}{M^2\, s\, (s + M^2 - 4\, m^2)}\, C,$$

$$\begin{aligned}
C \ = \ & M^2\, (1 - \frac{s}{m^2})\, C_{0\delta}(-1,-1,0)\, \delta \\
+ \ & \frac{sM^2}{m^2}\, C_{\delta}(-1,0,-1)\, \delta \\
+ \ & 2\, C_{0\delta}(1,-1,-1)\, (1 - 2\, \delta) \\
- \ & C_{0\delta}(1,0,-2)\, (\delta + 1) \\
+ \ & (2 - \frac{s}{m^2})\, C_{0\delta}(0,-2,0)\, (\delta + 1) \\
+ \ & (1 - 2\, \frac{s}{m^2})\, M^2\, C_{0\delta}(0,-1,-1)\, \delta \\
+ \ & \frac{sM^2}{m^2}\, C_{0\delta}(0,-1,-1) \\
+ \ & C_{0\delta}(0,0,-1)\, (1 - \delta)).
\end{aligned}$$

After this we only have to compute δ-integrals in parametric space with $\Sigma\,\delta_i = -2 - 3\,\delta$,

$$I(\delta_1\,,\,\delta_2\,,\,\delta_3) = \delta^{-1}\,\frac{\Gamma\,(3\,\delta)}{\Gamma^3(\delta)}$$
$$\times\;\int_0^1 dx\,\int_0^x dy\,(x-y)^{-1-\delta_1}\,y^{-1-\delta_2}\,(1-x)^{-1-\delta_3}\,\chi^{-3\,\delta}(x,y)$$

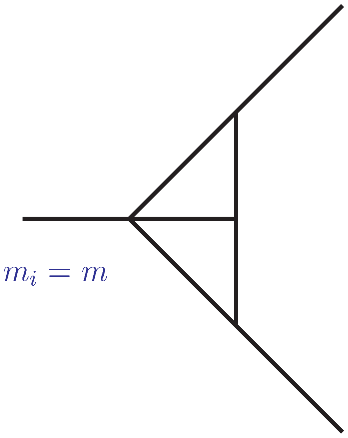
An example:

$$I(-1\,,\,-1\,,\,0) = \frac{\Gamma\,(3\,\delta)}{\delta^2\,\Gamma^3(\delta)}\,[I_{-2}\,\delta^{-2} + I_{-1}\,\delta^{-1} + I_0 + \mathcal{O}\,(\delta)],$$

$$\begin{aligned}
I_{-2} &= 1,\\
I_{-1} &= -3 - 3\,\int_0^1 dx\,\ln\chi(1,x),\\
I_0 &= -3\,\int_0^1 dx\,\int_0^x dy\,[\frac{\ln\chi(x,y)}{1-x}]_+\\
&+ \int_0^1 dx\,\int_0^x dy\,[\frac{\ln(x-y)}{1-x}]_+\\
&- 6\,\int_0^1 dx\,\ln(1-x)\,\ln\chi(1,x) - 3\,\int_0^1 dx\,\ln x\,\ln\chi(1,x)\\
&+ \frac{9}{2}\,\int_0^1 dx\,\ln^2\chi(1,x) + 9 - 2\,\zeta(2).
\end{aligned}$$

An important check on the calculation is that all poles in δ cancel in the total.

At the two-loop level the # of terms increases considerably.



$V^{221}\; m_i = m$

There are 2 **completely irreducible scalar product that we choose to be** $q_2 \cdot q_2 - q_1 \cdot p_2$ **and one** δ **-irreducible one,** q_1^2**. It follows**

$$V^{221}_{\delta} = \delta^{-1}\,\frac{1}{(s-4\,m^2)\,(s-m^2)}\,\sum_{i,j,k=0}^{2} \times \sum_{\mathcal{C}=S}^{D}\sum_{\overline{\mathcal{C}}} P^{ijk}_{\mathcal{C},\overline{\mathcal{C}}}\,V^{221}_{\mathcal{C},\overline{\mathcal{C}}},$$

where

- $\mathcal{C}$ **denotes simple** (S) **and double** (D) **contractions,**
- $\overline{\mathcal{C}}$ **sums over the remaining powers such that**

$$\mathcal{D}\,[V^{221}_{\mathcal{C},\overline{\mathcal{C}}}] = \epsilon\,(\mathbf{mod}\,\delta).$$

– **the remaining sum is over powers of** $q_2 \cdot q_2, q_1 \cdot p_2$ **and of** q_1^2**. Example of** D **is**

$$
\begin{aligned}
& \frac{7}{6}\, m^2\, V^{221}_{\delta}(0,0;0,0,-1,-1,-1) \\
- & \frac{5}{6}\, m^2\, V^{221}_{\delta}(0,0;0,0,-1,-1,-1)\, \delta + \frac{\delta+1}{m^2}\, [\\
+ & (-\frac{23}{6}\, s m^2 + \frac{5}{6}\, s^2 + 5\, m^4)\, V^{221}_{\delta}(0,0;0,0,-2,-1,-1) \\
- & \frac{1}{6}\, m^2\, (s - m^2)\, V^{221}_{\delta}(0,0;0,0,-1,-2,-1) \\
- & \frac{1}{6}\, m^2\, (s - m^2)\, V^{221}_{\delta}(0,0;0,0,-1,-1,-2) \\
- & (\frac{7}{3}\, s - 6\, m^2)\, V^{221}_{\delta}(0,1;0,0,-2,-1,-1) \\
+ & \frac{4}{3}\, V^{221}_{\delta}(0,2;0,0,-2,-1,-1) \\
- & (\frac{1}{6}\, s - m^2)\, V^{221}_{\delta}(1,0;0,0,-2,-1,-1) \\
- & \frac{1}{3}\, m^2\, V^{221}_{\delta}(1,0;0,0,-1,-2,-1) \\
- & \frac{1}{3}\, m^2\, V^{221}_{\delta}(1,0;0,0,-1,-1,-2) \\
+ & \frac{1}{3}\, V^{221}_{\delta}(1,1;0,0,-2,-1,-1)]
\end{aligned}
$$

with a grand total of 103 terms therefore proving a sort of unspoken law

Any algorithm aimed to *reduce* the analytical complexity of a multi - loop Feynman diagrams is generally bound to

– replace the original integral with a sum of many simpler diagrams,

– introducing denominators that show zeros.

An algorithm is optimal when

– there is a minimal number of terms,

– zeros of denominators correspond to solutions of Landau equations and

– the nature of the singularities is not badly over-estimated.

Conclusions: goals of TopSide

GraphShot Two-Loop Two-Point Functions and WST identities $100\,\%$ **completed;**

Numerics† **up to Three-Point Functions**
One-Loop $100\,\%$ **completed,**
Two-Loop IR $90\,\%$ **completed,**
Two-Loop NIR $90\,\%$ **completed;**

Renormalization $50\,\%$ **completed;**

γ^5 **One-Loop,** $100\,\%$ **completed;**

† **We require *at least* two independent algorithms per diagram**