THE INFRARED STRUCTURE OF GAUGE AMPLITUDES IN THE HIGH-ENERGY LIMIT

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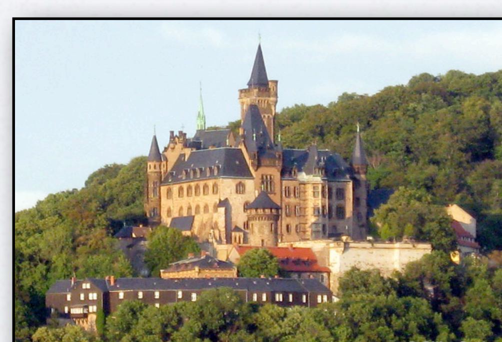
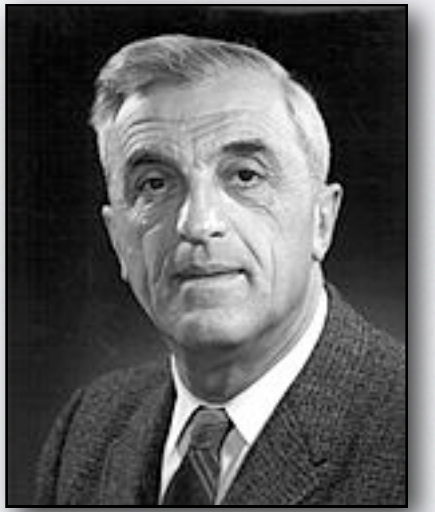
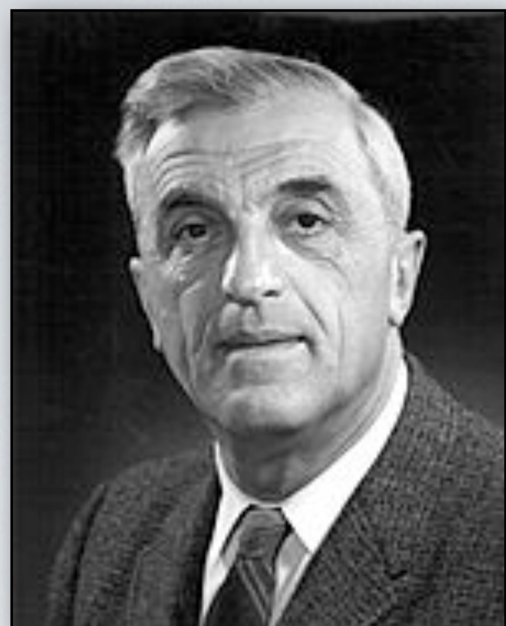


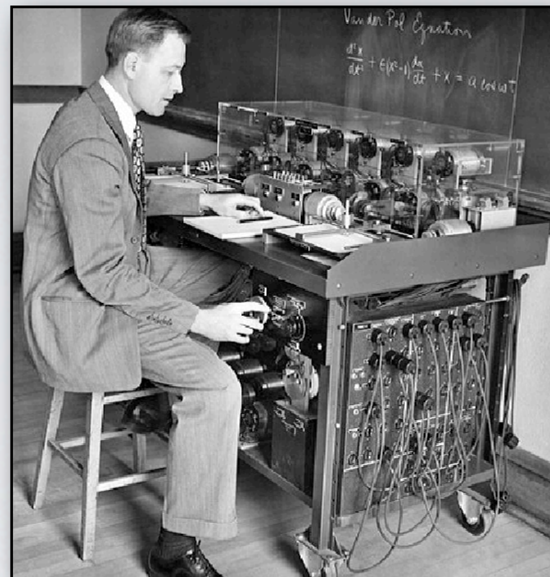
Outline

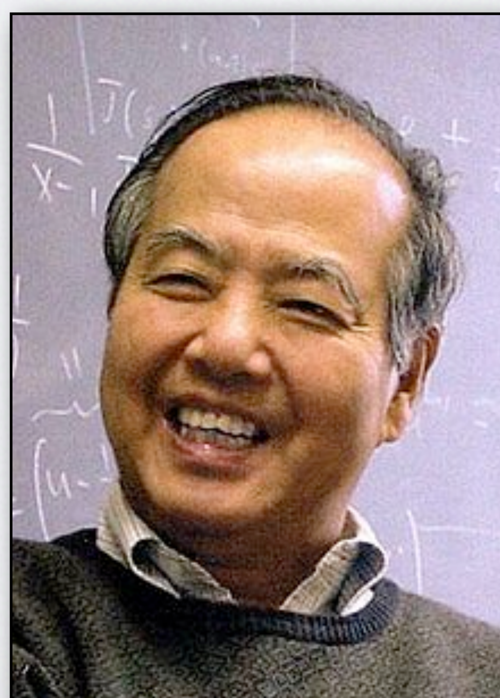
- Soft-collinear factorization
- The dipole formula (with E. Gardi)
- Reggeization and beyond
- Dipoles at high-energy (with V. Del Duca, C. Duhr, E. Gardi, C. White)
- Outlook

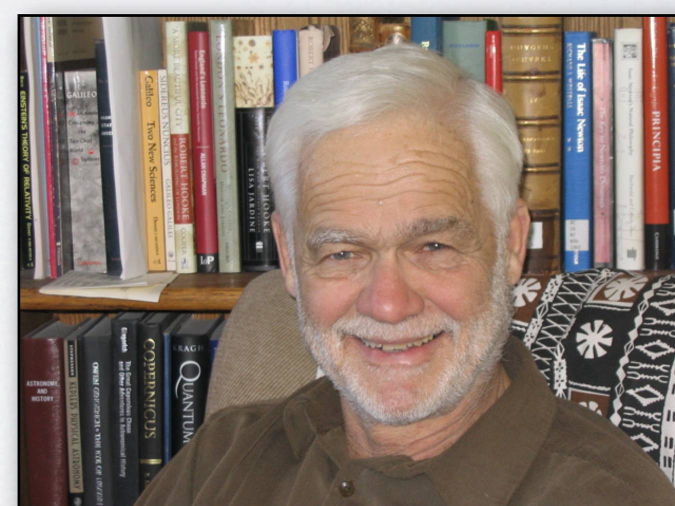
SOFT-COLLINEAR FACTORIZATION






Van der Pol Equation

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GALILEO
Galileo Two New Sciences
SIDEREUS NUNCIUS GALILEO GALILEI
A MORE BEAUTIFUL CITY ROBERT HOOKE
England's Leonardo
LONDON'S LEONARDO
LISA JARDINE
The Life of Isaac Newton
PRINCIPIA
ASTRONOMY AND HISTORY
KEPLER'S PHYSICAL ASTRONOMY
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Brief motivation

- Higher order QCD calculations at colliders hinge upon cancellation of divergences between virtual corrections and real emission contributions.
  - Cancellation must be performed analytically before numerical integrations.
  - State of the art: general NLO, NNLO for processes with color-singlet Born.
  - All-order understanding may yield systematic approach.

- Cancellations leave behind large logarithms: they must be resummed

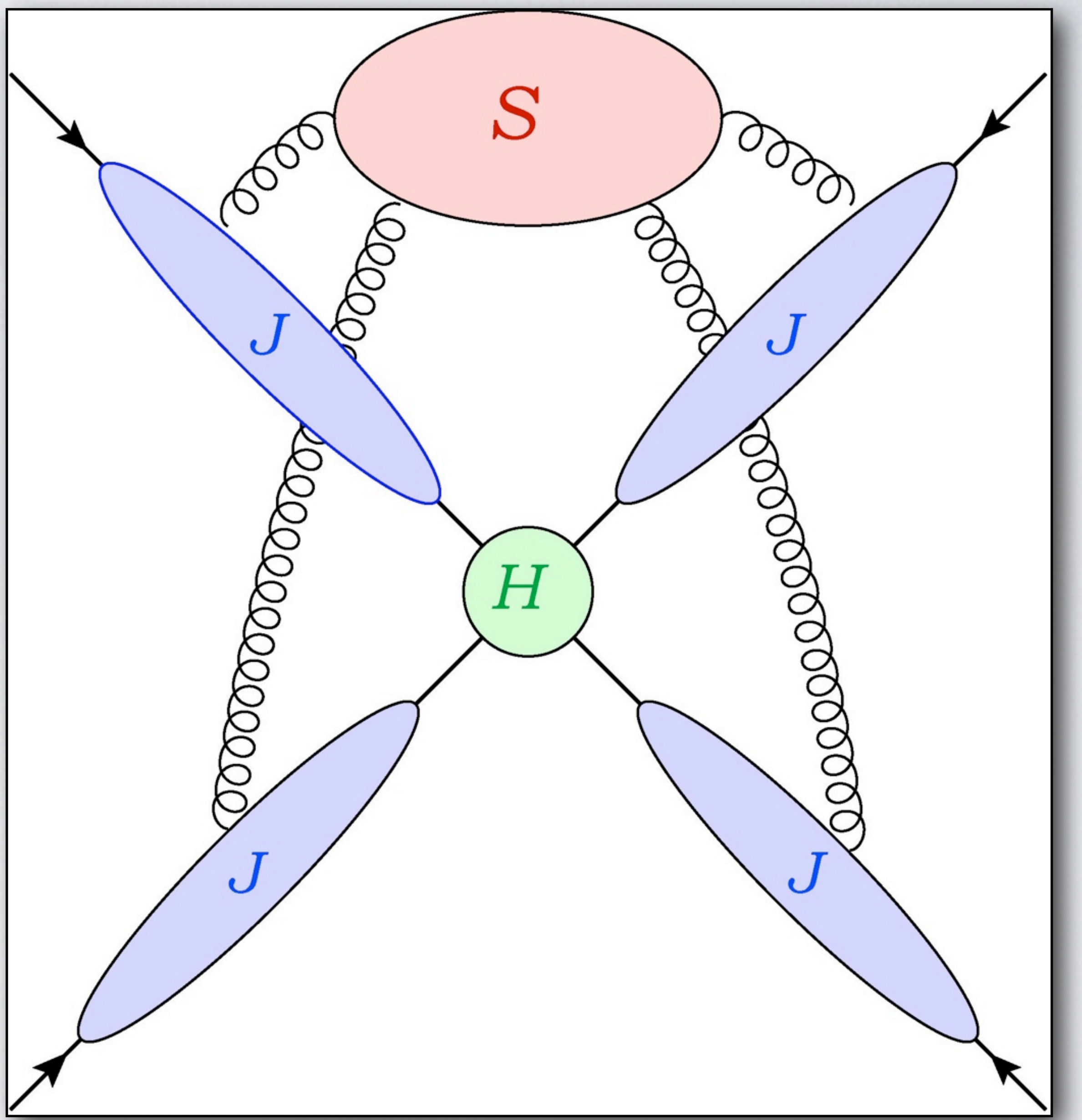
$$\underbrace{\frac{1}{\epsilon}}_{\text{virtual}} \; + \underbrace{(Q^2)^\epsilon \int_0^{m^2} \frac{dk^2}{(k^2)^{1+\epsilon}}}_{\text{real}} \qquad \Longrightarrow \qquad \ln(m^2/Q^2)$$

  - For inclusive observables: analytic resummation to high (N^3LL) logarithmic accuracy.
  - For exclusive final states: parton shower event generators, (N)LL accuracy.

- Resummation probes the all-order structure of perturbation theory.
  - Power-suppressed corrections to QCD cross sections can be studied.
  - Links to the strong coupling regime can be established for SUSY gauge theories.
  - The perturbative structure of conformal gauge theories is IR-dominated.

Soft-collinear factorization

- Divergences arise in scattering amplitudes from leading regions in loop momentum space.
- Power-counting arguments show that soft gluons decouple from the hard subgraph.
- Ward identities decouple soft gluons from jets and restrict color transfer to the hard part.
- Jet functions J represent color singlet evolution of external hard partons.
- The soft function S is a matrix mixing the available color representations.
- In the planar limit soft exchanges are confined to wedges: S is proportional to the identity.
- Beyond the planar limit S is determined by an anomalous dimension matrix Γ_S.
- The matrix Γ_S correlates color exchange with kinematic dependence.

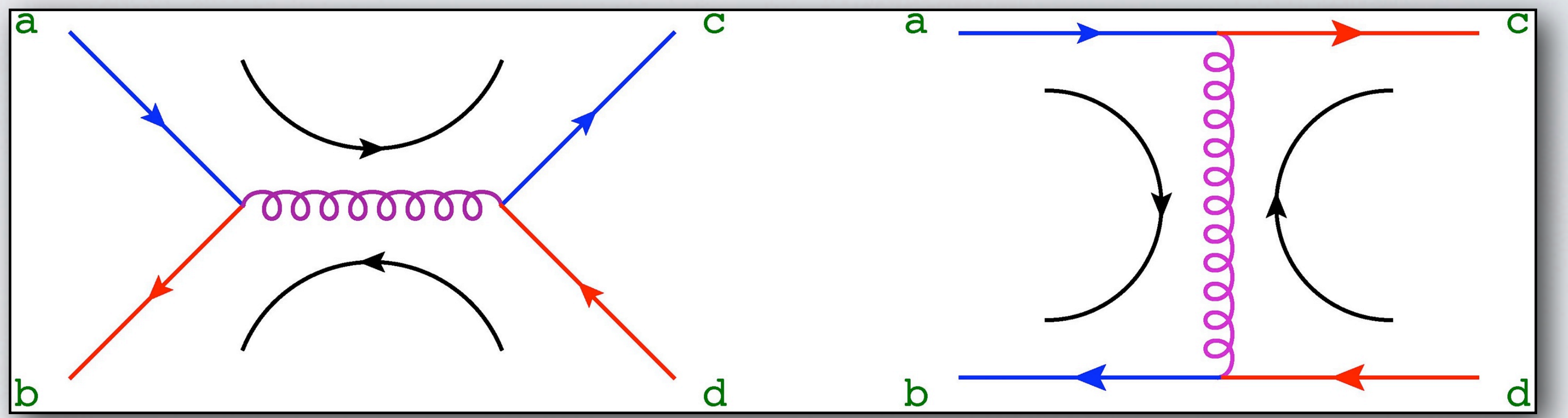


Leading integration regions in loop momentum space for Sudakov factorization

Color flow

In order to understand the matrix structure of the soft function it is sufficient to consider the simple case of quark-antiquark scattering.

At tree level



Tree-level diagrams and color flows for quark-antiquark scattering

For this process only two color structures are possible. A basis in the space of available color tensors is

$$c^{(1)}_{abcd} = \delta_{ab}\delta_{cd}\,, \qquad c^{(2)}_{abcd} = \delta_{ac}\delta_{bd}$$

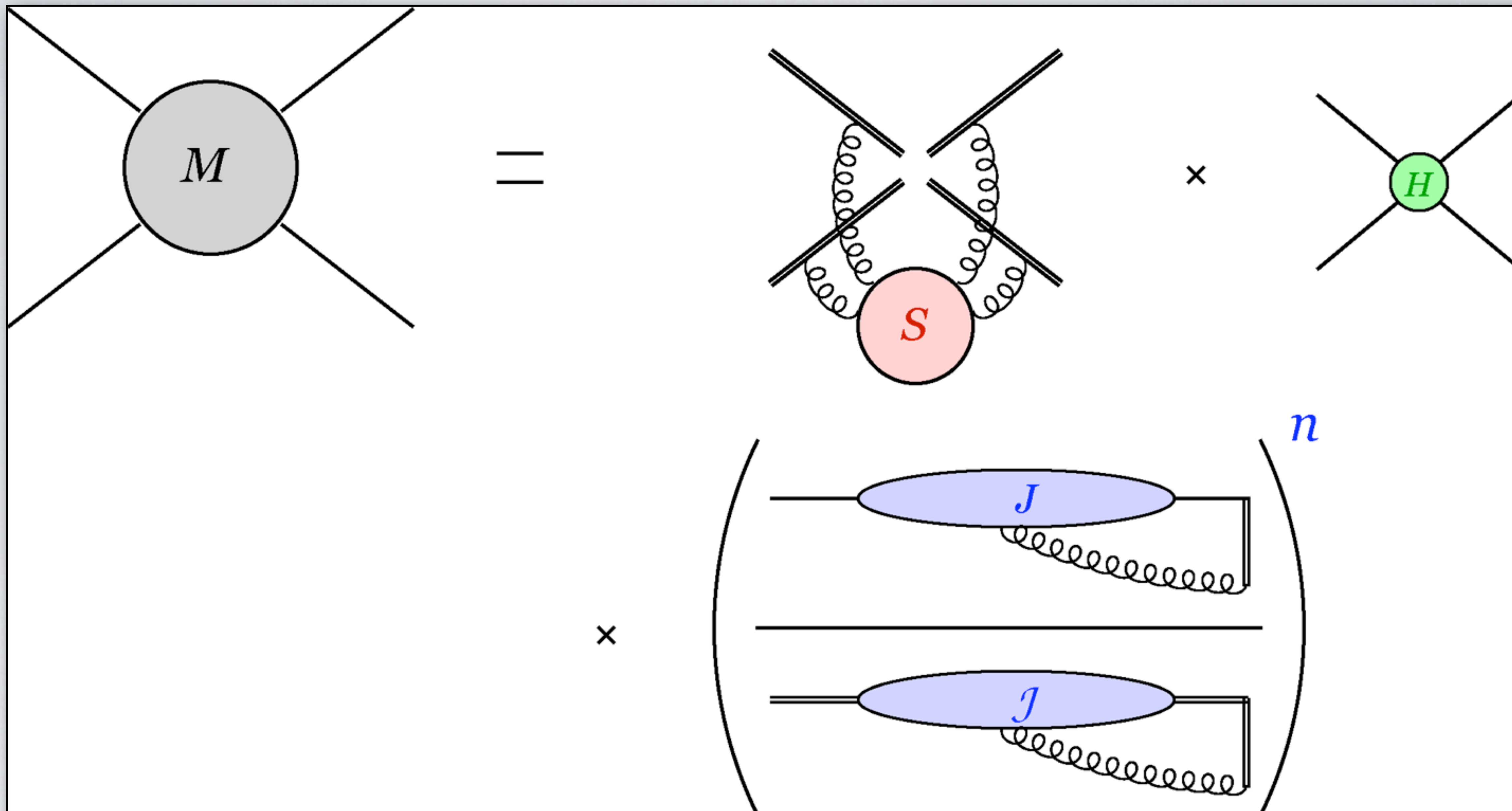
The matrix element is a vector in this space, and the Born cross section is

$$\mathcal{M}_{abcd} = \mathcal{M}_1\, c^{(1)}_{abcd} + \mathcal{M}_2\, c^{(2)}_{abcd} \longrightarrow \sum_{\text{color}} |\mathcal{M}|^2 = \sum_{J,L} \mathcal{M}_J\, \mathcal{M}^*_L \;\text{tr}\left[c^{(J)}_{abcd}\left(c^{(L)}_{abcd}\right)^\dagger\right] \equiv \text{Tr}\Big[HS\Big]_0$$

A virtual soft gluon will reshuffle color and mix the components of this vector

$$\text{QED}: \quad \mathcal{M}_{\text{div}} = S_{\text{div}}\mathcal{M}_{\text{Born}}\,; \qquad \text{QCD}: \quad [\mathcal{M}_{\text{div}}]_J = [S_{\text{div}}]_{JL}\,[\mathcal{M}_{\text{Born}}]_L$$

Soft-collinear factorization: pictorial



A pictorial representation of Sudakov factorization for fixed-angle scattering amplitudes

Operator Definitions

The precise functional form of this graphical factorization is

$$\mathcal{M}_L\left(p_i/\mu, \alpha_s(\mu^2), \epsilon\right) \;=\; \mathcal{S}_{LK}\left(\beta_i\cdot\beta_j, \alpha_s(\mu^2), \epsilon\right)\, H_K\left(\frac{p_i\cdot p_j}{\mu^2}, \frac{(p_i\cdot n_i)^2}{n_i^2\mu^2}, \alpha_s(\mu^2)\right)$$
$$\times\prod_{i=1}^{n}\left[J_i\left(\frac{(p_i\cdot n_i)^2}{n_i^2\mu^2}, \alpha_s(\mu^2), \epsilon\right)\Big/\mathcal{J}_i\left(\frac{(\beta_i\cdot n_i)^2}{n_i^2}, \alpha_s(\mu^2), \epsilon\right)\right]\,,$$

We introduced factorization vectors $n_i^\mu\,,\quad n_i^2\neq 0$ to define the jets,

$$J\left(\frac{(p\cdot n)^2}{n^2\mu^2}, \alpha_s(\mu^2), \epsilon\right) u(p) \;=\; \langle 0\,|\Phi_n(\infty,0)\,\psi(0)\,|p\rangle\,.$$

where Φ_n is the Wilson line operator along the direction n^μ ,

$$\Phi_n(\lambda_2,\lambda_1) = P\exp\left[\mathrm{i}g\int_{\lambda_1}^{\lambda_2} d\lambda\, n\cdot A(\lambda n)\right]\,.$$

The vectors n^μ :

- Ensure gauge invariance of the jets.
- Separate collinear gluons from wide-angle soft ones.
- Replace other hard partons with a collinear-safe absorber.

Wilson line correlators

The soft function S is a matrix, mixing the available color tensors. It is defined by a correlator of Wilson lines.

$$(c_L)_{\{\alpha_k\}}\,\mathcal{S}_{LK}\left(\beta_i\cdot\beta_j,\alpha_s(\mu^2),\epsilon\right) = \sum_{\{\eta_k\}}\langle 0|\prod_{i=1}^{n}\Big[\Phi_{\beta_i}(\infty,0)_{\alpha_k,\eta_k}\Big]|0\rangle\;(c_K)_{\{\eta_k\}}\;,$$

To avoid double counting, soft-collinear regions are subtracted dividing by eikonal jets $\mathcal{J}$.

$$\mathcal{J}\left(\frac{(\beta\cdot n)^2}{n^2},\alpha_s(\mu^2),\epsilon\right) = \langle 0|\Phi_n(\infty,0)\,\Phi_\beta(0,-\infty)\,|0\rangle\;,$$

- Wilson line correlators are pure counterterms in dimensional regularization.
  - Infrared poles are mapped to ultraviolet singularities.
- Their functional dependence on the vectors n^μ_i is restricted by the classical invariance of Wilson lines under velocity rescalings, $n^\mu_i \rightarrow \kappa_i\, n^\mu_i$.
- Rescaling invariance for light-like velocities, $\beta_i^2 = 0$, is broken by quantum corrections.
  - UV counterterms contain collinear poles, corresponding to soft-collinear singularities.
- Double poles are determined by the cusp anomalous dimension $\gamma_K(\alpha_s)$.
  - $\gamma_K(\alpha_s)$ governs the renormalization of Wilson lines with light-like cusps.

Soft anomalous dimensions

The soft function S obeys a matrix RG evolution equation

$$\mu \frac{d}{d\mu} \mathcal{S}_{IK} \left(\beta_i \cdot \beta_j, \alpha_s(\mu^2), \epsilon\right) = - \, \mathcal{S}_{IJ} \left(\beta_i \cdot \beta_j, \alpha_s(\mu^2), \epsilon\right) \, \Gamma^{\mathcal{S}}_{JK} \left(\beta_i \cdot \beta_j, \alpha_s(\mu^2), \epsilon\right)$$

- Γ^S is singular due to overlapping UV and collinear poles.

In dimensional regularization, using $\alpha_s(\mu^2 = 0, \varepsilon < 0) = 0$, one finds

$$\mathcal{S} \left(\beta_i \cdot \beta_j, \alpha_s(\mu^2), \epsilon\right) = \, P \, \exp \left[- \frac{1}{2} \int_0^{\mu^2} \frac{d\xi^2}{\xi^2} \Gamma^{\mathcal{S}} \left(\beta_i \cdot \beta_j, \alpha_s(\xi^2, \epsilon), \epsilon\right) \right] \, .$$

Double poles cancel in the reduced soft function

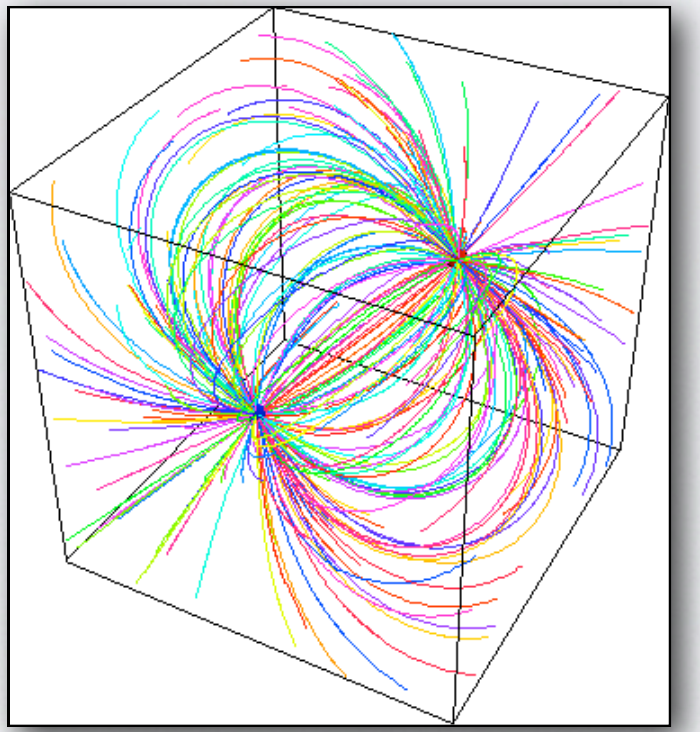
$$\overline{\mathcal{S}}_{LK} \left(\rho_{ij}, \alpha_s(\mu^2), \epsilon\right) = \frac{\mathcal{S}_{LK} \left(\beta_i \cdot \beta_j, \alpha_s(\mu^2), \epsilon\right)}{\prod_{i=1}^{n} \mathcal{J}_i \left(\frac{(\beta_i \cdot n_i)^2}{n_i^2}, \alpha_s(\mu^2), \epsilon \right)}$$

- The matrix $\overline{\mathcal{S}}$ must depend on rescaling invariant variables

$$\rho_{ij} \equiv \frac{n_i^2 \, n_j^2 \, \left(\beta_i \cdot \beta_j\right)^2}{(\beta_i \cdot n_i)^2 (\beta_j \cdot n_j)^2} \, .$$

- The anomalous dimension $\Gamma^{\overline{\mathcal{S}}}(\rho_{ij}, \alpha_s)$ for the evolution of $\overline{\mathcal{S}}$ is finite.

THE DIPOLE FORMULA



The Dipole Formula

For massless partons, the soft anomalous dimension matrix obeys a set of exact equations that correlate color exchange with kinematics.

The simplest solution to these equations is a sum over color dipoles (Becher, Neubert; Gardi, LM, 09). It gives an ansatz for the all-order singularity structure of all multiparton fixed-angle massless scattering amplitudes: the dipole formula.

- All soft and collinear singularities can be collected in a multiplicative operator Z

$$\mathcal{M}\left(\frac{p_i}{\mu}, \alpha_s(\mu^2), \epsilon\right) = Z\left(\frac{p_i}{\mu_f}, \alpha_s(\mu_f^2), \epsilon\right) \, \mathcal{H}\left(\frac{p_i}{\mu}, \frac{\mu_f}{\mu}, \alpha_s(\mu^2), \epsilon\right) \, ,$$

- Z contains both soft singularities from S, and collinear ones from the jet functions. It must satisfy its own matrix RG equation

$$\frac{d}{d \ln \mu} \, Z\left(\frac{p_i}{\mu}, \alpha_s(\mu^2), \epsilon\right) \; = \; - \, Z\left(\frac{p_i}{\mu}, \alpha_s(\mu^2), \epsilon\right) \, \Gamma\left(\frac{p_i}{\mu}, \alpha_s(\mu^2)\right) \, .$$

The matrix Γ inherits the dipole structure from the soft matrix. It reads

$$\boxed{\Gamma_{\rm dip}\left(\frac{p_i}{\mu}, \alpha_s(\mu^2)\right) = -\frac{1}{4} \, \widehat{\gamma}_K \left(\alpha_s(\mu^2)\right) \sum_{j \neq i} \ln\left(\frac{-2 \, p_i \cdot p_j}{\mu^2}\right) \mathbf{T}_i \cdot \mathbf{T}_j \, + \sum_{i=1}^{n} \gamma_{J_i} \left(\alpha_s(\mu^2)\right) \, .}$$

Note that all singularities are generated by integration over the scale of the coupling.

Features of the dipole formula

- All known results for IR divergences of massless gauge theory amplitudes are recovered.
- The absence of multiparton correlations implies remarkable diagrammatic cancellations.
- The color matrix structure is fixed at one loop: path-ordering is not needed.
- All divergences are determined by a handful of anomalous dimensions.
- The cusp anomalous dimension plays a very special role: a universal IR coupling.

Can this be the definitive answer for IR divergences in massless non-abelian gauge theories?

- There are precisely two sources of possible corrections.
  - Quadrupole correlations may enter starting at three loops: they must be tightly constrained functions of conformal cross ratios of parton momenta.

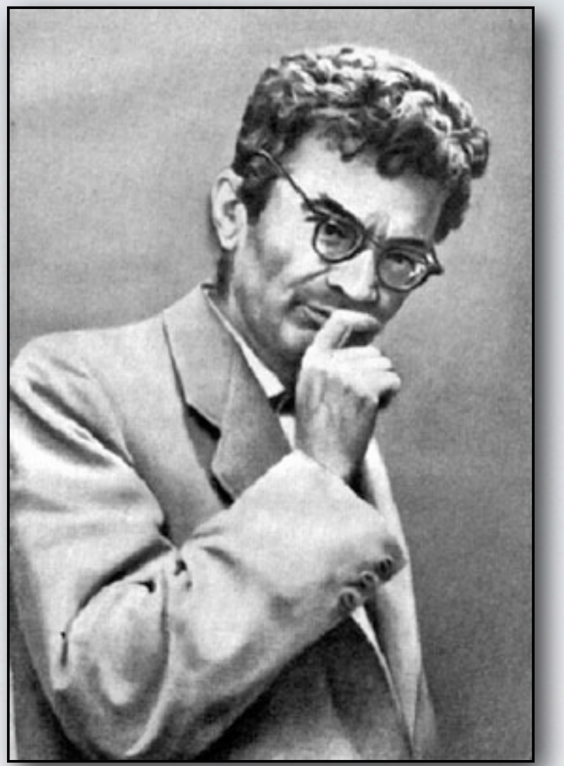
$$\Gamma\left(\frac{p_i}{\mu}, \alpha_s(\mu^2)\right) = \Gamma_{\rm dip}\left(\frac{p_i}{\mu}, \alpha_s(\mu^2)\right) + \Delta\left(\rho_{ijkl}, \alpha_s(\mu^2)\right) \,, \qquad \rho_{ijkl} = \frac{p_i \cdot p_j \, p_k \cdot p_l}{p_i \cdot p_k \, p_j \cdot p_l}$$

  - The cusp anomalous dimension may violate Casimir scaling beyond three loops.

$$\gamma_K^{(i)}(\alpha_s) = C_i \, \widehat{\gamma}_K(\alpha_s) + \widetilde{\gamma}_K^{(i)}(\alpha_s)$$

  - The functional form of Δ is further constrained by: collinear limits, Bose symmetry and transcendentality bounds (Becher, Neubert; Dixon, Gardi, LM, 09).
  - A four-loop analysis indicates that Casimir scaling holds (Becher, Neubert, Vernazza).

REGGEIZATION AND BEYOND

Regge Poles

- Studies of the high-energy limit of scattering amplitudes predate the construction of the Standard Model of particle physics.
- A powerful tool in S-matrix theory is the analytic continuation to complex angular momentum. Start with the well known partial wave expansion

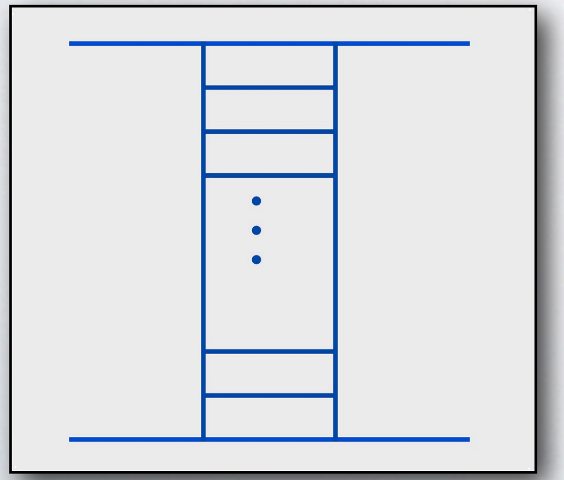
$$A(s,t) = 16\,\pi \sum_{l=0}^{\infty} (2l+1)\; a_l(s)\, P_l(\cos\theta_t)$$

- Moving to the crossed (t-) channel, using dispersion relations and overcoming several technical subtleties one finds a representation for the t-channel partial wave amplitude

$$a_l^{\mathcal{S}}(t) \;=\; \frac{1}{16\pi^2} \int_{\cos\theta_s^0}^{\infty} D^{\mathcal{S}}\left(\cos_{\theta_s}, t\right)\; Q_l(\cos\theta_s)\, d\cos\theta_s$$

- Singularities of $a_l(t)$ in the L plane determine the high-energy behavior of the amplitude: In the case of simple poles one gets

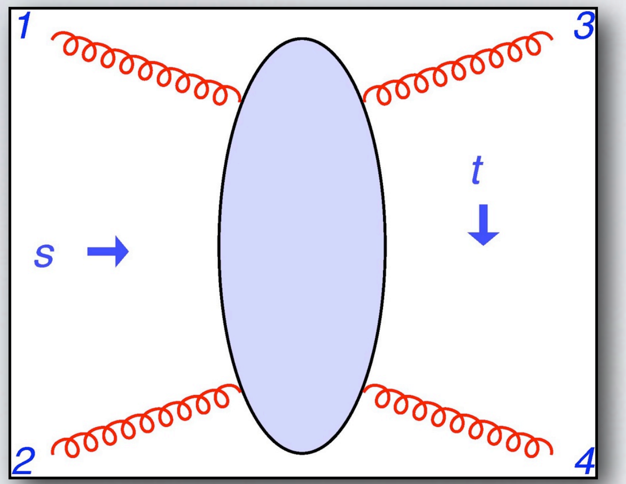
$$a_l^{\mathcal{S}}(t) \sim \frac{1}{l - \alpha(t)} \qquad \longrightarrow \qquad A\,(s,t) \xrightarrow{s\to\infty} f(t)\, s^{\alpha(t)}\,,$$



- The above is derived from the analiticity of the S-matrix, with no reference to a Lagrangian field theory.
- In perturbation theory, the same high-energy behavior is recovered through the summation of ladder diagrams.
- The Regge trajectory $\alpha(t)$ is computed from the one-loop diagram at vanishing longitudinal momentum.

Perturbative Reggeization

- In perturbative QCD the high-energy limit is governed by t-channel parton exchange.
- In the $t/s \to 0$ limit gluons in the t-channel `Reggeize' with a computable trajectory.



Gluon-gluon scattering: the t-channel gluon Reggeizes

- Large logarithms of s/t are generated by a simple replacement of the t-channel propagator,

$$\frac{1}{t} \longrightarrow \frac{1}{t}\left(\frac{s}{-t}\right)^{\alpha(t)}$$

- The Regge trajectory has a perturbative expansion, with IR divergent coefficients

$$\alpha(t) = \frac{\alpha_s(-t,\epsilon)}{4\pi}\,\alpha^{(1)} + \left(\frac{\alpha_s(-t,\epsilon)}{4\pi}\right)^2 \alpha^{(2)} + \mathcal{O}\left(\alpha_s^3\right)$$

- The gluon has been shown to Reggeize at NLL, and the two-loop Regge trajectory is known.
- For example, for gluon-gluon scattering the matrix element obeys Regge factorization

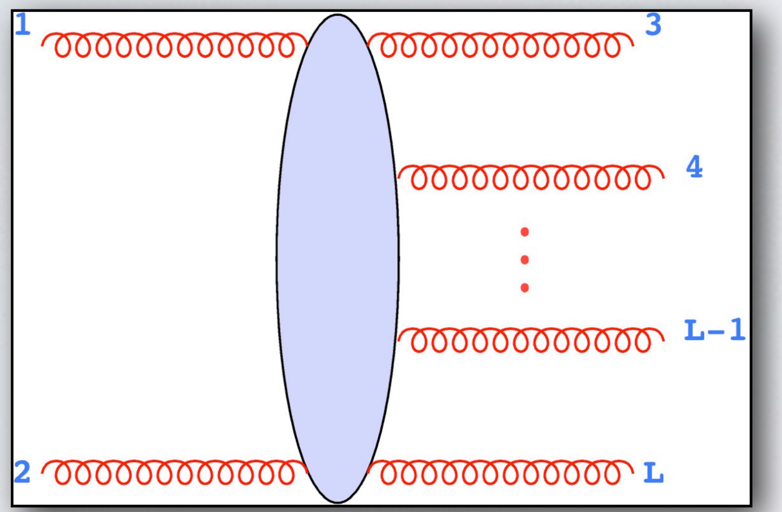
$$\mathcal{M}^{gg\to gg}_{a_1a_2a_3a_4}(s,t) = 2\,g_s^2\,\frac{s}{t}\left[(T^b)_{a_1a_3}C_{\lambda_1\lambda_3}(k_1,k_3)\right]\left(\frac{s}{-t}\right)^{\alpha(t)}\left[(T_b)_{a_2a_4}C_{\lambda_2\lambda_4}(k_2,k_4)\right]$$

with the perturbative coefficients

$$\alpha^{(1)} = C_A\,\frac{\widehat{\gamma}_K^{(1)}}{\epsilon} = C_A\,\frac{2}{\epsilon} \qquad \alpha^{(2)} = C_A\left[-\frac{b_0}{\epsilon^2} + \widehat{\gamma}_K^{(2)}\,\frac{2}{\epsilon} + C_A\left(\frac{404}{27} - 2\zeta_3\right) + n_f\left(-\frac{56}{27}\right)\right]$$

Multi-Regge kinematics

- Reggeization follows form the dominance of t-channel ladder diagrams as t/s → 0.
- By unitarity, multi-gluon emission must similarly simplify in the high-energy limit.
- Regge factorization extends to multi-particle emission in `Multi-Regge' kinematics.



Multi-gluon emission and Multi-Regge kinematics

$$y_3 \gg y_4 \gg \ldots \gg y_L \,, \qquad |k_i^\perp| \simeq |k_j^\perp| \,, \qquad \forall i, j$$

$$-s \;\equiv\; -s_{12} \;\simeq\; |k_3^\perp|\,|k_L^\perp|\, \mathrm{e}^{y_3 - y_L}\, \mathrm{e}^{\mathrm{i}\pi}$$

$$-s_{ij} \;\simeq\; |k_i^\perp|\,|k_j^\perp|\, \mathrm{e}^{y_i - y_j}\, \mathrm{e}^{\mathrm{i}\pi} \,, \quad 3 \le i < j \le L$$

- Large logarithms of s/t_i are generated by the Reggeization of t-channel propagators, as

$$\begin{aligned} \mathcal{M}^{gg \to (L-2)g}_{a_1 \ldots a_L} \;=\;& 2\, g_s^3\, s \left[(T^b)_{a_1 a_3}\, C_{\lambda_1 \lambda_3}(k_1, k_3) \right] \left[\frac{1}{t_1} \left(\frac{s_{34}}{-t_1} \right)^{\alpha(t_1)} \right] \\ & \times \left[(T^{a_4})_{bc}\, V_{\lambda_4}(q_1, q_2) \right] \left[\frac{1}{t_2} \left(\frac{s_{45}}{-t_2} \right)^{\alpha(t_2)} \right] \ldots \left[(T^c)_{a_2 a_L}\, C_{\lambda_2 \lambda_L}(k_2, k_L) \right] \end{aligned}$$

- The impact factors C and the Lipatov vertices V are universal and independent of s.

The dipole formula at high energy

- Introducing `Mandelstam' color operators, and using color and momentum conservation

$$\begin{aligned} \mathbf{T}_s &= \mathbf{T}_1 + \mathbf{T}_2 = -(\mathbf{T}_3 + \mathbf{T}_4)\,, \\ \mathbf{T}_t &= \mathbf{T}_1 + \mathbf{T}_3 = -(\mathbf{T}_2 + \mathbf{T}_4)\,, \\ \mathbf{T}_u &= \mathbf{T}_1 + \mathbf{T}_4 = -(\mathbf{T}_2 + \mathbf{T}_3) \end{aligned}$$

$$s + t + u = 0$$

$$\mathbf{T}_s^2 + \mathbf{T}_t^2 + \mathbf{T}_u^2 = \sum_{i=1}^{4} C_i$$

it is easy to see that the infrared dipole operator Z factorizes in the high-energy limit

$$Z\left(\frac{p_i}{\mu}, \alpha_s(\mu^2), \epsilon\right) = \widetilde{Z}\left(\frac{s}{t}, \alpha_s(\mu^2), \epsilon\right) Z_{\mathbf{1}}\left(\frac{t}{\mu^2}, \alpha_s(\mu^2), \epsilon\right)$$

- The operator Z_1 is s-independent and proportional to the unit matrix in color space.
- Color dependence and s dependence are collected in the factor

$$\widetilde{Z}\left(\frac{s}{t}, \alpha_s(\mu^2), \epsilon\right) = \exp\left\{K\Big(\alpha_s(\mu^2), \epsilon\Big)\left[\ln\left(\frac{s}{-t}\right)\mathbf{T}_t^2 + i\pi\,\mathbf{T}_s^2\right]\right\},$$

where the coupling dependence is (once again!) completely determined by the cusp anomalous dimension and by the β function, through the function (Korchemsky 94-96)

$$K\Big(\alpha_s(\mu^2), \epsilon\Big) \equiv -\frac{1}{4}\int_0^{\mu^2} \frac{d\lambda^2}{\lambda^2}\, \widehat{\gamma}_K\left(\alpha_s(\lambda^2, \epsilon)\right)$$

- The simple structure of the high-energy operator governs Reggeization and its breaking.

Reggeization of leading logarithms

- At leading logarithmic accuracy, the (imaginary) s-channel contribution can be dropped, and the dipole operator becomes diagonal in a t-channel basis.

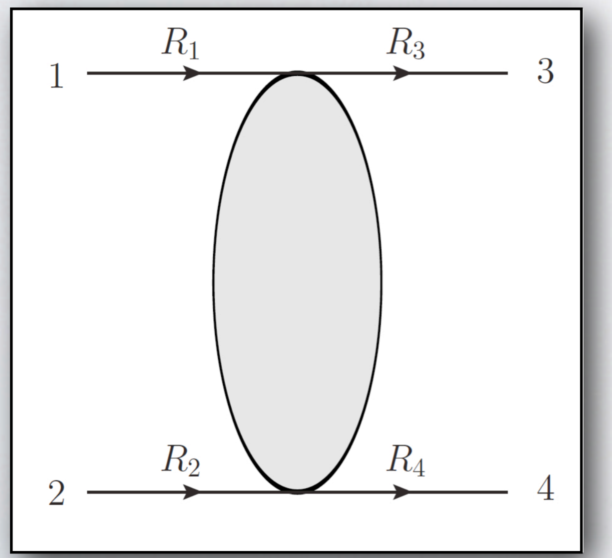
$$\mathcal{M}\left(\frac{p_i}{\mu}, \alpha_s(\mu^2), \epsilon\right) = \exp\left\{K\big(\alpha_s(\mu^2), \epsilon\big) \ln\left(\frac{s}{-t}\right) \mathbf{T}_t^2\right\} Z_\mathbf{1}\, \mathcal{H}\left(\frac{p_i}{\mu}, \alpha_s(\mu^2), \epsilon\right)$$

- If, at LO and at leading power in t/s, the scattering is dominated by t-channel exchange, then the hard function is an eigenstate of the color operator $\mathbf{T}_t^2$

$$\mathbf{T}_t^2\, \mathcal{H}^{gg\to gg} \xrightarrow{|t/s|\to 0} C_t\, \mathcal{H}_t^{gg\to gg}$$

- Leading-logarithmic Reggeization for arbitrary t-channel color representations follows

$$\mathcal{M}^{gg\to gg} = \left(\frac{s}{-t}\right)^{C_A\, K\left(\alpha_s(\mu^2),\epsilon\right)} Z_\mathbf{1}\, \mathcal{H}_t^{gg\to gg}$$



Scattering for generic color exchange

- The LL Regge trajectory is universal and obeys Casimir scaling.
- Scattering of arbitrary color representations can be analyzed
  Example: let 1 and 2 be antiquarks, 4 a gluon and 3 a sextet; use

$$\overline{\mathbf{3}} \otimes \mathbf{6} = \mathbf{3} \oplus \mathbf{15} \qquad\qquad \overline{\mathbf{3}} \otimes \mathbf{8}_a = \overline{\mathbf{3}} \oplus \mathbf{6} \oplus \overline{\mathbf{15}}$$

LL Reggeization of the $\mathbf{3}$ and $\mathbf{15}$ t-channel exchanges follows.

Beyond leading logarithms

- The high-energy infrared operator can be systematically expanded beyond LL, using the Baker-Campbell-Hausdorff formula. At NLL one finds a series of commutators

$$\widetilde{Z}\left(\frac{s}{t},\alpha_s,\epsilon\right)\Big|_{\rm NLL} = \left(\frac{s}{-t}\right)^{K(\alpha_s,\epsilon)\,\mathbf{T}_t^2}\Bigg\{1+{\rm i}\,\pi K\left(\alpha_s,\epsilon\right)\Bigg[\mathbf{T}_s^2-\frac{K\left(\alpha_s,\epsilon\right)}{2!}\ln\left(\frac{s}{-t}\right)\left[\mathbf{T}_t^2,\mathbf{T}_s^2\right] + \frac{K^2\left(\alpha_s,\epsilon\right)}{3!}\ln^2\left(\frac{s}{-t}\right)\left[\mathbf{T}_t^2,\left[\mathbf{T}_t^2,\mathbf{T}_s^2\right]\right]+\ldots\Bigg]\Bigg\}$$

- The real part of the amplitude Reggeizes also at NLL for arbitrary t-channel exchanges.
- At NNLL Reggeization generically breaks down also for the real part of the amplitude.
  - At two loops, terms that are non-logarithmic and non-diagonal in a t-channel basis arise

$$\mathcal{E}_0\left(\alpha_s,\epsilon\right) \equiv -\frac{1}{2}\pi^2\,K^2\left(\alpha_s,\epsilon\right)\,\left(\mathbf{T}_s^2\right)^2$$

  - At three loops, the first Reggeization-breaking logarithms of s/t arise, generated by

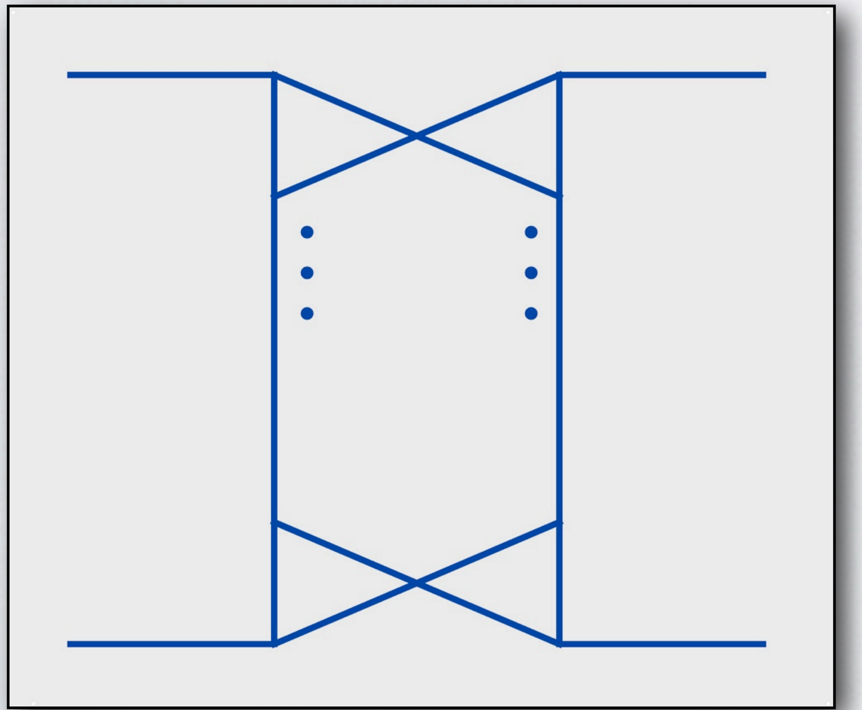
$$\mathcal{E}_1\left(\frac{s}{t},\alpha_s,\epsilon\right) \equiv -\frac{\pi^2}{3}\,K^3(\alpha_s,\epsilon)\,\ln\left(\frac{s}{-t}\right)\left[\mathbf{T}_s^2,\left[\mathbf{T}_t^2,\mathbf{T}_s^2\right]\right]$$

- **NOTE**
  - In the planar limit ($N_C \to \infty$) all commutators vanish and Reggeization holds also beyond NLL (as perhaps expected from string theory).
  - Possible quadrupole corrections to the dipole formula cannot come to the rescue.

Regge Cuts

- One may wonder how the breakdown of simple Regge factorization can be put in the context of the general results of Regge theory.
- Reggeization follows from the assumption that the only singularities in the complex angular momentum plane are isolated poles.
- From the early days of Regge theory it was understood that the the picture would become more intricate in the presence of cuts in the L plane
- Regge cuts can arise when at least two `Reggeons' are exchanged in the t channel (two ladders in perturbation theory)

On general grounds one can show that:



Mandelstam's `double-cross' diagram

- Regge cuts do not arise in the physical region from planar diagrams.
- The first nontrivial contribution from a Regge cut arises from the three-loop non-planar Mandelstam `double-cross' diagram.
- Regge cuts in the physical region arise at leading power in s only if the high energy limit picks up the discontinuity of an energy logarithm.

These properties are in agreement with our findings at three loops and beyond.

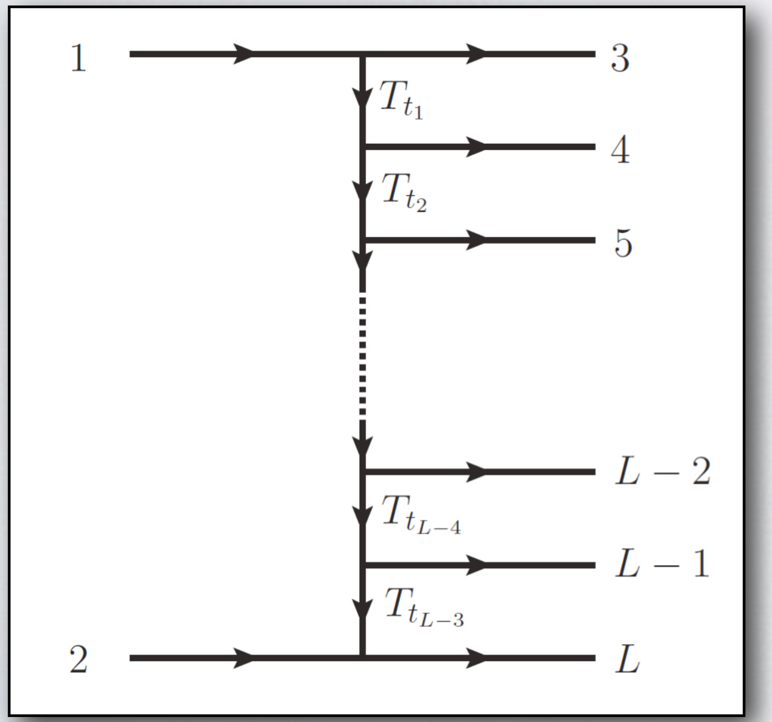
Multi-Regge kinematics

- The dipole formula applies for any number of particles: we expect similar simplifications in Multi-Regge kinematics, and similar results concerning Reggeization.
- Indeed, one can prove recursively that the dipole operator Z factorizes in MR kinematics, as

$$Z\left(\frac{p_l}{\mu}, \alpha_s(\mu^2), \epsilon\right) = \widetilde{Z}^{\rm MR}\Big(\Delta y_k, \alpha_s(\mu^2), \epsilon\Big) \; Z_{\mathbf{1}}^{\rm MR}\left(\frac{|k_i^\perp|}{\mu}, \alpha_s(\mu^2), \epsilon\right)$$

- The Multi-Regge high-energy operator has again a simple structure.

$$\widetilde{Z}^{\rm MR}\Big(\Delta y_k, \alpha_s(\mu^2), \epsilon\Big) = \exp\left\{K\Big(\alpha_s(\mu^2), \epsilon\Big)\left[\sum_{k=3}^{L-1} \mathbf{T}^2_{t_{k-2}} \, \Delta y_k + {\rm i}\pi \mathbf{T}_s^2\right]\right\}$$



Color structure in Multi-Regge kinematics

- We have defined the t-channel color operators

$$\mathbf{T}_{t_k} = \mathbf{T}_1 + \sum_{p=1}^{k} \mathbf{T}_{p+2}$$

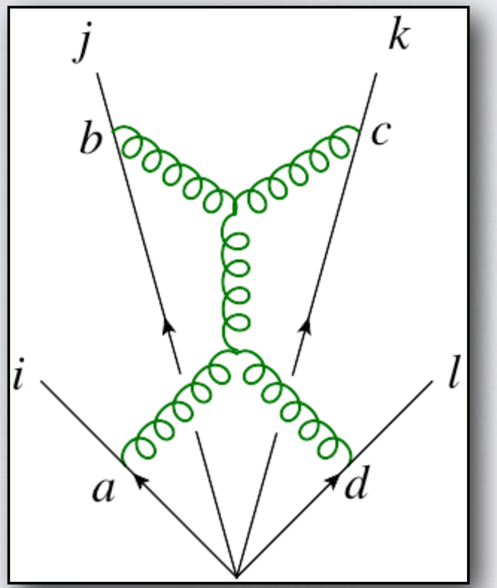
- A t-channel basis of common eigenstates of $\mathbf{T}_{t_k}$ can be constructed using Clebsch-Gordan coefficients.
- The operators $\mathbf{T}_{t_k}$ thus commute, and each color representation contributing to the hard function in the high-energy limit Reggeizes separately at LL.

Constraining quadrupoles

- Known results on the high-energy limit of QCD amplitudes imply new constraints on quadrupole corrections to the dipole formula at three loops and beyond.
- Previous analyses using collinear constraint, Bose symmetry and transcendentality bounds could not exclude a class of correction, including for example

$$\Delta^{(212)}(\rho_{ijkl}, \alpha_s) \, = \, \left(\frac{\alpha_s}{\pi}\right)^3 \mathbf{T}_1^a \mathbf{T}_2^b \mathbf{T}_3^c \mathbf{T}_4^d \left[f^{ade} f^{cbe} \, L_{1234}^2 \left(L_{1423} \, L_{1342}^2 \, + \, L_{1423}^2 \, L_{1342} \right) \, + \, \text{cycl.} \right]$$

where $L_{ijkl} \, \equiv \, \log(\rho_{ijkl})$.



A three-loop diagram for Δ

- In the high-energy limit one finds (with L = log|s/t|)

$$\rho_{1234} \equiv \frac{(-s_{12})(-s_{34})}{(-s_{13})(-s_{24})} = \left(\frac{s}{-t}\right)^2 \mathrm{e}^{-2\mathrm{i}\pi}\,; \qquad L_{1234} = 2(L - \mathrm{i}\pi)\,;$$

$$\rho_{1342} \equiv \frac{(-s_{13})(-s_{24})}{(-s_{14})(-s_{23})} = \left(\frac{-t}{s+t}\right)^2 ; \qquad L_{1342} \simeq -2L\,;$$

$$\rho_{1423} \equiv \frac{(-s_{14})(-s_{23})}{(-s_{12})(-s_{34})} = \left(\frac{s+t}{s}\right)^2 \mathrm{e}^{2\mathrm{i}\pi}\,; \qquad L_{1423} \simeq 2\mathrm{i}\pi\,,$$

- Previously admissible corrections display superleading high-energy logarithms at three loops.

$$\Delta^{(212)}(\rho_{ijkl}, \alpha_s)) \, = \, \left(\frac{\alpha_s}{\pi}\right)^3 \mathbf{T}_1^a \mathbf{T}_2^b \mathbf{T}_3^c \mathbf{T}_4^d \; 32\,\mathrm{i}\,\pi \Big[\left(-L^4 - \mathrm{i}\pi L^3 - \pi^2 L^2 - \mathrm{i}\pi^3 L\right) f^{ade} f^{cbe} \, + \, \ldots \Big]$$

- No known explicit example of admissible quadrupole correction survives. A complete proof is still lacking: linear combinations might restore the proper Regge behavior.

OUTLOOK

Summary

- A definitive solution of the problem of infrared divergences of (massless) gauge theory amplitudes may be at hand.
  - ✓ We are probing the all-order structure of the nonabelian exponent.
  - ✓ All-order results constrain, test and complement fixed-order calculations.
  - ✓ Understanding singularities has phenomenological applications through resummation.
- Factorization theorems determine the all-order structures through evolution equations
- A simple dipole formula may encode all infrared singularites for any massless gauge theory, a natural generalization of the planar limit.
- The study of possible corrections to the dipole formula is under way.
- The high-energy limit of the dipole formula provides insights into Reggeization and beyond, at least for divergent contributions to the amplitude.
- Leading logarithmic Reggeization is proved for generic color representations exchanged in the t channel, and for any number of partons in Multi-Regge kinematics.
- Regge factorization generically breaks down at NNLL, with computable corrections which may be related to Regge cuts in the angular momentum plane.
- The high-energy limit further constrains quadrupole corrections to the dipole formula: no known examples survive.

Claude

Einan

Chris

Congratulations Vittorio!

THANK YOU!